New recursion relation for M2-brane matrix model

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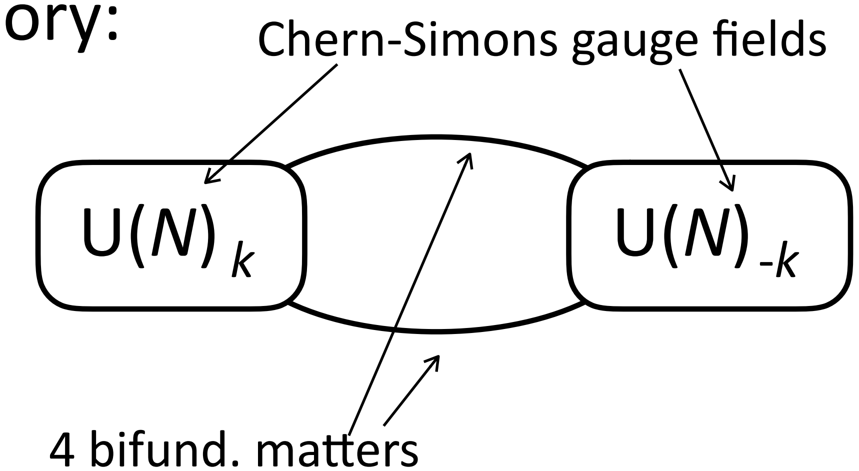
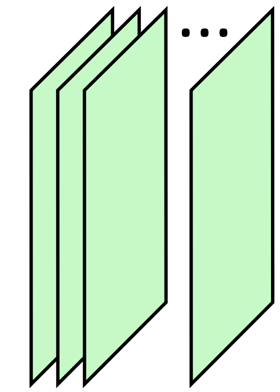
@Bootstrap, Localization and Holography (YITP, Kyoto) (2024/5/21)

ABJM theory:



= N M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$



[Hosomichi,Lee,Lee,Lee,Park,'08]
[Aharony,Bergman,Jafferis,Maldacena,'08]

Can add mass term $\longleftrightarrow$ background flux [Gomis,Rodriguez-Gomez,van Raamsdonk,Verlinde,'08]

(i) Useful for precision holography [Bobev,Charles,Hong,Hristov,Reys,...]

ABJM $\longleftrightarrow \mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ ➡ can study M-theory through large N expansion

mass = non-conformal deformation of $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ (small pure imaginary mass)

[Freedman,Pufu,'13][Gautason,Puletti,van Muiden,'23]

(ii) Useful as generating function of correlators

$$\partial^n_{m_i} \log Z^{\mathrm{mABJM}}_{S^3} \to \langle J_{R_{i_1}} J_{R_{i_2}} \cdots J_{R_{i_n}} \rangle$$ ➡ input for conformal bootstrap

(J_R: $SO(6)_R$ currents)

[Agmon,Chester,Pufu,'17][Alday,Chester,Raj,'22]...

Beautiful mathematical structures

$Z_{S^3}^{N\,\mathrm{M2}}$ —SUSY localization→ $\frac{1}{N!}\int d^N x(\cdots)$: "M2-matrix model"

M2 on different orbifolds

	ABJM (& circular quiver SCCS)	mABJM
Fermi gas formalism	YES	YES
all order 1/*N* = Airy	YES	YES (small mass only)
1/*N* non-pert. = topological string	YES	?
q-difference system	YES	YES

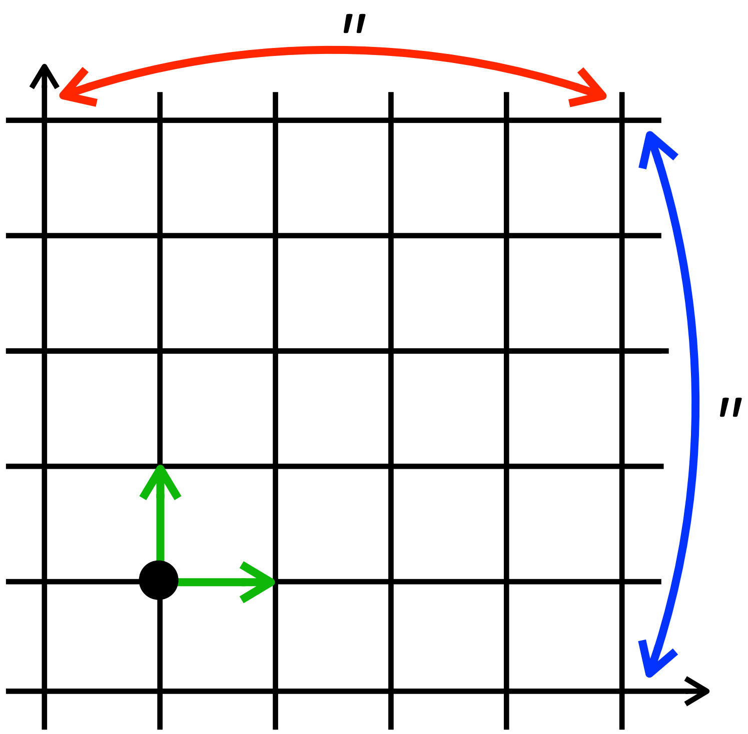
↓

New recursion relation !

Rough idea for recursion relation

M2 matrix models have discrete moduli (typically relative ranks of gauge group).

q-difference system = evolution on theory lattice space



lattice is periodic: dualities (typically Seiberg-like duality) act as large translations

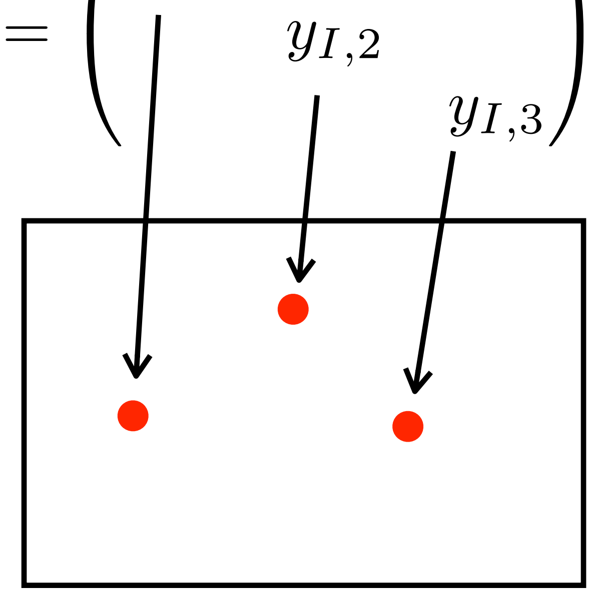
self-consistency condition → recursion relation at same point

An application: phase transition in mABJM

Mass deformation changes vacuum structure in flat space

m=0 $[Y,Y,Y]=0$

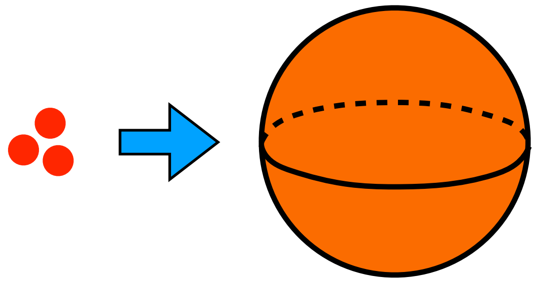
$$Y_I = \begin{pmatrix} y_{I,1} & & \\ & y_{I,2} & \\ & & y_{I,3} \end{pmatrix}$$



m>0 $[Y,Y,Y]=mY$

$$Y_1 = \sqrt{m}\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ Y_2 = \sqrt{m}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

: non-commutative



= fuzzy M5

[Gomis,Rodriguez-Gomez,van Raamsdonk,Verlinde,'08][Kim,Kim,'10]

