Review of supersymmetric localization

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A ship crossing the sea of theories, holding the compass of bootstrap, embarks on a journey to seek the islands of quantum gravity on AdS.

Bootstrap, Localization and Holography

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Introduction

- Supersymmetric localization is a vast subject – Nekrasov's 2003 paper reproducing the instanton contributions to Seiberg-Witten has 1577 cites, while Pestun's landmark 2007 paper has 1387 cites.
- Obviously, I will not be able to review everything.
- Instead I will focus on localizations that are relevant for holography.
- I will first consider spheres in various dimensions that preserve a maximal amount of supersymmetry.
- I will then consider deformations of the sphere and mass deformations which are of interest for computing integrated correlators.
- I will be working with a generalization of Pestun's original formalism and be considering theories with 8 or 16 supersymmetries.
- I will discuss solutions to the relevant matrix models (mainly ignoring instantons).

- Localization of gauge theories on spheres or other compact manifolds has been successfully applied to many situations.
- Examples:

  $\mathcal{N}=4$ and $\mathcal{N}=2$ SYM in 4d Pestun

  $\mathcal{N}=2$ and higher CS/SYM in 3d Kapustin, Willett, Yakov; Jafferis

  $\mathcal{N}=1,2$ SYM in 5d: Källén, Zabzine; Källén, Qiu, Zabzine; Kim, Kim

  $(2,2)$ SYM in 2d: Benini and Cremonisi; Daroud *et. al.*

  $\mathcal{N}=2$ 6d and $\mathcal{N}=1$ 7d SYM JAM, Zabzine
- Different techniques used for the different situations.
  - Index theorems to compute one-loop determinants for even and odd dimensions
    - Even spheres have vector fields with fixed points. (sec. 2.2 of 1608.02953)
    - Odd spheres have vector fields that act freely. (sec. 2.3 of 1608.02953)
  - At the end, the results are very similar
- This suggests the possibility to analytically continue the value of d and solve in one go.

Caveat emptor

SUPERSYMMETRIC DIMENSIONAL REGULARIZATION VIA DIMENSIONAL REDUCTION ☆

Warren SIEGEL
Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Received 12 March 1979

We introduce a modified form of dimensional regularization which manifestly preserves gauge invariance, unitarity, and global supersymmetry. The prescription is that the action which results from analytic continuation to lower dimensions is that found by dimensional reduction. We also consider its application to supergravity.

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INCONSISTENCY OF SUPERSYMMETRIC DIMENSIONAL REGULARIZATION

W. SIEGEL [1,2]
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We prove that the supersymmetric form of dimensional regularization proposed by us previously is mathematically inconsistent. We also discuss the kinds of modifications which would be necessary to make the regularization consistent.

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"The general difficulties described above for analytic dimensional reduction seem somewhat paradoxical in light of the fact that is has worked so well (with the possible exception to triangle anomalies) . . ."

Maximal SYM and the dimensional reduction procedure

- 10-dimensional flat-space Lagrangian: Brink, Scherk & Schwarz

$$\mathcal{L} = \frac{1}{g_{10}^2}\mathrm{Tr}\left(\tfrac{1}{2}F_{MN}F^{MN} - \Psi \not{D}\Psi\right) .$$

- Action is invariant (on-shell) under the susy transformations

$$\begin{aligned} \delta_\epsilon A_M &= \epsilon^\alpha \, \Gamma_{M\alpha\beta}\Psi^\beta \,, & M = 0,\dots 9 \\ \delta_\epsilon \Psi^\alpha &= \tfrac{1}{2}\Gamma^{MN\alpha}{}_\beta F_{MN}\,\epsilon^\beta \,, & \alpha,\beta = 1,\dots 16 \end{aligned}$$

ϵ^α are bosonic real chiral spinors; $\Gamma^{MN\alpha}{}_\beta = \tilde{\Gamma}^{[M\,\alpha\gamma}\Gamma^{N]}_{\gamma\beta}$

- Dimensionally reduce to d-dimensional Euclidean gauge theory.

$$A_\mu\,, \quad \mu = 1\dots, d \qquad \phi_I \equiv A_I\,, \quad I = 0, d+1, \dots 9\,.$$

- Derivatives along compactified directions are zero,

$$F_{\mu I} = [D_\mu, \phi_I] \qquad F_{IJ} = [\phi_I, \phi_J]\,.$$

- Scalars transform under vector rep. of $SO(1, 9-d)$ R-symmetry in flat Euclidean space. ϕ_0 has wrong-sign kinetic term.
- d-dimensional coupling: $g_{YM}^2 = g_{10}^2/V_{10-d}$.

The theory on spheres Blau '00, Zabzine and JM '15

- S^d with radius $\mathcal{R}$.

$$ds^2 = \frac{1}{(1+\beta^2 x^2)^2} dx_\mu dx^\mu, \qquad \beta = \frac{1}{2\mathcal{R}}$$

- $d = 4$: gauge theory is superconformal, $\Longrightarrow$ conformal mass term

$$S_{\phi\phi} = \frac{1}{g_{YM}^2} \int d^4x \sqrt{-g} \left(\frac{2}{\mathcal{R}^2} \mathrm{Tr} \phi_I \phi^I \right)$$

- $d \neq 4$: not conformal, but we include a similar term:

$$S_{\phi\phi} = \frac{1}{g_{YM}^2} \int d^dx \sqrt{-g} \left(\frac{d\,\Delta_I}{2\,\mathcal{R}^2} \mathrm{Tr} \phi_I \phi^I \right), \quad [I \text{ is summed over}]$$

  Δ_I is the analog of the dimension for ϕ_I.

- Need further terms to preserve the supersymmetry.

Conformal Killing spinors

- Supersymmetries defined by conformal Killing spinors (CKS)

$$\nabla_\mu \epsilon^\alpha = \tilde{\Gamma}_\mu^{\alpha\beta} \tilde{\epsilon}_\beta \,, \qquad \nabla_\mu \tilde{\epsilon}_\alpha = -\frac{1}{4r^2} \Gamma_{\mu\alpha\beta} \epsilon^\beta \,.$$

  $\tilde{\epsilon}_\alpha$ has opposite chirality to ϵ^α.

- General solution for $d \leq 10$:

$$\epsilon = \frac{1}{(1 + x^2\beta^2)^{1/2}} \left(\epsilon_s + x \cdot \tilde{\Gamma}\, \tilde{\epsilon}_c \right) \,,$$

  ϵ_s and $\tilde{\epsilon}_c$ are arbitrary constant spinors $\Longrightarrow$ 32 independent CKS's.

- Reduce to 16 spinors by further imposing

$$\begin{aligned} \tilde{\epsilon} \;&=\; \beta\Lambda\epsilon \,, \qquad \tilde{\Gamma}^\mu\Lambda = -\tilde{\Lambda}\Gamma^\mu \qquad \tilde{\Lambda}\Lambda = 1 \\ & d \neq 4 \text{ also need } \Lambda^T = -\Lambda \quad \Longrightarrow \quad \Lambda = \Gamma^8\tilde{\Gamma}^9\Gamma^0 \\ & \Longrightarrow \quad \tilde{\epsilon}_c = \beta\Lambda\epsilon_s \end{aligned}$$

- This construction can be used for spheres up to $d = 7$.

