Integrated correlators beyond localisation

iTHEMS-YITP Workshop: Bootstrap, Localization and Holography

Yukawa Institute for Theoretical Physics (YITP)

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Based on: 2308.07219 with A. Brown, C. Wen and H Xie

Outline of talk

1.) 4 point correlators <pqrs>

- Half BPS (single particle) in planar N=4 SYM
- = 4 point graviton amplitudes in IIB on $AdS_5 x S^5$
- Summarise current understanding
- Focus on results for all higher charges (KK modes)
- Perturbative (order by order) at weak and strong coupling

2.) Integrated 4-point correlators $\int < pqrs >$

- Localisation -> $\int < 2222 >$ and $\int < 22pp >$ exact all orders
- Conjecture $\int< pqrs >$ as a function of λ ('t Hooft coupling)
- "Beyond localisation"
- Checks, predictions (6 loop periods, strong coupling, octagon)

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1. 4 point correlators

N=4 SYM

Simplest 4d gauge theory

Breeding ground for new QFT techniqes esp for amplitudes

AdS/CFT

- Gauge theory with 6 scalars $\Phi_{I=1..6}$, 4 fermions, gauge field
- CFT
- Gauge group SU(N)
- $N \to \infty$ (planar = tree level string th) single coupling $\lambda = g_{YM}{}^2 N$
- $\Phi(x,y) = \Phi_I y^I, \quad y^{I=1..6}, \quad y^I y^I = 0$
- Half BPS operators: $O_p(x,y) = Tr(\Phi(x,y)^p)$
- 4 point correlators: $\langle pqrs \rangle = \langle O_p O_q O_r O_s \rangle$

Function of x,y

AdS/CFT

CFT = Quantum gravity on AdS space

Very precise formulation:

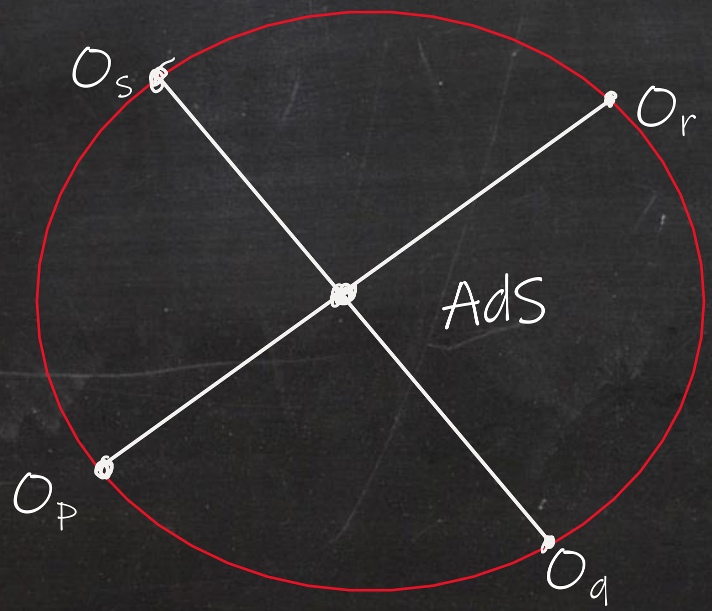
N=4 SYM = IIB string theory on $AdS_5 \times S^5$

Strong coupling N=4 SYM = quantum gravity

AdS/CFT

- $\mathcal{N}=4$ SYM
- Superconformal symmetry
- λ, N (also use $c=(N^2-1)/4$)
- $\mathcal{O}_p$
- $\langle \mathcal{O}_p \mathcal{O}_q \mathcal{O}_r \mathcal{O}_s \rangle$

- IIB string theory on $AdS_5 \times S^5$
- (super) isometry of $AdS_5 \times S^5$
- $\alpha' = \frac{1}{\sqrt{\lambda}}$, $G_N = \frac{1}{c}$
- Graviton multiplet ($p = S^5$ KK mode)
- 4 point graviton amplitude



