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Bootstrapping F-theory

Based on

arXiv: **2305.01016, 2403.17049** [hep-th] with **C. Behan, S. Chester**

iTHEMS-YITP Workshop
Bootstrap, Localization and Holography

May 20-24, 2024

Overview

F-theory

Vafa '96; Morrison, Vafa '96

D7 branes in type IIB string theory

0	1	2	3	4	5	6	7	z	$\bar{z}$
$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$		

are sources for the **axio-dilaton** field τ

$$\tau = C_0 + \frac{i}{g_s}\,, \qquad \tau = \tau(z, \bar{z})$$

Generally **strongly coupled** regions are present

Heavily exploited in string **model building**
(realistic string compactifications)

Q: effective action?

i.e. how do we compute the S-matrix at low energy?

SYM theory on D7 branes + HD corrections

$$S = -\frac{1}{4g_{YM}^2}\int \mathrm{d}^8x\,\sqrt{-g}\,\left[F^2 + c_1\,F^4 + c_2\,D^2F^4 + \ldots\right]$$

Goal: compute coefficients c_1, c_2 etc

Problem: strong coupling so no worldsheet methods available (*except for dualities*), similar to M-theory

A: bootstrap + free energy

Consider setups dual to CFTs and compute **holographic correlators**

$$\int \text{CFT correlator} = \frac{\partial^4}{\partial m^4} \text{Free energy}$$

Chester '22

Fix coefficients in **effective action**

Flat space limit gives **S-matrix**

Penedones '11

Developed for and applied to

Type IIB on $AdS_5 \times S^5$

Chester, Green, Puff, Wang, Wen '19,'20; Chester, Dempsey, Pufu '21

M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$

Binder, Chester, Pufu '19; Chester, Dempsey, Jerdee, Pufu '20

Plan of the talk

- § *The F-theory setup*
- § *Holographic correlators*
- § *The Lagrangian case*
- § *The non-Lagrangian cases*

The F-theory setup

Elliptic fibrations

For a D7 brane at $z = z_0$: **singularity** $\tau \simeq \frac{1}{2\pi i} \log(z - z_0) + \ldots$

