# A Differential Representation for Holographic Correlators


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**some preliminary observations on structures at loop level.**

arxiv:2403.10607

+ works to appear soon

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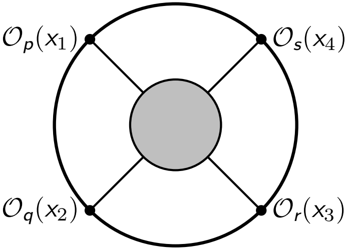
# Scattering half-BPS operators in AdS




typical examples: **supergravitons** in $\mathrm{AdS}_5 \times \mathrm{S}^5$,
**supergluons** in $\mathrm{AdS}_5 \times \mathrm{S}^3$, $\cdots$

[NUMEROUS literature, including papers of many in the audience]

[c.f., Zhou's talk]

$p$ in $\mathcal{O}_p$: Kaluza–Klein charge.

# Basic structure of the correlator

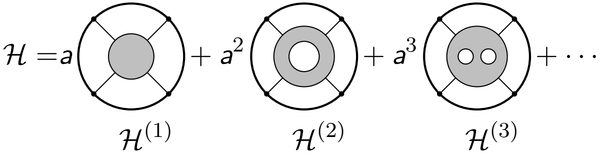
- Superconformal Ward identity [Nirschl, Osborn, '04]

$$\langle pqrs \rangle = (\text{protected}) + (\text{kinematics}) \times \underbrace{\mathcal{H}(U, V; \epsilon)}_{\text{reduced correlator}}$$

$U \equiv z\bar{z},\ V \equiv (1-z)(1-\bar{z})$: conformal cross ratios.

$\epsilon$: polarizations for internal symmetries.

- Bulk perturbation expansion $(N \gg 1,\ \lambda \gg 1)$




$\mathrm{AdS}_5 \times \mathrm{S}^5$: $a \propto 1/N^2$. $\mathrm{AdS}_5 \times \mathrm{S}^3$: $a \propto 1/N$.

$1/\lambda$ corrections (stringy effects) omitted.

# A tale of two representations

**position space** ⟺ **Mellin space**

$\mathcal{H}(U, V; \epsilon)$ $\mathcal{M}(s, t; \epsilon)$

Example: supergraviton $\langle 2222 \rangle$

$$\mathcal{H}_{2222} = \int \frac{\mathrm{d}s\mathrm{d}t}{(2\pi i)^2} U^{\frac{s+4}{2}} V^{\frac{t-4}{2}} \Gamma^2(\frac{4-s}{2})\Gamma^2(\frac{4-t}{2})\Gamma^2(\frac{4-\tilde{u}}{2}) \mathcal{M}_{2222}$$

# A tale of two representations

**position space** $\Longleftrightarrow$ **Mellin space**

$\mathcal{H}(U, V; \epsilon)$ $\qquad\qquad$ $\mathcal{M}(s, t; \epsilon)$

Example: supergraviton $\langle 2222 \rangle$ @ tree

$$\mathcal{H}^{(1)}_{2222} = \frac{P_0(z,\bar{z})}{(z-\bar{z})^4} + \frac{P_1(z,\bar{z})}{(z-\bar{z})^6}\log U + \frac{P_2(z,\bar{z})}{(z-\bar{z})^6}\log V + \frac{P_3(z,\bar{z})}{(z-\bar{z})^7}\underbrace{\left[2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right)\right]}_{W_2(z,\bar{z})}$$

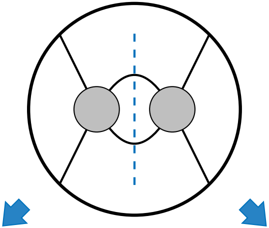
[Arutyunov, Frolov, '00], [Dolan, et al, '06]

$$\mathcal{M}^{(1)}_{2222} = \frac{2}{(s-2)(t-2)(\tilde{u}-2)}, \qquad s + t + \tilde{u} = 4$$

[Rastelli, Zhou '16]

