Analytic Bootstrap for Holographic Correlators

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Review Talk at Bootstrap, Localization and Holography

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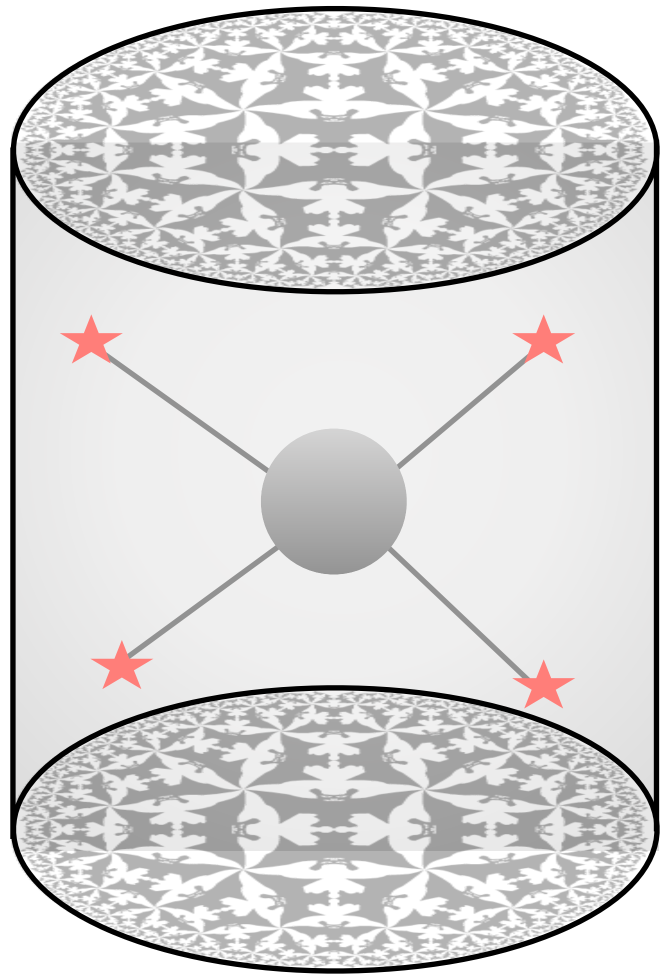
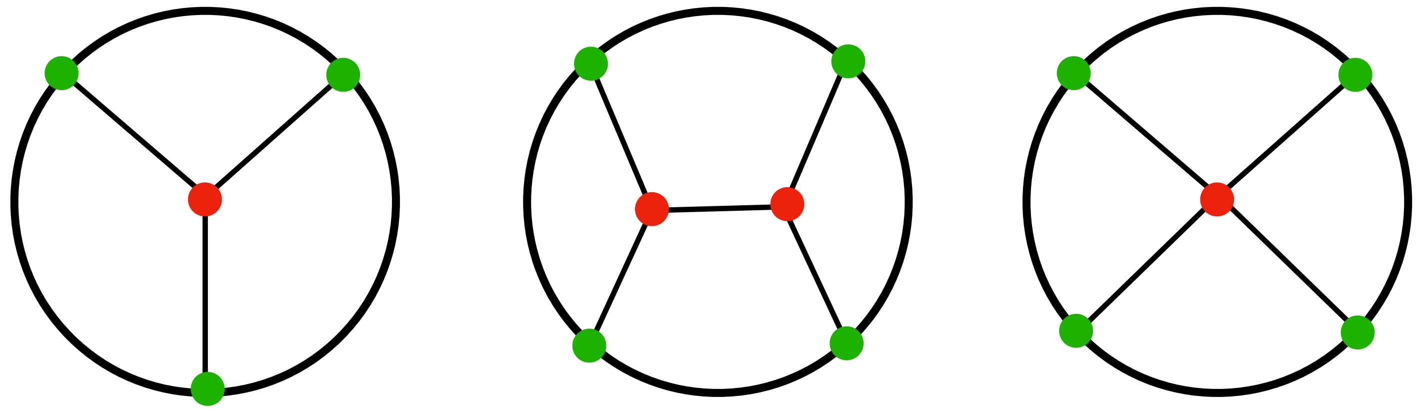
Motivations



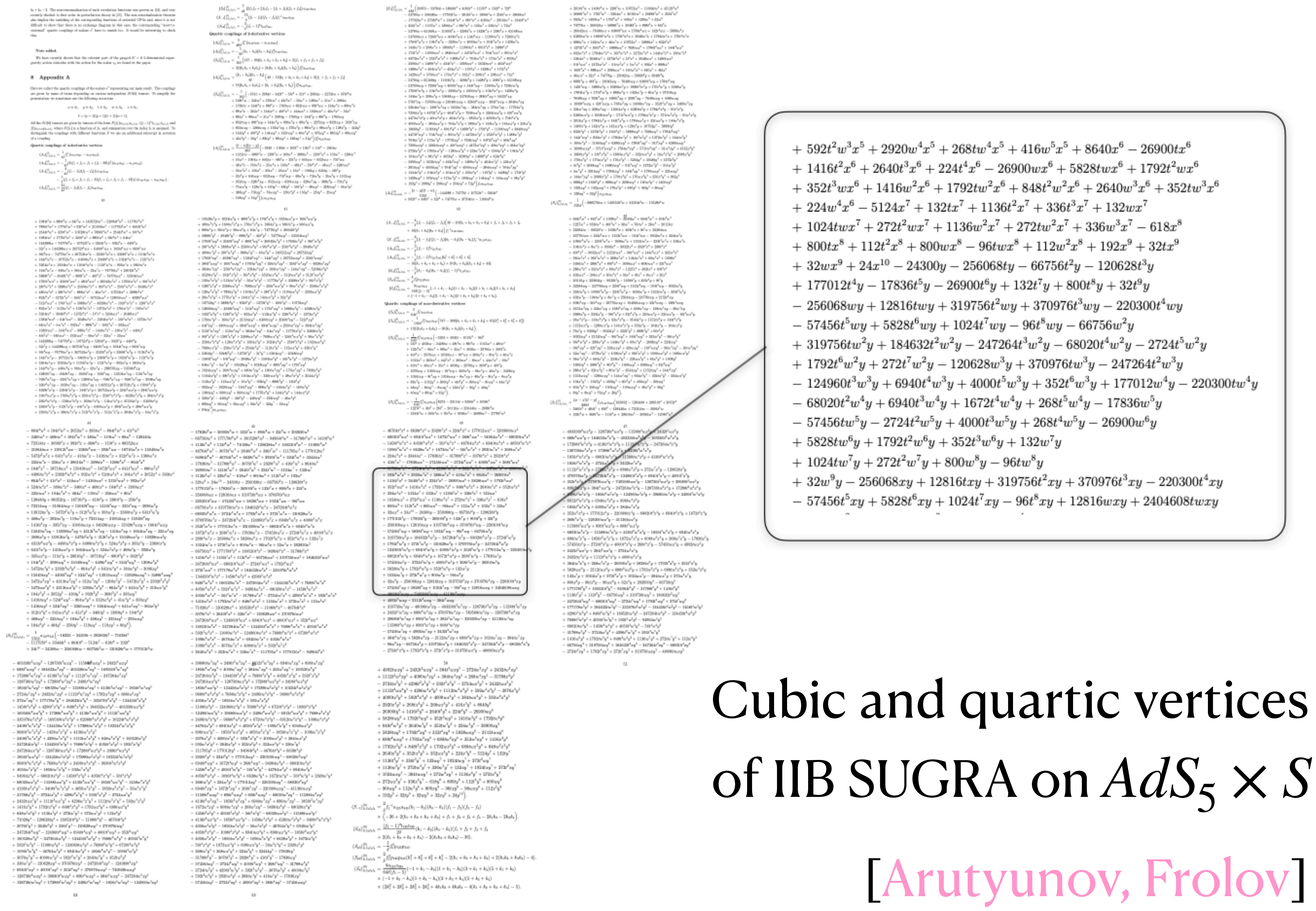
- Holographic correlators are the most basic observables of AdS/CFT
  - Boundary: encode all the CFT data.
  - Bulk: correspond to on-shell scattering amplitudes in curved backgrounds.
  - Important objects for analytically studying strong coupling and allow us to perform precision tests of AdS/CFT.

- The study of holographic correlators has a long history and dates back to the beginning of AdS/CFT
  - Witten diagrams ('98) [Witten]
  - 3pt functions ('98) [Freedman, Mathur, Matusis, Rastelli; Lee, Minwalla, Rangamani,Seiberg…]
  - First 4pt function for axion-dilatons ('99) [D'Hoker, Freedman, Mathur, Matusis, Rastelli]



Motivations

- However, going beyond turned out to be very difficult. The diagrammatic calculation of holographic correlators is notoriously complicated even at leading order in $1/N$ and for 4 pts
  - Curved backgrounds
  - Infinitely many particles from KK reduction
  - Very complicated vertices (15 pages!)
- Only a handful of explicit results after decades of hard work [Arutyunov, Frolov, Dolan, Osborn Sokatchev, Berdichevsky Naaijkens, Uruchurtu...]



$$\begin{aligned}
&+ 592t^2w^3x^5 + 2920w^4x^5 + 268tw^4x^5 + 416w^5x^5 + 8640x^6 - 26900tx^6\\
&+ 1416t^2x^6 + 2640t^3x^6 + 224t^4x^6 - 26900wx^6 + 5828twx^6 + 1792t^2wx^6\\
&+ 352t^3wx^6 + 1416w^2x^6 + 1792tw^2x^6 + 848t^2w^2x^6 + 2640w^3x^6 + 352tw^3x^6\\
&+ 224w^4x^6 - 5124x^7 + 132tx^7 + 1136t^2x^7 + 336t^3x^7 + 132wx^7\\
&+ 1024twx^7 + 272t^2wx^7 + 1136w^2x^7 + 272tw^2x^7 + 336w^3x^7 - 618x^8\\
&+ 800tx^8 + 112t^2x^8 + 800wx^8 - 96twx^8 + 112w^2x^8 + 192x^9 + 32tx^9\\
&+ 32wx^9 + 24x^{10} - 24300y - 256068ty - 66756t^2y - 120628t^3y\\
&+ 177012t^4y - 17836t^5y - 26900t^6y + 132t^7y + 800t^8y + 32t^9y\\
&- 256068wy + 12816twy + 319756t^2wy + 370976t^3wy - 220300t^4wy\\
&- 57456t^5wy + 5828t^6wy + 1024t^7wy - 96t^8wy - 66756w^2y\\
&+ 319756tw^2y + 184632t^2w^2y - 247264t^3w^2y - 68020t^4w^2y - 2724t^5w^2y\\
&+ 1792t^6w^2y + 272t^7w^2y - 120628w^3y + 370976tw^3y - 247264t^2w^3y\\
&- 124960t^3w^3y + 6940t^4w^3y + 4000t^5w^3y + 352t^6w^3y + 177012w^4y - 220300tw^4y\\
&- 68020t^2w^4y + 6940t^3w^4y + 1672t^4w^4y + 268t^5w^4y - 17836w^5y\\
&- 57456tw^5y - 2724t^2w^5y + 4000t^3w^5y + 268t^4w^5y - 26900w^6y\\
&+ 5828tw^6y + 1792t^2w^6y + 352t^3w^6y + 132w^7y\\
&+ 1024tw^7y + 272t^2w^7y + 800w^8y - 96tw^8y\\
&+ 32w^9y - 256068xy + 12816txy + 319756t^2xy + 370976t^3xy - 220300t^4xy\\
&- 57456t^5xy + 5828t^6xy + 1024t^7xy - 96t^8xy + 12816wxy + 2404608twxy
\end{aligned}$$

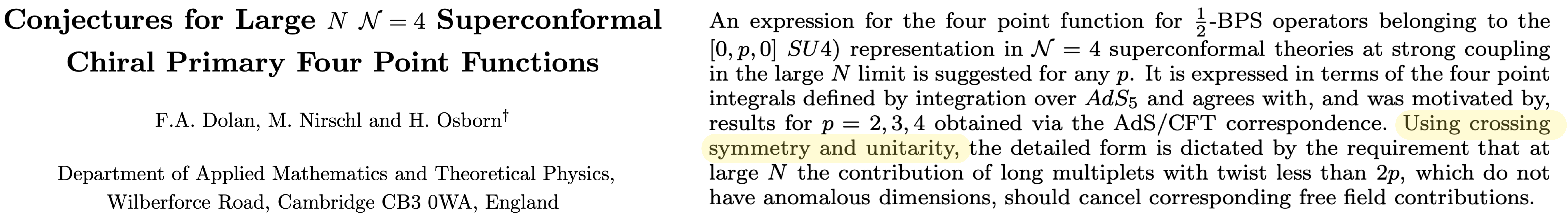
Cubic and quartic vertices of IIB SUGRA on $AdS_5 \times S^5$

[Arutyunov, Frolov]

- What are the principles that determine these vertices? Is there any underlying simplicity in holographic correlators?

Motivations

- Now we know that a more efficient approach is bootstrap
  - Inspired by both the conformal bootstrap and the scattering amplitude program in flat space
  - The idea: focus not on of-shell vertices but directly on "on-shell" correlators and compute them by imposing symmetries and consistency conditions.
  - An early version of these ideas appeared in a paper from 2006 [Dolan, Nirschl, Osborn].



Conjectures for Large N $\mathcal{N}=4$ Superconformal Chiral Primary Four Point Functions

F.A. Dolan, M. Nirschl and H. Osborn†

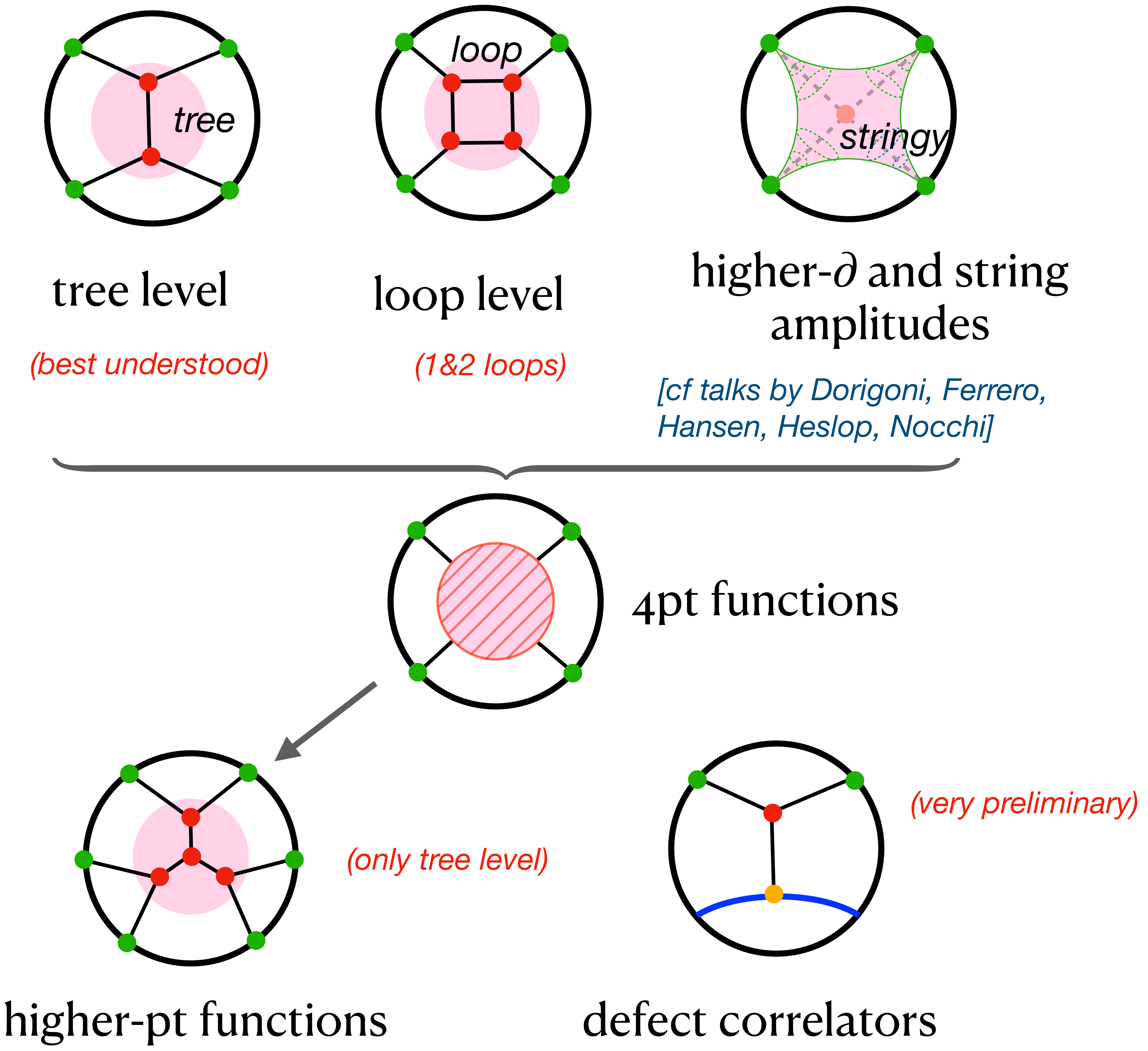
Department of Applied Mathematics and Theoretical Physics,
Wilberforce Road, Cambridge CB3 0WA, England

An expression for the four point function for $\frac{1}{2}$-BPS operators belonging to the $[0,p,0]$ $SU4)$ representation in $\mathcal{N}=4$ superconformal theories at strong coupling in the large N limit is suggested for any p. It is expressed in terms of the four point integrals defined by integration over AdS_5 and agrees with, and was motivated by, results for $p=2,3,4$ obtained via the AdS/CFT correspondence. Using crossing symmetry and unitarity, the detailed form is dictated by the requirement that at large N the contribution of long multiplets with twist less than $2p$, which do not have anomalous dimensions, should cancel corresponding free field contributions.