Modified SUSY Transformations

- SUSY transfs. need to be modified

$$\begin{aligned}\delta_\epsilon A_M &= \epsilon\,\Gamma_M \Psi \\ \delta_\epsilon \Psi &= \tfrac{1}{2}\Gamma^{MN} F_{MN}\,\epsilon + \frac{\alpha_I}{2}\Gamma^{\mu I}\phi_I \nabla_\mu\,\epsilon\end{aligned}$$

$$\alpha_A = \frac{4(d-3)}{d}\,, \quad A = 8, 9, 0\,, \qquad \alpha_i = \frac{4}{d}\,, \quad i = d+1, \ldots 7$$

- Set $\Delta_A = \alpha_A$, $\Delta_i = 2(d-2)/d$.
- Complete maximally SUSY Lagrangian:

$$\begin{aligned}\mathcal{L}_{ss} &= \frac{1}{g_{YM}^2}\mathrm{Tr}\Big[\big(\tfrac{1}{2}F_{MN}F^{MN} - \Psi \not{D}\Psi + \frac{(d-4)}{2\mathcal{R}}\Psi\Lambda\Psi + \frac{2(d-3)}{\mathcal{R}^2}\mathrm{Tr}\phi^A\phi_A \\ &\qquad + \frac{(d-2)}{\mathcal{R}^2}\mathrm{Tr}\phi^i\phi_i - \frac{4}{\mathcal{R}}(d-4)\mathrm{Tr}(\phi^0[\phi^8, \phi^9])\Big]\,.\end{aligned}$$

- Preserves 16 susys but R-symmetry explicitly broken ($d \neq 4, 7$):

$$SO(1, 9-d) \to SO(1,2) \times SO(7-d)$$

$d \leq 5$, can reduce to 8 supersymmetries

- If $d \leq 5$, $\epsilon = +\Gamma^{6789}\epsilon$; $\Psi \to \psi + \chi$

$$\psi = +\Gamma^{6789}\psi \qquad \chi = -\Gamma^{6789}\chi$$

- Vector multiplet: A_μ, ψ, ϕ^I, $I = 0, d+1\dots 5$
- Hypermultiplet: χ, ϕ^I, $I = 6\dots 9$
- Can give masses to the hypermultiplets

$$\alpha_I \quad \Rightarrow \quad \frac{2(d-2)}{d} + \frac{4i\sigma_I\, m\mathcal{R}}{d} \qquad I = 6\dots 9$$

$$\Delta_I \quad \Rightarrow \quad \frac{2}{d}\left(mr(mr + i\sigma_I) + \frac{d(d-2)}{4}\right), \quad \sigma_I = +1\,(-1) \quad I = 6,7\;(8,9)\,.$$

$$\frac{1}{g_{YM}^2}(d-4)\beta\mathrm{Tr}\Psi\Lambda\Psi \Rightarrow \frac{1}{g_{YM}^2}\left((d-4)\beta\mathrm{Tr}\psi\Lambda\psi + i\,m\,\mathrm{Tr}\,\chi\Lambda\chi\right)$$

$$-\frac{1}{g_{YM}^2}\frac{4}{\mathcal{R}}(d-4)\mathrm{Tr}(\phi^0[\phi^8,\phi^9]) \Rightarrow$$
$$\frac{1}{g_{YM}^2}\left(\left(\frac{2(d-4)}{\mathcal{R}} + 4i\,m\right)\mathrm{Tr}(\phi^0[\phi^6,\phi^7]) - \left(\frac{2(d-4)}{\mathcal{R}} - 4i\,m\right)\mathrm{Tr}(\phi^0[\phi^8,\phi^9])\right)$$

Off-shell formulation

- Choose ϵ to select a convenient vector field $v^M \equiv \epsilon\Gamma^M\epsilon$.

$$v^M v_M = 0, \qquad \text{set} \quad v^0 = 1, \quad v^{8,9} = 0$$

- Introduce 7 bosonic spinors ν_m and auxilliary fields K^m, m=1...7.

$$\epsilon\Gamma^M\nu_m = 0 \qquad \nu_m\Gamma^M\nu_n = \delta_{nm}v^M \,.$$

- Off-shell SUSY transformations:

$$\begin{aligned}
\delta_\epsilon A_M &= \epsilon\Gamma_M\Psi\,, \\
\delta_\epsilon\Psi &= \tfrac{1}{2}\Gamma^{MN}F_{MN}\epsilon + \frac{\alpha_I}{2}\Gamma^{\mu I}\phi_I\nabla_\mu\epsilon + K^m\nu_m \\
\delta_\epsilon K^m &= -\nu^m(\not{D}\Psi - (d-4)\beta\Lambda\Psi)\,,
\end{aligned}$$

- α_I are same as before. $\delta_\epsilon K^m = 0$ on-shell.
- Algebra closes up to symmetries of the Lagrangian:

$$e.g.: \quad \delta_\epsilon^2 A_\mu = -v^\mu F_{\mu\nu} + [D_\mu, v^I\phi_I]$$

- Off-shell Lagrangian

$$\mathcal{L}_{aux} = -\frac{1}{g_{YM}^2}\,\mathrm{Tr}K^m K_m$$

Localization

- Localizing the off-shell action. Modify the path integral to

$$Z = \int \mathcal{D}\Phi e^{-S-tQV} \,,$$

Q is a fermionic symmetry generator. QV positive definite.

- Take $t \to \infty$ so fields localize onto fixed loci of V under Q.

$$\mathcal{Z} = \sum_{k \in \text{fixed loci}} \int \mathcal{D}\Phi_0 e^{-S_k} \text{Det}_k$$

- For Q choose δ_ϵ, while

$$V = \int d^d x \sqrt{-g}\, \Psi\, \overline{\delta_\epsilon \Psi} \,.$$

$$\overline{\delta_\epsilon \Psi} = \tfrac{1}{2} \tilde{\Gamma}^{MN} F_{MN} \Gamma^0 \epsilon + \frac{\alpha_I}{2} \tilde{\Gamma}^{\mu I} \phi_I \Gamma^0 \nabla_\mu \epsilon - K^m \Gamma^0 \nu_m \,.$$

Bosonic part of $\delta_\epsilon V$

$$\delta_\epsilon V \Big|_{bos} = \int d^d x \sqrt{-g}\, \mathrm{Tr}(\delta_\epsilon \Psi\, \overline{\delta_\epsilon \Psi}) \,.$$

Localization (continued)

- Many terms are zero. Left-over terms (assume $v^0 = 1$, $v^{8,9} = 0$)

$$\delta_\epsilon \Psi \overline{\delta_\epsilon \Psi} = \frac{1}{2} F_{MN} F^{MN} - \frac{1}{4} F_{MN} F_{M'N'} (\epsilon \Gamma^{MNM'N'0} \epsilon) + \frac{\beta d \alpha_I}{4} F_{MN} \phi_I (\epsilon \Lambda (\tilde{\Gamma}^I \tilde{\Gamma}^{MN} \Gamma^0 - \tilde{\Gamma}^0 \Gamma^I \Gamma^{MN}) \epsilon) - [K^m + 2\beta(d-3)\phi_0(\nu_m \Lambda \epsilon)]^2 + \frac{\beta^2 d^2}{4} \sum_{J \neq 0} (\alpha_J)^2 \phi_J \phi^J \,.$$

- Fixed-point locus (zero instanton sector and after analytically continuing $K^m \to i\, K^m$ and $\phi_0 \to i\, \phi_0$):

$$K^m = -2\beta(d-3)\phi_0(\nu_m \Lambda \epsilon)\,, \qquad \phi_J = 0 \quad J \neq 0\,.$$

$$\nabla_\mu \phi_0 = 0$$

- Substitute the fixed locus into $\mathcal{L}$, (zero instanton sector)

$$\mathcal{L}_{fp} = \frac{1}{g_{YM}^2} \frac{(d-1)(d-3)}{\mathcal{R}^2} \mathrm{Tr}(\phi_0 \phi_0)\,.$$

Localization (continued)

- Define dimensionless variable: $\sigma = \mathcal{R}\phi_0$.

$$S_{fp} = V_d \mathcal{L}_{fp} = \frac{\mathcal{R}^{d-4}(d-1)(d-3)S_d}{g_{YM}^2}\mathrm{Tr}\sigma^2\,,$$

$$d=3: \quad S_{fp} = 0$$

$$d=4: \quad S_{fp} = \frac{8\pi^2}{g_{YM}^2}\mathrm{Tr}\sigma^2$$

$$d=5: \quad S_{fp} = \frac{8\pi^3\mathcal{R}}{g_{YM}^2}\mathrm{Tr}\sigma^2$$

$$d=6: \quad S_{fp} = \frac{16\pi^3\mathcal{R}^2}{g_{YM}^2}\mathrm{Tr}\sigma^2$$

$$d=7: \quad S_{fp} = \frac{8\pi^4\mathcal{R}^3}{g_{YM}^2}\mathrm{Tr}\sigma^2$$

Does not change when breaking the susy's, since only the vector multiplet field ϕ_0 contributes.