On S^3 ➡ phase transition in dimensionless parameter $m \times r_{S^3}$

't Hooft limit: $k, N \to \infty$ with $\lambda = \frac{N}{k}$ fixed [Anderson,Zarembo,'14][Anderson,Russo,'15]

M-theory limit: $N \to \infty$ with k fixed [TN,Shimizu,Terashima,'16] [Honda,TN,Shimizu,Terashima,'18] ⬅ **can study with recursion relation**

Plan of talk

1. All order $1/N$ expansion

2. q-difference system and recursion relation

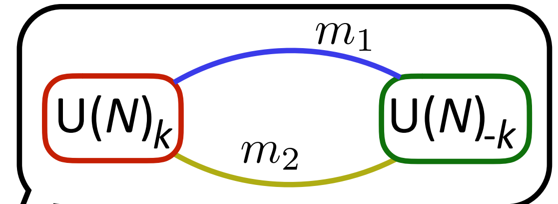
3. Application: large N expansion beyond phase transition

Fermi gas formalism

SUSY localization:

t-independent

$$Z=\int \mathcal{D}e^{-S^{\mathrm{mABJM}}-t\delta(\Psi,\delta\Psi)} \underset{t\to\infty}{=} \sum_{\mathrm{locus}:\delta\Psi=0} e^{-S^{\mathrm{mABJM}}|_{\mathrm{locus}}} Z_{\text{1-loop}}(\delta^2|_{\mathrm{locus}})$$



$$Z_{S^3}(N;m_1,m_2)=\frac{1}{(N!)^2}\int\frac{d^Nx}{(2\pi)^N}\frac{d^Ny}{(2\pi)^N}e^{\frac{ik}{4\pi}\sum_i(x_i^2-y_i^2)}\frac{\prod_{i<j}^N\left(2\sinh\frac{x_i-x_j}{2}\right)^2\prod_{i<j}^N\left(2\sinh\frac{y_i-y_j}{2}\right)^2}{\prod_{i,j}2\cosh\frac{x_i-y_j-m_1}{2}\,2\cosh\frac{y_i-x_j-m_2}{2}}$$

[Kapustin,Willett,Yaakov,'09]

Cauchy det $\frac{\prod_{i<j}(x_i-x_j)(y_i-y_j)}{\prod_{i,j}(x_i+y_j)}=\det\frac{1}{x_i+y_j}$ & $\frac{1}{2k\cosh\frac{x-y-mk}{2k}}=\langle x|\frac{e^{-\frac{im\hat{p}}{2\pi}}}{2\cosh\frac{\hat{p}}{2}}|y\rangle$

$$[\hat{x},\hat{p}]=2\pi ik$$

$$Z_{S^3}(N;m_1,m_2)=\frac{1}{N!}\int\frac{d^Nx}{(2\pi)^N}\det\langle x_i|\hat{\rho}|x_j\rangle$$

$$\hat{\rho}=e^{\frac{i\hat{x}^2}{4\pi k}}\frac{e^{-\frac{im_1\hat{p}}{2\pi}}}{2\cosh\frac{\hat{p}}{2}}e^{-\frac{i\hat{x}^2}{4\pi k}}\frac{e^{-\frac{im_2\hat{p}}{2\pi}}}{2\cosh\frac{\hat{p}}{2}}\sim\frac{e^{-\frac{im_2\hat{x}}{2\pi}}}{2\cosh\frac{\hat{x}}{2}}\frac{e^{-\frac{im_1\hat{p}}{2\pi}}}{2\cosh\frac{\hat{p}}{2}}$$

[Marino,Putrov,'11]

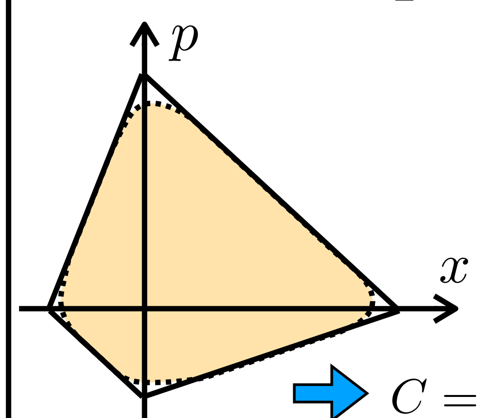
All order $1/N$ expansion

To study large N expansion, it is useful to consider grand potential $J(\mu)$:

$$\sum_{n\in\mathbb{Z}} e^{J(\mu+2\pi i n)} = \sum_{N=0}^{\infty} e^{\mu N} Z(N) = \mathrm{Det}(1+e^{\mu}\hat{\rho}) \qquad \left(Z(N) = \int_{-i\infty}^{i\infty} \frac{d\mu}{2\pi i} e^{J(\mu)-\mu N} \right)$$

$$\partial_\mu J(\mu) \sim \langle N \rangle_{\mathrm{G.C.}(\mu)} \sim \frac{\mathrm{vol}(H(x,p)<\mu)}{2\pi\hbar} \sim C\mu^2$$

$$H = -\log\rho \sim \frac{|x|}{2} + \frac{i m_2 x}{2\pi} + \frac{|p|}{2} + \frac{i m_1 p}{2\pi}$$



$$C = \frac{2}{\pi^2 k (1+\frac{m_1^2}{\pi^2})(1+\frac{m_2^2}{\pi^2})}$$

integrate → $J = \frac{C}{3}\mu^3 + B\mu + A + (\text{non-pert. in } \mu^{-1})$

B,*A* can also be fixed by semiclassical expansion

$$Z(N) \approx Z_{\mathrm{pert}}(N) = e^{A} C^{-\frac{1}{3}} \mathrm{Ai}[C^{-\frac{1}{3}}(N-B)]$$

$$-\log Z_{\mathrm{pert}}(N) = \frac{\pi\sqrt{2k}}{3}\sqrt{\left(1+\frac{m_1^2}{\pi^2}\right)\left(1+\frac{m_2^2}{\pi^2}\right)} N^{\frac{3}{2}} + \bigcirc N^{\frac{1}{2}} + \frac{1}{4}\log N + (\text{all order in } N^{-1})$$

[TN,'15]

1/*N* non-perturbative effects

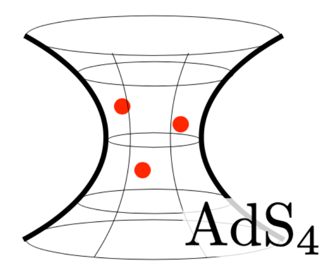
ABJM is dual to AdS$_4$ × S_7/Z_k → length scale $R_{\mathrm{AdS}} \sim (kN)^{\frac{1}{6}}$

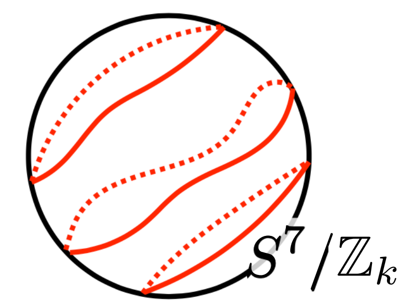
$$-\log Z^{\mathrm{ABJM}}_{\mathrm{pert}}(N) = \frac{\pi\sqrt{2k}}{3} N^{\frac{3}{2}} + \cdots = S_{\mathrm{11d\ SUGRA}} \sim k^{-1} R^{11-2}_{\mathrm{AdS}}$$

Non-perturbative effects in 1/*N* = closed M2 wrapped on $S^7/\mathbb{Z}_k$ (M2-instanton)

$$Z_{\mathrm{11d}} \sim e^{-S_{\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k}} + e^{-S_{\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k} + S_{\mathrm{M2inst}}} + \cdots$$