# “Boundary conditions” for loop-level bootstrap

**(leading) logarithmic singularities**



**position space**

- max power of $\log U$
- explicit function in front of $\log^2 U$
- function type of the entire $\mathcal{H}^{(2)}$

**Mellin space**

- max power of $S$ pole
- explicit residues of $\log^2 U$
- pole structure of the entire $\mathcal{M}^{(2)}$

# Position vs Mellin

| position space | ⟺ | Mellin space |
|---|---|---|
| $\sum_i \frac{P_i(z,\bar z)}{(z-\bar z)^{\#}} \left\{ \begin{matrix} 1 \\ \log U,\ \log V \\ W_2 \end{matrix} \right\}$ | tree<br>any KK | $\sum_{m,n,l=2}^{\text{truncate}} \frac{R_{mnl}}{(s-m)(t-n)(\tilde u - l)}$ |
| $\sum_{\text{weight}\le 4} \frac{P_i(z,\bar z)}{(z-\bar z)^{\#}} G_i^{\text{SV}}(z,\bar z)$ | one loop<br>$\langle 2222 \rangle$ | $\sum_{m,n=4}^{\infty} \frac{a_{mn}}{(s-m)(t-n)} + (\text{crossing})$ |
| $\sum_{\text{weight}\le 4} \frac{P_i(z,\bar z)}{(z-\bar z)^{\#}} G_i^{\text{SV}}(z,\bar z)$ | one loop<br>higher KK | $\sum_{m,n=4}^{\infty} \frac{a_{mn}}{(s-m)(t-n)} + (\text{crossing})$ |
| $\sum_{\text{weight}\le 6} \frac{P_i(z,\bar z)}{(z-\bar z)^{\#}} G_i^{\text{SV}}(z,\bar z)$ | two loops<br>$\langle 2222 \rangle$ | |

# Back to tree level

**position space** ⟺ **Mellin space**

$$\mathcal{W}_2 = \frac{W_2}{(z-\bar{z})} \equiv \bar{D}_{1111}$$

tree
$\langle 2222 \rangle$

?

$$-\partial_U \partial_V (1 + U\partial_U + V\partial_V)$$

$$\sum_i \frac{P_i(z,\bar{z})}{(z-\bar{z})^{\#}} \left\{ \begin{matrix} 1 \\ \log U,\ \log V \\ W_2 \end{matrix} \right\}$$

$$\frac{2}{(s-2)(t-2)(\tilde{u}-2)}$$

# Back to tree level

**standard form of Mellin transform**

$$\mathcal{H} = \int \frac{\mathrm{d}S\mathrm{d}T}{(2\pi i)^2} U^S V^T \underbrace{\Gamma(-S)^2\Gamma(-T)^2\Gamma(1+S+T)^2}_{\widetilde{\Gamma}(S,T)} \mathcal{M}$$

**position space** ⟷ **Mellin space**

tree $\langle 2222 \rangle$

$$\mathcal{W}_2 = \frac{W_2}{(z-\bar{z})} \equiv \bar{D}_{1111}$$

$$\mathcal{M}_2 \equiv 1$$

$$-\partial_U \partial_V (1 + U\partial_U + V\partial_V)$$

$$\sum_i \frac{P_i(z,\bar{z})}{(z-\bar{z})^{\#}} \begin{Bmatrix} 1 \\ \log U, \ \log V \\ W_2 \end{Bmatrix}$$

$$-\frac{((S+T+1)_2)^2}{(S+1)(T+1)(S+T+3)}$$

# The differential operators

$$-\partial_U\partial_V(1+U\partial_U+V\partial_V)$$

can be decomposed as polynomials of

$$\mathcal{D}_U \equiv U\partial_U, \quad \mathcal{D}_V \equiv V\partial_V, \quad U^{\pm 1}, \quad V^{\pm 1}$$

first type

$$\mathcal{D}_U\,\mathcal{H}(U,V) = \int \frac{\mathrm{d}S\mathrm{d}T}{(2\pi i)^2} U^S V^T \widetilde{\Gamma}(S,T) \times S\,\mathcal{M}(S,T)$$