One-loop determinants

- Compute quadratic fluctuations about fixed point locus
- Instead of index theorems we will compute determinants for bosons and fermions separately
- Generalization of 5D for 8 susy's (Kim & Kim) (and 3D for 4 susy's (Kapustin, Willet & Yaakov))
- Strategy is to find sets of basis vectors for bosons and fermions
- Directly doing 16 susy's is harder this way (But we can find the results indirectly)

One-loop determinants (8 SUSYs) Gorantis, Naseer, JAM '17

Following Kim and Kim we introduce basis states in terms of spherical harmonics Y^k_m

Basis vectors for vector multiplet bosons:

$$\mathcal{A}^1_{\tilde{M}} = v_{\tilde{M}} Y^k_m + c^1 \nabla_{\tilde{M}} Y^k_m$$
$$\mathcal{A}^2_{\tilde{M}} = \epsilon \Gamma_{\tilde{M}}{}^{\mu} \Lambda \epsilon \nabla_{\mu} Y^k_m + c^2 \nabla_{\tilde{M}} Y^k_m$$
$$\mathcal{A}^3_{\tilde{M}} = \epsilon \Gamma_{\tilde{M}}{}^{\mu} \Gamma^{079} \epsilon \nabla_{\mu} Y^k_m$$
$$\mathcal{A}^4_{\tilde{M}} = \epsilon \Gamma_{\tilde{M}}{}^{\mu} \Gamma^{069} \epsilon \nabla_{\mu} Y^k_m \qquad \tilde{M} = 1 \ldots 5$$

$$v^{\tilde{M}} v_{\tilde{M}} = 1 \qquad v^{\mu} \nabla_{\mu} Y^k_m = 2im\beta Y^k_m,$$

Basis spinors for vector multiplet fermions:

$$\mathcal{X}^1 = Y^k_m \eta \qquad \mathcal{X}^2 = \tilde{\Gamma}_0 \Gamma^{\tilde{M}} \hat{\nabla}_{\tilde{M}} Y^k_m \eta$$
$$\tilde{\mathcal{X}}^1 = Y^k_m \tilde{\eta} \qquad \tilde{\mathcal{X}}^2 = \tilde{\Gamma}_0 \Gamma^{\tilde{M}} \hat{\nabla}_{\tilde{M}} Y^k_m \tilde{\eta}$$

$$\eta = (1 + i\Gamma^{67})\epsilon \qquad \tilde{\eta} = \left(\Gamma^{68} + i\Gamma^{69}\right)\epsilon$$

One-loop determinants (8 SUSYs) JAM '15; Gorantis, Naseer, JAM '17

Large cancellation between bosons and fermions:

$$\frac{Det_{f,v}}{Det_{b,v}} = \prod_{\gamma\in\text{roots}} \prod_{k=1}^{\infty} (k + i\langle\gamma, \phi_0\rangle)^{D(k,k,d)} \prod_{k=0}^{\infty} (k + d - 2 + i\langle\gamma, \phi_0\rangle)^{D(k,k,d)}$$

where $D(k, m, d)$ is the degeneracy of Y^k_m in d dimensions.

Only contributions from $Y^k_{\pm k}$ survive.

$$D(k, +k, d) = D(k, -k, d)$$

Counting factor $D(k,k,d)$

- Fixed point set for $v^{\tilde{M}}$. Note that $v^{\tilde{M}} v_{\tilde{M}} = 1$.
  - On S^5 $v^{\tilde{M}}$ has fixed S^{-1}.
  - On S^4 $v^{\tilde{M}}$ has fixed S^0.
  - On S^3 $v^{\tilde{M}}$ has fixed S^1.
  - On S^2 $v^{\tilde{M}}$ has fixed S^2.
  - On S^d $v^{\tilde{M}}$ has fixed S^{4-d}.
- Counting of Y_k^k polynomials
  - S^5: $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$
  - S^4: $|z_1|^2 + |z_2|^2 + x_1^2 = 1$
  - S^3: $|z_1|^2 + x_1^2 + x_2^2 = 1$
  - S^2: $x_1^2 + x_2^2 + x_2^2 = 1$

  $Y_k^k \sim z_{i_1} z_{i_2} \ldots z_{i_k}$
- $D(k,k,d) = \frac{\Gamma(k+d-2)}{\Gamma(k+1)\Gamma(d-2)}$

$$\frac{Det_{f,v}}{Det_{b,v}} = \prod_{\gamma} \prod_{k=1}^{\infty} (k + i\langle\gamma,\phi_0\rangle)^{\frac{\Gamma(k+d-2)}{\Gamma(k+1)\Gamma(d-2)}} \prod_{k=0}^{\infty} (k + d - 2 + i\langle\gamma,\phi_0\rangle)^{\frac{\Gamma(k+d-2)}{\Gamma(k+1)\Gamma(d-2)}}$$

A similar story holds for the hypermultiplet:

$$\frac{Det_{f,h}}{Det_{b,h}} = \prod_{\gamma} \prod_{k=0}^{\infty} \left[\left(k+i\langle\gamma,\sigma\rangle+i\mu+\frac{d-2}{2} \right) \left(k-i\langle\gamma,\sigma\rangle-i\mu+\frac{d-2}{2} \right) \right]^{-\frac{\Gamma(k+d-2)}{\Gamma(k+1)\Gamma(d-2)}}$$

$$\mu = m\mathcal{R}$$

- For $d \leq 5$ we can combine a vector multiplet with an adjoint hyper with mass $\mu = i(d-4)/2$ to give 16 supercharges.
  - For general d (after shifting some k) the combined determinant factor is

$$\prod_{\gamma>0} \frac{1}{\langle\gamma,\sigma\rangle^2} \prod_{k=0}^{\infty} \left(\frac{k^2+\langle\gamma,\sigma\rangle^2}{(k+d-3)^2+\langle\gamma,\sigma\rangle^2} \right)^{\frac{\Gamma(k+d-3)}{\Gamma(k+1)\Gamma(d-3)}}$$

- It is possible to put 16 susys on S^6 and S^7 JAM, Zabzine.
- Analytically continuing the 16 susy expression to $d > 5$ we find agreement with the $d=6$ and $d=7$ cases.