$\sim \mathrm{vol(M2)} \sim R^3_{\mathrm{AdS}} \sim \sqrt{kN}$





$$-\log Z \sim -\log Z_{\mathrm{pert}} + \mathcal{O}(e^{-\bigcirc\sqrt{kN}})$$

Two different kinds of wrapping:

$M_3 \subset S^7/\mathbb{Z}_k$ ➡ $e^{-\bigcirc\sqrt{kN}}$: D2-instanton

$M_2 \times S^1/\mathbb{Z}_k \subset S^7/\mathbb{Z}_k$ ➡ $e^{-\bigcirc\sqrt{\frac{N}{k}}}$: F1-instanton

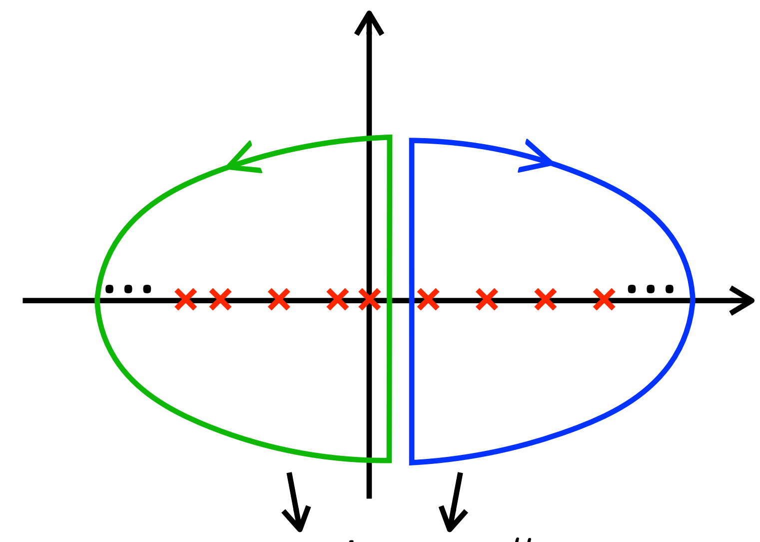
M2-instantons in mABJM

$$\frac{Z(N)}{Z_{\rm pert}(N)} - 1 \sim e^{-\omega\sqrt{\frac{N-B}{C}}} \longleftrightarrow J = J_{\rm pert} + \bigcirc e^{-\omega\mu}$$

D2-inst exponent:

➡ WKB (=small *k*) expansion & resummation of

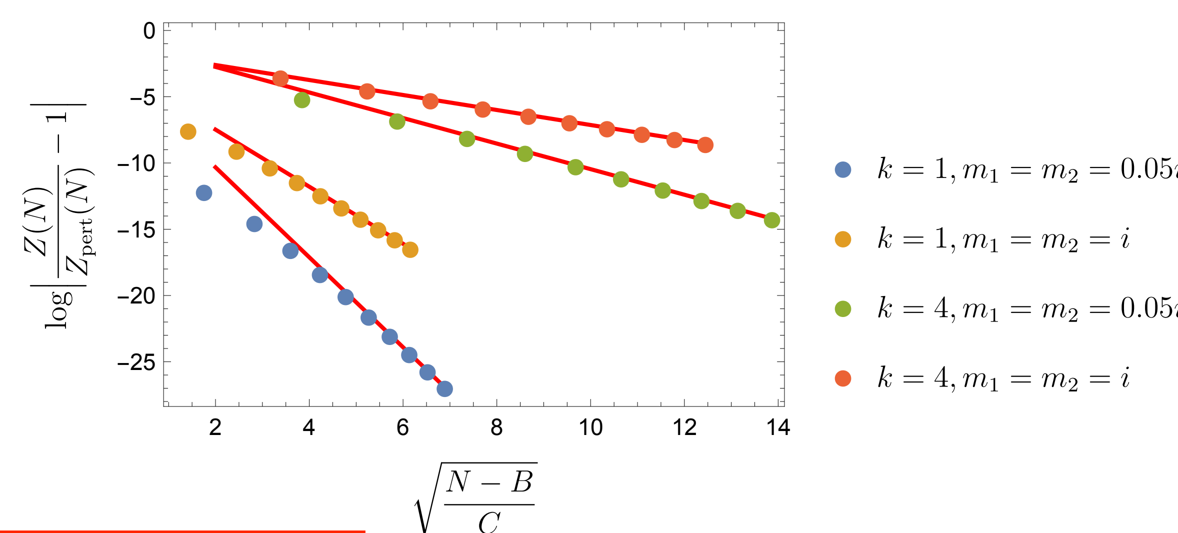
$$J = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{n\mu} {\rm Tr}\hat{\rho}^n = \int \frac{dt}{2\pi i}\Gamma(t)\Gamma(-t)e^{t\mu}{\rm Tr}\hat{\rho}^t$$



F1-inst exponent:

➡ fitting exact values of *Z*(*N*)

$$(k = 1, 2, \cdots,\ N = 1, 2, \cdots \lesssim 10)$$



$$\omega^{(\rm D2)}_{\pm,i} = \frac{2}{1 \pm \frac{im_i}{\pi}}, 1\,, \quad \omega^{(\rm F1)}_{\pm,\pm'} = \frac{4}{k(1 \pm \frac{im_1}{\pi})(1 \pm' \frac{im_2}{\pi})}$$

[TN,'15]

Instanton coefficients $\bigcirc$ can also be studied by WKB + fitting

M2-instantons and topological string ($m_1 = m_2 = 0$)

Topological String/Spectral Theory (TS/ST) correspondence:

$$\hat{\rho}^{-1} = \sum_{m,n} c_{mn} e^{m\hat{x}+n\hat{p}}$$ $\longrightarrow$ encodes a Calabi-Yau threefold *X*

(Newton polygon {(*m*,*n*)} = toric data)

$$\mathrm{Det}(1+\kappa\hat{\rho}) = \sum_{n\in\mathbb{Z}} e^{J(\mu+2\pi i n)}$$

$$J = J_{\mathrm{pert}} + F^{\mathrm{top}}\left(\boldsymbol{T}; \frac{2}{k}\right) + \frac{\partial}{\partial k}\left[\frac{1}{k} F^{\mathrm{NS}}\left(\frac{k\boldsymbol{T}}{2}; \frac{k}{2}\right)\right]$$

$$F^{\mathrm{top}}(\boldsymbol{T};\epsilon) = F^{\mathrm{ref}}(\boldsymbol{T};\epsilon,-\epsilon)$$

$$F^{\mathrm{NS}}(\boldsymbol{T};\epsilon) = \lim_{\epsilon_2\to 0} 2\pi i \epsilon_2 F^{\mathrm{ref}}(\boldsymbol{T};\epsilon,\epsilon_2)$$

$$F^{\mathrm{ref}}(\boldsymbol{T};\epsilon_1,\epsilon_2) = \sum_{j_L,j_R}\sum_{\boldsymbol{d}} N^{\boldsymbol{d}}_{j_L,j_R} \sum_n \frac{\chi_{j_L}(e^{\pi i n(\epsilon_1-\epsilon_2)})\chi_{j_R}(e^{\pi i n(\epsilon_1+\epsilon_2)})}{4n \sin \pi n\epsilon_1 \sin \pi n \epsilon_2} e^{-n\boldsymbol{d}\cdot\boldsymbol{T}}$$

$$\boldsymbol{T} \sim \frac{\mu}{k}$$

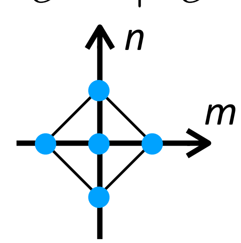
: refined topological string free energy on *X*

[Hatsuda,Marino,Moriyama,Okuyama,'13][Grassi,Hatsuda,Marino,'14]

