Doubled Hilbert space in double scaled SYK

Kazumi Okuyama (Shinshu University)

based on
KO [JHEP 04 (2024) 091] and [arXiv:2404.02833]
(see also KO [arXiv:2212.09213, 2305.12674, 2306.15981])

Bootstrap, Localization and Holography (May 21, 2024)

Holographic principle

- Quantum many body system without gravity is expected to be holographically dual to quantum gravity

Holographic duality

d-dim quantum system $\Leftrightarrow$ $(d+1)$-dim quantum gravity

- In particular, duality between a critical point of quantum many body system (CFT) and quantum gravity on AdS (AdS/CFT correspondence) has been widely studied
[Maldacena 1997]

SYK model

- Sachdev-Ye-Kitaev (SYK) model is an interesting toy model of holographic duality [Sachdev-Ye 1993, Kitaev 2015]
- SYK model is a quantum system of N Majorana fermions

$$H = i^{p/2} \sum_{1 \le i_1 < \cdots < i_p \le N} J_{i_1 \cdots i_p} \psi_{i_1} \cdots \psi_{i_p}$$

- Coupling constant J of p-body interaction is randomly distributed with Gaussian weight

Properties of SYK model

- 2-point function of fermions is exactly solvable in the large N limit
- Approximate conformal symmetry emerges at low energy [Sachdev-Ye 1993]
- This conformal symmetry is spontaneously broken
- Nambu-Goldstone mode for this SSB is called Schwarzian mode, which governs the low energy behavior

Relation to JT gravity

- Jackiw-Teitelboim (JT) gravity is a 2d dilaton gravity

$$S = -\frac{1}{2}\int_M \sqrt{g}\phi(R+2) - \int_{\partial M}\sqrt{h}\phi(K-1)$$

- Dynamical DOF of JT gravity is Schwarzian mode describing the fluctuation of AdS_2 boundary

Holographic duality

SYK model at low energy $\Leftrightarrow$ JT gravity on AdS_2

Double scaled SYK (DSSYK)

- Certain scaling limit of SYK model is exactly solvable without low energy approximation

  [Berkooz-Isachenkov-Narovlansky-Torrents 2018]

- Take a large N limit with p-body interaction $p \sim \sqrt{N}$

$$N, p \to \infty, \quad \lambda = \frac{2p^2}{N} = \text{fixed}$$

- This is called double scaled SYK model (DSSYK)

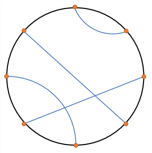
Chord diagram

- Average $\langle \cdots \rangle_J$ over random coupling $J_{i_1 \cdots i_p}$ boils down to the computation of chord diagrams

$$\left\langle \operatorname{tr} H^{2k} \right\rangle_J = \sum_{\text{chord diagrams}} q^{\#(\text{intersections})}, \quad q = e^{-\lambda}$$

- Average over J is computed by Wick contraction

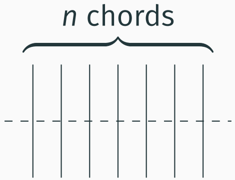
$\Rightarrow \quad \overbrace{HH} = \text{chord}$

$\left\langle \operatorname{tr} H^8 \right\rangle_J \supset$  $= q^2$

Transfer matrix

- Combinatorics of chord diagrams is solved by introducing the transfer matrix T

- Transfer matrix T acts on a chord number state $|n\rangle$

$|n\rangle =$ 

q-oscillator

- T is given by the q-deformed oscillator $A_\pm$

$$T = \frac{A_+ + A_-}{\sqrt{1-q}}$$

- $A_\pm$ creates/annihilates the chords

$$A_-|n\rangle = \sqrt{1-q^n}|n-1\rangle$$
$$A_+|n\rangle = \sqrt{1-q^{n+1}}|n+1\rangle$$

Partition function of DSSYK

- Partition function of DSSYK is written in terms of the transfer matrix T

$$\left\langle \operatorname{tr} e^{-\beta H} \right\rangle_J = \langle 0 | e^{-\beta T} | 0 \rangle$$

- 0-chord state $|0\rangle$ is interpreted as Hartle-Hawking vacuum of bulk quantum gravity [Lin 2022]