  - A modern program of analytic bootstrap for holographic correlators was developed more recently.

In this talk, I will survey some of the progress in studying holographic correlators.

Holographic correlators



tree level

(best understood)

loop level

(1&2 loops)

higher-∂ and string amplitudes

[cf talks by Dorigoni, Ferrero, Hansen, Heslop, Nocchi]

4pt functions

higher-pt functions

(only tree level)

defect correlators

(very preliminary)

Techniques used

- Position space
- Mellin space
- Momentum space
- Lorentzian inversion formula
- Localization
- Flat-space limit
- Unitarity
- Factorization
- Worldsheet
- Hidden symmetries
- ...

Testing ground

- ***IIB string theory on*** $AdS_5 \times S^5$ ***dual to 4d*** $\mathcal{N}=4$ ***SYM***
  - Corresponds to supergraviton scattering
  - Simple solution to superconformal Ward identities
  - Spectrum leads to simple analytic structure at tree level
  - Supersymmetric localization & integrability

The simplest maximally superconformal theory

- $\mathcal{N}=1$ ***SYM on*** $AdS_5 \times S^3$
  - A subsector of 4d $\mathcal{N}=2$ SCFTs constructed from D3 branes with probe D7 branes (or probing F-theory singularities) [Aharony, Fayyazuddin, Maldacena; Karch, Katz]
  - Corresponds to supergluon scattering
  - The simplest among theories with half maximal susy
  - Localization is also possible in some realizations

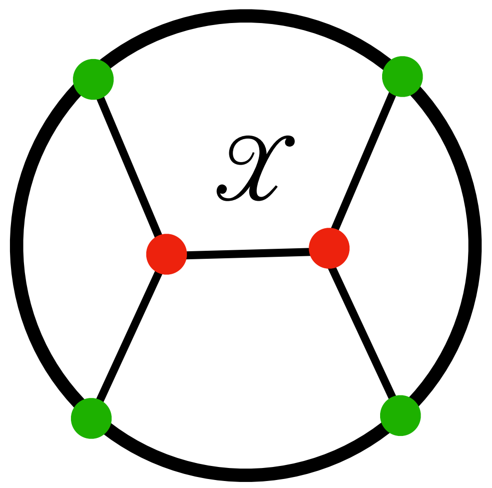
Although many results for many other theories!

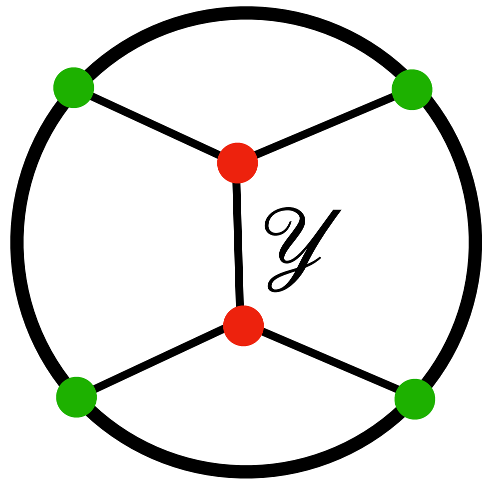
Outline

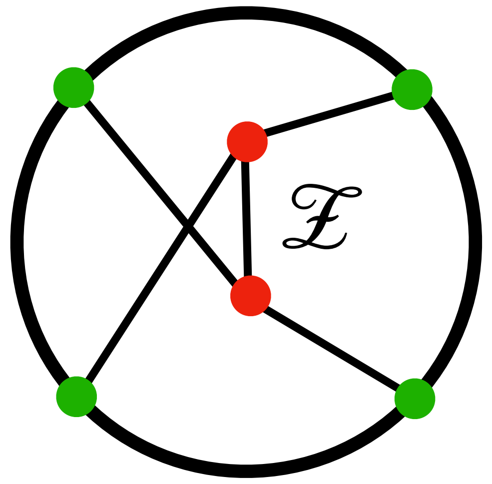
- Four-pt functions (position space)
- Four-pt functions (Mellin space)
- Higher-pt functions
- Hidden symmetries
- Stringy corrections
- Defect correlators
- Outlook

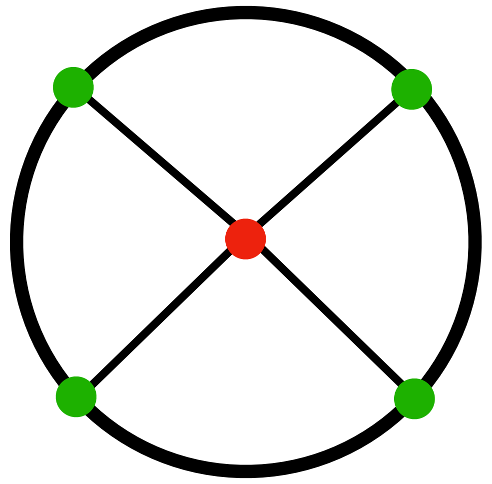
4pt functions (position space)

- Are the 15 pages of SUGRA vertices arbitrary? No, they are fixed by superconformal symmetry!
- A concrete way to efficiently implement this is the "position space method" [Rastelli, XZ].
- The starting point is an ansatz which is the linear combination of all possible Witten diagrams

$\mathscr{A} = \sum \lambda_{\mathscr{X}}^{(s)}$ 

$+ \lambda_{\mathscr{Y}}^{(t)}$ 

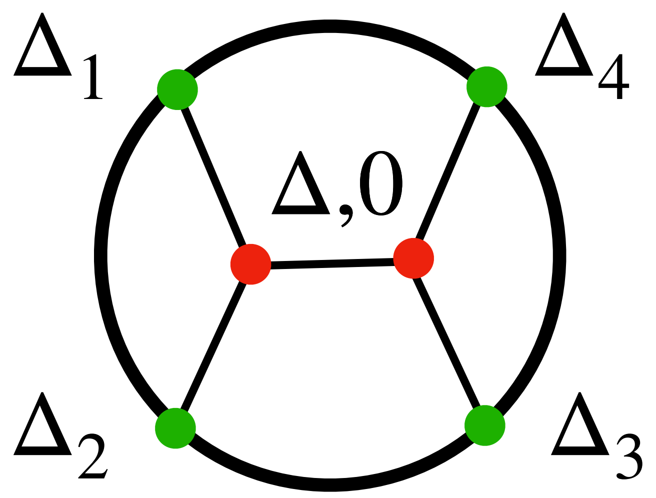
$+ \lambda_{\mathscr{Z}}^{(u)}$ 

$+ \lambda_c$ 

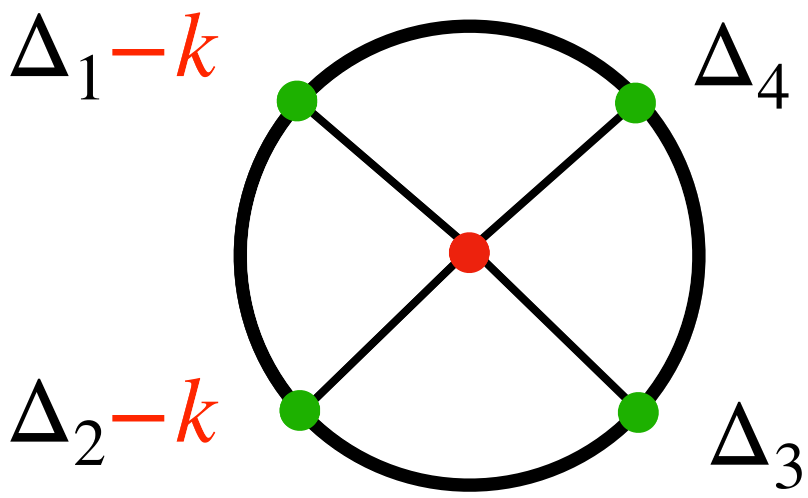
  - Determined by selection rules from consistency conditions (R-symmetry, ...)
  - $\lambda_{\mathscr{X}}^{(s)}, \lambda_{\mathscr{Y}}^{(t)}, \lambda_{\mathscr{Z}}^{(u)}$ are R-symmetry polynomials for exchanged R-symmetry irreps with unfixed overall coefficients.
  - λ_c includes all R-symmetry structures and at most two derivatives can appear in the contact part.

4pt functions (position space)

- The next step is to evaluate all the Witten diagrams in the ansatz
  - For certain “nice” spectra, exchange Witten diagrams can be written as finite sums of contact Witten diagrams [D’Hoker, Freedman, Rastelli]



$$= \sum_{k=1}^{\frac{\Delta_1+\Delta_2-\Delta}{2}} a_k x_{12}^{-2k}$$



D-functions

if $\Delta_1+\Delta_2-\Delta \in 2\mathbb{Z}_+$

(more generally: $\Delta_1+\Delta_2-\Delta+\ell \in 2\mathbb{Z}_+$)

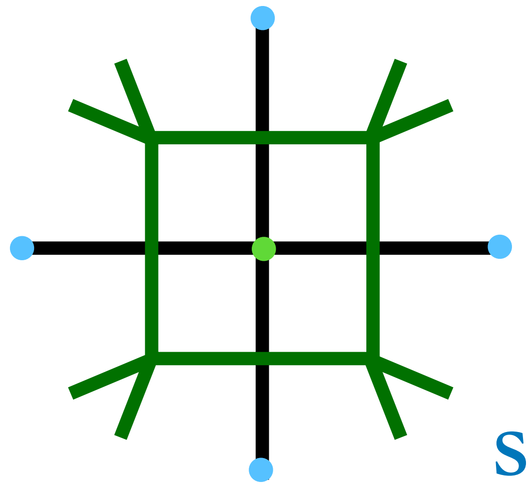
  - Different D-functions are related by differential recursion relations

$$D_{\Delta_1\ldots\Delta_i+1\ldots\Delta_j+1\ldots\Delta_n} = \frac{d-\Sigma}{2\Delta_i\Delta_j}\frac{\partial}{\partial x_{ij}^2} D_{\Delta_1\ldots\Delta_i\ldots\Delta_j\ldots\Delta_n}$$

$$\Sigma = \Delta_1 + \ldots + \Delta_n$$

(Consistency conditions are related to the Yangian symmetry [Rigatos, XZ])

$$D_{1111} = \frac{\Phi(z,\bar{z})}{x_{13}^2 x_{24}^2} \propto$$



scalar one-loop box diagram

$$\Phi(z,\bar{z}) = \frac{1}{(z-\bar{z})}\left(2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right)\right)$$

[Usykina, Davydychev]

4pt functions (position space)

- The box diagram satisfies

$$\partial_z\Phi(z,\bar{z}) = -\frac{1}{z-\bar{z}}\Phi(z,\bar{z})+\frac{1}{(z-1)(z-\bar{z})}\log U-\frac{1}{z(z-\bar{z})}\log V$$

$$\partial_{\bar{z}}\Phi(z,\bar{z}) = \frac{1}{z-\bar{z}}\Phi(z,\bar{z})-\frac{1}{(\bar{z}-1)(z-\bar{z})}\log U+\frac{1}{z(z-\bar{z})}\log V$$

- Then clearly $\{\Phi, \log U, \log V, 1\}$ form a basis, on which we can decompose the ansatz

$$\mathscr{A} = \mathscr{A}_\Phi\Phi(z,\bar{z}) + \mathscr{A}_{\log U}\log U + \mathscr{A}_{\log V}\log V + \mathscr{A}_1$$

The coefficients are rational functions.

- Finally, we impose the superconformal Ward identity [Nirschl, Osborn]

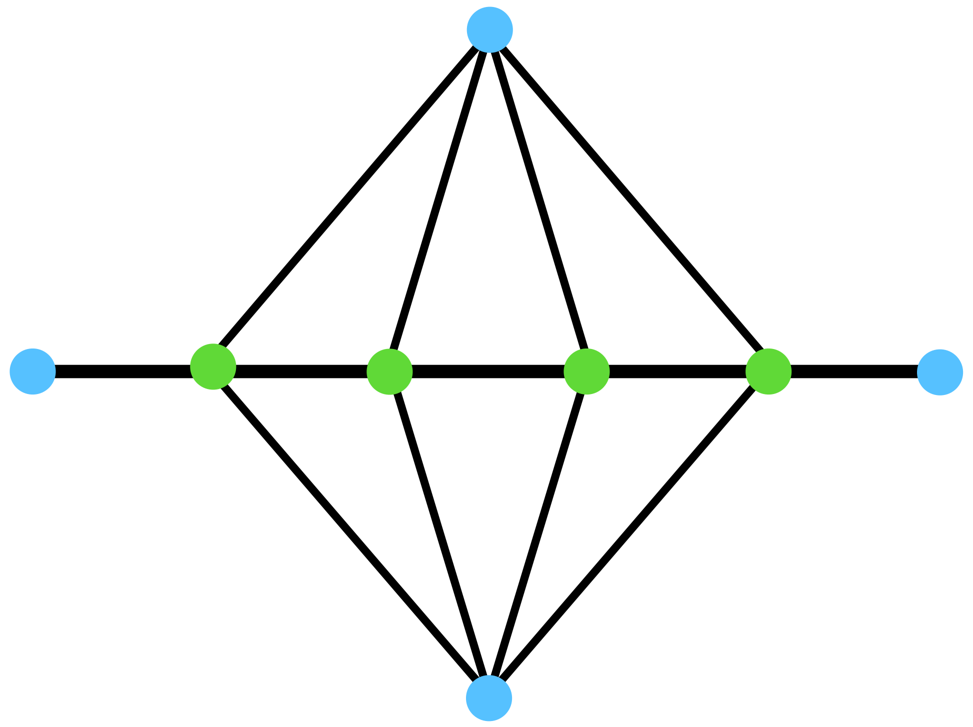
$$(z\partial_z - \alpha\partial_\alpha)\mathscr{A}(z,\bar{z};\alpha,\bar{\alpha})\,|_{\alpha=1/z} = 0$$

This fixes all the unknown coefficients in the ansatz up to an overall factor.