The large N limit for MSYM Bobev, Bomans, Gautason, Nedelin, JAM '19

- In the large N-limit (where we can ignore instantons) the theory can be localized onto constant ϕ_0, with all other fields zero.
- Taking our previous results:

$$\mathcal{Z} = \int d\sigma \exp\left(-\frac{C_1 N}{2\lambda}\operatorname{tr}\sigma^2\right)\prod_{\gamma>0}\frac{1}{\langle\gamma,\sigma\rangle^2}\prod_{k=0}^{\infty}\left(\frac{k^2+\langle\gamma,\sigma\rangle^2}{(k+d-3)^2+\langle\gamma,\sigma\rangle^2}\right)^{\frac{\Gamma(k+d-3)}{\Gamma(k+1)\Gamma(d-3)}}$$

where σ is an $N\times N$ matrix. We can diagonalize σ to get

$$\mathcal{Z} = \int \prod_{i=1}^{N} d\sigma_i \exp\left(-\frac{C_1 N}{2\lambda}\sum_i \sigma_i^2\right)\prod_{i<j}^{N}\prod_{k=0}^{\infty}\left(\frac{k^2+\sigma_{ij}^2}{(k+d-3)^2+\sigma_{ij}^2}\right)^{\frac{\Gamma(k+d-3)}{\Gamma(k+1)\Gamma(d-3)}}$$

$$C_1 = \frac{16\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d-3}{2})}\,, \qquad \lambda \equiv g_{YM}^2 N \mathcal{R}^{4-d}$$

- Large N limit $\longrightarrow$ solve by saddle point (eigenvalue force equation)

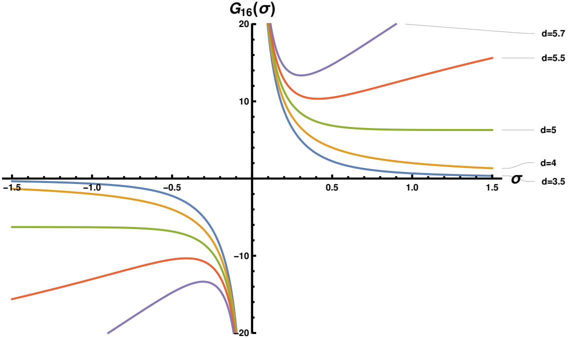
$$\frac{C_1 N}{\lambda}\sigma_i = \sum_{j\neq i} G(\sigma_{ij})$$

$$G(\sigma) \equiv -i\,\Gamma(4-d)\left(\frac{\Gamma(-i\sigma)}{\Gamma(4-d-i\sigma)} - \frac{\Gamma(i\sigma)}{\Gamma(4-d+i\sigma)} - \frac{\Gamma(d-3-i\sigma)}{\Gamma(1-i\sigma)} + \frac{\Gamma(d-3+i\sigma)}{\Gamma(1+i\sigma)}\right)$$

Saddle point equations

$$\frac{C_1 N}{\lambda} \sigma_i \quad = \quad \sum_{j \neq i} G(\sigma_{ij})$$

- $|\sigma_{ij}| \ll 1$ (weak coupling): $G(\sigma_{ij}) \approx \frac{2}{\sigma_{ij}}$ (Gaussian m.m. approx.)
- $|\sigma_{ij}| \gg 1$ (strong coupling): $G(\sigma_{ij}) \approx C_2 |\sigma_{ij}|^{d-5} \, \mathrm{sign}(\sigma_{ij})$



Free energy for maximal SYM ($d < 6$)

- We are interested in $\lambda \gg 1 \implies$ weak eigenvalue central potential, $\implies$ repulsive force terms in $G(\sigma_{ij})$ push the eigenvalues far apart.
- We write the saddle point equation as an integral equation

$$\frac{C_1}{\lambda}\sigma = C_2 \int_{-b}^{b} d\sigma' \rho(\sigma') |\sigma - \sigma'|^{d-5} \operatorname{sign}(\sigma - \sigma') \,.$$

- Scaling: $\sigma \sim \lambda^{\frac{1}{6-d}} \Rightarrow F \sim \lambda^{\frac{d-4}{6-d}} N^2$
- We can solve the integral equation to find the density

$$\rho(\sigma) = \mathcal{K}(b^2 - \sigma^2)^{\frac{5-d}{2}} \quad -b \le \sigma \le b \,, \quad \mathcal{K} = \frac{\Gamma(4 - \frac{d}{2})}{b^{6-d}\pi^{1/2}\Gamma(\frac{7-d}{2})}$$

$$\rho(\sigma) = 0 \quad |\sigma| > b$$

$$b = \left(\frac{\lambda \sin \frac{\pi(d-3)}{2} \Gamma(5-d)\Gamma(\frac{d-3}{2})\Gamma(\frac{d-1}{2})\Gamma(4 - \frac{d}{2})}{2\pi^{\frac{d+2}{2}}} \right)^{\frac{1}{6-d}} \,.$$

Free energy($d < 6$)

- Substitute $\rho(\sigma)$ into $\mathcal{Z}$ to find the free energy in terms of d:

$$\frac{F}{N^2} = -\left(\frac{\lambda}{2}\right)^{\frac{d-4}{6-d}} \frac{16\pi^{\frac{(d+1)(4-d)}{2(6-d)}}(6-d)}{\Gamma(\frac{d-3}{2})(8-d)(d-4)} \left(\frac{1}{4}\Gamma\left(\frac{8-d}{2}\right)\Gamma\left(\frac{6-d}{2}\right)\Gamma\left(\frac{d-1}{2}\right)\right)^{\frac{2}{6-d}}$$

- $d = 4$ and $d = 5$ reproduces previous results:

$$d = 4-\epsilon: \quad F_4 \approx -\frac{8\pi^2 N^2}{\lambda\,\epsilon}\left(\frac{\lambda}{8\pi^2}\right)^{1+\epsilon/2} \approx -\frac{N^2}{\epsilon} - \frac{N^2}{2}\log\lambda$$

$$d = 5: \quad F_5 = -\frac{16\pi^3 N^2}{3\lambda}\left(\frac{\lambda}{16\pi^2}\right)^2 = -\frac{\lambda N^2}{48\pi}$$

- Consider $d = 3$

$$d \to 3_+: \quad F_3 = 0$$

BPS Wilson loops

- BPS Maldacena-Wilson loop is along the equator of S^d

$$\langle W\rangle = \mathrm{Tr}\left(\mathbf{P}e^{i\oint dx^\mu A_\mu + i\oint ds\cdot\phi_0}\right) \approx \int_{-b}^{b} d\sigma\rho(\sigma)e^{2\pi\sigma}\,.$$

- Using the previous distribution $\rho(\sigma)$

$$\langle W\rangle = (\pi b)^{\frac{d-6}{2}}\Gamma\left(\frac{8-d}{2}\right) I_{\frac{6-d}{2}}(2\pi b)$$

- $d=4$ and $d=5$ reproduces previous results:

$$d=4: \qquad \langle W\rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

Erickson, Semenoff, Zarembo '00

- $d=3$ has nontrivial behavior

$$d\to 3_+: \qquad \langle W\rangle = \frac{1}{\pi^2\lambda}\left(\xi\cosh(\xi)-\sinh(\xi)\right)\,,\quad \xi = (3\pi^2\lambda)^{1/3}$$

▸ $d=3$ result is exact for $\lambda>0$. $\quad \lambda\ll 1$: $\langle W\rangle = 1+\frac{(3\pi^2\lambda)^{2/3}}{10}+\dots$

Verified holographically

$d = 7$

- For $d \geq 6$ the determinant is divergent and needs to be regularized.

$$\log Z_{1-\mathrm{loop}}(\sigma) = \sum_{i<j}\sum_{k=1}^{\infty} 2(k^2+1)\log\left(1+\frac{\sigma_{ij}^2}{k^2}\right)$$
$$= \frac{1}{2}\sum_{i,j}\sum_{n=1}^{\infty} 2\sigma_{ij}^2\left(1+k^{-2}\right) - \sigma_{ij}^4\left(k^{-2}+k^{-4}\right) + \dots .$$

- Divergent piece with mode # cutoff k_0 has same form as action:

$$2k_0\, N \sum_i \sigma_i^2$$

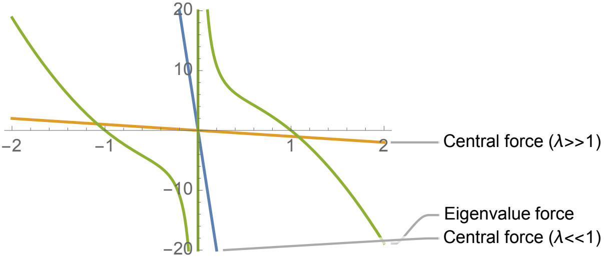
- Renormalized coupling:

$$\frac{1}{\lambda_{ren}} = \frac{1}{\lambda_{bare}} - \frac{k_0}{4\pi^4}$$

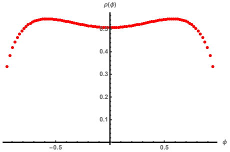
$d = 7$ (continued)