**position space** ⟷ **Mellin space**

| position space | Mellin space |
| --- | --- |
| $\mathcal{D}_U^m\,\mathcal{H}(U,V)$ | $S^m\,\mathcal{M}(S,T)$ |
| $\mathcal{D}_V^n\,\mathcal{H}(U,V)$ | $T^n\,\mathcal{M}(S,T)$ |

# The differential operators

$$-\partial_U\partial_V(1+U\partial_U+V\partial_V)$$

can be decomposed as polynomials of

$$\mathcal{D}_U \equiv U\partial_U, \quad \mathcal{D}_V \equiv V\partial_V, \quad U^{\pm 1}, \quad V^{\pm 1}$$

<u>second type</u>

$$\begin{aligned} U^a\,\mathcal{H}(U,V) &= \int \frac{\mathrm{d}S\mathrm{d}T}{(2\pi i)^2} U^{S+a} V^T \widetilde{\Gamma}(S,T)\mathcal{M}(S,T) \\ &= \int \frac{\mathrm{d}S\mathrm{d}T}{(2\pi i)^2} U^S V^T \widetilde{\Gamma}(S,T)\left[(-S)_a(1+S+T)_{-a}\right]^2 \mathcal{M}(S-a,T) \end{aligned}$$

**position space** ⟷ **Mellin space**

| | |
|---|---|
| $U^a\,\mathcal{H}(U,V)$ | $\left[(-S)_a(1+S+T)_{-a}\right]^2 \mathcal{M}(S-a,T)$ |
| $V^b\,\mathcal{H}(U,V)$ | $\left[(-T)_b(1+S+T)_{-b}\right]^2 \mathcal{M}(S,T-b)$ |

# The differential operators

$$-\partial_U \partial_V (1 + U\partial_U + V\partial_V)$$

can be decomposed as polynomials of

$$\mathcal{D}_U \equiv U\partial_U, \quad \mathcal{D}_V \equiv V\partial_V, \quad U^{\pm 1}, \quad V^{\pm 1}$$

**position space** ⟷ **Mellin space**

| position space | Mellin space |
|---|---|
| $\mathcal{D}_U^m \mathcal{H}(U,V)$ | $S^m \mathcal{M}(S,T)$ |
| $\mathcal{D}_V^n \mathcal{H}(U,V)$ | $T^n \mathcal{M}(S,T)$ |
| $U^a \mathcal{H}(U,V)$ | $\left[(-S)_a (1+S+T)_{-a}\right]^2 \mathcal{M}(S-a,T)$ |
| $V^b \mathcal{H}(U,V)$ | $\left[(-T)_b (1+S+T)_{-b}\right]^2 \mathcal{M}(S,T-b)$ |

# Tree-level $\mathcal{H}^{(1)}$ as differentials acting on $\mathcal{W}_2$

case of $\langle 2222 \rangle$ supergraviton

**position space**

$$\mathcal{H}_{2222} = (U^{-1})(V^{-1}) \underbrace{\left( -\mathcal{D}_U \mathcal{D}_V (1 + \mathcal{D}_U + \mathcal{D}_V) \right)}_{\partial^3} \mathcal{W}_2$$

**Mellin space**

$$\begin{aligned}
\mathcal{M}_2 \equiv 1 &\underset{\partial^3}{\longrightarrow} -S\,T(1+S+T) \\
&\underset{V^{-1}}{\longrightarrow} -\left(\frac{S+T+1}{T+1}\right)^2 S\,(T+1)\,(S+T+2) \\
&\underset{U^{-1}}{\longrightarrow} -\left(\frac{S+T+1}{S+1}\right)^2 \left(\frac{S+T+2}{T+1}\right)^2 (S+1)\,(T+1)\,(S+T+3) \\
&= -\frac{(S+T+1)^2(S+T+2)^2(S+T+3)^2}{(S+1)(T+1)(S+T+3)}
\end{aligned}$$