$\mathrm{U}(N)_k \times \mathrm{U}(N+M)_{-k}$ ABJM theory:

$$\hat{\rho}^{-1} = e^{\hat{x}'} + e^{\pi i(k-2M)} e^{-\hat{x}'} + e^{\hat{p}'} + e^{-\hat{p}'}$$ $\longrightarrow$ *X* = local $\mathbb{P}^1 \times \mathbb{P}^1$

c.f. $\hat{\rho}_{M=0} = \frac{1}{2\cosh\frac{\hat{x}}{2}} \frac{1}{2\cosh\frac{\hat{p}}{2}}$

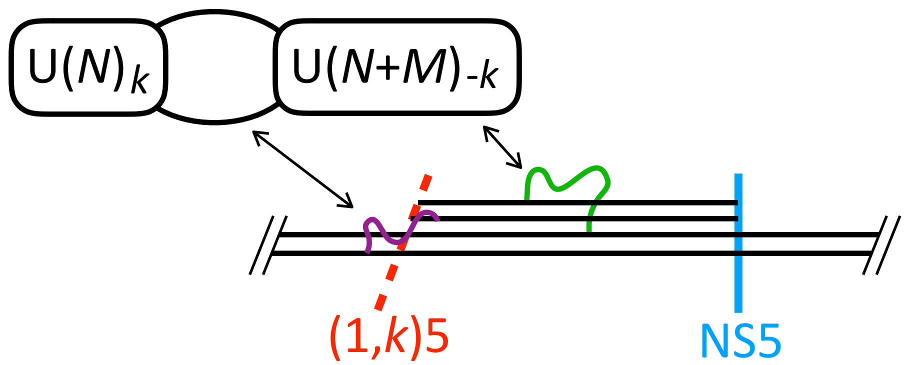


$$\boldsymbol{T} = \frac{4\mu}{k} \pm 2\pi i\left(\frac{1}{2} - \frac{M}{k}\right)$$

F^{top}: F1-inst, $\partial_k[k^{-1}F^{\mathrm{NS}}]$: D2-inst

Other M2 theories

M2 theories can be constructed systematically by type IIB brane systems

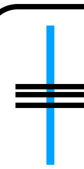
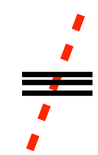


IIB	0	1	2	3(S^1)	4	5	6	7	8	9
D3	-	-	-	-						
NS5	-	-	-		-	-	-			
D5	-	-	-					-	-	-

(1,*k*)5 = NS5 + *k* D5

In general,

$$Z_{S^3} = \frac{1}{N!}\int d^N x \langle x_i|\hat{\rho}|x_j\rangle$$

 $\to \dfrac{1}{2\cosh\frac{\hat{p}}{2}}$  $\to \dfrac{1}{2\cosh\frac{\hat{x}}{2}}$

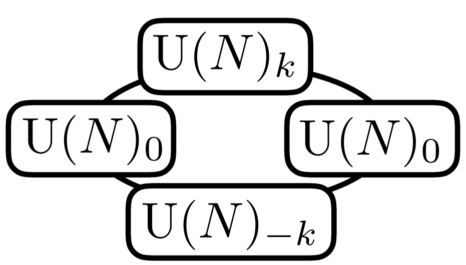
$$\Rightarrow \quad \hat{\rho}^{-1} = \sum_{m,n} c_{mn} e^{m\hat{x}+n\hat{p}}$$

↓ fits with TS/ST

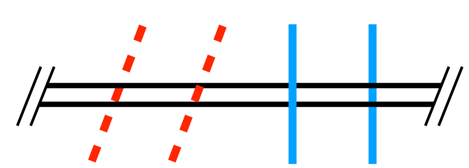
Moduli of curve ρ^{-1} = relative ranks & FI parameters

⇒ Given a Newton polygon, can guess ρ for general moduli from 3d

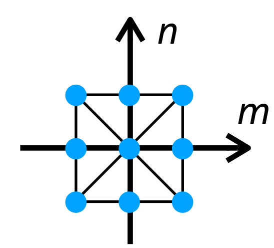
example:



⇒



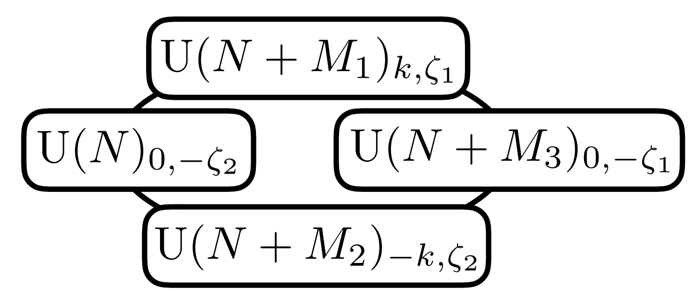
⇒



X = local D_5 del Pezzo

6 moduli = 1(μ) + 5(relative ranks & FI)

9(coefs) - 3(*x*,*p* shift & overall)



[Moriyama,Nakayama,TN,'17][Furukawa,Matsumura,Moriyama,Nakanishi,'21]

Plan of talk

✓ 1. All order $1/N$ expansion

2. q-difference system and recursion relation

3. Application: large N expansion beyond phase transition

ABJM and q-discrete Painleve

$\mathrm{U}(N)_k \times \mathrm{U}(N+M)_{-k}$ ABJM partition functions satisfy q-Painlevé III_3 equation $(\mathrm{q} = e^{\frac{\pi i}{k}})$

$$\Xi_{M+1}(\kappa)\Xi_{M-1}(\kappa) + e^{-\frac{2\pi i M}{k}}\Xi_M(-\kappa)^2 - \Xi_M(\kappa)^2 = 0 \qquad \Xi_M(\kappa) = \sum_{N=0}^{\infty} \kappa^N Z_{k,M}(N)$$

[Grassi,Hatsuda,Marino,'14][Bonelli,Grassi,Tanzini,'17]

Found experimentally against exact values of $Z_{k,M}(N)$

example:

$$Z_{2,0}(0) = 1 \qquad Z_{2,1}(0) = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{2}} \qquad Z_{2,2}(0) = -i$$

$$Z_{2,0}(1) = \frac{1}{8} \qquad Z_{2,1}(1) = \frac{e^{\frac{\pi i}{4}}}{4\sqrt{2}\pi} \qquad Z_{2,2}(1) = \frac{i}{8}$$

$$Z_{2,0}(2) = \frac{1}{32\pi^2} \qquad Z_{2,1}(2) = e^{\frac{3\pi i}{4}}\left(\frac{1}{128\sqrt{2}} - \frac{1}{16\sqrt{2}\pi^2}\right) \qquad Z_{2,2}(2) = -\frac{i}{32\pi^2}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

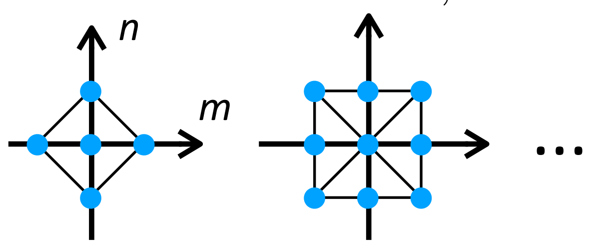
Special cases:

$k \to \infty$ with $M' = k e^{\frac{\pi i M}{k}}$ kept: reduces to PIII_3 differential eq → proved [Tracy,Widom,'95]

k=2,4: can prove against closed forms of $\Xi_M^{\mathrm{TS}}(\kappa)$ [Okuyama,'16]