Monodromy $\tau \to \tau + 1 \quad \leftrightarrow \quad SL(2,\mathbb{Z})$

τ: modular parameter of a **torus**, but z-dependent

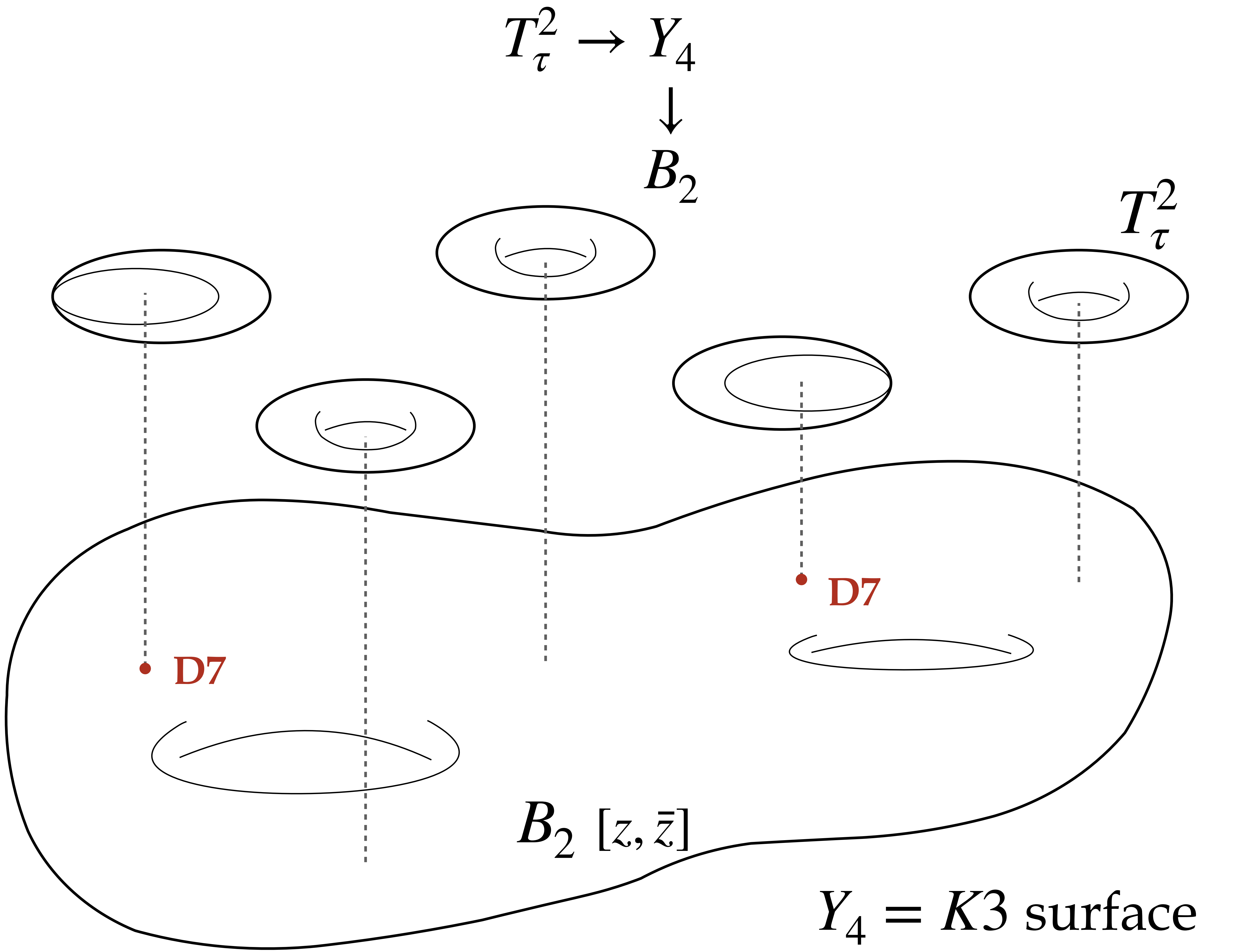
Elliptic fibration

$$y^2 = 4x^3 - g_2(z)\, x - g_3(z)$$

The curve is singular when its **discriminant** vanishes

$$\Delta \equiv 16\left[g_2(z)^3 - 27 g_3(z)^2\right] = 0$$

Location of D7 branes


$T^2_\tau \to Y_4$
$\downarrow$
B_2
T^2_τ
D7
D7
$B_2\ [z, \bar{z}]$
$Y_4 = K3$ surface

Constant coupling

Simplest examples have **constant axio-dilaton**

$$j(\tau) = 4(24 g_2(z)^3)/\Delta$$

Solutions completely classified (**ADE singularities**)

$\mathfrak{g}$	A_1	A_2	D_4	E_6	E_7	E_8
$h^\vee$	2	3	6	12	18	30
τ	i	$e^{i\pi/3}$	any	$e^{i\pi/3}$	i	$e^{i\pi/3}$

Sen '96; Dasgupta, Mukhi '96; Banks, Douglas, Seiberg '96; Douglas, Lowe, Schwarz '96

Can be probed by N D3 branes ($N \to \infty$)

4d $\mathcal{N} = 2$ SCFTs with flavor algebra $\mathfrak{g}$

AdS

$$ds^2 = L^2 \left[ds^2(\mathrm{AdS}_5) + ds^2(\tilde{S}^5)\right] \qquad ds^2(\tilde{S}^5) = d\theta^2 + \sin^2\theta\, d\varphi^2 + \cos^2\theta\, ds^2(S^3)$$

$$\frac{\Delta\varphi}{2\pi} = \frac{1}{\bar{\Delta}} = \frac{6}{6+h^\vee} < 1$$

$U(1)_r \quad SU(2)_L \times SU(2)_R$

Fayyazuddin, Spalinski '98; Aharony, Fayyazuddin, Maldacena '98

AdS_5 (0–4) $\quad S^5$ (5–7, z, $\bar{z}$)

	0	1	2	3	4	5	6	7	z	$\bar{z}$
D3	×	×	×	×						
D7	×	×	×	×	×	×	×	×		

S^3 (5–7) $\quad S^1_\varphi$

D7 branes: 8d SYM with flavor algebra $\mathfrak{g}$ on $\mathrm{AdS}_5 \times S^3$

CFT

R-symmetry: $SU(2)_R \times U(1)_r$ **+ Global symmetry:** $SU(2)_L \times \mathfrak{g}$

Two cases

1) $D_4 = SO(8)$: ***Lagrangian*** description available.
$USp(2N)$ gauge theory w/ 8 fund & 2 antisymm half-hypers

2) ***non-Lagrangian*** theories.
A_n : rank N Argyres-Douglas
E_n : rank N Minahan-Nemeschansky

KK modes of gluon on S^3 $\leftrightarrow$ 1/2 BPS multiplets in SCFT

Massless AdS_5 **gluon** $\leftrightarrow$ **flavor current** multiplet

Holographic correlators

Flavor current 4-pt function

Superprimary of **𝔤-current multiplet:** ϕ^A_α

- *Adj of flavor* $\mathfrak{g}$
- *Adj of* $SU(2)_R$
- *Dimension* $\Delta = 2$

$SU(2)_R$ cross ratio

$$\langle \phi^A \phi^B \phi^C \phi^D \rangle = \text{Pref} \times \sum_{\mathbf{r} \in \mathfrak{g} \otimes \mathfrak{g}} G_{\mathbf{r}}(U, V; w)\, P^{ABCD}_{\mathbf{r}}$$