# Tree-level $\mathcal{H}^{(1)}$ as differentials acting on $\mathcal{W}_2$

similar fact applies to ALL tree-level reduced correlators
in $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathrm{AdS}_5 \times \mathrm{S}^3$

$$\mathcal{H}^{(1)}_{pqrs} = \mathcal{P}_{pqrs}(U, V, U^{-1}, V^{-1}, \mathcal{D}_U, \mathcal{D}_V)\mathcal{W}_2$$

a consequence of recursion relations amoung $\bar{D}$ functions

[c.f., Zhou's talk]

call $\mathcal{W}_2$ or its counterpart (via standard Mellin transform) $\mathcal{M}_2$
a **seed function** in position space or Mellin space

a SINGLE seed function is sufficient at tree level

# Does this continue to hold at loop level?

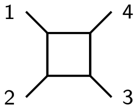
$\ast_\ast$ same type of differential operators

$\ast_\ast$ can have extra seed functions
but same set of seed functions for all correlators

# Test on $\mathcal{H}_{2222}$ of supergluons

**Mellin space**

[Alday, Bissi, Zhou, '21]




$$\mathcal{M}^{(2)} = \underbrace{\text{[box diagram]}}_{\text{color}} \; \underbrace{\sum_{m,n=0}^{\infty} \frac{a_{m,n}}{(S-m)(T-n)}}_{\mathcal{M}^{(2)}_{st}(S,T)} + (\text{crossing})$$

$$a_{m,n} = \frac{3m^2 n + 2m^2 + 3mn^2 + 8mn + 3m + 2n^2 + 3n}{3(m+n)(m+n+1)(m+n+2)}$$

(convention: $S = 0 \Leftrightarrow$ twist 4 in S channel)

**position space**

[ZH, Wang, Yuan, Zhou, '23]

(up to transcendental weight 4, too long to fit here)

# Find a closed-form expression for Mellin amplitude

observation

$\ast_{\ast}$ at tree level $\mathcal{M}_2 = 1$
poles of $\mathcal{M}^{(1)}_{pqrs}$ are created by acting with $U$ or $V$

$\ast_{\ast}$ this is fine since $\mathcal{M}^{(1)}_{pqrs}$ only has **finitely** many poles

$\ast_{\ast}$ $\mathcal{M}^{(2)}_{pqrs}$ has **infinitely** many poles
cannot be derived in the same way

# Find a closed-form expression for Mellin amplitude

intuition

we want some function that provide a grid of poles at
$S = m$ and $T = n$ $(m, n \in \mathbb{N})$

a simple choice

$$\mathcal{M}_3(S) \equiv \xi(S) = \psi^{(0)}(-S) + \gamma_{\mathrm{E}}$$

$$\mathcal{M}_4(S, T) \equiv \Phi(S, T) = -\frac{1}{2}\left((\xi(S) + \xi(T))^2 + \xi'(S) + \xi'(T) + \pi^2\right)$$

they have simple residues

$$\operatorname*{Res}_{S=m} \xi(S) = 1, \qquad \operatorname*{Res}_{S=m}\operatorname*{Res}_{T=n} \Phi(S, T) = 1, \qquad m, n \in \mathbb{N}$$

# Find a closed-form expression for Mellin amplitude

$$\mathcal{M}^{(2)}_{\mathrm{YM,st}}(S,T) =$$

$$\sum_{m,n=0}^{\infty} \underbrace{\frac{3m^2n + 2m^2 + 3mn^2 + 8mn + 3m + 2n^2 + 3n}{3(m+n)(m+n+1)(m+n+2)}} \underbrace{\frac{1}{(S-m)(T-n)}}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{3S^2T + 2S^2 + 3ST^2 + 8ST + 3S + 2T^2 + 3T}{3(S+T)(S+T+1)(S+T+2)} \qquad \Phi(S,T)$$