Motivation of the correspondence

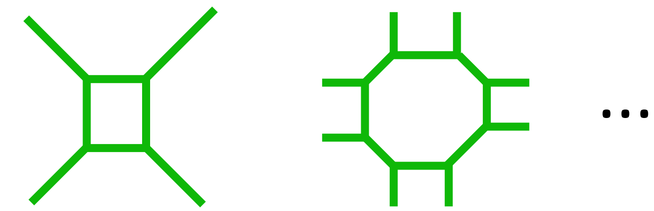
Fermi gas with $\hat{\rho}^{-1} = \sum_{m,n} c_{mn} e^{m\hat{x}+n\hat{p}}$ ⟷ (TS/ST) topological string on local CY_3 X



↕ M-theory on X

5d $\mathcal{N}=1$ SYM on (*p*,*q*)5-brane web

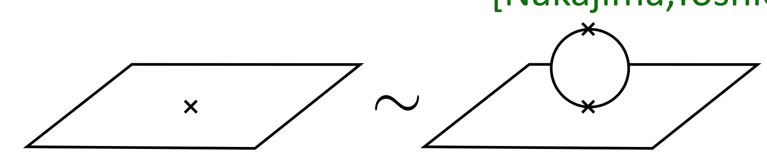
dual diagram
(ρ^{-1} = 5d SW curve)



$$\Xi_{k;M,\zeta,\cdots}(\kappa) \sim Z_{\rm top}(T_1, T_2, \cdots) \sim Z^{\rm 5d}(t = e^{\frac{1}{g^2_{\rm YM}}}, m_i, \cdots; \sigma)$$

In 5d, ∃self-consistency relation among q-shifted parameters $Z^{\rm 5d} \sim Z'^{\rm 5d} \times Z''^{\rm 5d}$

[Nakajima,Yoshioka,'03]



➡ $\Xi(\kappa)$ should satisfy q-shift relation in $(t, \cdots)$ = moduli of curve = relative ranks/FI

examples:

ABJM	5d SU(2) pure	q-Painlevé III_3	[Bershtein,Shchechkin,'18]
4-node quiver	5d SU(2) + N_f =4	q-Painlevé VI	[Jimbo,Nagoya,Sakai,'17]

[Bonelli,Globlek,Kubo,TN,Tanzini,'22][Moriyama,TN,'23]

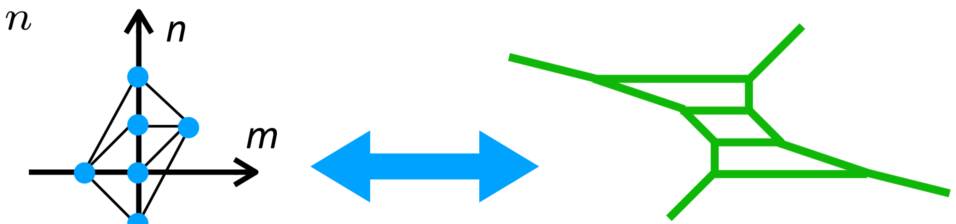
mass deformed ABJM

$$\rho^{-1} = e^{(1+\frac{i(m_1+m_2)}{2\pi})x+\frac{i(m_1-m_2)}{2\pi}p} + e^{\pi i(k-2M)}e^{(-1+\frac{i(m_1+m_2)}{2\pi})x+\frac{i(m_1-m_2)}{2\pi}p} + e^{\frac{i(m_1+m_2)}{2\pi}x+(-1+\frac{i(m_1-m_2)}{2\pi})p} + e^{\frac{i(m_1+m_2)}{2\pi}x+(1+\frac{i(m_1-m_2)}{2\pi})p}$$

is not a Laurent polynomial in a pair of $\mathbb{C}^\times$ coordinates

Exceptions: $m_1, m_2 \in \pi i \mathbb{Q}$

example: $m_1 = -m_2 = -\frac{\pi i}{3} \longrightarrow \rho^{-1} = \sum_{m,n} c_{mn} e^{mx'+np'}$ $\left(x' = x - \frac{p}{3},\ p' = \frac{2p}{3}\right)$



5d: SU(3)

$\Xi_M(\kappa)$ should satisfy some q-difference eq for a dense set of (m_1, m_2)

➡ can guess q-diff eq from exact expressions of $Z_{k,M}(1; m_1, m_2)$ and $Z_{k,M}(2; m_1, m_2)$

$$\Xi_{M+1}(-e^{-\frac{m_1+m_2}{2}}\kappa)\Xi_{M-1}(-e^{\frac{m_1+m_2}{2}}\kappa) + e^{-\frac{2\pi i M}{k}}\Xi_M(e^{\frac{m_1-m_2}{2}}\kappa)\Xi_M(e^{-\frac{m_1-m_2}{2}}\kappa) - \Xi_M(-e^{-\frac{m_1+m_2}{2}}\kappa)\Xi_M(-e^{\frac{m_1+m_2}{2}}\kappa) = 0$$

[TN,'24]

Relation among different *N*'s

$$\Xi_{M+1}(-e^{-\frac{m_1+m_2}{2}}\kappa)\Xi_{M-1}(-e^{\frac{m_1+m_2}{2}}\kappa)+e^{-\frac{2\pi iM}{k}}\Xi_M(e^{\frac{m_1-m_2}{2}}\kappa)\Xi_M(e^{-\frac{m_1-m_2}{2}}\kappa) -\Xi_M(-e^{-\frac{m_1+m_2}{2}}\kappa)\Xi_M(-e^{\frac{m_1+m_2}{2}}\kappa)=0$$

By expanding in *κ*, we obtain relation between $Z(N)$ and $\{Z(N')\}_{N'<N}$

$$\bigcirc Z_{M-1}(N)+\bigcirc Z_M(N)+\bigcirc Z_{M+1}(N)=(\cdots)$$

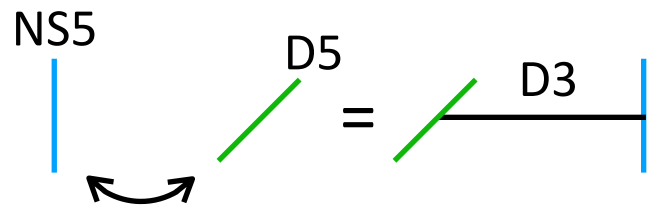
Problem: we cannot solve them in *N* because #(eqs) < #(new variables)

difference eq at $M=M_1,M_1+1,\cdots,M_2$ ➡ $Z_{M_1-1}(N),\cdots,Z_{M_2+1}(N)$

<u>Solution</u>: additional constraints from 3d duality

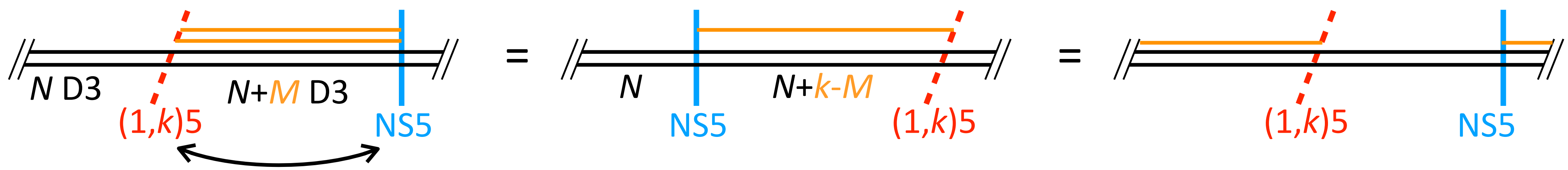
Additional constraint from duality

Hanany-Witten effect:



(1,*k*)5 = NS5 + *k*D5

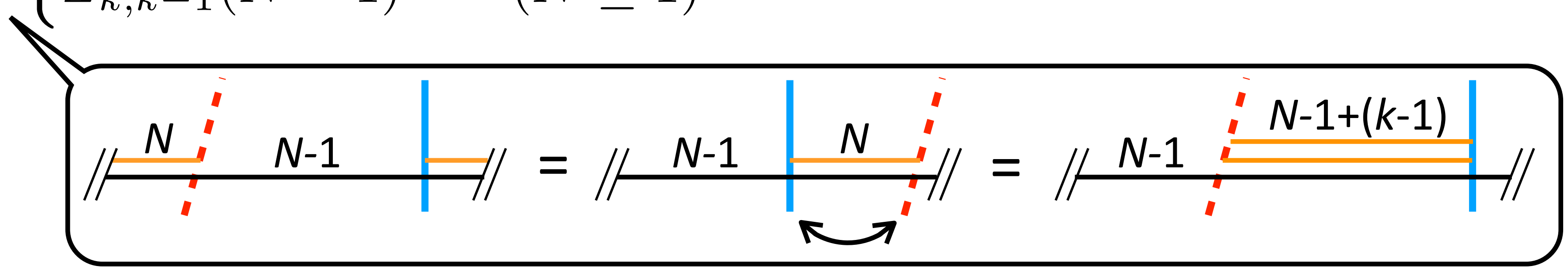
$\mathrm{U}(N)_k \times \mathrm{U}(N+M)_{-k}$ ABJM $\qquad\qquad$ $\mathrm{U}(N+k-M)_k \times \mathrm{U}(N)_{-k}$



[Assel,'14][Honda,Kubo,'20]

$$Z_{k,k}(N) = Z_{k,0}(N)$$

$$Z_{k,-1}(N) = \begin{cases} 0 & (N=0) \\ Z_{k,k-1}(N-1) & (N \geq 1) \end{cases}$$



$$\Xi_{M=k}(\kappa) = \Xi_{M=0}(\kappa), \quad \Xi_{M=-1}(\kappa) = \kappa\,\Xi_{M=k-1}(\kappa)$$

Example: *k*=2

M = 0: $\Xi_1(\kappa){\color{blue}\Xi_{-1}(\kappa)} + \Xi_0(\kappa)\Xi_0(\kappa) = 0$

$\kappa\Xi_1(\kappa)$

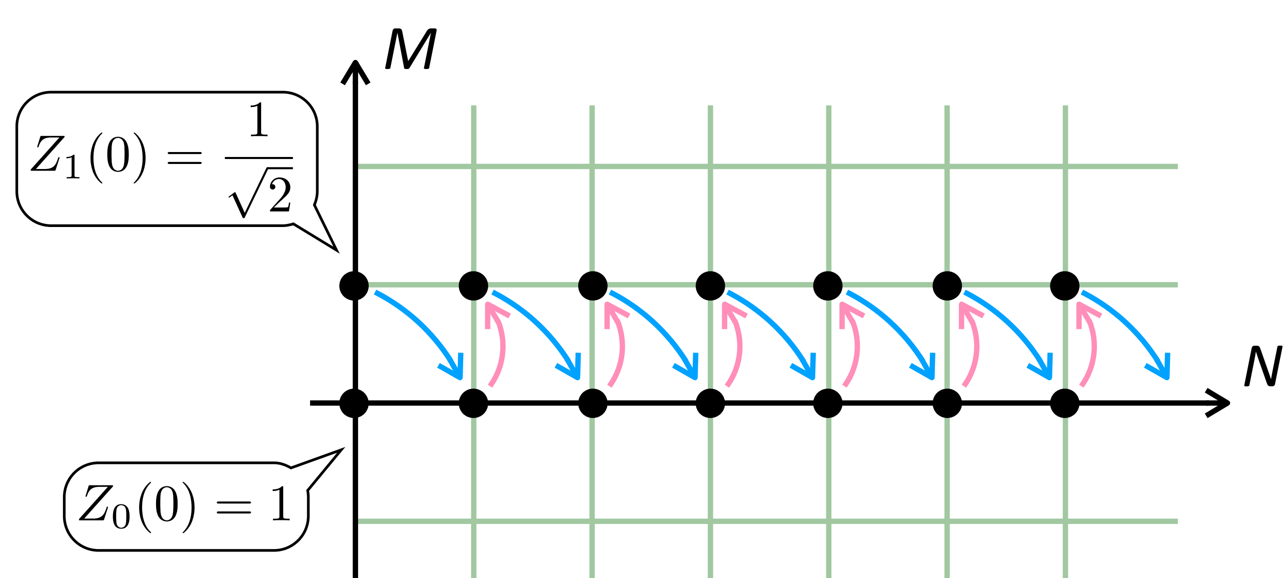
$$\sum_{\substack{n_1,n_2\geq 0\\(n_1+n_2=N-1)}} Z_1(n_1)Z_1(n_2) + Z_0(N)Z_0(0) + \sum_{\substack{n_1,n_2\geq 1\\(n_1+n_2=N)}} Z_0(n_1)Z_0(n_2) = 0$$

M = 1: ${\color{blue}\Xi_2(\kappa)}\Xi_0(\kappa) + \Xi_1(\kappa)\Xi_1(\kappa) = 0$

$\Xi_0(\kappa)$

$$\sum_{\substack{n_1,n_2\geq 0\\(n_1+n_2=N)}} Z_0(n_1)Z_0(n_2) + Z_1(N)Z_1(0) + \sum_{\substack{n_1,n_2\geq 1\\(n_1+n_2=N)}} Z_0(n_1)Z_0(n_2) = 0$$

Can solve recursively in *N* (unless coefficient of *Z*(*N*) vanishes)



[TN,'24]

Plan of talk

✓ 1. All order $1/N$ expansion

✓ 2. q-difference system and recursion relation

3. Application: large N expansion beyond phase transition

M2-instanton condensation

$$Z(N) = Z_{\text{pert}}(N)\left[1 + \mathcal{O}\left(e^{-\omega^{(\text{D2})}\sqrt{\frac{N-B}{c}}}, e^{-\omega^{(\text{F1})}\sqrt{\frac{N-B}{c}}}\right)\right]$$

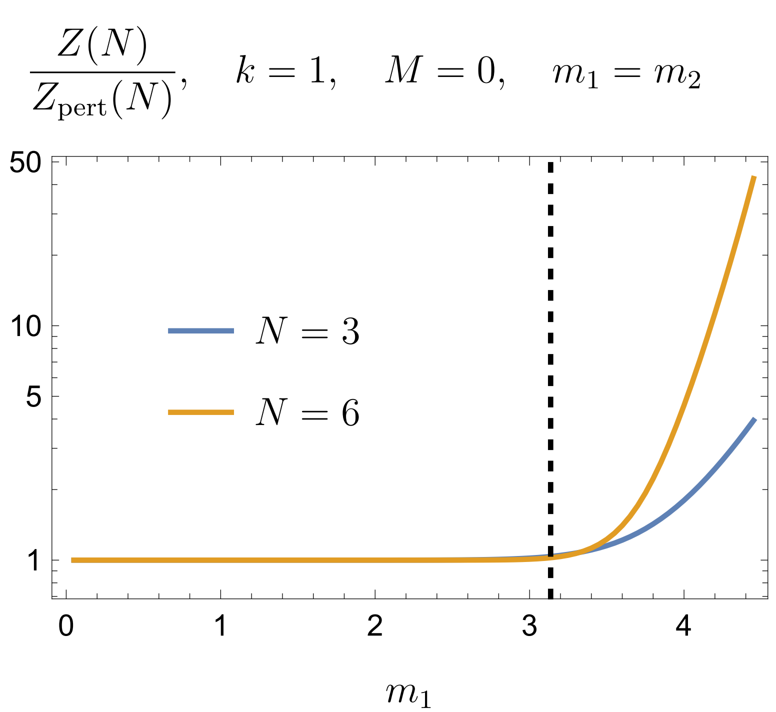
$$Z_{\text{pert}}(N) = e^{A} C^{-\frac{1}{3}} \text{Ai}[C^{-\frac{1}{3}}(N-B)]$$

$$\omega^{(\text{D2})} = \frac{2}{1 \pm \frac{i m_i}{\pi}}, 1 \quad \omega^{(\text{F1})} = \frac{4}{k(1 \pm \frac{i m_1}{\pi})(1 \pm' \frac{i m_2}{\pi})}$$

Negative when $\sqrt{m_1 m_2} > \pi$

F1-instanton is "non-perturbatively large" !