Projectors on **r**

Spacetime cross ratios

$U = z\bar{z}\,, \quad V = (1-z)(1-\bar{z})$

Superconformal Ward identity Dolan, Osborn '02

$$G_{\mathbf{r}}(U, V; w) = \mathcal{G}_{\text{short}} + (1 - z/w)(1 - \bar{z}/w)\mathcal{G}_{\mathbf{r}}(U, V)$$

Integrated constraints

Deform 4d $\mathcal{N} = 2$ SCFT by **masses** m^A coupled to ϕ^A

Flavor central charge $k = 2\bar{\Delta}N$

Aharony, Tachikawa '07

$$-\partial_{m^A}\partial_{m^B}\partial_{m^C}\partial_{m^D} F = k^2 \sum_{\mathbf{r}} P_{\mathbf{r}}^{ABCD} I[\mathcal{G}_{\mathbf{r}}]$$

Chester '22

Free energy $Z_{S^4} = e^{-F}$

$I[\mathcal{G}_{\mathbf{r}}] \equiv \int [\mathrm{d}U\mathrm{d}V]\mathcal{G}_{\mathbf{r}}(U, V)$

 # constraints = # quartic Casimirs of $\mathfrak{g}$

D_4 : 3 quartic Casimirs

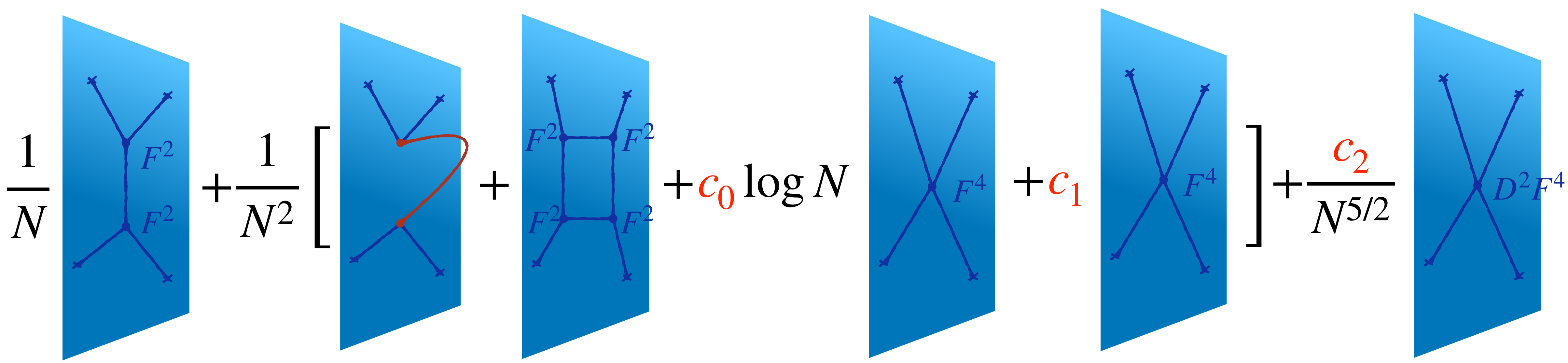
A_1, A_2, E_n : 1 quartic Casimir

Gluon amplitude at large N

$$S = -\frac{1}{4g_{YM}^2}\int d^8x\sqrt{-g}\,\left[F^2 + c_1\,F^4 + c_2\,D^2F^4 + \ldots\right] + \frac{1}{2\kappa^2}\int d^{10}X\sqrt{-G}\,\left[R + \ldots\right]$$

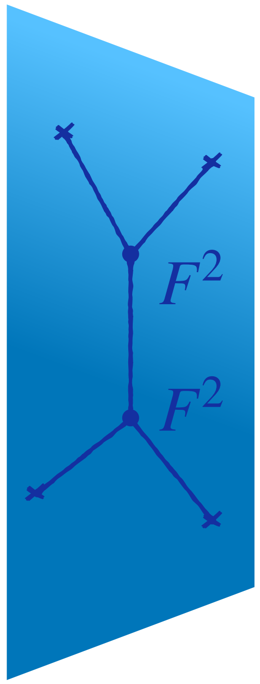
Main problem:

How do we compute c_1, c_2 at finite τ?



