Integrated correlators in a $\mathcal{N} = 2$ SYM theory

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Introduction

- $\mathcal{N} = 2$ conformal SYM theories in $d = 4$ (of which $\mathcal{N} = 4$ is a particular case) are at te crossroads of the topics of this workshop: localization, holographic duality, conformal bootstrap
- Localization allows to compute the partition function and the vev of BPS Wilson loops also for theories "deformed" by mass terms and coupling to chiral operators.
- Derivatives w.r.t. these parameter corresponds to integrated correlators
  - 4pt integrated correlators
  - 2 pt integrated correlators in presence of a WL defect

Introduction

4 pt integrated correlators in $\mathcal{N} = 4$

- We heard (and are going to hear) quite a bit in this workshop about localization (review by Minahan) and about integrated correlators.
  - Holograhic dual to (integrated) scattering processes in AdS. They place constraints on holographic correlators
  - Provide data for a CFT bootstrap approach to the latter (in the WL case, in DCFT)
- 4pt integrated correlators first introduced in $\mathcal{N} = 4$

  Binder et al, 2019; Chester, 2019; Chester et al 2020; ...
  - Modular properties (talk by Dorigoni)

    Dorigoni et al, 2021, 2022; Paul et al, 2022, Wen et al, 2022; Alday et al, 2023; ...
  - Different gauge groups Dorigoni et al, 2022; ...
  - Generic or large charge insertions

    Brown et al, 2023; Paul et al, 2023; Caetano et al, 2023
  - Similar definition and rôle of integrated correlators also in $d = 3$ ABJM (talk by Nosaka)

Introduction

Integrated correletors with a Wilson loop

- 2pt integrated correlators with a WL were introduced for $\mathcal{N} = 4$ in Pufu et al, 2023. Give inputs on scattering processes off extended strings in AdS
  - Measure fixed partially in Billo et al, 2023, finalized in Dempsey et al, 2024; Billo et al, 2024
- The abstract of my talk included this issue; however I will not discuss it, since Yifan Wang will
- However, the matrix model techniques I will illustrate could be useful for computing such observables in $\mathcal{N} = 2$ theories

Introduction

4 pt integrated correlators in $\mathcal{N} = 2$

4pt integrated correlators have been considered also in $\mathcal{N} = 2$ contexts

- In $\mathcal{N} = 2$ SQCD Fiol et al, 2023
- In a particular theory with Usp($2N$) gauge group, dual to type IIB on $AdS_5 \times S_5/\mathbb{Z}_2$ with D7 branes Behan et al, 2023
  - Inputs for the construction of open string scattering amplitudes in AdS Alday et al, 2024 (talks by Zhou and Hansen)
- In non-lagrangian $\mathcal{N} = 2$ theories, dual to F-theory setups Behan et al, 2024. Cannot rely on localization (talk by Ferrero)
- In the so-called E theory Billo et al,2023 (I will focus on this!) and in a quiver theory Pini et al, 2024

Motivations

and scope of the talk

- Pestun's localization of an observable reduces its path integral to a finite-dimensional matrix integral: it is a huge step forward!
- While for $\mathcal{N} = 4$ the matrix model is gaussian, it is interacting for $\mathcal{N} = 2$ theories.
- Computations might not be straight-forward, since to obtain info on the holographic dual one needs to extrapolate results to strong coupling

Motivations

and scope of the talk

- Our group (Turin U. and others) developed an approach which has proven useful in several cases
- My talk will be thus limited in scope, focusing on the matrix model side of the $\mathcal{N} = 2$ integrated correlators
- Based on arXiV:2311.17178 (M.B., Frau, Lerda and Pini)

Content of $\mathcal{N} = 2$ theories

The first slides will be very basic – especially after this morning review. I apologize if too basic!

- Vector multiplet in the adjoint of the gauge group – for us it will be SU(N):

$$A_\mu,\ \phi\ +\ \text{gauginos}$$

- Hypermultiplet in a representation $\mathcal{R}$ (can be reducible)

$$q,\ \tilde{q}\ +\ \text{hyperinos}$$

  These matter fields can be massive.

- Massless theories are conformal iff the one-loop β-function coefficient vanishes:

$$i_{\mathcal{R}} = N$$

Classes of conformal $\mathcal{N} = 2$ theories

- For the fundamental and the two-index symmetric and antisymmetric reps one has

$$i_{fun} = 1/2 \,, \quad i_{sym} = (N+2)/2 \,, \quad i_{asym} = (N-2)/2$$

- The following theories are superconformal ($i_{\mathcal{R}} = N$):

theory	representation $\mathcal{R}$	massive version
$\mathcal{N} = 4$	$adj = fun \otimes \overline{fun} - \cdot$	$\mathcal{N} = 2^*$
A	$2N\, fun$	
B	$(N-2)\, fun \oplus sym$	
C	$(N+2) fun \oplus asym$	
D	$4 fun \oplus 2 asym$	
E	$sym \oplus asym = fun \otimes fun - \cdot$	E*