$Z_{\text{pert}}(N)$ is wrong expansion point when $\sqrt{m_1 m_2} > \pi$



WKB expansion → cannot capture F1-instanton.

New: finite but (very) large *N* by recursion relation.

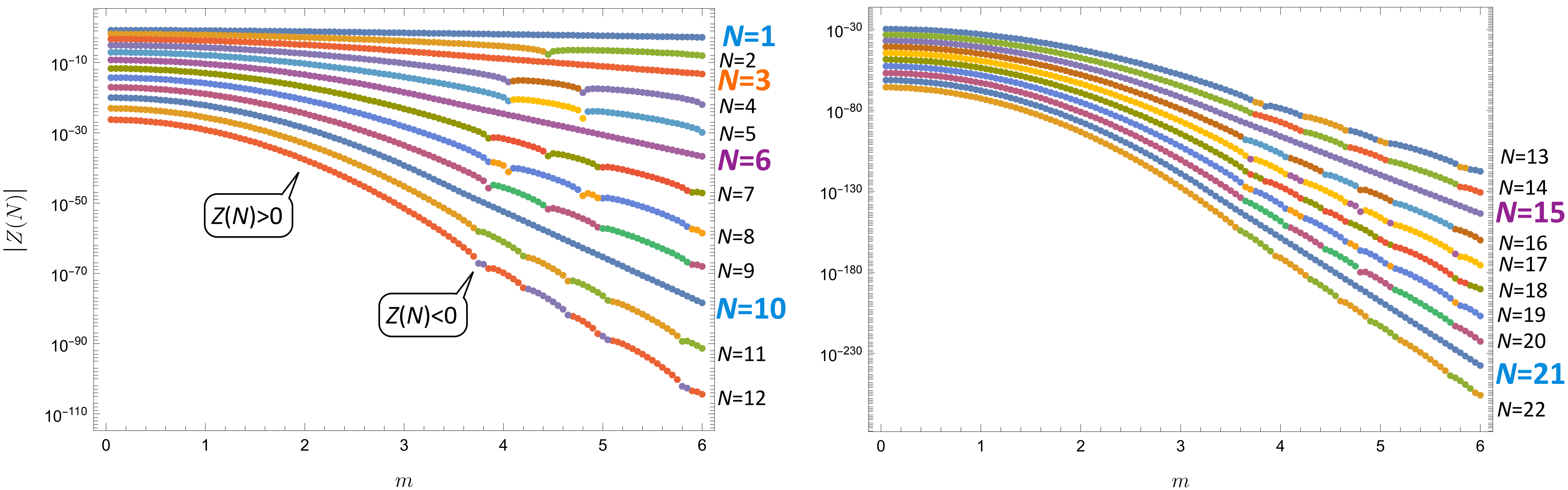
Example: *k*=1, *M*=0, $m_1 = m_2 = m$

$$\kappa \Xi(-ie^{-\frac{im^2}{2\pi}}\kappa)\Xi(ie^{\frac{im^2}{2\pi}}\kappa) - \Xi(\kappa)^2 + \Xi(-e^{m}\kappa)\Xi(-e^{-m}\kappa) = 0$$

$$\Rightarrow \boxed{Z(N) = \frac{\sum_{n=0}^{N-1} R^{2n-N+1} Z(n)Z(N-1-n) - \sum_{n=1}^{N-1}(1-(-1)^N e^{m(2n-N)})Z(n)Z(N-n)}{2(1-(-1)^N \cosh Nm)}}$$

$$R = ie^{\frac{im^2}{2\pi}}$$

General structure: $Z(N) = \sum_{a=-L_N}^{L_N} R^a f_a^{(N)}(m)$ (rational in e^m) $\Rightarrow$ oscillates around *Z*(*N*)=0



Oscillation is (almost) absent when $N = N_n = \frac{n(n+1)}{2}$

Large m asymptotics

Keeping only largest $f_a^{(N)}(m)$ at $m \to \infty$ (i.e. slowest decay), we find

$$Z_{1,0}(1) \to e^{-m}, \quad Z_{1,0}(2) \to e^{-3m} \cdot 2\sin\frac{m^2}{2\pi}, \quad Z_{1,0}(3) \to e^{-5m}, \quad Z_{1,0}(4) \to e^{-8m}\left(1 - 2\cos\frac{m^2}{\pi}\right),$$
$$Z_{1,0}(5) \to e^{-11m}\left(1 - 2\cos\frac{m^2}{\pi}\right), \quad Z_{1,0}(6) \to e^{-14m}, \quad Z_{1,0}(7) \to e^{-18m}\left(2\sin\frac{m^2}{2\pi} - 2\sin\frac{3m^2}{2\pi}\right),$$
$$Z_{1,0}(8) \to e^{-22m}\left(2 - 2\cos\frac{m^2}{\pi} + 2\cos\frac{2m^2}{\pi}\right), \quad Z_{1,0}(9) \to e^{-26m}\left(2\sin\frac{m^2}{2\pi} - 2\sin\frac{3m^2}{2\pi}\right), \quad Z_{1,0}(10) \to e^{-30m}, \cdots$$

$$\Rightarrow \quad Z_{1,0}\left(N_n = \frac{n(n+1)}{2}\right) \sim Z_{1,0}^{\text{asym}}(N_n) = e^{-\frac{n(n+1)(2n+1)m}{6}}$$

k	$N_n^{(k)}$	$Z_{k,0}^{\text{asym}}(N_n^{(k)})$
1	$\frac{n(n+1)}{2}$	$e^{-\frac{n(n+1)(2n+1)}{6}\cdot\frac{m_1+m_2}{2}}$
2	n^2	$2^{-n}e^{-\frac{n(4n^2-1)}{3}\cdot\frac{m_1+m_2}{2}}$
3	$\left\lceil\frac{(n+1)(n+2)}{6}\right\rceil$	$3^{-\frac{n+3}{3}}e^{-(\frac{n^3}{9}+\frac{n^2}{2}+\frac{7n}{6}+1)\cdot\frac{m_1+m_2}{2}}$ $(n \equiv 0 \bmod 3)$
		$3^{-\frac{n+2}{3}}e^{-(\frac{n^3}{9}+\frac{n^2}{2}+\frac{n}{2}-\frac{1}{9})\cdot\frac{m_1+m_2}{2}}$ $(n \equiv 1 \bmod 3)$
		$3^{-\frac{n+1}{3}}e^{-(\frac{n^3}{9}+\frac{n^2}{2}+\frac{n}{2}+\frac{1}{9})\cdot\frac{m_1+m_2}{2}}$ $(n \equiv 2 \bmod 3)$
4	$\left\lceil\frac{n^2}{2}\right\rceil$	$2^{-\frac{3n}{2}}e^{-(\frac{2n^3}{3}-\frac{2n}{3})\cdot\frac{m_1+m_2}{2}}$ $(n\text{: even})$
		$2^{-\frac{3n}{2}-\frac{1}{2}}e^{-(\frac{2n^3}{3}+\frac{n}{3})\cdot\frac{m_1+m_2}{2}}$ $(n\text{: odd})$

(might be related to fuzzy M5 vacua)