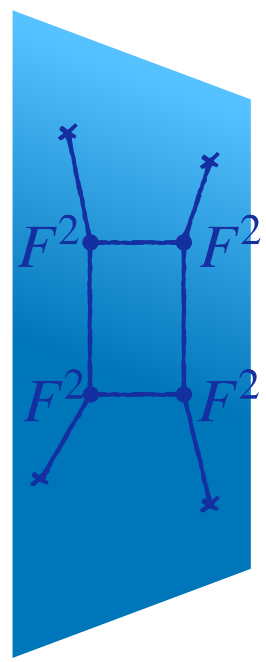
$$\mathcal{G} = \frac{1}{N}\mathcal{G}_{F^2} + \frac{1}{N^2}\left[\mathcal{G}_R + \mathcal{G}_{F^2|F^2} + c_0 \log N\,\mathcal{G}_{F^4} + c_1\mathcal{G}_{F^4}\right] + \frac{1}{N^{5/2}}\left[c_2\mathcal{G}_{D^2F^4}\right] + \ldots$$

Known terms



$\mathcal{G}_{F^2}$: *tree level*

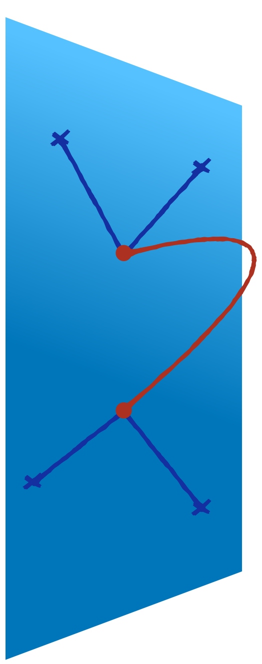
Alday, Behan, PF, Zhou '21



$\mathcal{G}_{F^2|F^2}$: *one loop*

Alday, Bissi, Zhou '21

Divergent, regulated by string theory
(Log term in flat space)



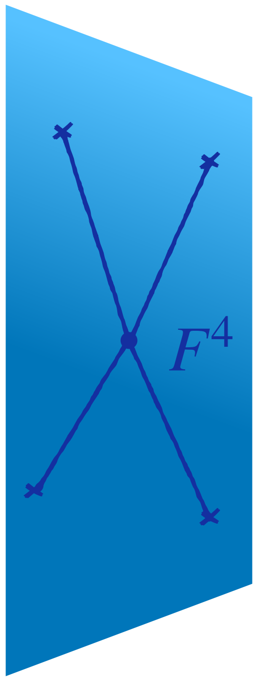
$\mathcal{G}_R$: *graviton exchange*

Alday, Bissi, Zhou '21

Exchange of short (computed) + ∞ long multiplets (unknown)
(Log term in flat space)

Chester, Pufu, Wang, Yin '23

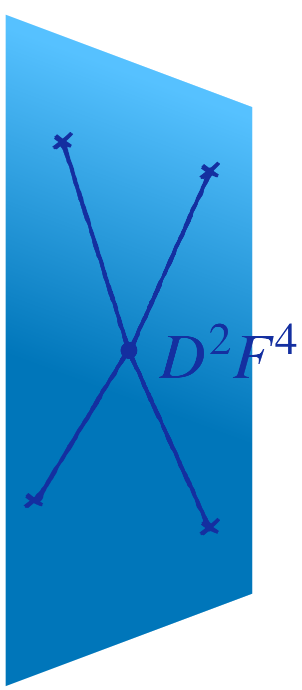
Contact terms [this talk]



As many as quartic Casimirs of $\mathfrak{g}$

$c_0 \log N\, \mathcal{G}_{F^4}$: regulates log terms in $\mathcal{G}_{F^2|F^2}$ and $\mathcal{G}_R$

$c_1\, \mathcal{G}_{F^4}$: fixed by first HD correction in effective action



We only study $c_2\, \mathcal{G}_{D^2F^4}$ for D_4 theory

5 structures: more than quartic Casimirs!

Solution: **integrated constraint + flat space limit**

The Lagrangian case

The D_4 theory

4d $\mathcal{N}=2$ $USp(2N)$ gauge theory with

	$USp(2N)$	$SO(8)$	$SU(2)_L$
8 half-hypers	Fund	$\mathbf{8_v}$	$\mathbf{1}$
2 half-hypers	Anti	$\mathbf{1}$	$\mathbf{2}$

N.B. For $N=1$: $SU(2)$ $\mathcal{N}=2$ **SQCD**

$$\mathfrak{g}\otimes\mathfrak{g}=\mathbf{28}\otimes\mathbf{28}=\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35_v}\oplus\mathbf{35_c}\oplus\mathbf{35_s}\oplus\mathbf{300}\oplus\mathbf{350}$$

$\tau_{YM}=\frac{\theta}{2\pi}+\mathrm{i}\frac{4\pi}{g_{YM}^2}$, but IR/string coupling τ fixed by $e^{\mathrm{i}\pi\tau_{YM}}=2\frac{\vartheta_2(\tau/2)}{\vartheta_3(\tau/2)}$