4pt functions (position space)

- A useful lesson from tree-level bootstrap: holographic correlators in position space can be decomposed into a basis of building block functions.
- What are the basis functions at loop level? These are the ladder integrals [Aprile, Drummond, Heslop, Paul]

Also cf talk by Yuan



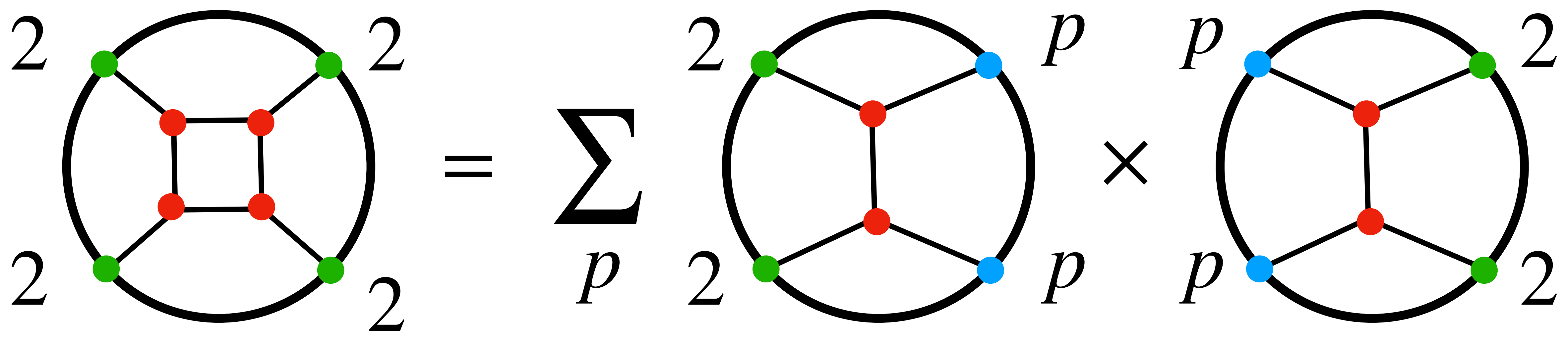
$$\Phi^{(l)} = -\frac{1}{z-\bar{z}}\phi^{(l)}\left(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}\right)$$

$$\phi^{(l)} = \sum_{r=0}^{l} (-1)^r \frac{(2l-r)!}{r!(l-r)!l!} \log^r(z\bar{z})\left(\mathrm{Li}_{2l-r}(z) - \mathrm{Li}_{2l-r}(\bar{z})\right)$$

- This allows us to extend the position space method to higher genus.
- Make an ansatz in terms of the basis functions with rational coefficients. At one loop, the highest transcendental weight is 4 [Aprile, Drummond, Heslop, Paul].

4pt functions (position space)

- Fix the unknowns in the ansatz by imposing physical conditions (focusing on $\langle 2222 \rangle$)
  - Leading logarithmic singularities (double-discontinuity [Caron-Huot])



AdS unitarity method [Aharony, Alday, Bissi, Perlmutter]

$$g_{\Delta,\ell} = U^{\tau/2}(f_0(V) + f_1(V)U + \ldots) \supset \frac{\gamma}{2} \log U \; g_{\Delta,\ell}$$

γ: anomalous dimension

Different orders of $1/c$, maximal powers of $\log U$:

Disconnected: $\mathscr{H}^{(0)}|_{\text{long}} = \sum_{n,\ell} \langle a^{(0)}_{n,\ell} \rangle g_{\Delta_{m,\ell},\ell}$

Tree ($\log U$): $\mathscr{H}^{(1)}|_{\log U} = \sum_{n,\ell} \frac{1}{2} \langle a^{(0)}_{n,\ell} \gamma^{(1)}_{n,\ell} \rangle g_{\Delta_{m,\ell},\ell}$

1-loop ($\log^2 U$): $\mathscr{H}^{(2)}|_{\log^2 U} = \sum_{n,\ell} \frac{1}{8} \langle a^{(0)}_{n,\ell} (\gamma^{(1)}_{n,\ell})^2 \rangle g_{\Delta_{m,\ell},\ell}$

In the SUGRA limit, all long operators are double-particle operators.

(For trace v.s. particle, see [Arutynov, Frolov; Aprile, Drummond, Heslop, Pual; Alday, XZ])

At one loop, only data from tree level and disconnected level are needed.

4pt functions (position space)

- A complication is operator mixing

$$:\mathcal{O}_2\,\Box^n\,\partial^\ell\,\mathcal{O}_2: \quad :\mathcal{O}_3\,\Box^{n-1}\,\partial^\ell\,\mathcal{O}_3: \quad :\mathcal{O}_4\,\Box^{n-2}\,\partial^\ell\,\mathcal{O}_4: \quad :\mathcal{O}_5\,\Box^{n-3}\,\partial^\ell\,\mathcal{O}_5: \;\ldots$$

- So the coefficients do not correspond to a particular operator.
- But we can still unmix the data by considering correlators of different KK modes [Alday, Bissi; Aprile, Drummond, Heslop, Paul].
- Studying the unmixing problem leads to interesting results
  - Surprisingly simple rational anomalous dimensions [Aprile, Drummond, Heslop, Paul]

$$\Delta_{pq} = \tau + \ell - \frac{2}{N^2}\frac{2M_t^{(4)}M_{t+\ell+1}^{(4)}}{\left(\ell + 2p - 2 - a - \frac{1+(-1)^{a+\ell}}{2}\right)_6} \qquad M_t^{(4)} = (t-1)(t+a)(t+a+b+1)(t+2a+b+2)$$

  - Hidden higher dimensional conformal symmetry [Caron-Huot Trinh] (more about this later).

4pt functions (position space)

- To compute $\langle 2222\rangle^{(2)}$ unmixing is actually not necessary. We only need to use $\langle 22pp\rangle^{(1)}$ and $\langle pppp\rangle^{(0)}$. This is easy to see by organizing CFT data into matrices.

Define:

$$\boldsymbol{\Lambda}^{(0)} = \begin{bmatrix} \lambda_{22O_1} & \lambda_{22O_2} & \cdots & \lambda_{22O_{M-1}} \\ \lambda_{33O_1} & \lambda_{33O_2} & \cdots & \lambda_{33O_{M-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{MM,O_1} & \lambda_{MMO_2} & \cdots & \lambda_{MMO_{M-1}} \end{bmatrix} \Big\downarrow \text{different correlators}$$

$$\xrightarrow{\hspace{6em}}$$

operator degeneracy

$$\boldsymbol{\Gamma}^{(1)} = \text{diag}(\gamma_1^{(1)}, \gamma_2^{(1)}, \ldots, \gamma_{M-1}^{(1)})$$

Disconnected:

$$\boldsymbol{\Lambda}^{(0)}(\boldsymbol{\Lambda}^{(0)})^T = \mathbf{N}^{(0)} = \begin{bmatrix} \langle 2222\rangle^{(0)} & & & \\ & \langle 3333\rangle^{(0)} & & \\ & & \cdots & \\ & & & \langle MMMM\rangle^{(0)} \end{bmatrix}$$

Tree-level:

$$\boldsymbol{\Lambda}^{(0)}\boldsymbol{\Gamma}^{(1)}(\boldsymbol{\Lambda}^{(0)})^T = \boldsymbol{\Omega}^{(1)} = \begin{bmatrix} \langle 2222\rangle^{(1)} & \langle 2233\rangle^{(1)} \ldots & & \langle 22MM\rangle^{(1)} \\ \langle 3322\rangle^{(1)} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \langle MM22\rangle^{(1)} & \cdots & \cdots & \langle MMMM\rangle^{(1)} \end{bmatrix}$$

One-loop:

$$\begin{aligned} \boldsymbol{\Lambda}^{(0)}(\boldsymbol{\Gamma}^{(1)})^2(\boldsymbol{\Lambda}^{(0)})^T &= \boldsymbol{\Lambda}^{(0)}\boldsymbol{\Gamma}^{(1)}(\boldsymbol{\Lambda}^{(0)})^T(\mathbf{N}^{(0)})^{-1}\boldsymbol{\Lambda}^{(0)}\boldsymbol{\Gamma}^{(1)}(\boldsymbol{\Lambda}^{(0)})^T \\ &= \boldsymbol{\Omega}^{(1)}(\mathbf{N}^{(0)})^{-1}\boldsymbol{\Omega}^{(1)} \end{aligned}$$

Need only the 11 component.

4pt functions (position space)

- Crossing symmetry
- Regular as $z \to \bar{z}$ (putting operators on a line)
- No twist-2 operators (no stringy states)

- These conditions completely fix the one-loop correlator up to a contact Witten diagram counter term [Aprile, Drummond, Heslop, Paul]. This algorithm also works for correlators of higher KK modes [Aprile, Drummond, Heslop, Paul].
- A similar strategy can now be used to compute correlators of supergravitons and supergluons up to two loops [Huang, Yuan; Drummond, Paul; Huang, Wang, Yuan, XZ].
- On the other hand, it should be noted that not much has been studied for individual loop-level Witten diagrams in position space, apart from very limited special examples [Bertan, Sachs; Heckelbacher, Sachs, Skvortsov, Vanhove; Stawinski]

4pt functions (Mellin space)

- "Momentum space" for AdS: Mellin space makes manifest the scattering amplitude nature of holographic correlators [Mack; Penedones; Paulos; Fitzpatrick, Kaplan, Penedones, Raju, van Rees]

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = \int [d\delta_{ij}]\Big(\prod_{i<j} x_{ij}^{-2\delta_{ij}}\Gamma(\delta_{ij})\Big)\mathcal{M}(\delta_{ij})$$

- Conformal symmetry:

$$\delta_{ij} = \delta_{ji}\,, \qquad \delta_{ii} = -\Delta_i\,, \qquad \sum_{j=1}^{n}\delta_{ij} = 0$$

- Solved by $\delta_{ij} = p_i \cdot p_j$ satisfying

$$p_i^2 = -\Delta_i\,, \qquad \sum_i p_i = 0 \qquad \Rightarrow \qquad \delta_{ij} \text{ are "Mandelstam variables"}$$

- The Mellin amplitude of contact diagrams are just constants.

- The Mellin amplitude of exchange diagrams have simple poles with polynomial residues.

4pt functions (Mellin space)

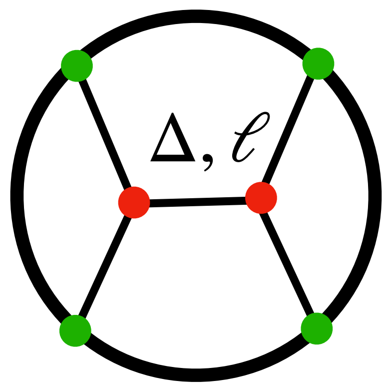
- For 4pt there are only two independent Mandelstam variables

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_4(x_n)\rangle = (\mathrm{kin\,.\ factor})\int \frac{dsdt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{t}{2}} \mathcal{M}(s,t)\Gamma_{1234}(s,t)$$

$$\Gamma_{1234}(s,t) = \Gamma(\frac{\Delta_1+\Delta_2-s}{2})\Gamma(\frac{\Delta_3+\Delta_4-s}{2})\Gamma(\frac{\Delta_1+\Delta_4-t}{2})\Gamma(\frac{\Delta_2+\Delta_3-t}{2})\Gamma(\frac{\Delta_1+\Delta_3-u}{2})\Gamma(\frac{\Delta_2+\Delta_4-u}{2})$$

$$s+t+u = \Delta_1+\Delta_2+\Delta_3+\Delta_4$$

- Exchange Witten diagram



$$\mathcal{M}\ [\quad] = \sum_m \frac{Q_{\ell,m}(t)}{s-(\Delta-\ell+2m)} + P_{\ell-1}(s,t)$$

The Mack polynomials $Q_{\ell,m}$ are known from solving EOM (Casimir equation) in Mellin space.

- Truncation in position space is equivalent to truncation of the poles.

- But we can now also deal with non-truncating cases.

E.g., in AdS_4,
$\Delta_i = \Delta = 1, \ell = 0$

$$\mathcal{M}_{1,0} = \frac{\Gamma(\frac{1-s}{2})}{\Gamma(1-\frac{s}{2})}$$

4pt functions (Mellin space)

- Extremely useful for $AdS_5 \times S^5$ IIB SUGRA. Here we illustrate it with tree-level 4pt functions. All 4pt functions can be obtained from solving an algebraic bootstrap problem in Mellin space [Rastelli, XZ]

  - *Superconformal symmetry*

$$\begin{tikzcd}
G \arrow[dd, dashed, bend right, "\text{full correlator}"'] & = & G_{\text{free}} \arrow[dl, dotted] & + & R \arrow[ddll, dotted] & H \arrow[ddl, dashed, bend left, "\text{reduced correlator}"] \\
& 0 & & & & \\
\mathscr{M} & = & \widehat{R} & \circ & \widetilde{\mathscr{M}} &
\end{tikzcd}$$

(solution to scf Ward id)

R is a polynomial in U, V, becomes a difference operator in Mellin space

$$U^m V^n \int U^{\frac{s}{2}} V^{\frac{t}{2}} f(s,t) = \int U^{\frac{s}{2}} V^{\frac{t}{2}} f(s-2m, t-2n)$$

  - *Analytic structure*

    $\mathscr{M}$ has only simple poles with polynomial residues.

  - *Bose symmetry*

    Invariant under permuting the particles.

  - *Asymptotic growth*

    $\mathscr{M}$ should grow no faster than linearly at high energy.

These conditions fix the amplitudes completely up to overall constants!

4pt functions (Mellin space)

- Mellin space is also useful for other theories. To obtain general results, we need two new ingredients
  - ***Superconformal Ward identities in Mellin space*** [XZ]
    - In general dimensions the scf Ward ids are [Dolan, Gallot, Sokatchev]

$$(z\partial_z - \epsilon\alpha\partial_\alpha)\mathcal{G}(z,\bar{z};\alpha,\bar{\alpha})\big|_{z=1/\alpha} = 0$$

$$(\bar{z}\partial_{\bar{z}} - \epsilon\alpha\partial_\alpha)\mathcal{G}(z,\bar{z};\alpha,\bar{\alpha})\big|_{\bar{z}=1/\alpha} = 0$$

$$\epsilon = (d-2)/2$$

The solution is quite complicated if $\epsilon \neq 1$.

    - Each equation is asymmetric in $z \leftrightarrow \bar{z} \Rightarrow$ not good for Mellin
    - But we can take the sum and the difference to write it in polynomials of U, V

$$z^n + \bar{z}^n = 2^{1-n}\sum_{k=0}^{\lfloor n/2 \rfloor}\binom{n}{2k}((1+U+V)^2-4U)^k(1+U-V)^{n-2k}$$

$$\frac{z^n - \bar{z}^n}{z-\bar{z}} = 2^{1-n}\sum_{k=0}^{\lfloor n/2 \rfloor}\binom{n}{2k+1}((1+U+V)^2-4U)^k(1+U-V)^{n-2k-1}$$

    - Scf Ward ids become difference equations for Mellin amplitudes.