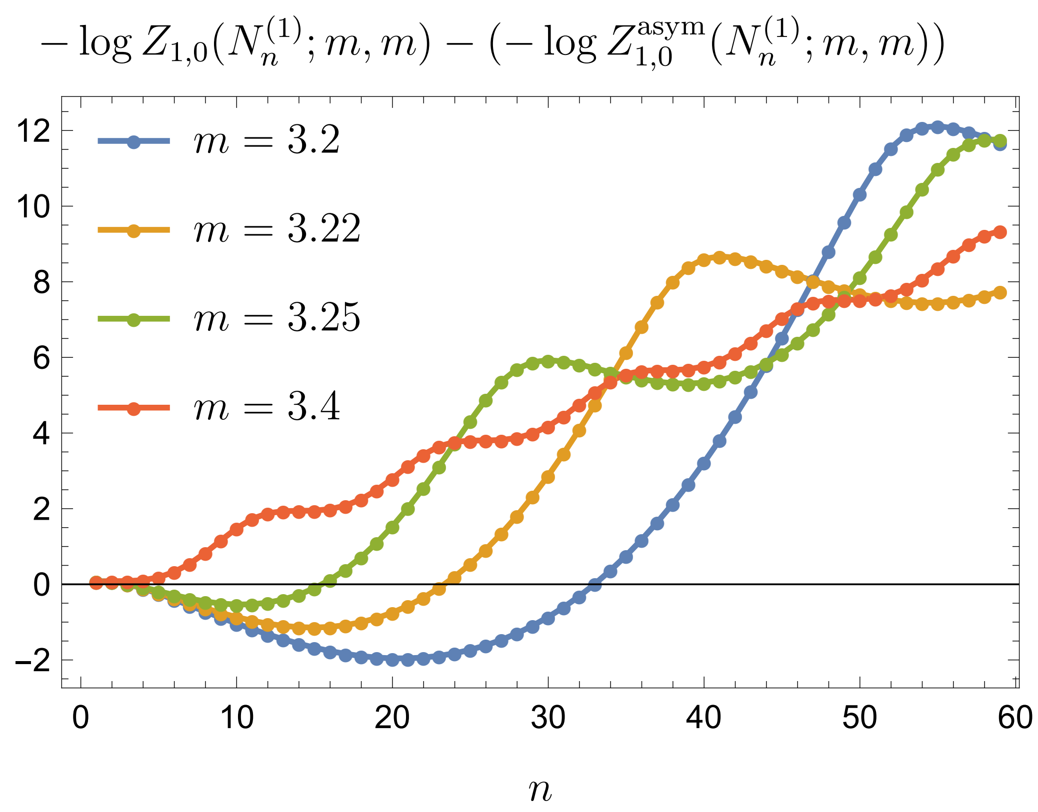
$$\Rightarrow \quad -\log Z_{k,0}^{\text{asym}}(N_n^{(k)}) = \frac{\sqrt{2k}}{3}(m_1+m_2)(N_n^{(k)})^{\frac{3}{2}} + \cdots$$

Finite m correction at large N

Finite m correction:

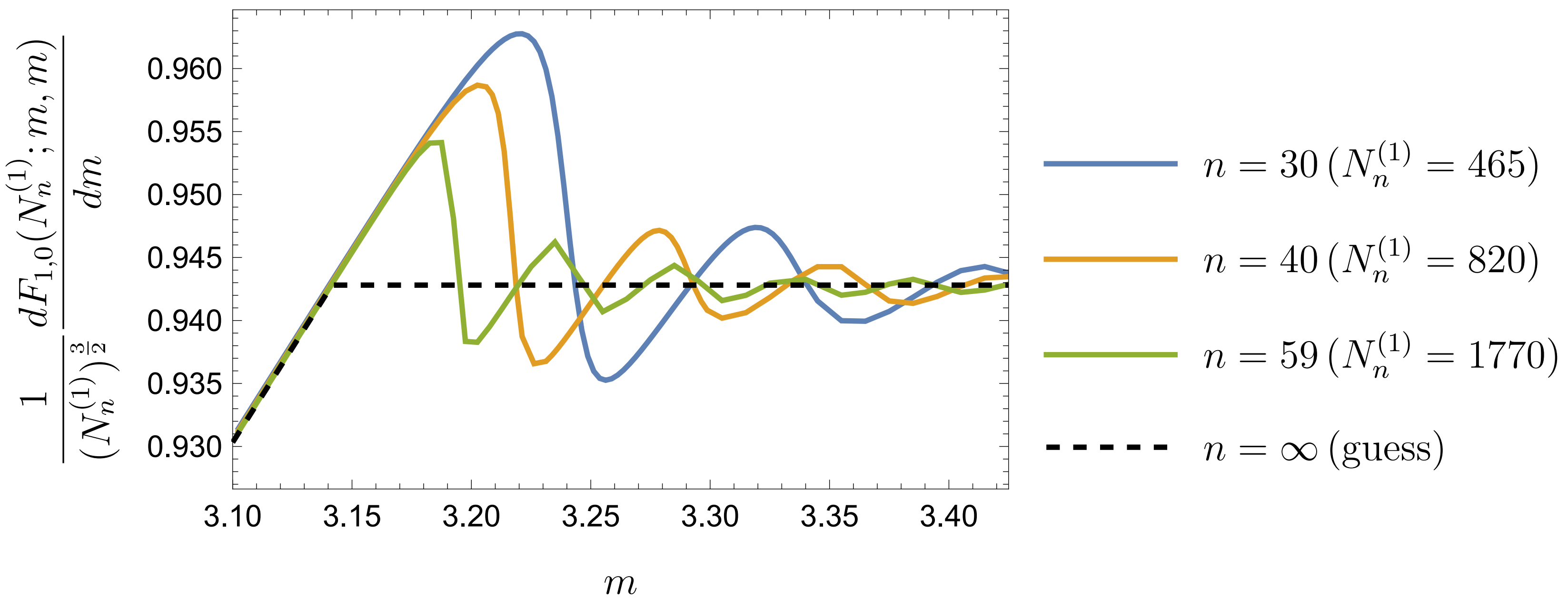
$$-\log Z_{k,0}(N_n^{(k)}) - (-\log Z_{k,0}^{\mathrm{asym}}(N_n^{(k)})) \sim \bigcirc n + \bigcirc e^{\pm i \bigcirc n}$$

$\mathcal{O}(n^3), \mathcal{O}(n^2)$ are exact in Z^{asym}



Conjectured large N free energy:

$$-\log Z_{k,0}(N_n^{(k)}) \approx \begin{cases} \frac{\pi\sqrt{2k(1+\pi^{-2}m_1^2)(1+\pi^{-2}m_2^2)}}{3}(N_n^{(k)})^{\frac{3}{2}} & (\sqrt{m_1 m_2} < \pi) \\ \frac{\sqrt{2k}(m_1+m_2)}{3}(N_n^{(k)})^{\frac{3}{2}} & (\sqrt{m_1 m_2} > \pi) \end{cases}$$



Summary & Future work

M2-matrix model = Fermi gas ⇢ q-difference eq

↙ Airy ↘ topological string ⟶ q-difference eq

q-diff eq + 3d duality ➡ **recursion relation in *N*** : powerfull tool for finite but large *N*

(c.f. NY blowup eq → $Z^{\rm 5d}_{\rm inst}(\nu)$ recursively [Keller,Song,'12][Kim,Kim,Lee,Lee,Song,'19])

application: large *N* expansion beyond M2-instanton condensation

Future works:

Other observables such as Wilson loops?

Direct derivation of q-difference eq from matrix model (not through TS/ST)?

Derivation/interpretation from 3d QFT?

- q-Painlevé ⟷ affine Weyl group (=Weyl+translation) $\overset{?}{\longleftrightarrow}$ 3d duality cascade

(Moriyama's talk)

M2-matrix models without Fermi gas?