problem

extra poles of $S+T$

# Find a closed-form expression for Mellin amplitude

$$\mathcal{M}^{(2)}_{\rm YM,st}(S,T) = \sum_{m,n=0}^{\infty} \underbrace{\frac{3m^2n+2m^2+3mn^2+8mn+3m+2n^2+3n}{3(m+n)(m+n+1)(m+n+2)}} \underbrace{\frac{1}{(S-m)(T-n)}}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\frac{3S^2T+2S^2+3ST^2+8ST+3S+2T^2+3T}{3(S+T)(S+T+1)(S+T+2)} \qquad \Phi(S,T)$$

$$-\frac{\left(3S^2+3ST+5S+T\right)}{3(S+T)(S+T+2)}\xi(S)+(S\leftrightarrow T)$$

$$+\frac{2}{3(S+T)}+C$$

there seem to be three seed functions:

$$\mathcal{M}_4 \equiv \Phi(S,T), \quad \mathcal{M}_3 \equiv \xi(S), \quad \mathcal{M}_2 \equiv 1$$

# What justifies a differential representation?

observation 1

any poles in addition to the seed functions
can ONLY come from an action of $U^a$ or $V^b$

$$\Big[\frac{3S^2T+2S^2+3ST^2+8ST+3S+2T^2+3T}{3(S+T)(S+T+1)(S+T+2)}\Phi(S,T)$$
$$-\frac{\left(3S^2+3ST+5S+T\right)}{3(S+T)(S+T+2)}\xi(S)+(S\leftrightarrow T)$$
$$+\frac{2}{3(S+T)}+C\Big]\times\underbrace{(S+T+1)^2(S+T+2)^2}_{\Rightarrow\text{ standard Mellin transform}}$$

# What justifies a differential representation?

observation 2

the action of $U^a$ or $V^b$ is always accompanied
by corresponding double zeros and shifts

**position space** ⟷ **Mellin space**

$$V\mathcal{H}(U,V) \qquad \frac{T^2}{(S+T)^2}\,\mathcal{M}(S,T-1)$$

**double zero** ↓ ($T^2$)

**shift** ↑ ($-1$)

these structures are uniquely fixed by the power of $U$ (or $V$)

we call such phenomenon **double zero property**
a *necessary* condition for the existence of differential representation

# Differential representation for $\mathcal{H}_{2222}$ of supergluons

a strategy

$$\Big[\frac{3S^2T+2S^2+3ST^2+8ST+3S+2T^2+3T}{3(S+T)(S+T+1)(S+T+2)}\Phi(S,T) - \frac{\left(3S^2+3ST+5S+T\right)}{3(S+T)(S+T+2)}\xi(S) + (S\leftrightarrow T) + \frac{2}{3(S+T)} + C\Big] \times (S+T+1)^2(S+T+2)^2$$

partial fraction + identities among $\Phi$ and $\xi$ $\Longrightarrow$

$$\Big[-\frac{2T^2}{3(S+T)}\Phi(S,T-1)+\frac{\left(T^2+T+1\right)}{3(S+T+1)}\Phi(S,T)+\frac{(T+1)^2}{3(S+T+2)}\Phi(S,T+1) - \xi(S) + C\Big] \times (S+T+1)^2(S+T+2)^2$$

**goal**: each piece being separately free of $S+T$ poles

# Differential representation for $\mathcal{H}_{2222}$ of supergluons

two terms involving action of multiplications

$$-\frac{2\,T^2}{3(S+T)}(S+T+1)^2(S+T+2)^2\Phi(S,T-1)$$

$$\frac{(T+1)^2}{3(S+T+2)}(S+T+1)^2(S+T+2)^2\frac{(T+1)^2}{(T+1)^2}\Phi(S,T+1)$$

# Differential representation for $\mathcal{H}_{2222}$ of supergluons

$$\Big[-\frac{2}{3}(2+\mathcal{D}_U+\mathcal{D}_V)^2(1+\mathcal{D}_U+\mathcal{D}_V)^2(\mathcal{D}_U+\mathcal{D}_V)\,V$$
$$+\frac{1}{3}(2+\mathcal{D}_U+\mathcal{D}_V)^2(1+\mathcal{D}_U+\mathcal{D}_V)(1+\mathcal{D}_V+\mathcal{D}_V^2)$$
$$+\frac{1}{3}(2+\mathcal{D}_U+\mathcal{D}_V)(1+\mathcal{D}_V)^4\,V^{-1}\Big]\,\mathcal{W}_4$$
$$+(2+\mathcal{D}_U+\mathcal{D}_V)^2(1+\mathcal{D}_U+\mathcal{D}_V)^2\left(-\mathcal{W}_3+C\mathcal{W}_2\right)$$