4pt functions (Mellin space)

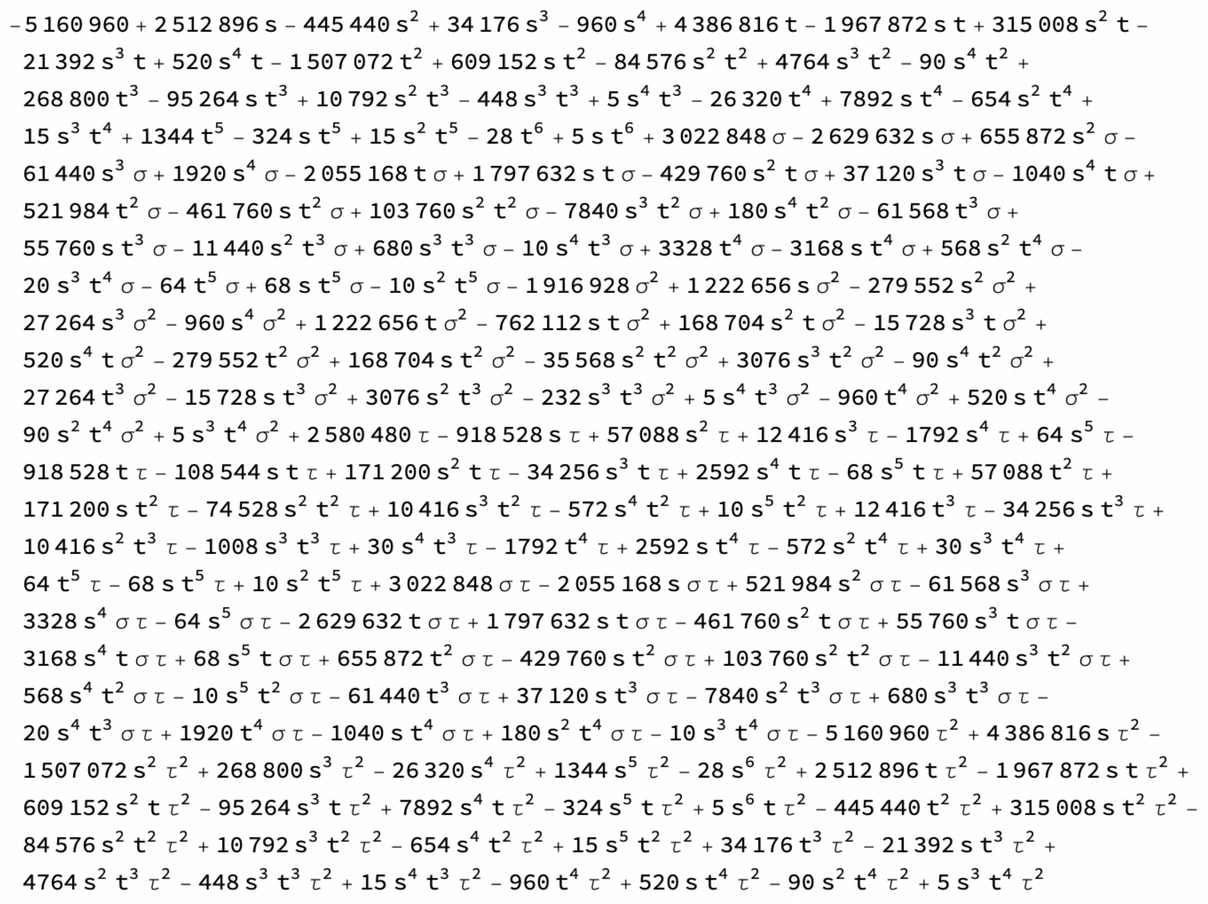
- ***"Maximally R-symmetry violating" configuration*** [Alday, XZ]
  - Amplitudes drastically simplify when some of the R-symmetry polarizations are aligned $t_1 = t_3$

E.g., simplest correlator in $AdS_7 \times S^4$

$$\mathcal{M}_{2222}(s,t;\sigma,\tau) = \frac{P_{2222}(s,t;\sigma,\tau)}{4n^3(s-4)(s-6)(t-4)(t-6)(u-4)(u-6)}$$

$$\Big\downarrow \quad t_1 = t_3$$

$$\mathbf{MRV}_{2222}(s,t) = \frac{(u-8)(u-10)}{n^3}\left(\frac{1}{s-4} + \frac{1}{4(s-6)} + \frac{1}{t-4} + \frac{1}{4(t-6)}\right)$$



$$\begin{aligned}
&-5\,160\,960 + 2\,512\,896\, s - 445\,440\, s^2 + 34\,176\, s^3 - 960\, s^4 + 4\,386\,816\, t - 1\,967\,872\, s\, t + 315\,008\, s^2\, t -\\
&21\,392\, s^3\, t + 520\, s^4\, t - 1\,507\,072\, t^2 + 609\,152\, s\, t^2 - 84\,576\, s^2\, t^2 + 4764\, s^3\, t^2 - 90\, s^4\, t^2 +\\
&268\,800\, t^3 - 95\,264\, s\, t^3 + 10\,792\, s^2\, t^3 - 448\, s^3\, t^3 + 5\, s^4\, t^3 - 26\,320\, t^4 + 7892\, s\, t^4 - 654\, s^2\, t^4 +\\
&15\, s^3\, t^4 + 1344\, t^5 - 324\, s\, t^5 + 15\, s^2\, t^5 - 28\, t^6 + 5\, s\, t^6 + 3\,022\,848\, \sigma - 2\,629\,632\, s\, \sigma + 655\,872\, s^2\, \sigma -\\
&61\,440\, s^3\, \sigma + 1920\, s^4\, \sigma - 2\,055\,168\, t\, \sigma + 1\,797\,632\, s\, t\, \sigma - 429\,760\, s^2\, t\, \sigma + 37\,120\, s^3\, t\, \sigma - 1040\, s^4\, t\, \sigma +\\
&521\,984\, t^2\, \sigma - 461\,760\, s\, t^2\, \sigma + 103\,760\, s^2\, t^2\, \sigma - 7840\, s^3\, t^2\, \sigma + 180\, s^4\, t^2\, \sigma - 61\,568\, t^3\, \sigma +\\
&55\,760\, s\, t^3\, \sigma - 11\,440\, s^2\, t^3\, \sigma + 680\, s^3\, t^3\, \sigma - 10\, s^4\, t^3\, \sigma + 3328\, t^4\, \sigma - 3168\, s\, t^4\, \sigma + 568\, s^2\, t^4\, \sigma -\\
&20\, s^3\, t^4\, \sigma - 64\, t^5\, \sigma + 68\, s\, t^5\, \sigma - 10\, s^2\, t^5\, \sigma - 1\,916\,928\, \sigma^2 + 1\,222\,656\, s\, \sigma^2 - 279\,552\, s^2\, \sigma^2 +\\
&27\,264\, s^3\, \sigma^2 - 960\, s^4\, \sigma^2 + 1\,222\,656\, t\, \sigma^2 - 762\,112\, s\, t\, \sigma^2 + 168\,704\, s^2\, t\, \sigma^2 - 15\,728\, s^3\, t\, \sigma^2 +\\
&520\, s^4\, t\, \sigma^2 - 279\,552\, t^2\, \sigma^2 + 168\,704\, s\, t^2\, \sigma^2 - 35\,568\, s^2\, t^2\, \sigma^2 + 3076\, s^3\, t^2\, \sigma^2 - 90\, s^4\, t^2\, \sigma^2 +\\
&27\,264\, t^3\, \sigma^2 - 15\,728\, s\, t^3\, \sigma^2 + 3076\, s^2\, t^3\, \sigma^2 - 232\, s^3\, t^3\, \sigma^2 + 5\, s^4\, t^3\, \sigma^2 - 960\, t^4\, \sigma^2 + 520\, s\, t^4\, \sigma^2 -\\
&90\, s^2\, t^4\, \sigma^2 + 5\, s^3\, t^4\, \sigma^2 + 2\,580\,480\, \tau - 918\,528\, s\, \tau + 57\,088\, s^2\, \tau + 12\,416\, s^3\, \tau - 1792\, s^4\, \tau + 64\, s^5\, \tau -\\
&918\,528\, t\, \tau - 108\,544\, s\, t\, \tau + 171\,200\, s^2\, t\, \tau - 34\,256\, s^3\, t\, \tau + 2592\, s^4\, t\, \tau - 68\, s^5\, t\, \tau + 57\,088\, t^2\, \tau +\\
&171\,200\, s\, t^2\, \tau - 74\,528\, s^2\, t^2\, \tau + 10\,416\, s^3\, t^2\, \tau - 572\, s^4\, t^2\, \tau + 10\, s^5\, t^2\, \tau + 12\,416\, t^3\, \tau - 34\,256\, s\, t^3\, \tau +\\
&10\,416\, s^2\, t^3\, \tau - 1008\, s^3\, t^3\, \tau + 30\, s^4\, t^3\, \tau - 1792\, t^4\, \tau + 2592\, s\, t^4\, \tau - 572\, s^2\, t^4\, \tau + 30\, s^3\, t^4\, \tau +\\
&64\, t^5\, \tau - 68\, s\, t^5\, \tau + 10\, s^2\, t^5\, \tau + 3\,022\,848\, \sigma\, \tau - 2\,055\,168\, s\, \sigma\, \tau + 521\,984\, s^2\, \sigma\, \tau - 61\,568\, s^3\, \sigma\, \tau +\\
&3328\, s^4\, \sigma\, \tau - 64\, s^5\, \sigma\, \tau - 2\,629\,632\, t\, \sigma\, \tau + 1\,797\,632\, s\, t\, \sigma\, \tau - 461\,760\, s^2\, t\, \sigma\, \tau + 55\,760\, s^3\, t\, \sigma\, \tau -\\
&3168\, s^4\, t\, \sigma\, \tau + 68\, s^5\, t\, \sigma\, \tau + 655\,872\, t^2\, \sigma\, \tau - 429\,760\, s\, t^2\, \sigma\, \tau + 103\,760\, s^2\, t^2\, \sigma\, \tau - 11\,440\, s^3\, t^2\, \sigma\, \tau +\\
&568\, s^4\, t^2\, \sigma\, \tau - 10\, s^5\, t^2\, \sigma\, \tau - 61\,440\, t^3\, \sigma\, \tau + 37\,120\, s\, t^3\, \sigma\, \tau - 7840\, s^2\, t^3\, \sigma\, \tau + 680\, s^3\, t^3\, \sigma\, \tau -\\
&20\, s^4\, t^3\, \sigma\, \tau + 1920\, t^4\, \sigma\, \tau - 1040\, s\, t^4\, \sigma\, \tau + 180\, s^2\, t^4\, \sigma\, \tau - 10\, s^3\, t^4\, \sigma\, \tau - 5\,160\,960\, \tau^2 + 4\,386\,816\, s\, \tau^2 -\\
&1\,507\,072\, s^2\, \tau^2 + 268\,800\, s^3\, \tau^2 - 26\,320\, s^4\, \tau^2 + 1344\, s^5\, \tau^2 - 28\, s^6\, \tau^2 + 2\,512\,896\, t\, \tau^2 - 1\,967\,872\, s\, t\, \tau^2 +\\
&609\,152\, s^2\, t\, \tau^2 - 95\,264\, s^3\, t\, \tau^2 + 7892\, s^4\, t\, \tau^2 - 324\, s^5\, t\, \tau^2 + 5\, s^6\, t\, \tau^2 - 445\,440\, t^2\, \tau^2 + 315\,008\, s\, t^2\, \tau^2 -\\
&84\,576\, s^2\, t^2\, \tau^2 + 10\,792\, s^3\, t^2\, \tau^2 - 654\, s^4\, t^2\, \tau^2 + 15\, s^5\, t^2\, \tau^2 + 34\,176\, t^3\, \tau^2 - 21\,392\, s\, t^3\, \tau^2 +\\
&4764\, s^2\, t^3\, \tau^2 - 448\, s^3\, t^3\, \tau^2 + 15\, s^4\, t^3\, \tau^2 - 960\, t^4\, \tau^2 + 520\, s\, t^4\, \tau^2 - 90\, s^2\, t^4\, \tau^2 + 5\, s^3\, t^4\, \tau^2
\end{aligned}$$

  - The MRV limit allows only a single R-symmetry irrep to propagate in the u-channel.
  - Together with scf symmetry requires the decoupling of certain long operators, manifested as the zeroes in the amplitude.

4pt functions (Mellin space)

- The MRV zeroes exist for the exchange of each individual scf multiplet. This allows us to fix the coefficients within each multiplet → implementing scf symmetry at a more local level.
- Moreover, there is a general prescription to go away from MRV limit and restore the general R-symmetry dependence.
- Remarkably, this prescription leads to no explicit contact diagrams in the final amplitude!

$$\mathcal{M} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u$$

over finitely many multiplets

$$\mathcal{M}_s = \sum_{i,j} \sigma^i \tau^j \sum_{s_0} \frac{R_s^{i,j}(t,u)}{s - s_0} \qquad s_0 = \epsilon p + 2m \ , \ \ m \in \mathbb{N}$$

$$p - \max\{\,|k_1 - k_2|, |k_3 - k_4|\,\} = 2,4,\ldots 2\mathscr{E} - 2$$

- This method gives all 4pt amplitudes in maximally scf theories $AdS_5 \times S^5, AdS_7 \times S^4, AdS_4 \times S^7$ [Alday, XZ].
- It also gives all 4pt amplitudes in half maximally scf theories of the form $AdS_{d+1} \times S^3$ corresponding to superglouon scattering [Alday, Behan, Ferrero XZ].

4pt functions (Mellin space)

- The Mellin formalism is also useful for studying correlators at loop level
  - Consider $\langle 2222\rangle$ at one loop for $AdS_5\times S^5$. The correlator has the structure

fixed by tree-level data

$$\begin{aligned}\mathscr{H}(U,V) \;=\; & \boxed{f_{2,2}(U,V)}\log^2 U\log^2 V \;+\; f_{2,1}(U,V)\log^2 U\log V \;+\; f_{2,0}(U,V)\log^2 U\\ & +f_{1,2}(U,V)\;\log U\log^2 V \;+\; f_{1,1}(U,V)\;\log U\log V\; + f_{1,0}(U,V)\;\log U\\ & +f_{0,2}(U,V)\log^2 V \;+\; f_{0,1}(U,V)\;\log V \;+\; f_{0,0}(U,V)\end{aligned}$$

$$f_{2,2}(U,V)=\sum_{m,n=0}^{\infty} d_{nm}\,U^{2+n}V^m$$

The task is to complete it into a full correlator.

$$\mathscr{H}=\int\frac{dsdt}{(4\pi i)^2}U^{\frac{s}{2}}V^{\frac{t}{2}-2}\underbrace{\widetilde{\mathscr{M}}(s,t)\Gamma^2(\frac{4-s}{2})\Gamma^2(\frac{4-t}{2})\Gamma^2(\frac{4-\tilde{u}}{2})}$$

$\log^2 U\log^2 V$ implies the Mellin integrand must

have triple poles at $s=4+2n,\,t=4+2m$

$\Rightarrow$ The reduced Mellin amplitude should have simple poles in s and t.

4pt functions (Mellin space)

- Make a minimal assumption: only simultaneous poles with constant numerators [Alday]

$$\widetilde{\mathcal{M}}(s,t) = \sum_{m,n=0}^{\infty} \frac{c_{mn}}{(s-4-2n)(t-4-2m)} + \underbrace{(su) + (tu)}_{\text{crossing symmetry}}$$

- The poles are easy to justify: we only expect long operators from double-particle operators at 1 loop.

- That c_{mn} are constants is an assumption that must be checked by calculation.