Localization formulæ

- The partition function of a generic $\mathcal{N} = 2$ theory on S_4 of radius r localizes Pestun 2007

$$A_\mu(x) \to \text{instanton solution} , \quad \phi(x) \to a$$

where a is constant matrix in $su(N)$. Then

$$\mathcal{Z} = \int da \, e^{-\frac{8\pi^2 r^2}{g^2} \text{tr}\, a^2} \, |Z_{inst}|^2 \, |Z_{1-loop}|^2$$

where we highlighted the saddle point value and the fluctuation determinant about it

- In the 't Hoof limit with $N \to \infty$ and $\lambda = g^2 N$ fixed, instantons are suppressed and $Z_{inst} \to 1$

The determinant factor

- For a theory with $i_{\mathcal{R}} = N$, also massive, the infinite products in the 1-loop determinant can be expressed in terms of Barnes' G-functions

$$|Z_{1-loop}|^2 = \frac{\prod_\alpha H(\mathrm{i}\alpha \cdot \mathbf{a}r)}{\prod_{\mathbf{w}} \left[H(\mathrm{i}\mathbf{w} \cdot \mathbf{a}r + \mathrm{i}mr)\, H(\mathrm{i}\mathbf{w} \cdot \mathbf{a}r - \mathrm{i}mr) \right]^{1/2}}$$

- α are the roots, $\mathbf{w}$ the weights of $\mathcal{R}$ and $\mathbf{a}$ the vector of eigenvalues of a. Moreover

$$H(x) = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)^n \mathrm{e}^{\frac{x^2}{n}} \equiv \mathrm{e}^{(1+\gamma)x^2}\, G(1+x)\, G(1-x)$$

- We regard the 1-loop determinant as an interaction action for the matrix model

$$|Z_{1-loop}|^2 = \mathrm{e}^{-S_{int}}$$

A side remark

- Asymptotically free theories with $i_{\mathcal{R}} < N$ and a dynamically generated sale Λ must be embedded in a larger theory with $i_{\mathcal{R}^*} = N$ and mass M

  Pestun 2007; Russo, Zarembo 2013; M.B., Griguolo, Testa 2023, ...
- In the regime

$$\Lambda << 1/r << M\,,$$

  the matrix model is formally analogous, but the coupling in the Gaussian term becomes the running coupling $g^2(r)$
- Does it still provide some information on the theory on $\mathbb{R}^4$?
- For the WL vev at two loop order M.B., Griguolo, Testa 2023 and three loops (quite hard) it does in preparation

Mass expansion of the interaction action

- For the integrated correlators, one is interested in the mass expansion of the model Forward.

$$S_{int} = S_0 + m^2 S_2 + m^4 S_4 + \dots$$

- Since $S_{int} = \log \left| Z_{1-loop} \right|^2$ one finds

$$S_0 = \log |Z_{1-loop}|^2 = \mathrm{Tr}_{\mathcal{R}} \log H(\mathrm{i}\, a\, r) - \mathrm{Tr}_{adj} \log H(\mathrm{i}\, a\, r) \ ,$$

$$S_2 = -\frac{1}{2} \mathrm{Tr}_{\mathcal{R}} \partial^2 \log H(\mathrm{i}\, a\, r)$$

...

- For the $\mathcal{N} = 2^*$ theory $S_0 = 0$ since $\mathcal{R} = adj$.
- In the $N = 4$ theory $m = 0$, thus $S_{int} = 0$. The matrix model is gaussian

The interaction action in terms of traces

Explicit expressions follow from

- Expanding

$$\log H(x) = -\sum_{n=1}^{\infty} \frac{\zeta_{2n+1}}{n+1} x^{2n+2}$$

- Rewriting traces in the $\mathcal{R}$ and the adjoint rep in terms traces in the fundamental. In particular ▸ Forward.

$$\mathrm{Tr}_{adj}\, a^{2k} = \sum_{l=2}^{2k-2} (-1)^l \binom{2k}{l} \mathrm{tr}\, a^l\, \mathrm{tr}\, a^{2k-l} \,,$$

$$\mathrm{Tr}_{E}\, a^{2k} = \sum_{l=2}^{2k-2} \binom{2k}{l} \mathrm{tr}\, a^l\, \mathrm{tr}\, a^{2k-l}$$

Interacting matrix model

- Localization allows the computation of vevs of certain BPS operators (Wilson loop, correlators of chiral ops...)
- Such operators are mapped to "gauge invariant" matrix operators $f(a)$ and

$$\langle f(a) \rangle = \frac{1}{\mathcal{Z}} \int da\, f(a)\, e^{-\mathrm{tr}\, a^2} e^{-S_{int}(a)} = \frac{\langle f(a)\, e^{-S_{int}(a)} \rangle_0}{\langle e^{-S_{int}(a)} \rangle_0}$$

  - By $\langle \cdots \rangle_0$ we mean the vev in the Gaussian matrix model
  - We rescaled the matrix by

$$a \to \sqrt{\frac{g^2}{8\pi^2 r^2}}$$

- Typically, as is the case for $S_{int}(a)$, also $f(a)$ can be written in terms of traces of powers of a