Matter operator

- We can introduce matter operators in DSSYK
  [Berkooz-Isachenkov-Narovlansky-Torrents 2018]

$$\mathcal{O}_\Delta = i^{s/2} \sum_{1\leq i_1 < \cdots < i_s \leq N} K_{i_1 \cdots i_s} \psi_{i_1} \cdots \psi_{i_s}$$

- K is assumed to be Gaussian random and independent of J

- We also take a scaling limit $s \sim \sqrt{N}$

$$\Delta = \frac{2ps}{N} = \text{fixed}$$

Matter chord

- Two types of chord arise from Wick contraction of J and K

$$\contraction{}{H}{}{H}HH = H\text{-chord}$$

$$\contraction{}{\mathcal{O}_\Delta}{}{\mathcal{O}_\Delta}\mathcal{O}_\Delta\mathcal{O}_\Delta = \text{matter chord}$$

- Correlator of $\mathcal{O}_\Delta$ also reduces to the computation of chord diagrams

$$\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle = \sum_{\text{chord}} q^{\#(H\text{-}H \text{ intersections})} e^{-\Delta \#(H\text{-}\mathcal{O} \text{ intersections})}$$

Matter 2-point function

- Combinatorics of matter correlator is also solved by the technique of transfer matrix
- 2-point function of $\mathcal{O}_\Delta$

$$\left\langle \mathrm{tr}(e^{-\beta_1 H}\mathcal{O}_\Delta e^{-\beta_2 H}\mathcal{O}_\Delta)\right\rangle_{J,K} = \langle 0|e^{-\beta_1 T}e^{-\Delta\widehat{N}}e^{-\beta_2 T}|0\rangle$$

- $\widehat{N}$ is the number operator of chords

$$\widehat{N}|n\rangle = n|n\rangle$$

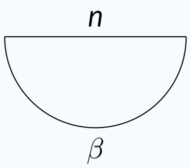
Hartle-Hawking wavefunction

- 2-point function is expanded as

$$\langle 0|e^{-\beta_1 T}e^{-\Delta\widehat{N}}e^{-\beta_2 T}|0\rangle = \sum_{n=0}^{\infty}\langle 0|e^{-\beta_1 T}|n\rangle e^{-\Delta n}\langle n|e^{-\beta_2 T}|0\rangle$$

- $\langle n|e^{-\beta T}|0\rangle$ is interpreted as the Hartle-Hawking wavefunction [KO 2022]

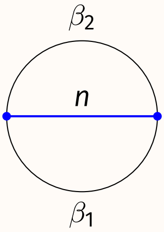
$$\langle n|e^{-\beta T}|0\rangle =$$



2-point function

- 2-point function $\langle 0|e^{-\beta_1 T} e^{-\Delta \widehat{N}} e^{-\beta_2 T}|0\rangle$ is obtained by gluing HH wavefunctions $\langle n|e^{-\beta_i T}|0\rangle$

$$\sum_{n=0}^{\infty} e^{-\Delta n}$$



- n is interpreted as a discretized bulk geodesic length

Diagonalization of T

- Transfer matrix T is diagonalized in the $|\theta\rangle$-basis

$$T|\theta\rangle = E_0 \cos\theta |\theta\rangle, \qquad E_0 = \frac{2}{\sqrt{1-q}}$$

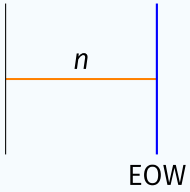
- θ-representation of chord number state $|n\rangle$ is q-Hermite polynomial $H_n(x|q)$

$$\langle\theta|n\rangle = \frac{H_n(\cos\theta|q)}{\sqrt{(q;q)_n}}$$

Big q-Hermite polynomial and EOW brane

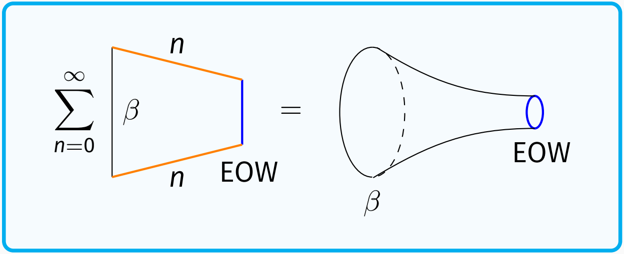
- Big q-Hermite polynomial $H_n(x,a|q)$ is a 1-parameter generalization of q-Hermite polynomial $H_n(x|q)$
- $H_n(x,a|q)$ is a wavefunction of bulk quantum gravity in the presence of end of the world (EOW) brane [KO 2023]