- We first compute the c_{mn} coefficients. We take residues at $s = 4 + 2n$, $t = 4 + 2m$. The result contains $\log^i U \log^j V$ with $i, j = 0,1,2$ and we focus on the $\log^2 U \log^2 V$ coefficient. Comparing it with the term $U^{2+n}V^m$ of $f_{2,2}(U, V)$, we can find c_{mn} in a closed form

$$c_{mn} = \frac{p^{(6)}(m,n)}{(m+n-1)_5}$$

4pt functions (Mellin space)

- Then we need to check the ansatz is correct. We insert c_{mn} into the ansatz and sum over all the poles in t and u. We take residue w.r.t. s and focus on the $\log^2 U$ term, but we keep general V dependence.
- We find that not only $\log^2 V$ is reproduced but also subleading powers of $\log V$

$$\mathscr{H}(U,V) \supset \underbrace{f_{2,2}(U,V)\log^2 U \log^2 V}_{\text{input}} + \underbrace{f_{2,1}(U,V)\log^2 U\log V + f_{2,0}(U,V)\log^2 U}_{\text{check}}$$

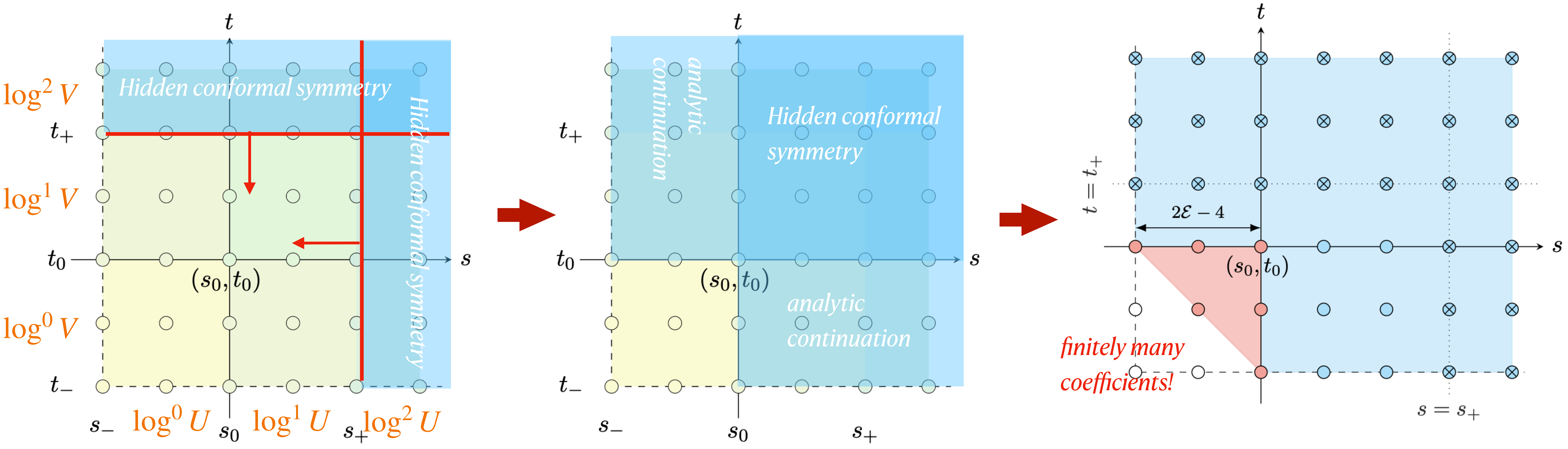
- Rules out single poles in s (and t, u by crossing). This fixes all singularities!
- Regular terms are ambiguous. But they can be fixed by the flat-space limit and can only be a constant.

$$\partial_s\partial_t \widetilde{\mathscr{M}}(s,t) \xrightarrow{\text{flat space limit}} \partial_s\partial_t I_{\text{10d box}}(s,t)$$

- The same analytic structure extends to higher KK modes [Alday, XZ], as well as to one-loop amplitudes of super gluons [Alday, Bissi, XZ; Huang, Wang, Yuan, XZ].

4pt functions (Mellin space)

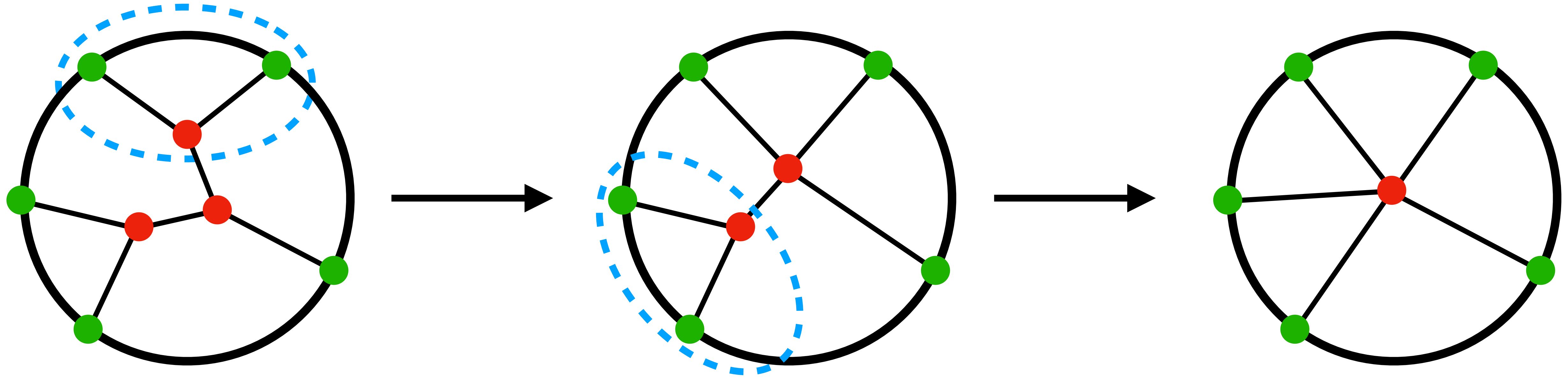
- The simplicity of Mellin makes it easier to spot patterns. Interesting higher dimensional structure at one loop [Huang, Wang, Yuan, XZ]: for higher KK modes, almost all simultaneous pole coefficients are fixed by tree-level hidden conformal symmetry [c.f. E. Yuan's talk].



General result for $\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle$ with $\mathcal{E} = \min\{p_1, \frac{1}{2}(p_1 + p_2 + p_3 - p_4)\} = 2{,}3$.

Higher-pt functions

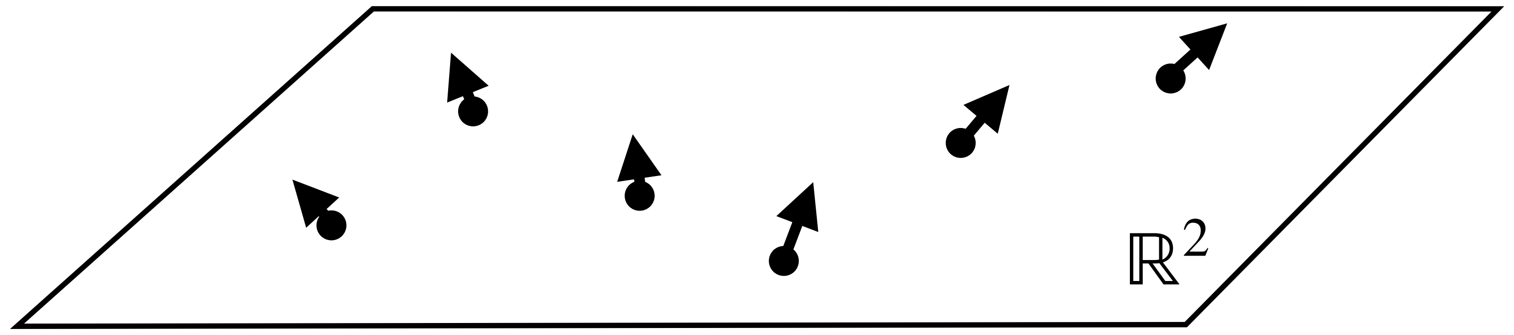
- Higher-pt holographic correlators is more challenging because of more complicated kinematics.
- For 5pt, it is still feasible to adapt the position space method and results were obtained for supergraviton and supergluon in AdS_5 [Goncalves, Pereira, XZ; Alday, Goncalves, XZ]
- As in 4pt, we can also reduce all exchange diagrams to contact diagrams. The reduction works when we have cubic vertex



- Ultimately boils down to D_{11112}, which is the same as the pentagon integral in flat space and can be expressed as box diagrams [Bern, Dixon, Kosower].
- We can make an ansatz in terms of all possible diagrams and evaluate it in terms of elementary functions.

Higher-pt functions

- Scf Ward ids for more than 4 pts are not known. So far only two weaker conditions are known.
  - Chiral algebra [Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees]: correlators of operators on a plane with special R-symmetry polarizations are meromorphic



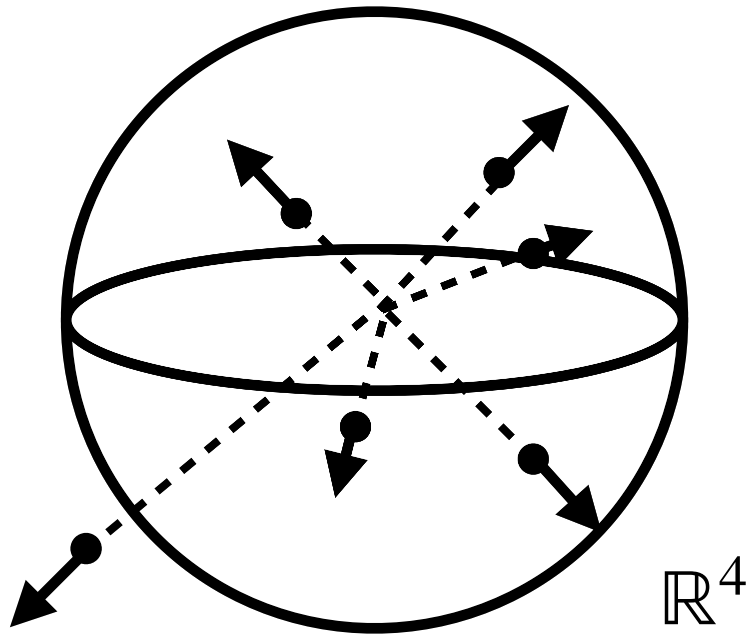
$$\langle S_1(z_1,\bar{z}_1;y_1)\ldots S_n(z_1,\bar{z}_n;y_n)\rangle\,|_{y_i=\bar{z}_i} = F(z_i)$$

y_i parameterizes an $SU(2)\subset SO(6)$ or $SO(4)$ subgroup of R-symmetry (for 4d $\mathcal{N}=4$ or $\mathcal{N}=2$)

  - Topological twisting (for 4d $\mathcal{N}=4$) [Drukker, Plefka]: identifying R-symmetry polarizations with the embedding space vectors make the correlator topological

$$\langle S_1(P_1;t_1)\ldots S_n(P_n;t_n)\rangle\,|_{t_i=P_i} = \text{const}$$

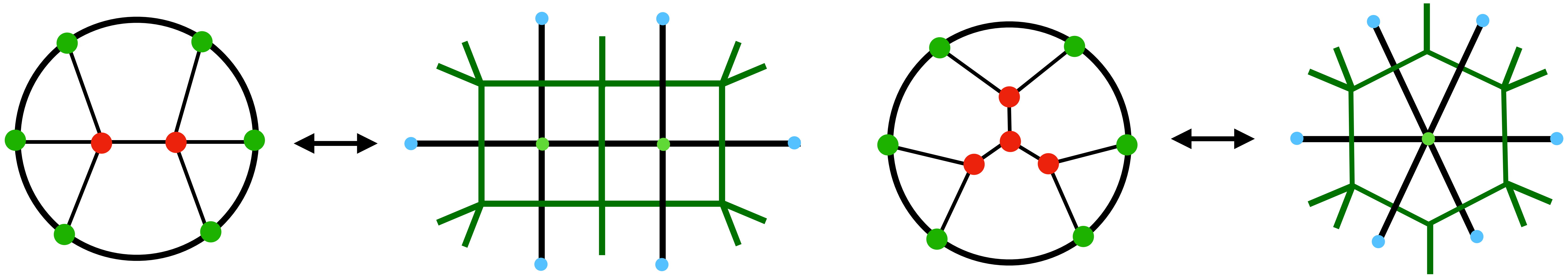
t_i are null vectors of $SO(6)_R$



  - Topological twisting $\subset$ chiral algebra for $n=4$ but not when $n\geq 5$.
- These conditions fix the 5pt functions uniquely [Goncalves, Pereira, XZ; Alday, Goncalves, XZ].

Higher-pt functions

- However, this method fails at 6pt because we cannot reduce all exchange diagrams to contact and the basic integrals are not known



Elliptic

- The more suitable formalism for higher-pt is Mellin space where the Mellin amplitude is always a rational function.

- However, the superconformal constraints in position space are difficult to translate to Mellin space. The Mellin version of the Drukker-Plefka twisting is known [Boas, Goncalves, Meneghelli, Pereira, XZ] but not the chiral algebra twisting.

- Need an alternative approaches where superconformal symmetry plays a less crucial role.

Higher-pt functions

- One such approach was developed in [Alday, Goncalves, Nocchi, XZ] which makes minimal use of superconformal symmetry. Instead, it relies on two different principles.

- ***Flat-space limit***
  - The flat-space limit is encoded in the high energy limit of the Mellin amplitude [Penedones]

$$T(S_{ij}) \approx \frac{\Gamma(\frac{1}{2}\sum_i \Delta_i - \frac{d}{2})}{R^{n(1-d)/2+d+1}} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} d\Lambda \Lambda^{-\frac{1}{2}\sum_i \Delta_i + \frac{d}{2}} e^{\Lambda} \mathcal{M}\left(s_{ij} = \frac{S_{ij}R^2}{2\Lambda}\right) \qquad R \to \infty$$

  - For tree-level amplitudes of massless fields the integral is trivial

$$T(s_{ij}) \propto \lim_{\beta\to\infty} \beta^p \mathcal{M}(\beta s_{ij})$$

  - But supergravitons and supergluons have polarizations in the flat-space limit. They come from higher dimensional gravity field and gauge field pointing in the internal directions.

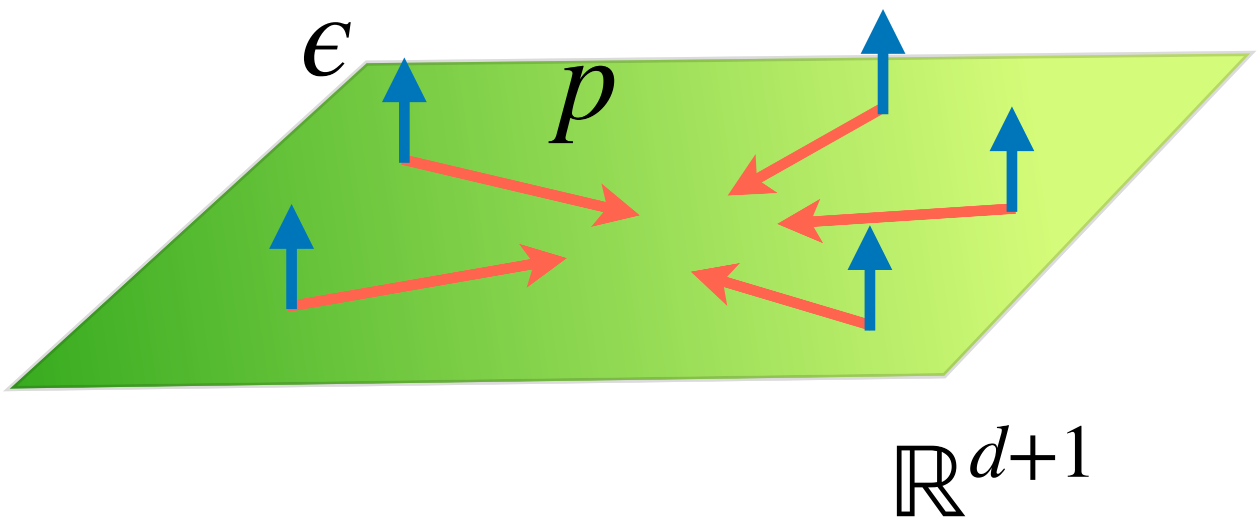
Higher-pt functions

- The flat-space polarizations are simply related to the R-symmetry polarizations

$$\epsilon \leftrightarrow t \qquad \text{(and } t \sim \sigma^{\mu}_{\alpha\beta} v^{\alpha} v^{\beta} \text{ for } AdS_5 \times S^3\text{)}$$