- Analytically continued eigenvalue equation:

$$\frac{16\pi^4 N}{\lambda_{ren}}\sigma_i = \sum_{j\neq i} G^{(7)}(\sigma_{ij}), \quad G^{(7)}(\sigma_{ij}) = 2\pi(1-(\sigma_{ij})^2)\coth(\pi\sigma_{ij})$$

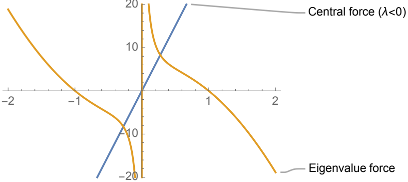


- $\lambda_{ren} = \infty$ eigenvalue distribution (solved numerically)

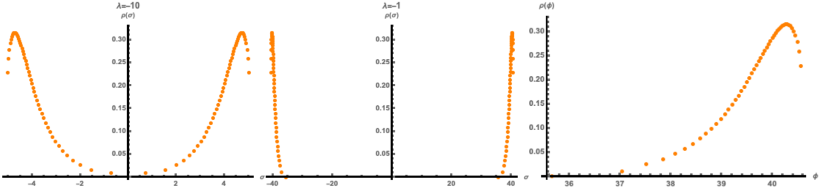


  - Eigenvalues not widely separated $\longrightarrow$ wide sep. approx. not valid

$d = 7$ (continued)



- Continue $\lambda_{ren}^{-1} < 0$
- C.O.M. is unstable
  $\Longrightarrow$ $SU(N)$ not $U(N)$



- Large separation: $d \to 7$ $\Longrightarrow$ $G(\sigma_{ij}) \to -2\pi(\sigma_{ij})^2 \,\mathrm{sign}(\sigma_{ij})$

$$\rho(\sigma) = \frac{1}{2}\left(\delta(\sigma + b) + \delta(\sigma - b)\right), \qquad b = -\frac{4\pi^3}{\lambda_{ren}}$$

- Free energy: $F_7 = \frac{128\pi^{10} N^2}{3\,\lambda_{ren}^3}$
- Wilson loop: $\langle W \rangle_7 = \cosh\left(\frac{8\pi^4}{\lambda_{ren}}\right) \sim \exp\left(-\frac{8\pi^4}{\lambda_{ren}}\right)$

$\Longleftarrow$ Valid for finite N

Comparison to supergravity: S^7

Supergravity dual for MSYM on S^7 : Bobev, Bomans, Gautason 2018

$$\mathrm{d}s_{10}^2 = \frac{\mathcal{R}^2\, e^{2\Phi/3}}{g_s^{2/3}}\left(\frac{1}{4}\mathrm{d}\rho^2 + \mathrm{d}\Omega_7^2 + \frac{1}{16}\sinh^2\rho\ \mathrm{d}\widetilde{\Omega}_2^2\right) \qquad g_s = \frac{g_{YM}^2}{(2\pi)^4\ell_s^3}$$

$$e^{2\Phi} = g_s^2\left(\frac{\mathcal{R}}{2g_sN\ell_s}\sinh\rho\right)^3, \qquad F_2 = \frac{\mathrm{i}N\ell_s}{2}\mathrm{vol}_2\,, \qquad H_3 = \frac{3N^2\ell_s^2}{4}e^{2\Phi}\,\mathrm{d}\rho\wedge\mathrm{vol}_2$$

- Puzzle (Peet &Polchinski): Weakly coupled SYM appears dual to weakly coupled SUGRA
- Resolution suggested by localization: $g_s < 0$, $\rho \to -\rho$, $H_3 \to -H_3$.

M^* theory uplift

- As $\rho \to \infty$, $e^{2\Phi} \to \infty$, uplift to M^* theory (M theory with a (2,9) signature Hull '98)
- Solution lies on $\mathbf{H}^{2,2}/\mathbf{Z}_N \times S^7$ (Bobev, Bomans, Gautason 2018)

$$
\begin{aligned}
ds^2 &= \frac{\mathcal{R}^2}{4}\left(ds_4^2 + 4d\Omega_7^2\right) \\
ds_4^2 &= d\rho^2 - \frac{\sinh^2\rho}{4}\left(dt^2 - \cosh^2 t\, d\psi^2 + (N^{-1}d\omega - \sinh t\, d\psi)^2\right) \\
G_4 &= -\frac{6\,\mathrm{i}}{\mathcal{R}}\,\mathrm{vol}_{\mathbf{H}^{2,2}}
\end{aligned}
$$

- ω is the direction along the time-like M* theory circle
- ds_4^2 has an A_{N-1} singularity at $\rho = 0$.
- Dictionary: $\ell_{11} = \left(-\frac{g^2_{YM,ren}}{(2\pi)^4}\right)^{1/3}$
- The usual holographic computations reproduce the free energy and the the Wilson loop from localization Bobev, Bomans, Gautason, Nedelin, JAM '19
- Finite N is valid in sugra Itzhaki *et. al.* '98
- We will see other examples of "negative coupling" later in this talk.

Deformations

- We can squash the spheres and still maintain some of the supersymmetry.

  S^{2p+1}: $\sum_{i=1}^{p} |z_i|^2 = 1$

  $$\text{3d}: \quad ds^2 = \mathcal{R}^2 \left(\omega_1^{-2}|dz_1|^2 + \omega_2^{-2}|dz_2|^2\right) \,, \quad \omega_1 + \omega_2 = 2$$

  $$\text{5d}: \quad ds^2 = \mathcal{R}^2 \left(\omega_1^{-2}|dz_1|^2 + \omega_2^{-2}|dz_2|^2 + \omega_3^{-2}|dz_3|^2\right) \,, \quad \omega_1+\omega_2+\omega_3 = 3$$

  The isometry group is broken from $SO(2p) \to U(1)^p$.
- In 4d we can consider the ellipsoid: $|z_1|^2 + |z_2|^2 + y^2 = 1$

  $$ds^2 = \mathcal{R}^2 \left(b^2|dz_1|^2 + b^{-2}|dz_2|^2 + dy^2\right)$$

  Isometry breaks $SO(5) \to U(1) \times U(1)$.
- The breaking of the isometry leads to more complicated partition functions.

$\mathcal{N} = 2^*$ on the 4D ellipsoid

- We let $\mu = M\mathcal{R}$ for mass of adjoint hypermultiplet.

$$\mathcal{Z} = \int d\sigma \exp\left(-\frac{8\pi^2 N}{\lambda}\operatorname{tr}\sigma^2\right)\mathcal{Z}_{vec}\mathcal{Z}_{hyp}\mathcal{Z}_{inst}$$

$$\begin{aligned}\mathcal{Z}_{vec} &= \prod_{\gamma>0}\frac{1}{\langle\gamma,\sigma\rangle^2}\prod_{m,n\geq 0}\left(\left(b(m+1)+b^{-1}(n+1)\right)^2+\langle\gamma,\sigma\rangle^2\right)\\ &\qquad\times\left(\left(bm+b^{-1}n\right)^2+\langle\gamma,\sigma\rangle^2\right)\end{aligned}$$

$$\begin{aligned}\mathcal{Z}_{hyp} &= \prod_{\gamma}\prod_{m,n\geq 0}\left(b(m+1/2)+b^{-1}(n+1/2)+i(\langle\gamma,\sigma\rangle+\mu)\right)^{-1}\\ &\qquad\times\left(b(m+1/2)+b^{-1}(n+1/2)-i(\langle\gamma,\sigma\rangle+\mu)\right)^{-1}\end{aligned}$$