Douglas, Lowe, Schwartz '97; Hollands, Keller, Song '11

$SL(2,\mathbb{Z})$ **duality** $\leftrightarrow$ $SO(8)$ **triality**

$S:$	$\tau\to-1/\tau$	$\mathbf{35_v}\leftrightarrow\mathbf{35_s}$
$T:$	$\tau\to\tau+1$	$\mathbf{35_c}\leftrightarrow\mathbf{35_s}$

Seiberg, Witten '94

Supersymmetric localization

For any 4d $\mathcal{N}=2$ gauge theory **Pestun '12**

$$Z_{S^4} = \mathrm{e}^{-F(m)} = \int [\mathrm{d}X]\, \mathrm{e}^{-\frac{\mathrm{tr}X^2}{g_{YM}^2}} Z_{1-\mathrm{loop}}(m, X)\, |Z_{\mathrm{inst}}(X, m, \tau_{YM})|^2$$

We consider $N \to \infty$ limit and

Finite $\lambda_{YM} \equiv g_{YM}^2 N$

[expand for large λ_{YM}]

Instantons exp suppressed

Finite g_{YM}

Instantons contribute

Three independent quartic mass derivatives

$$\mathcal{F}_v \equiv -4\partial_{\mu_1}^2 \partial_{\mu_2}^2 F|_{\mu=0}$$

$$\mathcal{F}_c \equiv -\partial_{\mu_1}^4 F|_{\mu=0} - \partial_{\mu_1}^2 \partial_{\mu_2}^2 F|_{\mu=0} + 2\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\partial_{\mu_4} F|_{\mu=0}$$

$$\mathcal{F}_s \equiv -\partial_{\mu_1}^4 F|_{\mu=0} - \partial_{\mu_1}^2 \partial_{\mu_2}^2 F|_{\mu=0} - 2\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\partial_{\mu_4} F|_{\mu=0}$$

Large N, large λ

Neglect instantons + single trace deformation → use Toda equations

Beccaria, Korchemsky, Tseytlin '22

$$\mathcal{F}_v = 8\log\lambda + 4\left(\frac{1}{N} - \frac{7}{48N^2} + \ldots\right)$$

$$\mathcal{F}_c = \mathcal{F}_s = \frac{32\pi^2}{\lambda}N + 8\log(\lambda/4) + 4\left(\frac{1}{N} - \frac{7}{48N^2} + \ldots\right)$$

$$\frac{1}{\lambda} = \frac{1}{\lambda_{YM}} + \frac{\log 2}{2\pi^2 N}$$

$\log\lambda \leftrightarrow \log N$ *regulating term with coefficient* c_0

$1/\lambda \leftrightarrow F^4$ *contact term with coefficient* c_1

The expressions are exact in $1/\lambda$

[vs $\mathcal{N}=4$ *case]*

Chester '19

Veneziano amplitude at O(1/N)

Veneziano amplitude in flat space

$$\mathcal{A}^V = \frac{F_{st}}{st}\,\mathrm{tr}(T^AT^BT^CT^D) + \frac{F_{su}}{su}\,\mathrm{tr}(T^AT^BT^CT^D) + \frac{F_{tu}}{tu}\,\mathrm{tr}(T^AT^BT^CT^D)$$

$$F_{st} = \frac{\Gamma[1-\alpha' s]\Gamma[1-\alpha' t]}{\Gamma[1-\alpha' s-\alpha' t]} \simeq 1 - s\,t\,\zeta(2)\,(\alpha')^2 + s\,t\,u\,\zeta(3)\,(\alpha')^3 + \ldots$$

Veneziano amplitude in AdS

$(\alpha')^{\mathbf{2}}$ **term:** matched with localization constraint alone.

$(\alpha')^{\mathbf{3}}$ **term:** localization constraint + flat space limit.

$$\mathcal{M} = \frac{\tilde{F}_{st}}{st}\,\mathrm{tr}(T^AT^BT^CT^D) + \frac{\tilde{F}_{su}}{su}\,\mathrm{tr}(T^AT^BT^CT^D) + \frac{\tilde{F}_{tu}}{tu}\,\mathrm{tr}(T^AT^BT^CT^D)$$

$$\tilde{F}_{st} = \simeq 1 - \frac{24}{\lambda}(s-2)\,(t-2)\,\zeta(2) + \frac{192}{\lambda^{3/2}}(s-2)\,(t-2)\,(u-2)\,\zeta(3)\,(\alpha')^3 + \ldots$$