with the important property that they are orthogonal to momenta

$$\epsilon_i \cdot p_j = 0 \qquad \text{for all } i \text{ and } j$$



- Therefore, the flat-space condition for tree-level Mellin amplitudes is

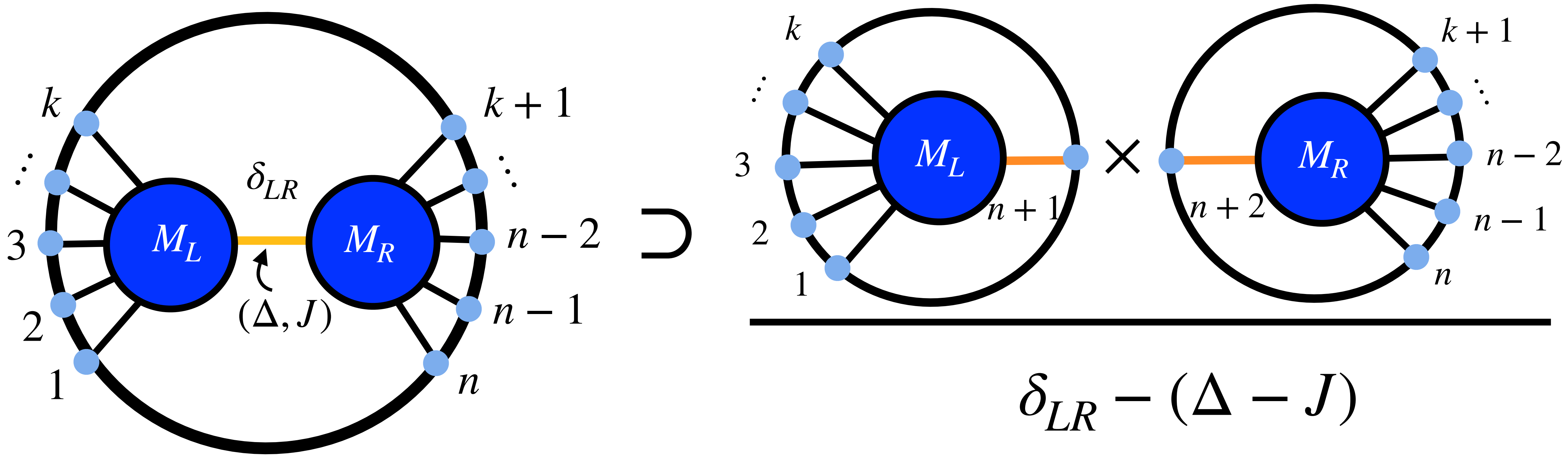
$$\lim_{\beta \to \infty} \beta^p \mathcal{M}(\beta s_{ij}; t_i) \propto$$ Flat-space amplitudes of bosonic gluons or gravitons with polarizations $\epsilon_i = t_i$ satisfying $\epsilon_i \cdot p_j = 0$

- Note the flat-space Feynman rules also simplify in this limit. Odd-pt gluon amplitudes vanish (cannot contract all ϵ with ϵ) and odd-pt graviton amplitudes also vanish (by double copy).

- Higher KK is proportional to the lowest one with a factor given by Wick contractions.

Higher-pt functions

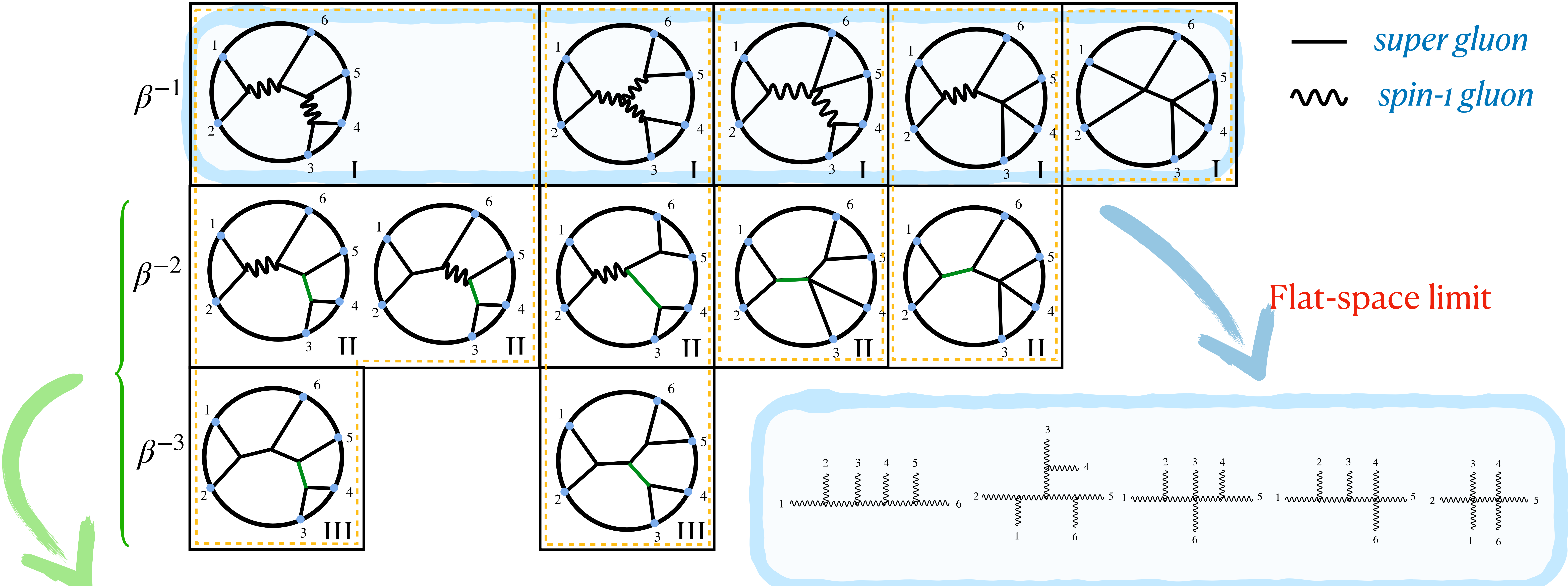
- ***Mellin amplitude factorization*** [Goncalves, Penedones, Trevisani]
  - Similar analytic structure as in flat space: exchange of single-trace operators of dimension Δ and spin J corresponds to poles at $\Delta - J + 2m$ with $m = 0,1,\ldots$
  - The residues factorize into “products” of lower-point Mellin amplitudes



  - The precise form of the “product” depends on spin J and the m.
  - Currently only known for correlators with at most one spinning correlator (a general Mellin formalism for multiple spinning operators is not known!).

Higher-pt functions

- A new strategy based on flat-space limit and factorization [Alday, Goncalves, Nocchi, XZ]



super gluon

spin-1 gluon

Flat-space limit

AdS factorization

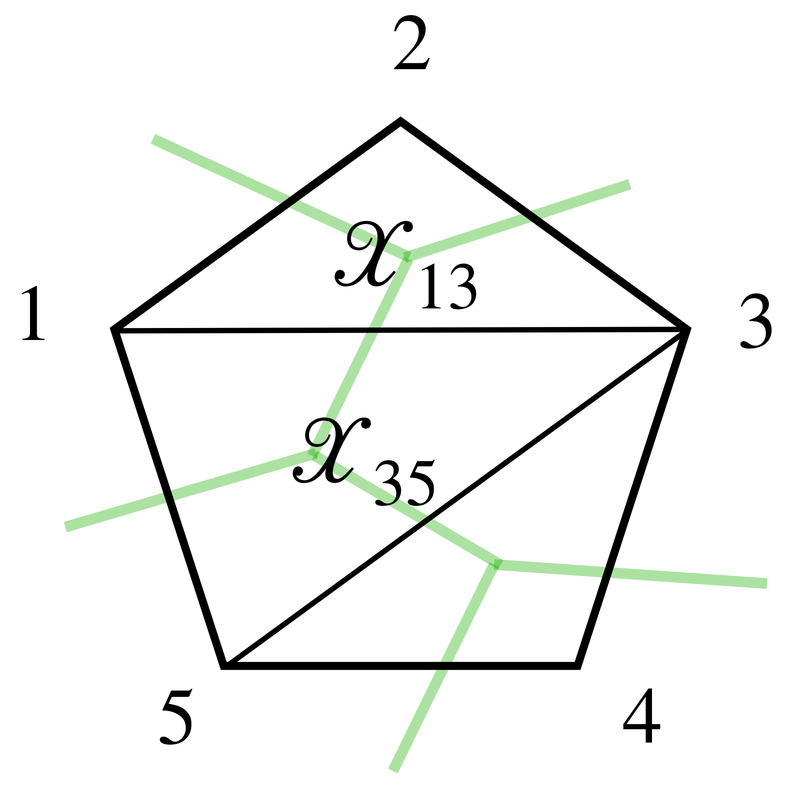
$\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle \times \langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle$

This algorithm led to the first 6pt super gluon amplitude in AdS_5.

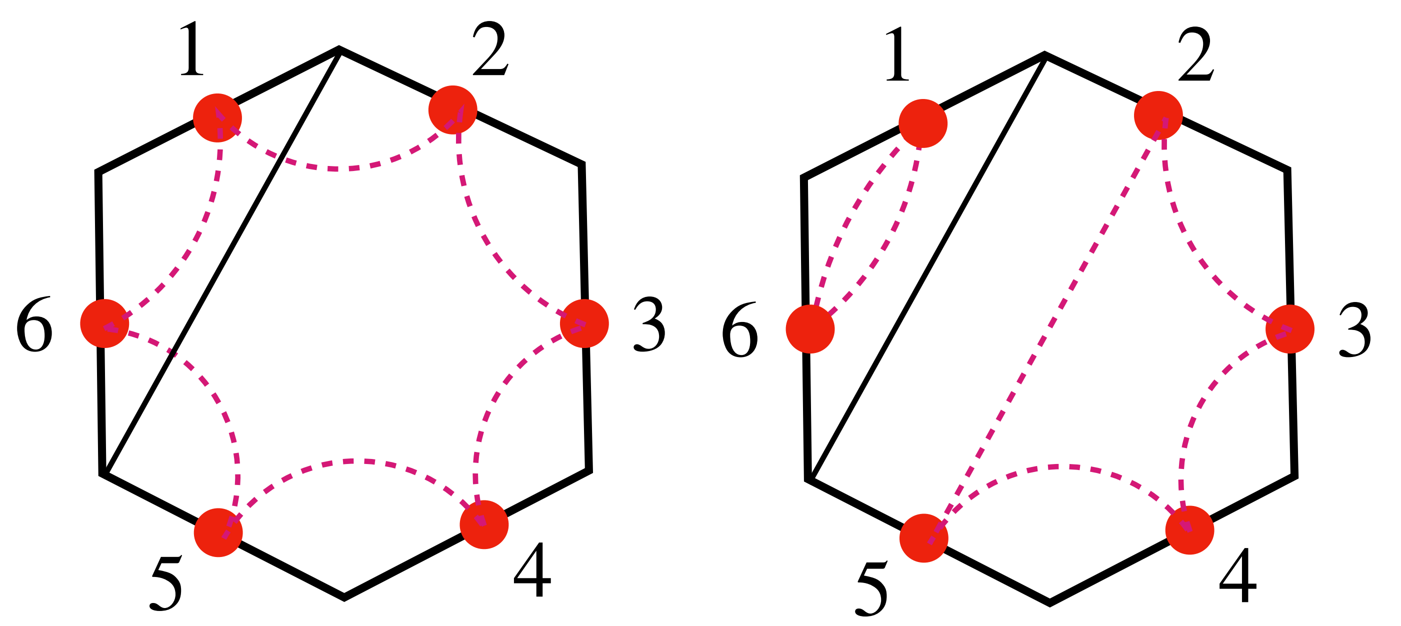
Higher-pt functions

- Amplitude constructibility was also recently proven for the lowest KK-level supergluons [Cao, He, Tang] by adapting more ideas and technologies from flat space
  - Planar variables

$$-2\delta_{ij} = \mathcal{X}_{i,j} + \mathcal{X}_{i+1,j+1} - \mathcal{X}_{i,j+1} - \mathcal{X}_{i+1,j}$$

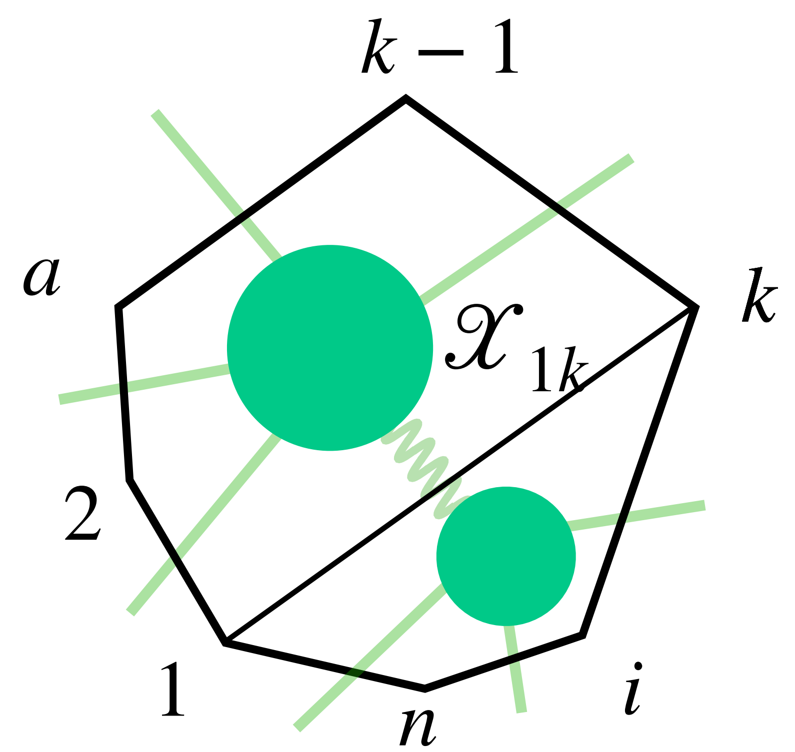


$$\mathcal{X}_{i,j} = 2 + (\sum_{i\le k<j} p_k)^2$$



  - A natural $SU(2)$ R-symmetry basis

$$V_{i_1 i_2 \ldots i_r} = \langle i_1 i_2\rangle\langle i_2 i_3\rangle\ldots\langle i_r i_1\rangle \ \rightarrow \text{products of traces}$$



  - “No-gluon” kinematics

$$\mathcal{X}_{ai} - m + 1 - \frac{\mathcal{X}_{1a} + \mathcal{X}_{1i} + \mathcal{X}_{ak} + \mathcal{X}_{ik}}{2} - \frac{(\mathcal{X}_{1a} - \mathcal{X}_{ak})(\mathcal{X}_{1i} - \mathcal{X}_{ik})}{4(m+1)} = 0$$

$$\mathcal{X}_{1k} = -2m$$

- But at the moment this proof seems difficult to generalize to SUGRA and higher KK modes.
- It would be nice to find recursion relations similar to BCFW in Mellin space (in momentum space [Raju]).

Hidden symmetries

- Holographic correlators also exhibit interesting hidden properties which can be used in their computations.
- ***Hidden conformal symmetry***
  - First found in tree-level 4pt functions in $AdS_5 \times S^5$ [Caron-Huot, Trinh] with hints from the simple form of anomalous dimensions [Aprile, Drummond, Heslop, Paul].
  - Probably related with conformal flatness of the background but the precise physical origin is not known.
  - All correlators can be packaged into a single generating function which comes from lifting the lowest weight 4pt function

$$\mathbf{H}(x_i, t_i) = H_{2222}(x_{ij}^2 - t_{ij})$$

  - The replacement $x_{ij}^2 \to x_{ij}^2 - t_{ij}$ lifts 4D distances to conformally invariance distances in 10D.
  - All correlators of higher KK modes are obtained by Taylor expanding in t_{ij} and collecting all the possible R-symmetry structures.