- $b=1,\ \mu=0 \Rightarrow \mathcal{Z}_{vec}\mathcal{Z}_{hyp}=1$, $\mathcal{N}=4$ on round S^4.
- $b\neq 1,\ \mu=\frac{i}{2}(b-b^{-1}) \Rightarrow \mathcal{Z}_{vec}\mathcal{Z}_{hyp}=1$. Why?
- Turns out that there are 2 extra supersymmetries at the fixed points.
  Naseer and Thull '21

$\mathcal{N} = 2^*$ on the 4D ellipsoid for large λ

$$Z(b,\mu) = \int \prod_i d\sigma_i \prod_{i<j} (\sigma_{ij})^2 e^{-\frac{8\pi^2}{\lambda} N \sum_i \sigma_i^2} Z_{\text{vec}} Z_{\text{hyp}}$$

- We can write the partition function in terms of $\Upsilon_b(i\sigma_{ij})$ functions (*cf.* Nakayama '04), but we keep it in product form

$$Z_{\text{vec}} Z_{\text{hyp}} =$$

$$\prod_{i\neq j} \prod_{n=1}^{\infty} \prod_{m=1}^{n} \left(1 - \frac{(n-2m)^2 \gamma'^2}{(n+i\sigma'_{ij})^2}\right) \left(1 - \frac{(n-2m+1+i\,\rho)^2 \gamma'^2}{(n+i\sigma'_{ij})^2}\right)^{-1}$$

$$= \quad \exp\left(-\sum_{i\neq j} \sum_{p=1}^{\infty} \frac{(\gamma')^{2p}}{p} \sum_{n=1}^{\infty} \left[\frac{1}{(n+i\sigma'_{ij})^{2p}} \times \sum_{m=1}^{n} \left((n-2m)^{2p} - (n-2m+1+i\,\rho)^{2p} \right) \right] \right)$$

$$\gamma' = \sqrt{1 - \frac{4}{Q^2}},\ \rho = \frac{2\mu}{Q\gamma'},\ \sigma'_i = 2\sigma_i/Q,\ Q = b + b^{-1}$$

- The sum over n is divergent and needs to be regularized.
- Cutoff: $n = \mathcal{R}\Lambda'$, $\Lambda' = \Lambda/Q$.

$\mathcal{N} = 2^*$ on the 4D ellipsoid for large λ

$$Z_{\text{vec}} Z_{\text{hyp}} = \exp\left((1+\rho^2) \sum_{i \neq j} \sum_{p=1}^{\infty} (\gamma')^{2p} f_p(\sigma'_{ij}, \rho) \right)$$

- For large $|\sigma'_{ij}|$, $f_p(\sigma'_{ij}, \rho) \approx -\log \Lambda' + \log \sigma'_{ij} = -\log \Lambda + \log \sigma_{ij}$.

$$Z\big|_{\text{reg.}} \approx \int \prod_i d\sigma_i \prod_{i<j} (\sigma_{ij}^2)^{\frac{Q^2}{4} + \mu^2} e^{-\frac{8\pi^2}{\lambda} N \sum_i \sigma_i^2}$$

- Saddle point equation

$$\frac{16\pi^2}{\lambda} N \sigma_i = 2\left(\frac{Q^2}{4} + \mu^2\right) \sum_{j \neq i} \frac{1}{\sigma_i - \sigma_j},$$

- Free energy

$$\log Z\big|_{\text{reg.}} \approx \frac{N^2}{2}\left(\frac{Q^2}{4} + \mu^2\right) \log\left(\lambda\left(\frac{Q^2}{4} + \mu^2\right)\right)$$

- Wilson loop

$$\langle W \rangle = e^{\frac{\sqrt{\lambda}}{2\pi}\sqrt{\frac{Q^2}{4} + \mu^2}}$$

Russo and Zarembo '12; Bobev, Elvang, Freedman, Pufu '13

Other examples of "negative couplings"

- $\mathcal{N} = 2^*$ on S^4, $\mu = M\mathcal{R}$

$$\mathcal{Z} = \int d\sigma \exp\left(-\frac{8\pi^2}{g_0^2}\,\mathrm{tr}\,\sigma^2\right) \mathcal{Z}_{vec}\mathcal{Z}_{hyp}\mathcal{Z}_{inst}$$

$$\mathcal{Z}_{vec} = \prod_{\gamma}\prod_{k=1}^{\infty}(k + i\langle\gamma,\phi_0\rangle)^{k+1}\prod_{k=0}^{\infty}(k+2+i\langle\gamma,\phi_0\rangle)^{k+1}$$

$$\mathcal{Z}_{hyp} = \prod_{\gamma}\prod_{k=0}^{\infty}\left[(k+i\langle\gamma,\sigma\rangle+i\mu+1)\,(k-i\langle\gamma,\sigma\rangle-i\mu+1)\right]^{-k-1}$$

$$\begin{aligned}\mathcal{Z} &= \int\prod_{i=1}^{N}d\sigma_i e^{-\frac{4\pi^2}{g_0^2}\sum_i\sigma_i^2} \\ &\times\prod_{i<j}^{N}\frac{\sigma_{ij}^2(G(1+i\,\sigma_{ij})G(1-i\,\sigma_{ij}))^2}{G(1+i\,(\sigma_{ij}+\mu))G(1-i\,(\sigma_{ij}+\mu))G(1+i\,(\sigma_{ij}-\mu))G(1-i\,(\sigma_{ij}-\mu))} \\ &\times Z_{\mathrm{N}}\,(i\sigma,i\mu)\,Z_{\mathrm{S}}\,(i\sigma,i\mu)\ ,\end{aligned}$$

$\mathcal{N} = 2^*$ on S^4 Russo '14; Naseer, Thull, JAM *unpub*

- Saddle point (ignoring instantons)

$$\frac{16\pi^2}{g_0^2}\sigma_i = \sum_{j\neq i}\Bigg(\frac{2}{\sigma_{ij}} - 2\sigma_{ij}\left(\psi(1+i\,\sigma_{ij})+\psi(1+i\,\sigma_{ij})\right) + (\sigma_{ij}+\mu)\left(\psi(1+i\,(\sigma_{ij}+\mu))+\psi(1-i\,(\sigma_{ij}+\mu))\right) + (\sigma_{ij}-\mu)\left(\psi(1+i\,(\sigma_{ij}-\mu))+\psi(1-i\,(\sigma_{ij}-\mu))\right)\Bigg),$$

- $|x| \gg 1 \Rightarrow \psi(1+i\,x)+\psi(1-i\,x) \approx 2\log(x)$
- Reduce to pure $\mathcal{N}=2$: Assume $\mu \gg |\sigma_{ij}| \gg 1$,

$$\frac{16\pi^2}{g_0^2}\sigma_i = 4N\sigma_i\log\mu + 4\sum_{j\neq i}\sigma_{ij}(1-\log|\sigma_{ij}|)$$

Rewrite as $\frac{16\pi^2}{g_{YM}^2}\sigma_i = 4\sum_{j\neq i}\sigma_{ij}(1-\log|\sigma_{ij}|)$,

$$\frac{4\pi^2}{g_{YM}^2} = \frac{4\pi^2}{g_0^2} - N\log\mu \equiv -N\log\frac{\Lambda\mathcal{R}}{2}$$

- Eigenvalues split in half: $\sigma_{ij} \approx \exp(-\frac{4\pi^2}{\lambda}) = \frac{\Lambda\mathcal{R}}{2}$

- Specialize to $SU(2)$

$$\log \sigma_{12} = 1 - \frac{2\pi^2}{g_{YM}^2} = 1 + \log \frac{\Lambda\mathcal{R}}{2}$$

- At strong coupling we should expect to reach the massless monopole point (where the magnetic theory is weakly coupled).

$$\text{Seiberg-Witten:} \quad \sigma_{12} = \frac{8}{\pi}\frac{\Lambda\mathcal{R}}{2} = (2.54565)\frac{\Lambda\mathcal{R}}{2}$$

$$\text{Localization:} \quad \sigma_{12} = e\,\frac{\Lambda\mathcal{R}}{2} = (2.71828)\frac{\Lambda\mathcal{R}}{2}$$

- Need to include instantons; modifies the saddle point equation

$$\text{1 inst.} \quad -4\log\left(\frac{\Lambda\mathcal{R}}{2}\right)\sigma_{12} + 8\left(\frac{\Lambda\mathcal{R}}{2}\right)^4 \sigma_{12}^{-3} = 4\sigma_{12}(1 - \log\sigma_{12})\,.$$

$\sigma_{12} = a\,\frac{\Lambda\mathcal{R}}{2}$, $\Rightarrow$ $2 = a^4(1 - \log a)$ $\Rightarrow$ $a \approx 2.6023$.