$$H_n(x,a|q) =$$



Half-wormhole

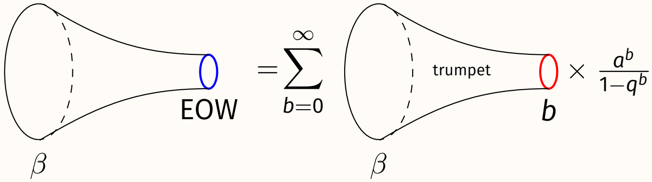
- Amplitude of half-wormhole ending on the EOW brane



- Sum over n represents a trace
  $\Rightarrow$ top and bottom of the LHS are identified

Trumpet

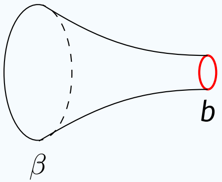
- Half-wormhole can be decomposed into trumpet and the factor coming from EOW brane



- Half-wormhole in JT gravity has a similar decomposition [Gao-Jafferis-Kolchmeyer 2021]

Trumpet of DSSYK

- Trumpet of DSSYK is given by the modified Bessel function [Jafferis-Kolchmeyer-Mukhametzhanov-Sonner 2022, KO 2023]

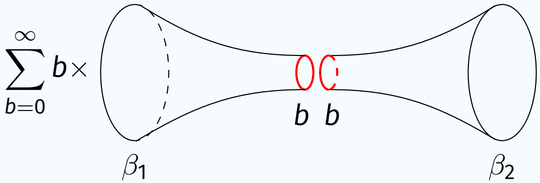


$$= I_b(\beta E_0)$$

- Trumpet also arises in JT gravity, but there is a difference
  - In DSSYK, length of geodesic loop b is discrete
  - In JT gravity, length b is continuous [Saad-Shenker-Stanford 2019]

Cylinder amplitude

- Cylinder amplitude is obtained by gluing two trumpets



- Sum over b can be performed in a closed form [KO 2023]

$$\text{above} = \frac{\beta_1 \beta_2 E_0}{2(\beta_1 + \beta_2)} \Big[I_0(\beta_1 E_0) I_1(\beta_2 E_0) + I_1(\beta_1 E_0) I_0(\beta_2 E_0) \Big]$$

More general topology

- Amplitude with more general topology is constructed by gluing trumpets and discrete volume of the moduli space of Riemann surfaces [KO 2023]

$$
\left\langle \prod_{i=1}^{n} \operatorname{tr} e^{-\beta_i H} \right\rangle_{\text{conn}}
= \sum_{g=0}^{\infty} g_s^{2g-2+n} \sum_{b_1,\cdots,b_n=0}^{\infty} \underbrace{N_{g,n}(b_1,\cdots,b_n)}_{\text{discrete volume}} \prod_{i=1}^{n} \underbrace{b_i I_{b_i}(\beta_i E_0)}_{\text{trumpet}}
$$

- This is a discrete version of the construction of amplitude in JT gravity [Saad-Shenker-Stanford 2019]

Doubled Hilbert space

- Matter correlator of DSSYK has a simple expression in terms of the doubled Hilbert space $\mathcal{H} \otimes \mathcal{H}$
- $\mathcal{H}$ is the Fock space of q-oscillators, spanned by the chord number state $|n\rangle$
- $\mathcal{H} \otimes \mathcal{H}$ is spanned by $|n, m\rangle = |n\rangle \otimes |m\rangle$

$$\mathcal{H} \otimes \mathcal{H} = \bigoplus_{n,m=0}^{\infty} \mathbb{C}|n, m\rangle$$