Full Lie algebra approach

- The core ingredients are the Gaussian vevs of multitraces:

$$t_{k_1,k_2,\dots} \equiv \langle \mathrm{tr}\, a^{k_1}\, \mathrm{tr}\, a^{k_2}\, \dots \rangle_0$$

- A standard approach is to rotate a to the Cartan subalgebra and express everything in terms of its eigenvalues, picking up a Vandermonde factor in the measure
- In the so-called "full Lie algebra approach" one expands the matrix a on all generators:

$$a = a_b T^b \,, \quad \mathrm{tr}\, T^b T^c = \frac{1}{2}\delta^{bc} \,, \quad \int da = \int \prod_b \frac{da_b}{\sqrt{2\pi}}$$

- In this way one gets the gaussian "propagator"

$$\langle a_b\, a_c \rangle_0 = \delta_{bc}$$

Recursion relations

- From the "fission/fusion" identities

$$\mathrm{tr}\, T^b B_1 T^b B_2 = \frac{1}{2} \mathrm{tr}\, B_1 \mathrm{tr}\, B_2 - \frac{1}{2N} \mathrm{tr}\, B_1 B_2 \;,$$
$$\mathrm{tr}\, T^b B_1 \mathrm{tr}\, T^b B_2 = \frac{1}{2} \mathrm{tr}\, B_1 B_2 - \frac{1}{2N} \mathrm{tr}\, B_1 \mathrm{tr}\, B_2$$

  and the gaussian propagator follow powerful recursive relations for the vevs of multitraces $t_{k_1,k_2,\ldots}$ Billo et al, 2017
- Such recursive relation simplify in the large-N limit

Correlators of chiral operators

- Extremal correlators of protected operators $O_k = \mathrm{tr}\,\phi^k(x)$, such as $\langle O_{k_1}(x1)\ldots O_{k_n}(x_n)\,\bar{O}_p(y)\rangle$, have a fixed coordinate dependence and are captured by localization

  Baggio et al, 2014; Gerkovits et al, 2016; ...

- Since in field theory $O_k(x)$ is normal ordered, at the operator level the map to the matrix model is

$$O_k(x) \to \mathcal{O}_k \equiv :\mathrm{tr}\,a^k:$$

- The operators $\mathcal{O}_k$, orthogonal to all those of lower dimension, can be determined by a Gram-Schmid procedure

Tree level normal ordering

- In the gaussian model, one can write in closed form at the leading order for large N Rodriguez-Gomez et al 2016 a set of normal ordered operators satisfying

$$\langle \mathcal{P}_k \rangle_0 = 0 \,, \quad \langle \mathcal{P}_k \, \mathcal{P}_l \rangle_0 = \delta_{k,l}$$

- The map form the field theory operators is

$$O_k(x) \to \mathcal{O}_k = \mathcal{G}_k^{(0)} \mathcal{P}_k \,, \quad \mathcal{G}_k^{(0)} = k \, (N/2)^k .$$

- The map to the basis of the traces is

$$\mathrm{tr}\, a^k = \left(\frac{N}{2}\right)^{\frac{k}{2}} \sum_{\ell=0}^{\lfloor \frac{k-1}{2} \rfloor} \sqrt{k - 2\ell} \binom{k}{\ell} \mathcal{P}_{k-2\ell} + \langle \mathrm{tr}\, a^k \rangle_0$$

- The $\mathcal{P}$ form a convenient basis of operators also when interactions are included but a further coupling-dependent normal ordering is needed.

Large N factorization

- In the planar limit, as a consequence of the recursion relations on the multitrace vevs, the gaussian correlators of many $\mathcal{P}_k$ operators are computed à la Wick with the propagator $\langle \mathcal{P}_k \mathcal{P}_l \rangle_0 = \delta_{k,l}$ Beccaria et al, 2020
- They behave as real variables p_k with a Gaussian weight:

$$\langle \mathcal{P}_{k_1} \mathcal{P}_{k_2} \dots \mathcal{P}_{k_n} \rangle_0 = \int \mathcal{D}\mathbf{p}\, p_{k_1} p_{k_2} \dots p_{k_n}\, e^{-\frac{1}{2}\mathbf{p}^T \mathbf{p}} + O(N^{-1})$$

- Correlators of an odd number of $\mathcal{P}$'s are subleading in $1/N$. In particular, the 3-point functions are (for $k + \ell + p$ even)

$$\langle \mathcal{P}_k \mathcal{P}_\ell \mathcal{P}_n \rangle_0 = \frac{1}{N} \sqrt{k\ell p} \equiv \frac{1}{N} d_{k,\ell,n} \,,$$