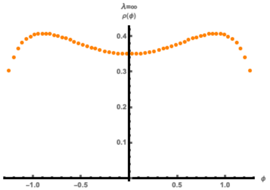
9 instantons: $a \approx 2.5595$

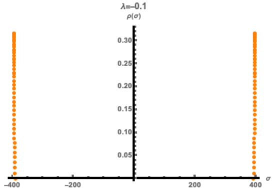
$\mathcal{N} = 1^*$ on S^5 Nedelin, JAM '21

- This is closer to the 7D case.
- $SU(N)$ $\mathcal{N} = 1^*$ on round S^5: Eigenvalue eq. in limit of large hypermultiplet mass M

$$\frac{8\pi^3 N}{\lambda}\sigma_i = \sum_{j\neq i} 2\pi(1 - \tfrac{1}{2}\,\sigma_{ij}^2)\coth \pi\sigma_{ij}$$

$$\boxed{\frac{4\pi^2}{\lambda} = \frac{4\pi^2}{g_{YM}^2 N} = \mathcal{R}\left(\frac{4\pi^2}{g_0^2 N} - M\right)}$$

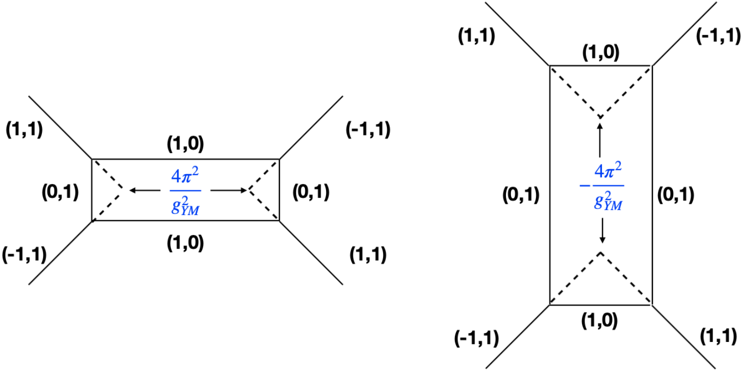




Nedelin, JM 2020

- $\lambda \to 0_-$: Coulomb branch $SU(N) \to SU(N/2) \times SU(N/2) \times U(1)$
- Generate an effective coupling for $U(1)$, $\frac{\mathcal{R}}{g_{U(1)}^2} = -\frac{1}{\lambda} > 0$
- $U(1)$ enhanced to $SU(2)$.

5D $\mathcal{N} = 1$ $SU(2)$ Pure SYM at negative coupling



(a) $g_{YM}^2 > 0$

(b) $g_{YM}^2 < 0$,

$$m_W = \phi,\ m_I = \phi + \frac{4\pi^2}{g_{YM}^2} \qquad m_W = \phi' - \frac{4\pi^2}{g_{YM}^2},\ m_I = \phi'$$

$$\phi' = \left(\sigma_{12} + \frac{4\pi^2}{g_{YM}^2}\right)\mathcal{R}^{-1}$$

$SU(2)$ in 7D

- In 7D instantons are membranes
  - Membrane tension $T = \frac{2}{\pi \mathcal{R}^3}\delta\sigma$.
  - $\mathcal{R}$ dependence suggests that $\delta\sigma$ is not part of a vector multiplet.
  - Membrane is minimally coupled to a three-form field
    $\Rightarrow$ Expect $\delta\sigma$ to be in the same multiplet.
  - Only such multiplet in 7D is the $\mathcal{N} = 2$ graviton multiplet.
  - Contains the graviton, the three-form C, an $SO(1,2)$ triplet of vector fields A^I, and a real scalar ρ.
- Suggests that "negative coupling" regime is actually weakly coupled supergravity.

Summary

- We have given a uniform description for putting MSYM and its mass deformed cousins on S^d.
- We can generalize this to deformed spheres
- While localization is limited to only supersymmetric observables, it can still provide a wealth of *exact* information.

THANKS!

Other stuff

Alternative 2D vector multiplet Lagrangian

- In 2 dimensions we can choose a different modification of the flat space Lagrangian.
- $(2,2)$ vector multiplet $[A_\mu, \phi^0, \phi^3, \psi]$:

$$\delta_\epsilon \psi = \tfrac{1}{2}\Gamma^{MN} F_{MN}\,\epsilon + \frac{\alpha_I}{2}\Gamma^{\mu I}\phi_I \nabla_\mu\,\epsilon$$

$$M, N = 0\ldots 3, \quad I, J = 0, 3 \qquad \alpha_3 = 2 \qquad \alpha_0 = 0$$

- The vector multiplet Lagrangian is modified to

$$\begin{aligned}\mathcal{L}_{ss} &= \frac{1}{g_{YM}^2}\mathrm{Tr}\Big[\tfrac{1}{2}F_{MN}F^{MN} - \psi \not{D}\psi + \frac{1}{r^2}\mathrm{Tr}\phi^3\phi^3 - \frac{2}{r}F_{12}\phi^3\Big] \\ &= \frac{1}{g_{YM}^2}\mathrm{Tr}\Big[\left(F_{12} - \frac{\phi_3}{r}\right)^2 + D_\mu\phi_I D^\mu\phi^I + \tfrac{1}{2}[\phi_I, \phi_J][\phi^I, \phi^J] - \psi\not{D}\psi\Big]\end{aligned}$$

This is the Q-exact Lagrangian

- No change in chiral multiplet Lagrangian

One-loop determinants (8 SUSYs)

- Instead of index theorems we will generalize Kim & Kim for 8 susy's and Kapustin, Willet & Yaakov for 4 susy's
- Directly doing 16 susy's is harder this way
- In this talk we only consider mass deformations of maximal SYM
- Fluctuations about fixed point locus (Bosons)

$$\mathcal{L}^{\text{bos}}_{\text{vm}} = A^{\tilde{M}}\,\mathcal{O}^{\tilde{N}}_{\tilde{M}}\,A_{\tilde{N}} - [A_{\tilde{M}},\phi^0_{cl}][A^{\tilde{M}},\phi^0_{cl}] \qquad \tilde{M} = 1\ldots 5$$

$$\mathcal{O}^{\tilde{N}}_{\tilde{M}} = -\delta^{\tilde{N}}_{\tilde{M}}\nabla^2 + \gamma^{\tilde{N}}_{\tilde{M}} + 2\beta(d-3)\epsilon\Gamma_{\tilde{M}}{}^{\nu\tilde{N}89}\epsilon\nabla_\nu .$$

$$\gamma^{\tilde{N}}_{\tilde{M}} = 4\beta^2\begin{pmatrix}(d-1)\,\delta^\nu_\mu & 0\\ 0 & \delta^j_i\end{pmatrix}.$$

$$\mathcal{L}_{a,b}(\mu) = \sum_{i=a,b}\left[\phi_i\left(-\nabla^2+\beta^2(d-2+2i\mu)^2\right)\phi_i - [\phi^0_{cl},\phi_i][\phi^0_{cl},\phi_i]\right]$$
$$-4\beta\,(1-2i\mu)\,\phi_a v^\mu\nabla_\mu\phi_b \qquad \mu = m\,r$$

$$\mathcal{L}^{\text{bos}}_{\text{hm}} = \mathcal{L}_{67}(\mu) + \mathcal{L}_{98}(-\mu).$$