Hidden symmetries

- Applications
  - A 6D version was used to compute 4pt functions in $AdS_3 \times S^3 \times K3$ [Rastelli, Roumpedakis, XZ] where scf symmetry is not enough to completely fix the higher KK correlators.
  - Useful for computing certain quantities which depends on only the tree-level data, such as the leading logarithmic singularity, to any loop level [Caron-Huot, Trinh; Bissi, Fardelli, Georgoudis].
  - Although is broken, hidden conformal symmetry was useful for studying higher-derivative corrections [Abl, Heslop, Lipstein; Aprile, Drummond, Paul, Santagata].
  - Also useful for integrated correlators [Brown, Heslop, Wen, Xie; cf talk by Heslop].
- Higher-pt?

Hidden symmetries

- *Double copy*
  - In flat space graviton amplitudes can be obtained from gluon amplitudes by squaring the numerators [Bern, Carrasco, Johansson]

$$\mathcal{A}_{\text{gluon}} = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \quad \xrightarrow{c_{s,t,u} \to n_{s,t,u}} \quad \mathcal{A}_{\text{graviton}} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

  - In AdS, there is an almost identical realization of double copy for 4pt functions at the level of the reduced Mellin amplitudes [XZ]. The $AdS_5 \times S^3$ supergluon amplitudes can be written as

$$\widetilde{\mathcal{M}}_{\mathcal{N}=2} = \sum_{i+j+k=\mathcal{E}-2} a_{jik}\sigma^i\tau^j \left(\frac{c_s n_s^{i,j}}{s - s_M + 2k} + \frac{c_t n_t^{i,j}}{t - t_M + 2j} + \frac{c_u n_u^{i,j}}{\tilde{u} - u_M + 2k} \right)$$

$$n_s^{i,j} = \frac{1}{t - t_M + 2j} - \frac{1}{\tilde{u} - u_M + 2i} \qquad n_u^{i,j} = \frac{1}{s - s_M + 2k} - \frac{1}{t - t_M + 2j}$$

$$n_t^{i,j} = \frac{1}{\tilde{u} - u_M + 2i} - \frac{1}{s - s_M + 2k}$$

AdS color-kinematic duality

$$n_s^{i,j} + n_t^{i,j} + n_u^{i,j} = 0$$

Hidden symmetries

- The same operation as in flat space leads to SUGRA 4pt amplitudes in $AdS_5 \times S^5$

$$c_{s,t,u} \to n^{i,j}_{s,t,u} : \qquad \widetilde{\mathcal{M}}_{\mathcal{N}=2} \to \widetilde{\mathcal{M}}_{\mathcal{N}=4}$$

- We can also consider $n^{i,j}_{s,t,u} \to c_{s,t,u}$, this gives all the 4pt functions in a bosonic theory of a conformally coupled bi-adjoint scalar in $AdS_5 \times S^1$.
- BCJ relation are also satisfied[Drummond, Santagata].
- But all properties require reduced amplitudes. How do they generalize to the full amplitudes?
- Do they also hold at loop level? This seems to require us to define an “integrand” for Mellin amplitudes.
- How to generalize to higher pt?
- Also investigations in momentum space [Farrow, Lipstein, McFadden; Armstrong, Lipstein, Mei; Albayrak, Kharel, Meltzer; Mei…] and in position space [Diwakar, Herderschee, Robin, Teng; Herderschee, Robin, Teng; Bissi, Fardelli, Manetti, XZ…], cf talk by Lipstein.

Hidden symmetries

- ***Parisi-Sourlas dimensional reduction***
  - In all the tree-level examples of 4pt functions, we find the contribution of a superconformal multiplet in AdS_{d+1} can be written in terms of a scalar exchange diagram in AdS_{d-1} (half maximal susy) or in AdS_{d-3} (maximal susy) acted by some difference operator [Behan, Ferrero, XZ; Alday, Behan, Ferrero, XZ].

$$\mathcal{M}^{AdS_{d+1}}_{\text{mult. } p} = \widehat{D}_1 \circ \mathcal{M}^{AdS_{d-3}}_{\epsilon p,0}$$

maximal susy: $AdS_4 \times S^7, AdS_5 \times S^5, AdS_7 \times S^4$

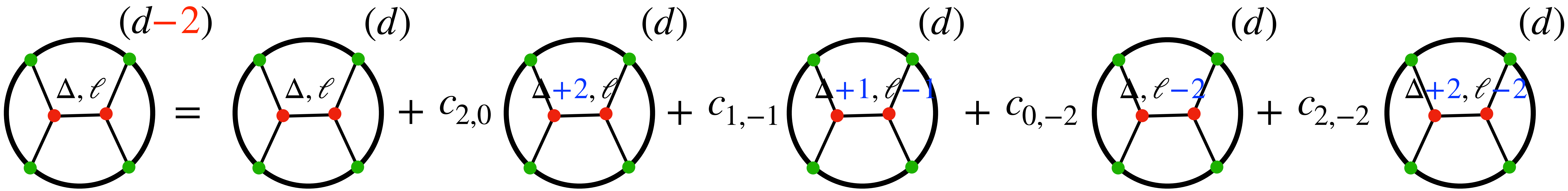
$$\mathcal{M}^{AdS_{d+1}}_{\text{mult. } p} = \widehat{D}_2 \circ \mathcal{M}^{AdS_{d-1}}_{\epsilon p,0}$$

half maximal susy: $AdS_7 \times S^4$

  - This phenomenon has been observed in all maximal and half maximal susy theories for all KK modes and is at the level of full Mellin amplitudes (not reduced).

Hidden symmetries

- Can be traced to the identities satisfied by the underlying Witten diagram, which reflects the Parisi-Sourlas supersymmetry [XZ]



Take the single-trace part

$$g^{(d-2)}_{\Delta,\ell} = g^{(d)}_{\Delta,\ell} + c_{2,0}g^{(d)}_{\Delta+2,\ell} + c_{1,-1}g^{(d)}_{\Delta+1,\ell-1} + c_{0,-2}g^{(d)}_{\Delta,\ell-2} + c_{2,-2}g^{(d)}_{\Delta+2,\ell-2}$$

A kinematic consequence of Parisi-Sourlas supersymmetry [Kaviraj, Rychkov, Trevisani]

- If it also persists at higher pt, it might be useful for computing higher KK correlators.

- Its appearance is quite curious: Parisi-Sourlas supersymmetry is non-unitary!

String corrections [cf talks by Dorigoni, Ferrero, Hansen, Heslop, Nocchi]

- At low energies, stringy or M-theory corrections appear as higher-derivative contact interactions. In Mellin space they are polynomials. The leading term at each order can be fixed by matching with the flat-space amplitude [Goncalves; Chester, Pufu, Yin]

$AdS_5 \times S^5$:

$$\widetilde{\mathcal{M}} = \frac{1}{N^2}\left[\frac{1}{stu} + \frac{\alpha^{(0)}_{R^4}}{\lambda^{3/2}} + \frac{\alpha^{(1)}_{D^2R^4}}{\lambda^2} + \frac{\alpha^{(0)}_{D^4R^4}(s^2+t^2+u^2) + \alpha^{(2)}_{D^4R^4}}{\lambda^{5/2}} + \frac{\alpha^{(0)}_{D^6R^4}stu + \alpha^{(1)}_{D^6R^4}(s^2+t^2+u^2) + \alpha^{(3)}_{D^6R^4}}{\lambda^3} + O(\lambda^{-7/2})\right] + O(N^{-4})$$

$$g_s = \frac{g_{YM}^2}{4\pi}$$

$$\frac{L^4}{\ell_s^4} = \lambda = g_{YM}^2 N$$

$$\Big\downarrow \quad (s,t,u) \to L^2(s,t,u)\ ,\ \ L/\ell_s \to \infty$$

Virasoro-Shapiro in flat space:

$$A^{(0)}(S,T) = -\frac{\Gamma(-\frac{S}{4})\Gamma(-\frac{T}{4})\Gamma(-\frac{U}{4})}{\Gamma(\frac{S}{4}+1)\Gamma(\frac{T}{4}+1)\Gamma(\frac{U}{4}+1)}$$

expanded in small S, T, U

String corrections

- To fix the subleading terms, we can use supersymmetric localization [Binder, Chester, Pufu, Wang]. Taking derivatives of the S^4 partition function of the mass-deformed $\mathcal{N} = 4$ SYM, one obtains integrated correlators [cf review talk by Minahan and also talk by Billo]

$$\tau_2^2 \partial_\tau \partial_{\bar{\tau}} \partial_m^2 \log Z \Big|_{m=0} = \int \prod_{i=1}^{4} dx_i \mu(\{x_i\}) \langle \mathcal{O}_2(x_1) \ldots \mathcal{O}_2(x_4) \rangle \qquad \tau = \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$$

  This gives constraints for the unfixed coefficients. A lot of work in this direction both at planar level and higher genus [Binder, Chester, Pufu, Wang; Chester; Chester, Pufu...]

- We also consider the "very strong coupling expansion" ($N \to \infty$ keeping g_{YM} fixed), revealing interesting $SL(2,\mathbb{Z})$ invariance properties involving non-holomorphic (generalized) Eisenstein series [Binder, Chester, Pufu, Wang; Chester, Green, Pufu, Wang, Wen; Dorigoni, Green, Wen ...].

- Many studies of interesting properties of integrated correlators (e.g., differential equations relating theories with different ranks [Dorigoni, Green, Wen ...]).

String corrections

- It is also possible to consider AdS Virasoro-Shapiro amplitudes at finite S, T, U [Alday, Hansen, Silva]

$$\widetilde{\mathcal{M}}(s,t) \to \lambda^{\frac{3}{2}}\widetilde{\mathcal{M}}(\sqrt{\lambda}s, \sqrt{\lambda}t) \qquad \text{expanded in large } R/\ell_s = \lambda^{1/4}$$

The leading order is just Virasoro-Shapiro amplitude. At subleading order, this would require fixing infinitely many coefficients $\alpha_{\#}^{(1)}$ and corresponds to a small curvature expansion

$$A(S,T) = A^{(0)}(S,T) + \frac{1}{\sqrt{\lambda}}A^{(1)}(S,T) + O(1/\lambda)$$

- Computed using two constraints
  - Dispersion relation relating the correlator to the single-trace massive string operators
  - Assuming a worldsheet integral with additional insertions of single-valued multiple polylogarithms

$$A^{(k)}(S,T) = \int d^2z\,|z|^{-2S-2}|1-z|^{-2T-2}G^{(k)}(S,T,z)$$

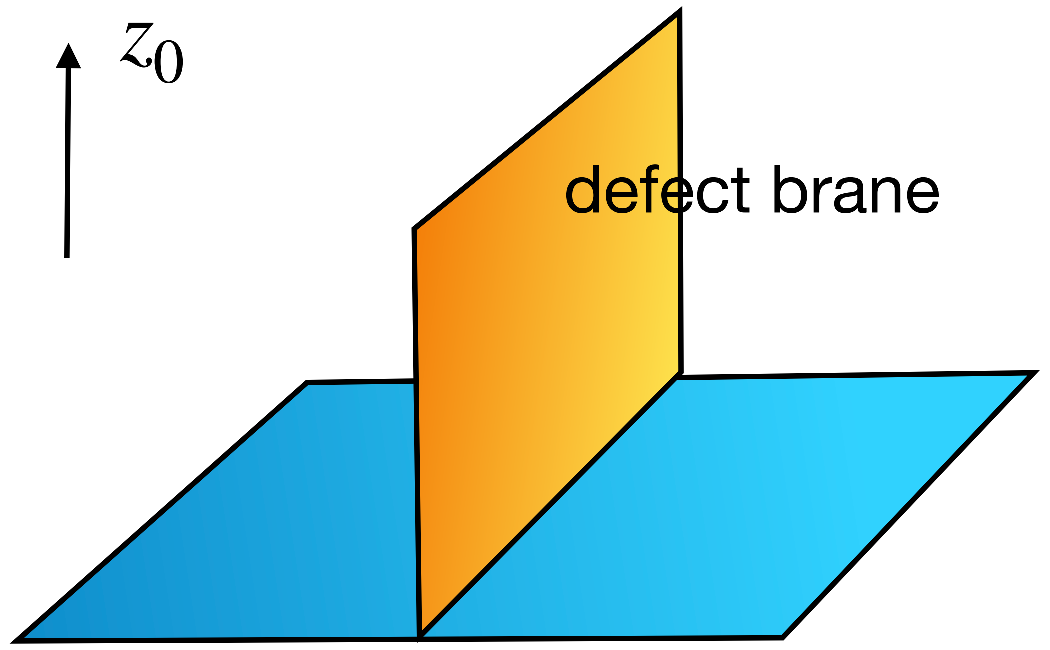
[cf talks by Hansen and Nocchi]

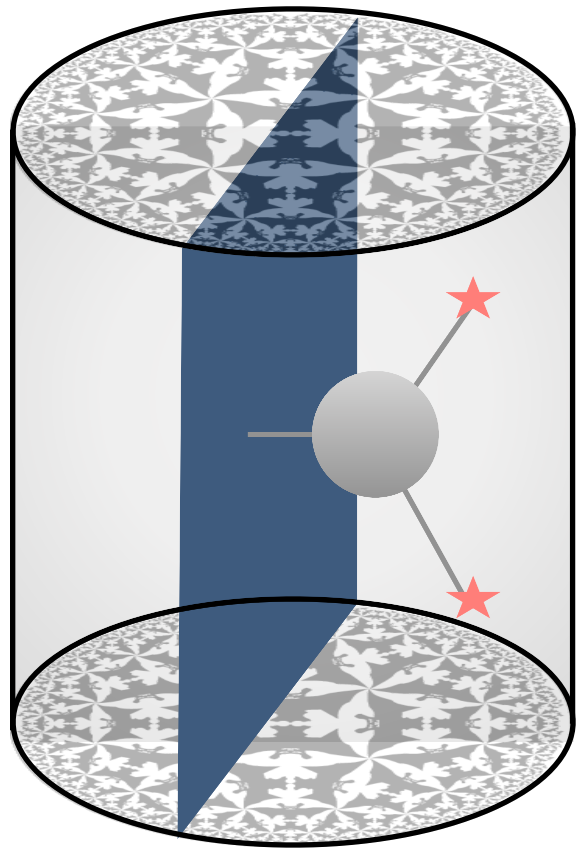
- The story was also be extended to open strings [Behan, Chester, Ferrero; Paul, Santagata; Alday, Chester, Hansen, Zhong; Alday, Hansen; talks by Hansen and Ferrero].