- At order $1/N$, just add this vertex to the above free theory

The E theory

- For the E theory using the above results Back, the interaction action reads Forward

$$S_0 = 4 \sum_{n,\ell=1}^{\infty} (-1)^{n+\ell} \frac{(2n+2\ell+1)!\,\zeta_{2n+2\ell+1}}{(2n+1)!\,(2\ell+1)!} \left(\frac{\lambda}{8\pi^2 N}\right)^{n+\ell+1} \mathrm{tr}\, a^{2n+1} \mathrm{tr}\, a^{2\ell+1}$$

  - Quadratic in odd traces only
  - Written as an expansion in the coupling λ
- In terms of the $\mathcal{P}_k$ operators it becomes

$$S_0 = -\frac{1}{2} \sum_{k,\ell=1}^{\infty} \mathcal{P}_{2k+1}\, \mathsf{X}_{2k+1,2\ell+1}\, \mathcal{P}_{2\ell+1}$$

where in the coefficients we can resum the perturbative expansion: Forward

$$\mathsf{X}_{n,m} = 2\,(-1)^{\frac{n+m+2nm}{2}+1} \sqrt{nm} \int_0^{\infty} \frac{dt}{t} \frac{1}{\sinh(t/2)^2} J_n\!\left(\frac{t\sqrt{\lambda}}{2\pi}\right) J_m\!\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

Even and odd

- The interactions only involve the odd operators $\mathcal{P}_{2k+1}$. Observables in the **E** theory involving only even operators are planar equivalent to $\mathcal{N} = 4$
- The dual holographic theory is an orbifold/orientifold of $AdS_5 \times S^5$ Ennes et al, 2000: the odd operators are dual to twisted fields
- Writing $(\mathrm{X}^{odd})_{k,\ell} \equiv \mathrm{X}_{2k+1,2\ell+1}$ the partition function is

$$\mathcal{Z}_E = \int \mathcal{D}\mathbf{p}_{even}\, e^{-\frac{1}{2}\mathbf{p}_{even}^T \mathbf{p}_{even}} \int \mathcal{D}\mathbf{p}_{odd}\, e^{-\frac{1}{2}\mathbf{p}_{odd}^T (\mathbf{1}-\mathrm{X}^{odd})\mathbf{p}_{odd}}$$

$$= \det\left(\mathbf{1} - \mathrm{X}^{odd}\right)^{-\frac{1}{2}}$$

so the free energy is $\mathcal{F}_E = -\frac{1}{2}\mathrm{tr}\,\log(\mathbf{1} - \mathrm{X}^{odd})$

Basic 2pt correlators in the E Theory

even case

- In the even case, $\mathcal{O}_{2q} = \mathcal{P}_{2q} - \langle \mathcal{P}_{2q} \rangle$, with $\langle \mathcal{P}_{2q} \rangle = O(1/N)$
- The 2-pt function of the even $\mathcal{P}$ operators is unchanged:

$$\langle \mathcal{P}_{2k} \, \mathcal{P}_{2\ell} \rangle = \delta_{k\ell} + O(N^{-2})$$

- The normal ordered even operators are simply

$$\mathcal{O}_{2q} = \mathcal{G}_{2q}^{(0)} \left(\mathcal{P}_{2q} - \langle \mathcal{P}_{2q} \rangle \right)$$

so that

$$\mathcal{G}_{2q} \equiv \langle \mathcal{O}_{2q} \, \mathcal{O}_{2q} \rangle = \mathcal{G}_{2q}^{(0)} \left((1 + O(1/N^2) \right)$$

Basic 2pt correlators in the E Theory
Odd case

- The 2pt function of odd $\mathcal{P}$'s is now

$$\langle \mathcal{P}_{2k+1}\, \mathcal{P}_{2\ell+1} \rangle = (\mathsf{D}^{odd})_{k,\ell} + O(N^{-2}) \,, \qquad \mathsf{D}^{odd} = \frac{1}{\mathbf{1} - \mathsf{X}^{odd}}$$

- The odd $\mathcal{P}$ operators are no longer normal ordered and

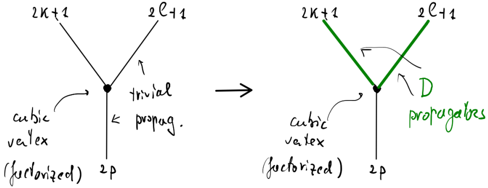
$$\mathcal{O}_{2q+1} = \sqrt{\mathcal{G}^{(0)}_{2q+1}} \Big(\mathcal{P}_{2q+1} - \sum_{q'<q} \mathsf{Q}_{q,q'} \mathcal{P}_{2q'+1} \Big)$$

where the λ-dependent coefficients $\mathsf{Q}_{q,q'}$ can be expressed in terms of the matrix D^{odd} via Gram-Schmid

Forward

Basic 3-pt correlators in the E Theory

- Taking into account the interaction action the correlator of three even $\mathcal{P}$ operators stays the same, while



$$\langle \mathcal{P}_{2q+1}\, \mathcal{P}_{2r+1}\, \mathcal{P}_{2k} \rangle = \frac{1}{N}\, \mathsf{d}_{2q+1}\, \mathsf{d}_{2r+1} \sqrt{2k} + O(N^{-2}) \ ,$$

$$\mathsf{d}_{2q+1} = \sum_{q'=1}^{\infty} (\mathsf{D}^{\text{odd}})_{q,q'}\ \sqrt{2q'+1}$$