$\mathcal{W}_i$ the seed functions in position space

$$\mathcal{W}_i=\int\frac{\mathrm{d}S\,\mathrm{d}T}{(2\pi i)^2}U^S V^T\underbrace{\Gamma(-S)^2\Gamma(-T)^2\Gamma(1+S+T)^2}_{\widetilde{\Gamma}(S,T)}\mathcal{M}_i$$

# More on the position space

multiple polylogarithms (MPL)

$$G_{a_1 a_2 \cdots a_n, z} \equiv G_{a_1 a_2 \cdots a_n}(z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G_{a_2 a_3 \cdots a_n}(t)$$

$$G(z) = 1, \quad G_{\vec{0}_n}(z) = \frac{1}{n!} \log^n z$$

[Goncharov '01] [also c.f., Hansen & Nocchi's talks]

$*_*$ branch points at $z, \bar{z} = 0, 1, \infty$

$*_*$ single-valued on the Euclidean slice $\bar{z} = z^*$ (SVMPL)

| # of independent SVMPLs | tree | one loop |
|---|---|---|
| ansatz for bootstrap | 8 | 42 |
| reduced correlator | 4 | 10 |
| seed functions | 1 | 3 |

[Huang, EYY, '21]

# More on the position space

Seed functions $\mathcal{W}_3, \ \mathcal{W}_4$

$$\mathcal{W}_i(z,\bar{z}) = \frac{W_i(z,\bar{z})}{z-\bar{z}}$$

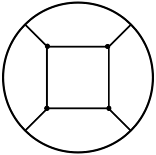
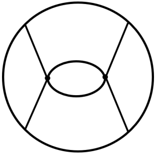
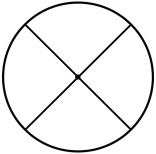
$$\begin{aligned}
W_3(z,\bar{z}) = \ & G_{00,\bar{z}}G_{1,z} - G_{1,\bar{z}}G_{00,z} + G_{01,\bar{z}}G_{0,z} + G_{0,\bar{z}}G_{01,z} - G_{10,\bar{z}}G_{0,z} \\
& - G_{10,\bar{z}}G_{1,z} + G_{0,\bar{z}}G_{10,z} - G_{1,\bar{z}}G_{10,z} + 2G_{10,\bar{z}}G_{\bar{z},z} - 2G_{01,\bar{z}}G_{\bar{z},z} \\
& + 2G_{1,\bar{z}}G_{\bar{z}0,z} - 2G_{0,\bar{z}}G_{\bar{z}1,z} + 2G_{\bar{z}01,z} - 2G_{\bar{z}10,z} - G_{001,\bar{z}} \\
& + G_{010,\bar{z}} - G_{100,\bar{z}} + G_{101,\bar{z}} - G_{001,z} + G_{010,z} + G_{100,z} - G_{101,z}
\end{aligned}$$

$$\begin{aligned}
W_4(z,\bar{z}) = \ & G_{01,\bar{z}}G_{00,z} + G_{10,\bar{z}}G_{01,z} + G_{11,\bar{z}}G_{10,z} + G_{00,\bar{z}}G_{11,z} + G_{001,\bar{z}}G_{1,z} \\
& + G_{0,\bar{z}}G_{001,z} + G_{010,\bar{z}}G_{0,z} + G_{1,\bar{z}}G_{010,z} + G_{101,\bar{z}}G_{0,z} \\
& + G_{1,\bar{z}}G_{101,z} + G_{110,\bar{z}}G_{1z} + G_{0\bar{z}}G_{110,z} + G_{0011,\bar{z}} + G_{0100,\bar{z}} \\
& + G_{1010,\bar{z}} + G_{1101,\bar{z}} + G_{0010,z} + G_{0101,z} + G_{1011,z} + G_{1100,z} \\
& - (z \leftrightarrow \bar{z}) \, .
\end{aligned}$$