Defect correlators

- An interesting generalization is correlators in holographic defect CFTs. In the simplest setup, the defect is dual to a probe brane inside the bulk

$$AdS_{p+1} \times S^r \subset AdS_{d+1} \times S^n$$





e.g.,

$(p, r, d, n) = (1,0,4,5)$: WL in $\mathcal{N} = 4$ SYM

$(p, r, d, n) = (3,2,4,5)$: interface in $\mathcal{N} = 4$ SYM

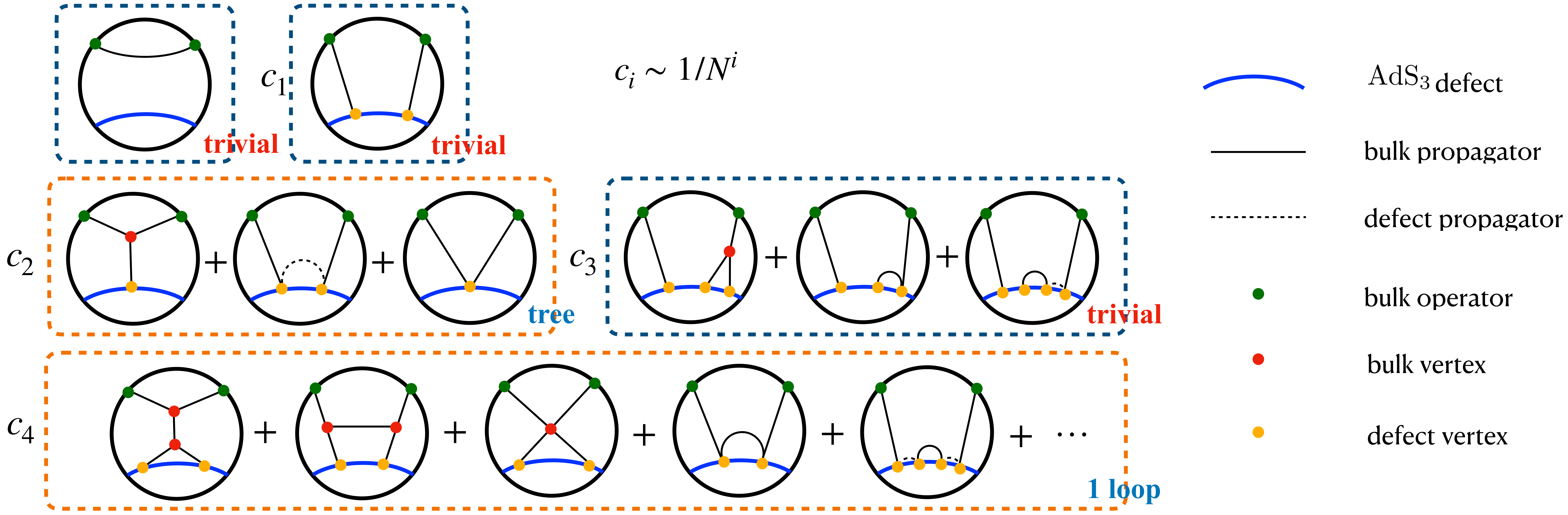
$(p, r, d, n) = (2,0,6,4)$: surface defect 6d (2,0) theory

- When operators are all inserted on the defect, holographic correlators are kinematically similar to defect-free CFTs and the simplest case is 4pt (e.g., WL in $\mathcal{N} = 4$ SYM [Giombi, Roiban, Tseytlin; Meneghelli, Ferrero], surface defect in 6d (2,0) theory [Drukker, Giombi, Tseytlin, XZ]).

- When operators are inserted away from the defect, we get interesting new observables corresponding to form factors of AdS particles scattering off an extended object.

Defect correlators

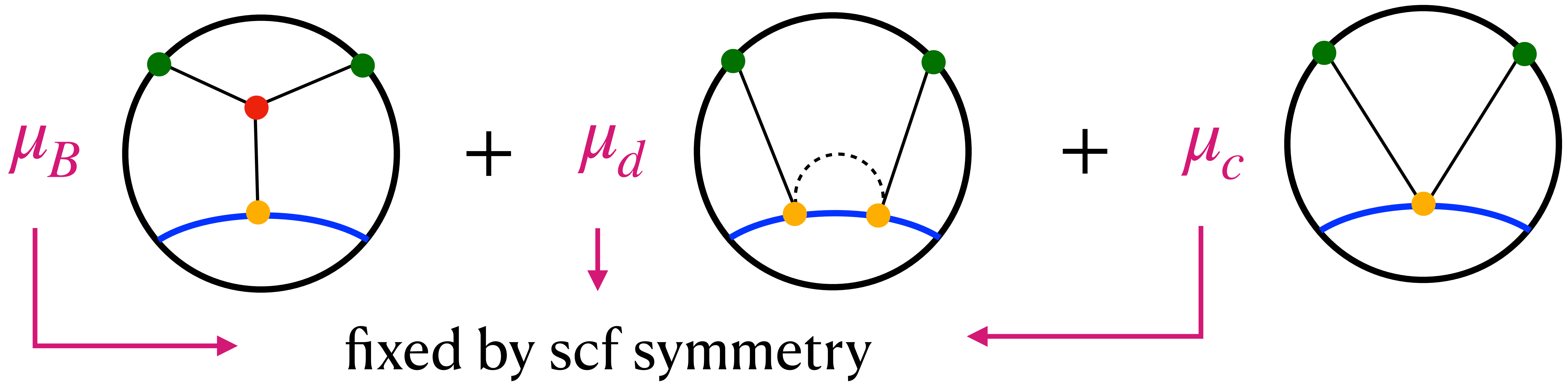
- The first nontrivial case is 2pt function of bulk operators. Such holographic correlators have been studied using analytic conformal bootstrap techniques (inversion formula etc) in [Barrat, Gimenez-Grau, Liendo; Meneghelli, Trepanier]. A more direct and systematic bootstrap analysis using Witten diagrams was initiated in [Gimenez-Grau].
- At large central charge (for half-BPS surface defects in 6d (2,0) theory)



Defect correlators

- At tree-level, we can apply the position space approach [Gimenez-Grau; Chen, Gimenez-Grau, XZ]

$$\langle S_{k_1} S_{k_2} V \rangle_{\text{tree}} = \sum$$



- The results are simplest in Mellin space [Rastelli, XZ '17, Goncalves, Itsios '18]. For example, for surface defects in 6d (2,0) theory [Chen, Gimenez-Grau, XZ]

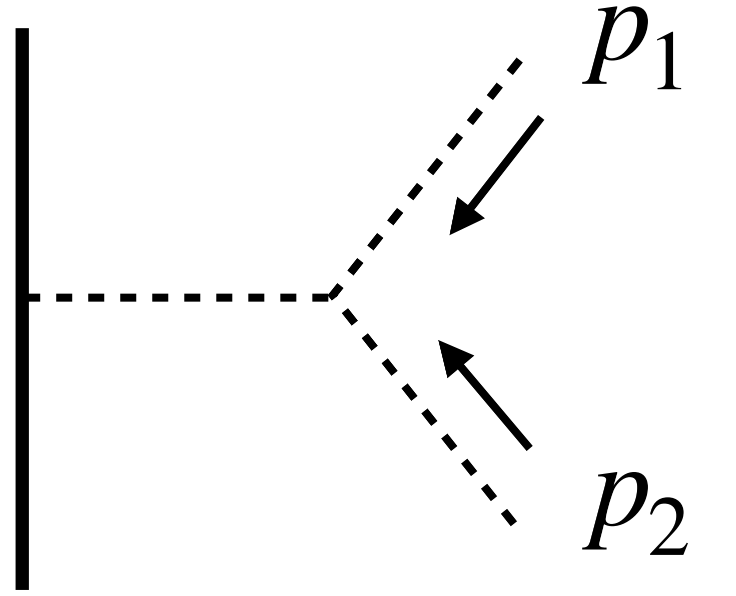
$$\mathscr{H} = \int \frac{d\delta\, d\gamma}{(2\pi i)^2} B^{-\delta} D^{\gamma} \widetilde{\mathscr{M}}(\delta,\gamma) \widetilde{\Gamma}_{k_1 k_2}(\delta,\gamma)$$

B, D: bulk & defect channel cross ratios

$$\widetilde{\mathscr{M}}(\delta,\gamma,\sigma) = \sum_{i=1}^{2k_m-2} \sum_{j=2}^{k_m} \frac{\mathfrak{R}_{ij}(\sigma)}{(\delta+i)(\gamma-2j)}$$

polynomial in σ

$k_m = \min\{k_1, k_2\}$

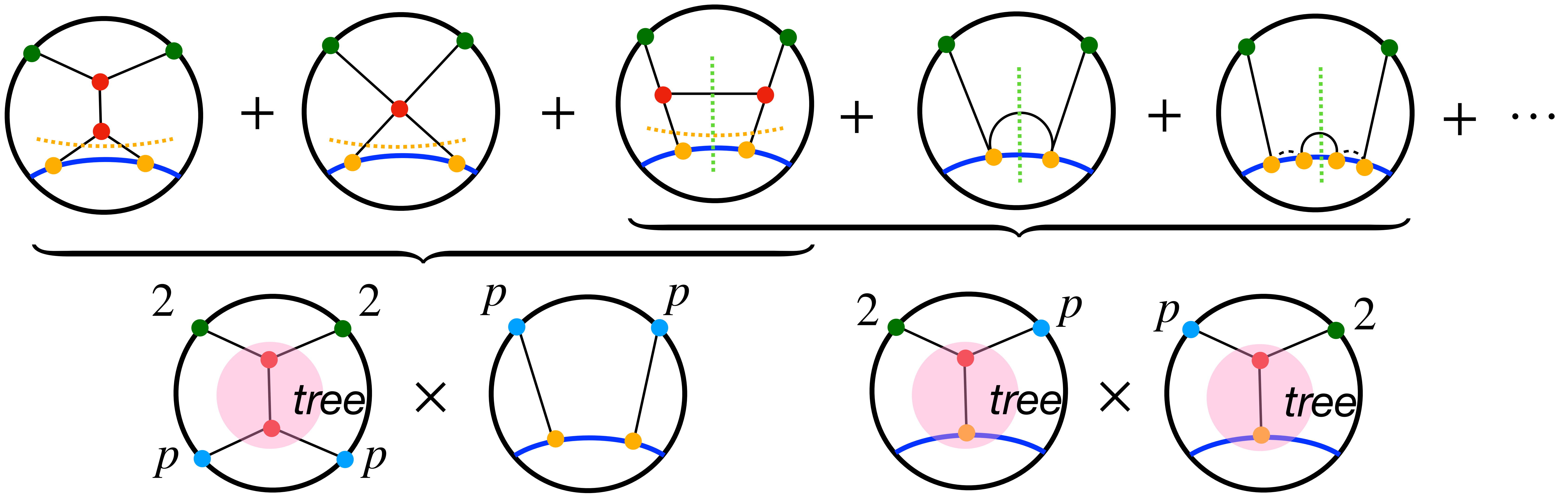


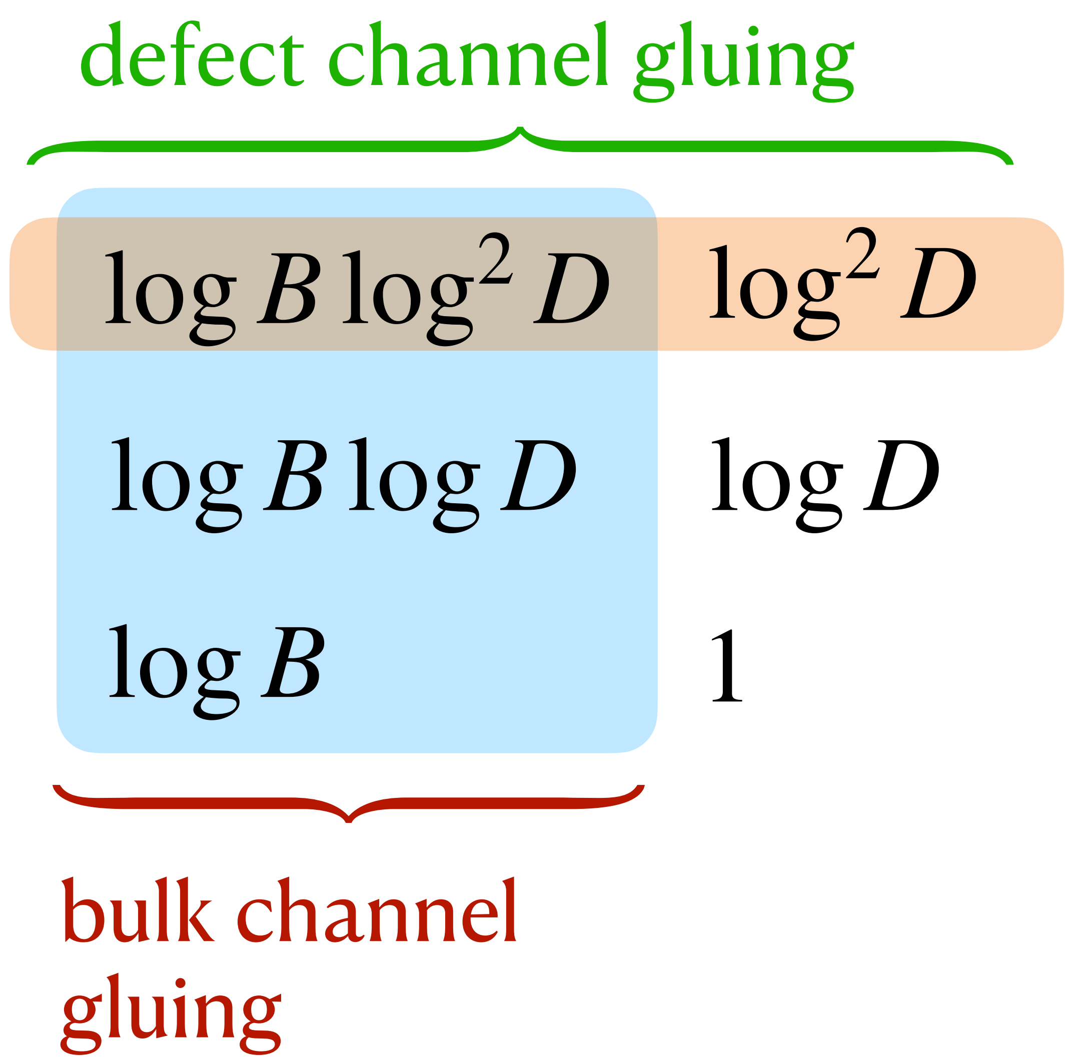
$$\delta = p_1 \cdot p_2$$

$$\gamma = -p_{1,\parallel}^2 = -p_{2,\parallel}^2$$

Defect correlators

- At 1-loop level, we can compute them by gluing together tree-level correlators following a similar strategy as in 4pt functions [Chen, Gimenez-Grau, Paul, XZ]




defect channel gluing

$\log B \log^2 D$	$\log^2 D$
$\log B \log D$	$\log D$
$\log B$	1

bulk channel gluing

- The Mellin amplitude are simultaneous poles with constant numerators.

$$\widetilde{\mathcal{M}}(\delta,\gamma)=\sum_{m,n=0}^{\infty}\frac{c_{mn}}{(\delta+n)(\gamma-6-2m)}$$

with c_{mn} given by ${}_3F_2$

Defect correlators

- Still quite preliminary
  - A zoo of theories to study: so far only WL in 4d $\mathcal{N} = 4$ SYM [Barrat, Gimenez-Grau, Liendo; Gimenez-Grau] and surface defect in 6d (2,0) theory [Meneghelli, Trepanier; Chen, Gimenez-Grau, XZ; Chen, Gimenez-Grau, Paul, XZ].
  - No analogue of position space method at loop level which will be useful for 2-loop and beyond.
  - We do not how to rigorously take the flat-space limit.
  - One can also consider constraints from localization and integrability [de Leeuw, Kristjansen, Zarembo; Wang, Komatsu, Wang...]
  - Integrated 2pt functions and scattering from (p,q)-strings [Pufu, Rodriguez, Wang; Billo, Galvagno, Frau, Lerda; cf talk by Rodriguez and Billo].

Outlook

Clearly much more to do…

- Higher pts: on-shell recursion relations a la BCFW?
- More efficient tools at loop level to go to 3 loops and beyond?
- A more direct use of integrability?
- Systematic generalization of flat-space amplitude properties: what formalism? How to include more spinning operators?
- AdS string amplitudes: all order curvature corrections?

Many other interesting topics

- Finite temperature and black holes [Iliesiu, Kologlu, Mahajan, Perlmutter, Simmons-Duffin; Alday, Kologlu, Zhiboedov; Dodelson, Ooguri; also talk by Vichi]
- Correlators involving determinant operators [Jiang, Komatsu, Vescovi; Jiang, Wu, Zhang; Brown, Galvagno, Wen]
- Scattering equations and AdS CHY formalism [Eberhardt, Komatsu, Mizera; Roehrig, Skinner]
- Bootstrability [Cavaglia, Gromov, Julius, Preti; talk by Gromov]
- Correlators in dS [Sleight, Taronna; Gomez, Jusinskas, Lipstein; talk by Lipstein]

Thank you!