Forward

“Exact” expressions

- All the basic computational ingredients in the interacting matrix model for the E theory can be expressed in terms of the infinite matrices X and $\mathsf{D} = 1/(\mathbf{1} - \mathsf{X})$
- The dependence on the coupling is resummed into Bessel functions. Back, One can easily derive long perturbative series or (less easily) asymptotic expansions for large λ
- This is crucial for comparisons with the dual holographic description
- Equipped with the above tools, let us now consider a specific class of observables in this theory

Integrated correlators

Foreword

- The coordinate dependence of non-extremal correlators with more than three operators is not fixed by conformal symmetry symmetry. They cannot be encoded in a matrix model.
- However, integrated 4-point functions can

  Binder et al, 2019; Chester, 2019; long list ...
- Such observables correspond to derivatives w.r.t. to parameters in the partition function.
- They were first introduced in the $\mathcal{N} = 4$ theory. We just recall briefly their definition then move to the E theory

Parameters in the partition function

- Even if we're interested in a massless $\mathcal{N} = 2$ theory, we can turn on masses for the hypers in the representation $\mathcal{R}$. Pestun's localization still applies.
- Deforming the Lagrangian on S^4 in a susy invariant way by adding with couplings τ_p the multiplet containing the chiral operators $O_p(x)$ has a matrix model description

Gerkovits et al, 2016

- It is thus possible to compute expressions such as

$$\partial_{\tau_p}\partial_{\bar{\tau}_p}\partial_m^2 \, Z(m,\tau_p,\bar{\tau}_p)|_{defs=0} \, ,$$
$$\partial_m^4 \, Z(m,\tau_p,\bar{\tau}_p)|_{defs=0}$$

exploiting the power of localization

The $\mathcal{N} = 2^*$ mass terms on the sphere

- These expressions are related to the integrated correlators of certain operators in the $\mathcal{N} = 2$ theory on flat space
- For instance, the mass terms that deform the $\mathcal{N} = 4$ action on S^4 into $\mathcal{N} = 2*$ are of the form see Binder et al, 2019

$$\int d^4x \sqrt{g(x)} \left(\frac{\mathrm{i}\, m}{r} \mathcal{J}(x) + m\, \mathcal{K}(x) + m^2\, \mathcal{L}(x) \right)$$

  - $m\,\mathcal{K}(x)$ and $m^2\,\mathcal{L}(x)$ are the usual mass terms for the fermions and the bosons in the hypermultiplet
  - $\mathcal{J}(x)$ is a moment map operator $\mathrm{tr}\,(q^2 + \tilde{q}^2 + (q^*)^2 + (\tilde{q}^*)^2)$
- Two mass derivatives bring down two integrated $\mathcal{J}$ or $\mathcal{K}$ operators (or one $\mathcal{L}$, but this has been shown not to contribute Binder et al,2019; ...)

Integrated 4pt correlators

in flat space for the $\mathcal{N} = 4$ theory

- Deriving w.r.t.the τ_p and $\bar{\tau}_p$ couplings brings down the operators $O_p(N)$, $\bar{O}_p(S)$ an the North, South pole

Gerkovitz et al, 2026

- The integrated $\langle \mathcal{J}\, \mathcal{J}\, O_p\, \bar{O}_p \rangle$ and $\langle \mathcal{K}\, \mathcal{K}\, O_p\, \bar{O}_p \rangle$ correlators
  - can be conformally mapped to $\mathbb{R}^4$;
  - are related by a a susy Ward identity
- One can write the result, with a specific measure μ, as

$$\partial_{\tau_p}\partial_{\bar{\tau}_p}\partial_m^2 \log \mathcal{Z}_{\mathcal{N}=2^*}\Big|_{defs=0}$$

$$= \int \prod_{i=1}^{4} dx_i\ \mu(\{x_i\})\, \langle \mathcal{O}_p(x_1)\, \bar{\mathcal{O}}_p(x_2)\, \mathcal{J}(x_3)\, \mathcal{J}(x_4) \rangle_{\mathcal{N}=4, flat}\ .$$

- The l.h.s. can be computed by localization. This gives a constraint on the 4pt correlator on the r.h.s.