# Loop reduction in AdS?

$$\mathcal{H}^{(2)}_{\text{YM,st}}(U,V) = \mathcal{D}_4\mathcal{W}_4 + \mathcal{D}_3\mathcal{W}_3 + \mathcal{D}_2\mathcal{W}_2$$

$$\Downarrow \ ?$$

$\mathcal{D}_4$  $+\,\mathcal{D}_3$  $+\,\mathcal{D}_2$ 

# $\langle 2222 \rangle$ of supergravitons

$$\mathcal{M}^{(2)} = \mathcal{M}_{st}^{(2)}(S, T) + (\text{two other channels})$$

[Alday, Zhou, '19]

$$\mathcal{M}_{st}^{(2)}(S, T) = \sum_{m,n=0}^{\infty} \frac{b_{m,n}}{(S-m)(T-n)}$$

with

$$b_{m,n} = \frac{16}{5(m+n-1)_5}(F_{m,n} + F_{n,m})$$

$$\begin{aligned} F_{m,n} = \; & 2(m-1)m(n+1)(n+2)(m+n+2)(m+n+3) \\ & + (m+1)(m+2)(n+1)(n+2)(m+n-1)(m+n) \\ & + 4m(m+1)n(n+1)(m+n+2)(m+n+3) \\ & + 8m(m+1)(n+1)(n+2)(m+n-1)(m+n+3). \end{aligned}$$

# $\langle 2222 \rangle$ of supergravitons

$$
\begin{aligned}
&\Big[ \frac{48}{5}\frac{(S-1)^2 S^2}{S+T-1}\Phi_{S-2,T} - \frac{8}{5}\frac{\left(4S^2+19S+20\right)S^2}{S+T}\Phi_{S-1,T} \\
&- \frac{4}{5}\frac{\left(18S^4-30S^3-64S^2+5ST-44S-22\right)}{S+T+1}\Phi_{S,T} \\
&+ \frac{16}{5}\frac{\left(3S^2-S+1\right)(S+1)^2}{S+T+2}\Phi_{S+1,T} + \frac{8}{5}\frac{(S+1)^2(S+2)^2}{S+T+3}\Phi_{S+2,T} \\
&- 16(S+2T+4)\xi(S) + (S\leftrightarrow T) + C \Big] \\
&\times (S+T+1)^2(S+T+2)^2(S+T+3)^2
\end{aligned}
$$

plus two other channels

straightforward to convert to position space

# $\langle 2222 \rangle$ of supergravitons

$$
\begin{array}{lll}
-16(S+2T+4)f_S & -16(T+2S+4)f_T & \\
-16(S+2\tilde{U}+4)f_S & & -16(\tilde{U}+2S+4)f_{\tilde{U}} \\
& -16(T+2\tilde{U}+4)f_T & -16(\tilde{U}+2T+4)f_{\tilde{U}}
\end{array}
$$

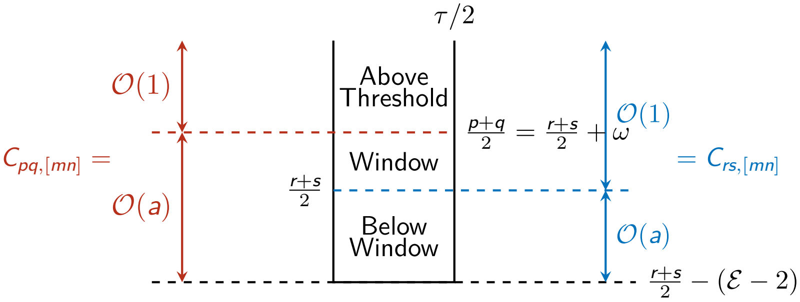
$$f_x = \xi(x), \qquad S + T + \tilde{U} = -4$$

$\xi$ (and hence $\mathcal{W}_3$) drops out in the final result

$\Rightarrow$ reduces to differentials on a “box”!