One-loop determinants (8 SUSYs)

- Fermion fluctuations

$$\mathcal{L}^{\text{ferm}}_{\text{vm}} = (\psi \nabla\!\!\!/ \psi) - \frac{1}{2}(d-3)\beta v^{\tilde{M}} \left(\psi \Gamma^0 \tilde{\Gamma}_{\tilde{M}} \Lambda \psi\right) - v^0 \left(\psi \Gamma^0 D_0 \psi\right)$$
$$- \frac{1}{4}(d-3)\beta \left(\epsilon \tilde{\Gamma}^{\tilde{M}\tilde{N}} \Lambda \epsilon\right) \left(\psi \Gamma^0 \Gamma_{\tilde{M}\tilde{N}} \psi\right) + \frac{d-1}{2} (\psi \Lambda \psi) .$$

$$\mathcal{L}^{\text{ferm}}_{\text{hm}} = (\chi \nabla\!\!\!/ \chi) + \left(\chi \Gamma^0 D_0 \chi\right) - \frac{1}{2}\beta \left(\epsilon \tilde{\Gamma}^{\tilde{M}\tilde{N}} \Lambda \epsilon\right) \left(\chi \Gamma^0 \Gamma_{\tilde{M}\tilde{N}} \chi\right)$$
$$+ 2i\mu\beta v^{\tilde{N}} \left(\chi \Gamma^0 \tilde{\Gamma}_{\tilde{N}} \Lambda \chi\right) .$$

One-loop determinants (8 SUSYs)

Following Kim and Kim we introduce basis vectors for the vm bosons:

$$\begin{aligned}
\mathcal{A}^1_{\tilde{M}} &= v_{\tilde{M}} Y^k_m + c^1 \nabla_{\tilde{M}} Y^k_m \\
\mathcal{A}^2_{\tilde{M}} &= \epsilon \Gamma_{\tilde{M}}{}^{\mu} \Lambda \epsilon \nabla_\mu Y^k_m + c^2 \nabla_{\tilde{M}} Y^k_m \\
\mathcal{A}^3_{\tilde{M}} &= \epsilon \Gamma_{\tilde{M}}{}^{\mu} \Gamma^{079} \epsilon \nabla_\mu Y^k_m \\
\mathcal{A}^4_{\tilde{M}} &= \epsilon \Gamma_{\tilde{M}}{}^{\mu} \Gamma^{069} \epsilon \nabla_\mu Y^k_m
\end{aligned}$$

$$v^{\tilde{M}} v_{\tilde{M}} = 1 \qquad v^\mu \nabla_\mu Y^k_m = 2im\beta Y^k_m,$$

$$(\mathcal{O}\mathcal{A})^1_{\tilde{M}} = 4\beta^2 \left[k(k+d-1) + (d-1)^2\right] \mathcal{A}^1_{\tilde{M}} - 4\beta \left(\frac{d-1}{2}\right) \mathcal{A}^2_{\tilde{M}}$$

$$(\mathcal{O}\mathcal{A})^2_{\tilde{M}} = -4\beta^3\, 2(d-1)k(k+d-1)\mathcal{A}^1_{\tilde{M}} + 4\beta^2 k(k+d-1)\mathcal{A}^2_{\tilde{M}}.$$

$$\mathcal{O}\mathcal{A}^I_{\tilde{M}} = 4\beta^2 \left[k(k+d-1) + (d-2)^2 \mathcal{A}^I_{\tilde{M}} + i\epsilon^{IJ}(d-3)m\mathcal{A}^J_{\tilde{M}}\right], \qquad I, J = 3, 4$$

eigenvalues:

$$4\beta^2 k^2, \qquad k \geq 2,\ m \neq \pm k \qquad \text{and} \qquad 4\beta^2 (k+d-1)^2.$$

$$4\beta^2 \left[k(k+d-1) + (d-2)^2 \pm (d-3)m\right]. \qquad (+)\ m \neq +k \quad (-)\ m \neq -k$$

One-loop determinants (8 SUSYs)

Basis spinors for vm fermions

$$\mathcal{X}^1 = Y^k_m \eta \qquad \mathcal{X}^2 = \tilde{\Gamma}_0 \Gamma^{\tilde{M}} \hat{\nabla}_{\tilde{M}} Y^k_m \eta$$
$$\tilde{\mathcal{X}}^1 = Y^k_m \tilde{\eta} \qquad \tilde{\mathcal{X}}^2 = \tilde{\Gamma}_0 \Gamma^{\tilde{M}} \hat{\nabla}_{\tilde{M}} Y^k_m \tilde{\eta}$$

$$\eta = (1 + i\Gamma^{67})\epsilon \qquad \tilde{\eta} = \left(\Gamma^{68} + i\Gamma^{69}\right)\epsilon$$

$$\Gamma^{89}\eta = i\eta, \quad \Gamma^{89}\tilde{\eta} = -i\Gamma^{89}\tilde{\eta}, \quad v^{\tilde{M}}\Gamma_{\tilde{M}}\eta, \quad v^{\tilde{M}}\Gamma_{\tilde{M}}\tilde{\eta} = \tilde{\eta},$$

For $k \geq 1,\ m \neq k$ determinants:
$4\beta^2\, k(k+d-1), \quad 4\beta^2\left[k(k+d-1) + (d-2)^2 + m(d-3)\right]$
For $k \geq 0,\ m = k$: eigenvalues
$2i\beta(k+d-1),\ 2i\beta(k+d-2)$

$$\frac{Det_{f,v}}{Det_{b,v}} = \prod_{\beta \in \mathrm{roots}} \prod_{k=1}^{\infty} (k + i\langle \beta, \phi_0 \rangle)^{D(k,k,d)} \prod_{k=0}^{\infty} (k + d - 2 + i\langle \beta, \phi_0 \rangle)^{D(k,k,d)}$$

where we have included the contribution of the constant bosonic field ϕ_0

Deformed Chern-Simons

$$Z(b;m_1,m_2,m_3)=$$

$$e^{\frac{i\pi}{12k}(b-b^{-1})M(M^2-1)}\int\frac{d^{N+M}\mu d^N\nu}{N!(N+M)!}e^{i\pi k\left(\sum_i\nu_i^2-\sum_a\mu_a^2\right)}$$

$$\times\prod_{a>b}4\sinh\left(\pi b(\mu_a-\mu_b)\right)\sinh\left(\pi b^{-1}(\mu_a-\mu_b)\right)$$

$$\prod_{i>j}4\sinh\left(\pi b(\nu_i-\nu_j)\right)\sinh\left(\pi b^{-1}(\nu_i-\nu_j)\right)$$

$$\times\prod_{i,a}\left\{s_b\left[\frac{iQ}{4}-\left(\mu_a-\nu_i+\frac{m_1+m_2+m_3}{2}\right)\right]s_b\left[\frac{iQ}{4}-\left(\mu_a-\nu_i+\frac{m_1-m_2-m_3}{2}\right)\right]\right.$$

$$\left.\times s_b\left[\frac{iQ}{4}-\left(-\mu_a+\nu_i+\frac{-m_1-m_2+m_3}{2}\right)\right]s_b\left[\frac{iQ}{4}-\left(-\mu_a+\nu_i+\frac{-m_1+m_2-m_3}{2}\right)\right]\right\}$$

$$s_b(\sigma)=\prod_{m,n\geq0}\frac{mb+nb^{-1}+\frac{Q}{2}-i\sigma}{mb+nb^{-1}+\frac{Q}{2}+i\sigma}$$

$$\sinh(z)=z\prod_{k\geq1}\left(1+\frac{z^2}{\pi^2k^2}\right).$$