Integrated 4pt correlators

for the E theory

- Only the details of the mass deformation change
  - One can give two different masses to the hypers in the symmetric and antisymmetric representation. Here for simplicity $m_S = m_A = m$
- Thus we have

$$\mathcal{I}_p \equiv \partial_{\tau_p}\partial_{\bar{\tau}_p}\partial_m^2 \log \mathcal{Z}_{E^*}\Big|_{defs=0}$$
$$= \int \prod_{i=1}^{4} dx_i \, \mu(\{x_i\}) \left\langle \mathcal{O}_p(x_1)\, \bar{\mathcal{O}}_p(x_2)\, \mathcal{J}_E(x_3)\, \mathcal{J}_E(x_4) \right\rangle_{E,flat} .$$

- The computation of the l.h.s. is now more difficult: the matrix model of the undeformed E theory is interacting
- We exploit the “full Lie algebra” method introduced above

Correlators in the matrix model

that capture the integrated correlators

- Recall the mass expansion of the matrix model Back
  It is rather direct to see that

$$\mathcal{I}_p = \langle \mathcal{O}_p \, \mathcal{O}_p \, S_2 \rangle - \langle \mathcal{O}_p \, \mathcal{O}_p \, S_2 \rangle = \langle\!\langle \mathcal{O}_p \, \mathcal{O}_p \, S_2 \rangle\!\rangle$$

  - The operators $\mathcal{O}_p \mathcal{O}_p$ must be connected with S_2
  - The vevs are in the E theory matrix model
- Just as the interaction action S_0 of the E theory Back also S_2 can be expanded in traces:

$$S_2 = \sum_{n=1}^{\infty} \sum_{\ell=0}^{2n} (-1)^n \frac{(2n+1)! \, \zeta_{2n+1}}{(2n-\ell)! \, \ell!} \left(\frac{\lambda}{8\pi^2 N} \right)^n \mathrm{tr}\, a^{2n-\ell} \mathrm{tr}\, a^{\ell}$$

but here both even and odd powers of the matrix a appear.

The mass deformation operator S_2

- Since the even traces have a vev, passing to the $\mathcal{P}$ basis the mass deformation operator S_2 gets terms with 0, 1 or 2 $\mathcal{P}$'s:

$$S_2 = S_2^{(0)} + S_2^{(1)} + S_2^{(2)}$$

- In each term the λ expansion can be resummed in terms of Bessel functions. The basic quantity is an infinite matrix M, similar to X:

$$\mathsf{M}_{0,0} = \int_0^\infty \frac{dt}{t} \frac{(t/2)^2}{\sinh(t/2)^2}\left[1 - \frac{16\pi^2}{t^2\lambda} J_1\left(\frac{t\sqrt{\lambda}}{2\pi}\right)^2\right],$$

$$\mathsf{M}_{0,n} = (-1)^{\frac{n}{2}+1}\sqrt{n}\int_0^\infty \frac{dt}{t} \frac{(t/2)^2}{\sinh(t/2)^2}\left(\frac{4\pi}{t\sqrt{\lambda}}\right) J_1\left(\frac{t\sqrt{\lambda}}{2\pi}\right) J_n\left(\frac{t\sqrt{\lambda}}{2\pi}\right),$$

$$\mathsf{M}_{n,m} = (-1)^{\frac{n+m+2nm}{2}+1}\sqrt{nm}\int_0^\infty \frac{dt}{t} \frac{(t/2)^2}{\sinh(t/2)^2} J_n\left(\frac{t\sqrt{\lambda}}{2\pi}\right) J_m\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

The mass deformation operator S_2

- In terms of the matrix M one finds in the E theory

$$S_2^{(0)} = N^2\, \mathsf{M}_{0,0} + \mathsf{M}_{1,1} - \frac{1}{6}\sum_{k=1}^{\infty}\sqrt{2k+1}\,\mathsf{M}_{1,2k+1} + O(N^{-2})\ ,$$

$$S_2^{(1)} = 2N\sum_{k=1}^{\infty}\mathsf{M}_{0,2k}\,\mathcal{P}_{2k} + O(N^{-1})\ ,$$

$$S_2^{(2)} = \sum_{k,\ell=1}^{\infty}\Big[\mathsf{M}_{2k,2\ell}\,\mathcal{P}_{2k}\mathcal{P}_{2\ell} - \mathsf{M}_{2k+1,2\ell+1}\,\mathcal{P}_{2k+1}\mathcal{P}_{2\ell+1}\Big]$$

- In the $\mathcal{N}=4$ theory there is just a $+$ between the even and odd part of $\mathcal{M}^{(2)}$

Mass corrections to the free energy

Just the results

- The first mass correction to the free energy is $m^2\langle S_2\rangle$.
- With the techniques discussed above, one finds

$$\langle S_2\rangle = N^2\,\mathsf{M}_{0,0} + \Big[\big(1+2\lambda\,\partial_\lambda\mathcal{F}\big)\mathsf{M}_{1,1} - \frac{1}{6}\sum_{k=1}^{\infty}\sqrt{2k+1}\,\mathsf{M}_{1,2k+1} + \mathrm{Tr}\,\mathsf{M}^{\rm even} - \mathrm{Tr}\left(\mathsf{M}^{\rm odd}\,\mathsf{D}^{\rm odd}\right)\Big] + O(N^{-2})$$