Later improved by Alday, Chester, Hansen, Zhong '24

Large N, finite g_{YM}

$$\tau = \tau_1 + \mathrm{i}\,\tau_2$$

$$\mathcal{F}_v = g_v(\tau,\bar\tau) + \tilde F(N) = 8\log[\sqrt{\tau_2}\,|\vartheta_2(\tau)|^2] - 24\log[\sqrt{\tau_2}\,|\eta(\tau)|^2] + \tilde F(N)$$
$$\mathcal{F}_c = g_c(\tau,\bar\tau) + \tilde F(N) = 8\log[\sqrt{\tau_2}\,|\vartheta_3(\tau)|^2] - 24\log[\sqrt{\tau_2}\,|\eta(\tau)|^2] + \tilde F(N)$$
$$\mathcal{F}_s = g_s(\tau,\bar\tau) + \tilde F(N) = 8\log[\sqrt{\tau_2}\,|\vartheta_4(\tau)|^2] - 24\log[\sqrt{\tau_2}\,|\eta(\tau)|^2] + \tilde F(N)$$

$$\tilde F(N) = 8\log(2\pi N) + 4\left(\frac{1}{N} - \frac{7}{48N^2} + \ldots\right)$$

$$\vartheta_2(\tau) = \sum_{n=-\infty}^{+\infty} q^{(n+1/2)^2} = q^{1/4}\left(2 + 2q^2 + 2q^6 + 2q^{12} + q^{20}\ldots\right)$$

$$q = \mathrm{e}^{\mathrm{i}\pi\tau}$$

$$\vartheta_3(\tau) = \sum_{n=-\infty}^{+\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \ldots$$

$$\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau) = 2\eta(\tau)^3$$

$$\vartheta_4(\tau) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2} = 1 - 2q + 2q^4 - 2q^9 + 2q^{16} + \ldots$$

$SL(2,\mathbb{Z})$ action $\leftrightarrow$ $SO(8)$ triality

$S:$	$\tau \to -1/\tau$	$\mathbf{35_v} \leftrightarrow \mathbf{35_s}$	$\vartheta_2 \leftrightarrow \vartheta_4$
$T:$	$\tau \to \tau + 1$	$\mathbf{35_c} \leftrightarrow \mathbf{35_s}$	$\vartheta_3 \leftrightarrow \vartheta_4$

Surprising simplicity

Compare with $\mathcal{N} = 4$ SYM results

$$
\begin{aligned}
\partial_m^4 \log Z\big|_{m=0,b=1} &= 6N^2 + \frac{6\sqrt{N}}{\pi^{\frac{3}{2}}} E(\tfrac{3}{2},\tau,\bar\tau) + C_0 - \frac{9}{2\sqrt{N}\pi^{\frac{5}{2}}} E(\tfrac{5}{2},\tau,\bar\tau) - \frac{27}{2^3\pi^3 N}\mathcal{E}(3,\tfrac{3}{2},\tfrac{3}{2},\tau,\bar\tau) \\
&+ \frac{1}{N^{\frac{3}{2}}}\left[\frac{117}{2^8\pi^{\frac{3}{2}}}E(\tfrac{3}{2},\tau,\bar\tau) - \frac{3375}{2^{10}\pi^{\frac{7}{2}}}E(\tfrac{7}{2},\tau,\bar\tau)\right] + \frac{1}{N^2}\left[C_1 + \frac{14175}{704\pi^4}\mathcal{E}(6,\tfrac{5}{2},\tfrac{3}{2},\tau,\bar\tau) - \frac{1215}{88\pi^4}\mathcal{E}(4,\tfrac{5}{2},\tfrac{3}{2},\tau,\bar\tau)\right] \\
&+ \frac{1}{N^{\frac{5}{2}}}\left[\frac{675}{2^{10}\pi^{\frac{5}{2}}}E(\tfrac{5}{2},\tau,\bar\tau) - \frac{33075}{2^{12}\pi^{\frac{9}{2}}}E(\tfrac{9}{2},\tau,\bar\tau)\right] + \frac{1}{N^3}\Big[\alpha_3\mathcal{E}(3,\tfrac{3}{2},\tfrac{3}{2},\tau,\bar\tau) \\
&\quad + \sum_{r=5,7,9}\left[\alpha_r\mathcal{E}(r,\tfrac{3}{2},\tfrac{3}{2},\tau,\bar\tau) + \beta_r\mathcal{E}(r,\tfrac{5}{2},\tfrac{5}{2},\tau,\bar\tau) + \gamma_r\mathcal{E}(r,\tfrac{7}{2},\tfrac{3}{2},\tau,\bar\tau)\right]\Big] + O(N^{-\frac{7}{2}}),
\end{aligned}
$$

Chester, Green, Pufu, Wang, Wen '21

In our case:

Only holomorphic functions
No τ dependence beyond leading order
No genus corrections

Contact terms at finite g_{YM}

From integrated constraints alone, fix order $1/N^2$

$c_0 \log N \, \mathcal{G}_{F^4}$ [*matches* $\log \ell_s$ *in flat space limit from string theory*]