Matter correlators

- Matter correlator takes the form $\langle 0|X|0\rangle$ with some operator $X \in \mathrm{End}(\mathcal{H})$
- $X \in \mathrm{End}(\mathcal{H})$ can be mapped to the state $|X\rangle \in \mathcal{H} \otimes \mathcal{H}$

$$X \quad \mapsto \quad |X\rangle = \sum_{n,m=0}^{\infty} |n, m\rangle\langle n|X|m\rangle$$

- Then the matter correlator $\langle 0|X|0\rangle$ is written as $\langle 0, 0|X\rangle$

Entangled state

- The operator $q^{\Delta \widehat{N}}$ corresponds to an entangled state $|q^{\Delta \widehat{N}}\rangle$

$$|q^{\Delta \widehat{N}}\rangle = \sum_{n=0}^{\infty} q^{\Delta n} |n, n\rangle = \mathcal{E}_\Delta |0, 0\rangle$$

- $\mathcal{E}_\Delta$ is the entangler

$$\mathcal{E}_\Delta = \frac{1}{(q^\Delta A_+ \otimes A_+; q)_\infty}$$

4-point function

- Matter 4-point function G_4 is written in terms of the $6j$-symbol of $\mathcal{U}_q(\mathfrak{sl}_2)$ [Berkooz-Isachenkov-Narovlansky-Torrents 2018]

- In the doubled Hilbert space formalism, G_4 is written as

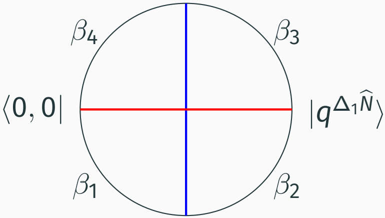
$$G_4 = \langle 0, 0|\mathcal{U}_{41}\mathcal{E}_{\Delta_1}(q^{\Delta_2\widehat{N}} \otimes q^{\Delta_2\widehat{N}})(\mathcal{E}_{\Delta_1})^{-1}\mathcal{U}_{32}|q^{\Delta_1\widehat{N}}\rangle$$

- Here $\mathcal{U}_{ij} = e^{-\beta_i T} \otimes e^{-\beta_j T}$ denotes the Euclidean time evolution

Entangler and disentangler

$$G_4 = \langle 0, 0 | \mathcal{U}_{41} \mathcal{E}_{\Delta_1} (q^{\Delta_2 \widehat{N}} \otimes q^{\Delta_2 \widehat{N}}) (\mathcal{E}_{\Delta_1})^{-1} \mathcal{U}_{32} | q^{\Delta_1 \widehat{N}} \rangle$$

- G_4 is schematically depicted as



- The intersection-counting operator $q^{\Delta_2 \widehat{N}} \otimes q^{\Delta_2 \widehat{N}}$ is conjugated by the entangler $\mathcal{E}_{\Delta_1}$ and disentangler $(\mathcal{E}_{\Delta_1})^{-1}$

Relation to Lin-Stanford

- Lin and Stanford introduced a 1-particle state [Lin-Stanford 2023]

$$|n, m\rangle^{\mathrm{LS}} \;=$$

- Their state $|n, m\rangle^{\mathrm{LS}}$ corresponds to a bivariate q-Hermite polynomial [Xu 2024]
- Their $|n, m\rangle^{\mathrm{LS}}$ and our $|n, m\rangle$ are related by [KO 2024]

$$|n, m\rangle^{\mathrm{LS}} = (q^{\Delta} A_- \otimes A_-; q)_\infty |n, m\rangle$$

Summary

- DSSYK is a solvable example of holography
- Combinatorics of chord diagrams is solved by the technique of transfer matrix
- Bulk geodesic length is discreteized in DSSYK
- One can define EOW brane, trumpet, and volume of moduli space in a similar manner as JT gravity
- In the doubled Hilbert space formalism, the entangler and disentangler should be inserted into the 4-point function

Future problem

- Curiously, transfer matrix of DSSYK is exactly same as that of ASEP (asymmetric simple exclusion process)
- ASEP is a lattice gas model, studied as a toy model of out-of-equilibrium system
- ASEP can be mapped to a matrix product state of open XXZ spin chain
- What is the role of spin chain in DSSYK?
  Relation to tensor network?