# Higher Kaluza–Klein charges

# Various contributions in the OPE




[Aprile, Drummond, Heslop, Paul, '19] [Huang, Wang, EYY, Zhou, '23]

$\omega$: size of the window

$\mathcal{E}$: extremality (amount of R structures)

$$\mathcal{E} = \frac{p+q+r+s}{4} - \frac{1}{2}(\omega_s + \omega_t + \omega_u)$$

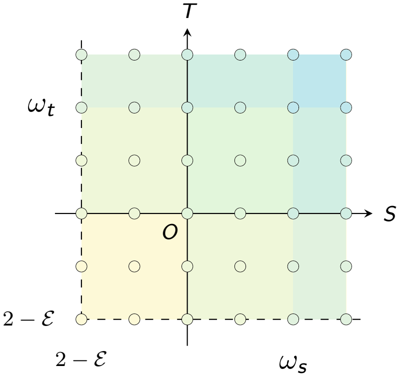
$\{\mathcal{E}, \omega_s, \omega_t, \omega_u\}$ provide equivalent parametrization to $\{p, q, r, s\}$

# Mellin amplitudes

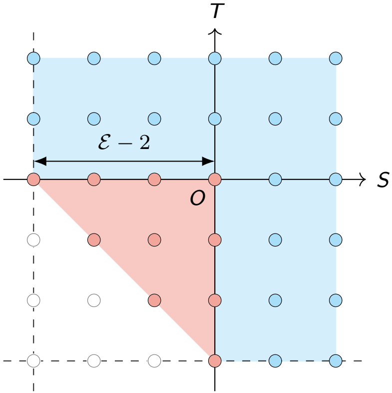
a natural def of Mellin amplitudes

$$\mathcal{H}_{\{p_i\}}(U,V) = \int \frac{\mathrm{d}S\mathrm{d}T}{(2\pi i)^2} U^S V^T \mathcal{M}_{\{p_i\}}(S,T)$$
$$\times \Gamma(-S)\Gamma(-T)\Gamma(-\tilde{U})\Gamma(\omega_s - S)\Gamma(\omega_t - T)\Gamma(\omega_u - \tilde{U})$$

with $S + T + \tilde{U} = -2 - \frac{\mathcal{N}}{2}$



# Pattern of residues in the Mellin amplitudes




Bulk + Edge Region

Corner Region

[Huang, Wang, EYY, Zhou, '23]

this analytic pattern does NOT reply on $\omega$'s

assuming the existence of a differential representation
the double zero property may provide an explanation to this pattern

(for details, see Zhongjie Huang's poster)

# Closed formulas for supergluons/supergravitons at fixed extremality

$*_*$ it is ideal to have closed formulas
for a whole class of correlators, like at the tree level

$*_*$ can incorporate differential representation into bootstrap,
which accelerates computations

$*_*$ we obtained formulas for both supergluons and supergravitons,
in the class of **next-next-to-extremal correlators**, i.e., $\mathcal{E} = 2$
(higher $\mathcal{E}$ in progress)

$*_*$ the results already manifest an interesting interplay between
the pattern of $\omega$ dependence and the double zero property

(for details, see Bo Wang's poster)

# Outlook

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- Higher points, higher loops, higher KK modes...
- Other backgrounds
  ($\mathrm{AdS}_3 \times \mathrm{S}^3, \ \mathrm{AdS}_4 \times \mathrm{S}^7, \ \mathrm{AdS}_7 \times \mathrm{S}^4$, etc....)
- More systematic Mellin-position hybrid bootstrap
  (bootstraping the differential operator?)
- Witten diagram description?
- Weak coupling?

# Thank you very much!

Questions & comments are welcome.