- At weak coupling,

$$\langle S_2\rangle = N^2\Big[\frac{3\,\zeta_3}{2}\,\hat\lambda - \frac{25\,\zeta_5}{8}\,\hat\lambda^2 + \frac{245\,\zeta_7}{32}\,\hat\lambda^3 - \frac{1323\,\zeta_9}{64}\,\hat\lambda^4 + O(\hat\lambda^5)\Big] - \Big[\frac{3\,\zeta_3}{2}\,\hat\lambda - \frac{25\,\zeta_5}{8}\,\hat\lambda^2 - \frac{175\,\zeta_7}{32}\,\hat\lambda^3 + \frac{6615\,\zeta_9 + 360\,\zeta_3\,\zeta_5}{64}\,\hat\lambda^4 + O(\hat\lambda^5)\Big] + O(N^{-2})$$

where we higlighted the terms differing from $\mathcal{N}=2^*$

Mass corrections to the free energy

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- The first mass correction to the free energy is $m^2 \langle S_2 \rangle$.
- With the techniques discussed above, one finds

$$\langle S_2 \rangle = N^2 \, \mathsf{M}_{0,0} + \Bigg[\left(1 + 2\lambda \, \partial_\lambda \mathcal{F}\right) \mathsf{M}_{1,1} - \frac{1}{6} \sum_{k=1}^{\infty} \sqrt{2k+1} \, \mathsf{M}_{1,2k+1} + \mathrm{Tr} \, \mathsf{M}^{\mathrm{even}} - \mathrm{Tr} \left(\mathsf{M}^{\mathrm{odd}} \, \mathsf{D}^{\mathrm{odd}} \right) \Bigg] + O(N^{-2})$$

- At strong coupling one finds

$$\langle S_2 \rangle \underset{\lambda \to \infty}{\sim} N^2 \frac{\log \lambda}{2} - \frac{5\sqrt{\lambda}}{48} + O(N^{-2})$$

  - In $\mathcal{N} = 2^*$ one has $\frac{\sqrt{\lambda}}{6}$ instead Russo Zarembo, 2013
  - For some terms, one needs Bessel Kernel techniques similar to those used for the octagon form factor in $\mathcal{N} = 4$ in

    Belitsky Korchemsky, 2020

Explicit expression

of the integrated correlators

- We need to compute the correlators

$$\mathcal{I}_p = \langle\!\langle \mathcal{O}_p\, \mathcal{O}_p\, S_2 \rangle\!\rangle$$

where $\mathcal{O}_p$ is the matrix image of the protected operator $O_p(x) = \mathrm{tr}\,[\phi(x)]^p$

- From the expression of S_2, we get

$$\mathcal{I}_p = 2N \sum_{k=1}^{\infty} \mathsf{M}_{0,2k} \langle\!\langle \mathcal{O}_p\, \mathcal{O}_p\, \mathcal{P}_{2k} \rangle\!\rangle + \sum_{k,\ell=1}^{\infty} \mathsf{M}_{2k,2\ell} \langle\!\langle \mathcal{O}_p\, \mathcal{O}_p\, \mathcal{P}_{2k}\, \mathcal{P}_{2\ell} \rangle\!\rangle$$

$$- \sum_{k,\ell=1}^{\infty} \mathsf{M}_{2k+1,2\ell+1} \langle\!\langle \mathcal{O}_p\, \mathcal{O}_p\, \mathcal{P}_{2k+1}\, \mathcal{P}_{2\ell+1} \rangle\!\rangle$$

Explicit expression

for even operators

- Recalling the simple relation between $\mathcal{O}_{2q}$ and $\mathcal{P}_{2q}$

$$\frac{\mathcal{I}_{2q}}{\mathcal{G}_{2q}} = 2N \sum_{k=1}^{\infty} \mathsf{M}_{0,2k} \langle \mathcal{P}_{2q}\, \mathcal{P}_{2q}\, \mathcal{P}_{2k} \rangle_c + 2 \sum_{k,\ell=1}^{\infty} \mathsf{M}_{2k,2\ell} \langle \mathcal{P}_{2q}\, \mathcal{P}_{2k} \rangle \, \langle \mathcal{P}_{2q} \mathcal{P}_{2\ell} \rangle$$

- From the properties of the even-$\mathcal{P}$ correlators, this gives

$$\frac{\mathcal{I}_{2q}}{\mathcal{G}_{2q}} = 4q \sum_{k=1}^{\infty} \mathsf{M}_{0,2k} \sqrt{2k} + 2\mathsf{M}_{2q,2q} = 2\, \mathsf{M}_{2q,2q} - 4q\, \mathsf{M}_{1,1}$$

  - The second step follows from Bessel function recursion id.s
  - The result is identical to the $\mathcal{N} = 4$ one!