$c_1 \mathcal{G}_{F^4}$ in terms of $g_{v,c,s}(\tau,\bar\tau)$ up to constant (require full $\mathcal{G}_R$)
[*matches known flat space limit*]
Lerche, Stieberger, Warner '98; Gutperle '99; Billò, Gallot, Lerda, Pesando '10

At order $1/N^{5/2}$ also use flat space limit:

$$\Delta_\tau c_2(\tau,\bar\tau) = \frac{3}{4} c_2(\tau,\bar\tau)$$ **Wang, Yin '15**

$c_2(\tau,\bar\tau) \sim E_{3/2}(\tau,\bar\tau)$ Non-holomorphic Eisenstein series

$$E_{3/2}(\tau) = 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3\sqrt{\tau_2}} + 4\pi\sqrt{\tau_2}\sum_{k\neq 0} |k|\,\sigma_{-2}(|k|)\,K_1(2\pi\tau_2|k|)\,e^{2\pi i k\tau_1}$$

More on instantons

Instantons from Nekrasov partition function: $Z_{\text{inst}} = \sum_{k=0}^{\infty} \mathrm{e}^{2\pi \mathrm{i} \tau_{YM}} Z_k$

Nekrasov '03

For $USp(2N)$, Z_k computed with **ADHM construction** in 5d + reduction

Nekrasov '03, Shadchin '05

$$Z_k(X, \mu_i) = \int \prod_{I=1}^{n} \frac{d\phi_I}{2\pi i} \, z_k^{\text{anti}}(X, \phi) \, z_k^{\text{vec}}(X, \phi) \prod_{i=1}^{4} z_k^{\text{hyper}(i)}(X, \phi, \mu_i)$$

Naive expression problematic due to extra d.o.f. in antisymmetric hyper ($SL(2,\mathbb{Z})$ invariance + reproduce SCQD for $N = 1$)

Solution: **remove extra d.o.f.** with Z_{extra}, first found in 5d

Hollands, Keller, Song '11; Kim, Kim, Lee '12; Hwang, Kim, Kim, Park '15

We fix Z_{extra} in 4d by assuming $N-$indep + reproduce SQCD for $N = 1$

The non-Lagrangian cases

Strong coupling

$\mathfrak{g}$	A_1	A_2	E_6	E_7	E_8
τ	i	$\mathrm{e}^{\mathrm{i}\pi/3}$	$\mathrm{e}^{\mathrm{i}\pi/3}$	i	$\mathrm{e}^{\mathrm{i}\pi/3}$

No Lagrangian description

Only **one quartic Casimir** (consider a single mass m)

Q: how do we compute $F(m)$?

A: Use SW description expanding for small r (radius of S^4)

Note: method proposed by other authors
but corrections not under control.
Here apply in large N limit where it is better defined.

Russo '14; Grassi, Komargodski, Tizzano '19; Bissi, Fucito, Manenti, Morales, Savelli '21

SW reminder

Most general effective action of $\mathcal{N} = 2$ SYM theory on CB

Prepotential

$$\mathcal{L}_{\text{eff}}^{\text{CB}} = \frac{1}{4\pi}\,\text{Im}\left[\int d^4\theta \frac{\partial \mathcal{F}_0}{\partial \Phi^i}\bar{\Phi}^i + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}_0}{\partial \Phi^i \partial \Phi^j} W^{\alpha i} W^j_\alpha\right]$$

$\mathcal{N} = 1$ chiral superfield

$\mathcal{N} = 1$ vector superfield

CB parameter

$$\textit{e.g. } SU(2) \textit{ theory: } \langle \Phi \rangle = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad u \equiv \langle \text{tr}\,\Phi^2 \rangle = 2a^2, \quad \tau = \frac{\partial^2 \mathcal{F}_0}{\partial a^2}$$

CB coordinate (gauge invariant)

Complexified gauge coupling

*Geometrically: τ is modular parameter of a torus, fibered over CB (**special Kahler**)*

SW curve and prepotential

Rank 1 SW curve $y^2 = 4x^3 - g_2(u,m)x - g_3(u,m)$

Rank 1 prepotential $\widetilde{\mathcal{F}}_0 = -2\pi \mathrm{i} \int^{u} \mathrm{d}u\, \omega_1(u',m) \int^{u'} \mathrm{d}u''\, \omega_1(u'',m)\, \tau(u'',m)$

$$\omega_1 = \frac{\partial a}{\partial u}, \qquad g_2(u,m) = \frac{E_4(\tau)}{3\omega_1^4}, \qquad g_3(u,m) = \frac{8E_6(\tau)}{27\omega_1^6}$$

Rank N prepotential $\mathcal{F}_0 = \sum_{i=1}^{N} \widetilde{\mathcal{F}}_0(u_i, m)$

Douglas, Lowe, Schwartz '97

Prepotential in F-theory

$\mathfrak{g}$	A_1	A_2	E_6	E_7	E_8
$g_2(u,m)$	$8u$	$4m^2$	$2m^2u^2$	$8u^3$	$2m^2u^3$
$g_3(u,m)$	$2m^2$	$8u^2$	$8u^4$	$\frac{2}{3}m^2u^4$	$16u^5$

Argyres, Douglas '95; Argyres, Plesser, Seiberg, Witten '95; Minahan, Nemeschansky '96

$$\widetilde{\mathcal{F}}_0 = -\frac{c_{\mathcal{F}}^2}{2}u^{2/\bar{\Delta}} - f_{\log}\, m^2 \log u + O(m^4)\,, \qquad f_{\log} = \frac{1}{2}$$

$$\mathrm{Re}\, c_{\mathcal{F}}^2 > 0\,; \quad \bar{\Delta} = \frac{4}{3}, \frac{3}{2}, 3, 4, 6$$

N.B. crucial to normalize mass consistently

Noguchi, Terashima, Yang '99

Free energy from prepotential

$$Z(m) = e^{-F(m)} = \int d^N a \left| e^{r^2 \mathcal{F}(a,m)} \right|^2, \quad \mathcal{F}(a,m) = \sum_{k=0}^{\infty} r^{-2k} \mathcal{F}_k(a,m)$$

Coulomb branch parameters

Prepotential on S_r^4

$$\mathcal{F}_1 = \frac{1}{2} \log \det \frac{\partial u}{\partial a} + \frac{1}{12} \log \Delta(u)$$

Nakajima, Yshioka '13; Witten '95; Moore, Witten '97; Shapere, Tachikawa '08; Manschot, Moore, Zhang '20

Using just $\mathcal{F}_0$ and $\mathcal{F}_1$:

$$Z(m) = e^{-F(m)} = \int d^N u \prod_{i<j} |u_i - u_j| \exp\left[-c_{\mathcal{F}}^2 \sum_{i=1}^{N} u_i^{2/\bar{\Delta}} - 2 f_{\log} m^2 \sum_{i=1}^{N} \log u_i + \ldots \right]$$

N.B. non-Gaussian: $\bar{\Delta} = \frac{4}{3}, \frac{3}{2}, 3, 4, 6$

The log term

$\mathscr{F}_0$ and $\mathscr{F}_1$ are sufficient to compute $c_0 \log N \,\mathscr{G}_{F^4}$:

$$F(m) = -\,2f_{\log}^2 m^4 \left\langle \sum_{i=1}^{N} \log(N^{\Delta/2}u_i) \sum_{j=1}^{N} \log(N^{\Delta/2}u_j) \right\rangle_{\text{conn}} + O(1)$$

2-pt function has **universal form** in *any* matrix model

Ambjorn, Jurkiewicz, Makeenko '90

$$F(m) = -\frac{\bar{\Delta}}{2}\, f_{\log}^2\, m^4 \log N + O(1) = -\frac{\bar{\Delta}}{8}\, m^4 \log N + O(1)$$

Fixes $c_0 \sim \frac{1}{\bar{\Delta}}$

Matches the prediction from flat space limit:

$\log(-s)$ *term from* $\mathscr{G}_{F^2|F^2} + \mathscr{G}_R$ *regulated by string theory* $\rightarrow \log(-\ell_s^2\, s)$

Other terms?

$\mathscr{F}_0$ and $\mathscr{F}_1$ are *not* enough to compute c_1 and c_2

Require all $\mathscr{F}_p \leftrightarrow$ **large** u **expansion** in $Z(m)$

For rank 1: $\mathscr{F}_p$ from holomorphic anomaly equations. Harder at rank N

Bershadski, Cecotti, Ooguri, Vafa '93; Huang, Klemm '07

Non-gaussian matrix model: computations are harder

+ require **analytical continuation** in u

Eynard, Kimura, Ribault '15

Interesting problem for the future!

Outlook

Lagrangian case

Compute $\partial_m^4 F(m)\,|_m = 0$ *for finite* N, τ; *use as input for numerical bootstrap*

Green, Dorigoni, Wen '21 Chester '22

Non-Lagrangian cases

Understand corrections to matrix model and how to use in practice

Graviton exchange

To compute $\mathcal{G}_{F^4}$ *we need full* $\mathcal{G}_R$, *so far only known in flat space limit*

Mixed gluon-graviton

New $\boldsymbol{R^2F^2}$ *HD correction in effective action from* $\langle AAgg \rangle$ + *localization*

Chester, Pufu, Wang, Yin '23

Integrability in D_4 theory?

Thank you for the attention!