Explicit expression

for odd operators

- The expansion of $\mathcal{O}_{2q+1}$ in terms of the odd $\mathcal{P}$'s is non trivial ▸ Back
- In this way $\mathcal{I}_{2q+1}$ is determined by the correlators

$$\Pi_{q,r} \equiv \langle\!\langle \mathcal{P}_{2q+1}\, \mathcal{P}_{2p+1}\, S_2 \rangle\!\rangle$$

- Using the epansion of S_2 and the large-N properties of the odd $\mathcal{P}$ correlators,one finds

$$\Pi_{q,r} = 2N \sum_{k=1}^{\infty} \mathrm{M}_{0,2k} \langle \mathcal{P}_{2q+1}\, \mathcal{P}_{2r+1}\, \mathcal{P}_{2k} \rangle_c \\ - 2 \sum_{k,\ell=1}^{\infty} \mathrm{M}_{2k,2\ell} \langle \mathcal{P}_{2q+1}\, \mathcal{P}_{2k+1} \rangle \, \langle \mathcal{P}_{2q+1} \mathcal{P}_{2\ell+1} \rangle$$

Explicit expression

for odd operators

- Using the results for the 2pt and 3pt functions of odd operators ▸ Back in the end

$$\Pi_{q,r} = -2\, \mathsf{d}_{2q+1}\, \mathsf{d}_{2r+1}\, \mathsf{M}_{1,1} - 2\left(\mathsf{D}^{\text{odd}}\, \mathsf{M}^{\text{odd}}\, \mathsf{D}^{\text{odd}}\right)_{q,r}$$

- All quantities here depend on the coupling through Bessel functions
- Working out the λ-dependent normal ordering gives finally the odd integrated correlator $\mathcal{I}_{2q+1}$ in terms of the $\Pi_{q,r}$
- At weak coupling one can generate easily very long perturbative series

Strong coupling behaviour

- At large λ, apply Mellin-Barnes methods
- To the elements of X, D Beccaria et al, 2021.This implies that the normal ordering simplifies at large λ

$$\mathcal{O}_{2q+1} \underset{\lambda\to\infty}{\sim} \sqrt{\mathcal{G}^{(0)}_{2q+1}}\left(\mathcal{P}_{2q+1} - \sqrt{\frac{2q+1}{2q-1}}\,\mathcal{P}_{2q-1}\right)$$

- To the coefficients d_{2q+1} Billo et al, 2022

$$\mathsf{d}_{2q+1} \underset{\lambda\to\infty}{\sim} \frac{2\pi}{\sqrt{\lambda}}\sqrt{2q+1}\,q\,(q+1) + \ldots$$

- To the mass deformation S_2:

$$(S_2)_{n,m} \underset{\lambda\to\infty}{\sim} -\frac{1}{2}\,\delta_{n,m} + \frac{\sqrt{n\,m}}{\sqrt{\lambda}} + \ldots$$

Strong coupling behaviour

of the integrated correlators

- In the expression of $\Pi_{q,r}$ also the matrix $(\mathsf{D}^{odd}\mathsf{M}^{odd}\mathsf{D}^{odd})_{q,r}$ appears. Difficult to study analytically, but numerically (conformal Padé approximants) appears to go like $\lambda^{-3/2}$
- Altogether one gets

$$\mathcal{I}_{2q} \underset{\lambda\to\infty}{\sim} \frac{2q-1}{2}\,\mathcal{G}^{(0)}_{2q} + O(\lambda^{-\frac{1}{2}})\,,$$

$$\mathcal{I}_{2q+1} \underset{\lambda\to\infty}{\sim} \frac{8\pi^2}{\lambda}(2q+1)q^2\,\mathcal{G}^{(0)}_{2q+1} + O(\lambda^{-\frac{3}{2}})$$

- In the odd case, the 2pt function also goes like Billo et al, 2022

$$\mathcal{G}_{2q+1} \underset{\lambda\to\infty}{\sim} \frac{8\pi^2}{\lambda}(2q+1)q\,\mathcal{G}^{(0)}_{2q+1}$$

Strong coupling behaviour

of the integrated correlators

- The following relation holds therefore for both even and odd integrated correlators:

$$\lim_{\lambda \to \infty} \frac{\mathcal{I}_p}{\mathcal{G}_p} = \frac{p-1}{2}$$

- It should be possible to exploit this constraint on the dual holographic correlators

Conclusions

remarks, perspectives

- The full Lie algebra method is rather efficient for dealing with integrated correlators in $\mathcal{N} = 2$ matrix models such as the E theory one
- At large N it allows to resum the perturbative expansion into integrals of Bessel functions and access in this way the strong coupling regime
  - Similar integrals (with different kernels) often appear: cusp anomaly and octagon form factor in $\mathcal{N} = 4$, chiral correlators in $\mathcal{N} = 2$, integrated correlators ...
  - What could be the deep origin of this structure?
- The method is also very efficient at finite N to generate perturbative corrections. However (not yet) to resum them. Maybe try to somehow implement topological recursion in this framework?

Conclusions

remarks, perspectives

- To explore: the theory D, with 4 fundamental and 2 antisymmetric hypers
  - The holographic dual Ennes et al, 2000 has a sector with *D7* branes on $AdS_5 \times S_3$, similarly to the theory considered by Behan et al, 2023. Certain integrated correlators of moment map operators should relate to open string scatterings

    work at initial stage with Torino-Humboldt group
- To explore: 2-pt integrated correlators with a Wilson loop in $\mathcal{N} = 2$ theories

THE END

Thank you very much for your attention!