Witten diagrams

Half BPS correlators

- 2 point functions $\langle O_p O_p \rangle(N)$ fixed in free theory (indep of λ)
- 3 point functions $\langle O_p O_q O_r \rangle(N)$ fixed in free theory (indep of λ)
- 4 point functions $\langle O_p O_q O_r O_s \rangle(N,\lambda)$ non-trivial function of λ
- Steady progress since AdS/CFT (1998) at both small and large λ

Superconformal symmetry

- Constraints of superconformal symmetry (non perturbative):

Function of $x_1,y_1,..x_4,y_4$ Homogeneous of degree p_i in y_i

$$\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle = \text{free} \; + \; C_{p_1p_2p_3p_4}\frac{I(x_i,y_j)}{\xi^{(4)}} \times F_{p_k}(x_i,y_j;\lambda,c)$$

$$C_{p_1p_2p_3p_4} = \frac{p_1p_2p_3p_4}{2c}\left(\frac{c}{16\pi^4}\right)^{\frac{1}{4}\sum p_i}$$

$$I(x_i,y_j) = x_{13}^4x_{24}^4y_{13}^4y_{24}^4(x-y)(x-\bar{y})(\bar{x}-y)(\bar{x}-\bar{y}) \qquad \xi^{(4)} = x_{13}^4x_{24}^4x_{12}^2x_{23}^2x_{34}^2x_{14}^2$$

$$F_{p_k}(x_i,y_j;\lambda,c) = \sum_{\{b_{ij}\}}\left(\prod_{i<j} g_{ij}^{b_{ij}}\right) F_{\{b_{ij}\}}(x,\bar{x};\lambda,c)$$

$$\{b_{ij}\} := \{b_{ij} = b_{ji} : b_{ii} = 0, \; \sum_i b_{ij} = p_j - 2\} \qquad g_{ij} := \frac{y_{ij}^2}{x_{ij}^2}$$

Superconformal symmetry -> Charge reduction: homogeneous of degree p_i-2 in y_i

$F_{\{b\}}$ weight 4 at each point, simple poles in x_{ij}^2

$$x\bar{x} = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2} \qquad (1-x)(1-\bar{x}) = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$$

$$y\bar{y} = \frac{y_{12}^2y_{34}^2}{y_{13}^2y_{24}^2} \qquad (1-y)(1-\bar{y}) = \frac{y_{14}^2y_{23}^2}{y_{13}^2y_{24}^2}\,.$$

Cross ratios

Perturbative correlator integrands

- Loop expansion:

$$F_{\{b_{ij}\}} = \sum_{l=1}^{\infty} \left(\frac{\lambda}{4\pi^2}\right)^l F^{(l)}_{\{b_{ij}\}}$$

$$F^{(l)}_{\{b_{ij}\}}(x_1,..,x_4) = \frac{\xi^{(4)}}{l!} \int \frac{d^4x_5}{(-4\pi^2)} .. \frac{d^4x_{4+l}}{(-4\pi^2)} f^{(l)}_{\{b_{ij}\}}(x_1,.,x_{4+l})$$

- $f^{(l)}_{\{b_{ij}\}}(x_1,.,x_{4+l})$

Known to 5 loops, l=5 using conformal symmetry, pole structure $1/x_{ij}^2$ and OPE [Chicherin, Drummond, Sokatchev,PH 2015; Chicherin, Georgudis, Goncalves, Pereira 2018]

Hidden 10d conformal symmetry (perturbative)

- Observed to possess a 10d conformal integrand symmetry
- All higher charge correlators $f^{(l)}_{\{b_{ij}\}}$ obtained directly from the <2222> correlator $f^{(l)}$ [Caron-Huot, Coronado 2021]

$$f^{(l)}_{\{b_{ij}\}}(x_{ij}^2) = f^{(l)}(\mathbf{x}_{ij}^2)|_{(g_{ij})^{b_{ij}}}$$

$$\mathbf{x}_{ij}^2 = x_{ij}^2 - y_{ij}^2 = x_{ij}^2(1 - g_{ij})$$

- <2222> correlator integrand known to 10 loops [Bourjaily, Tran, PH 2016]
- Therefore all charge <pqrs> known to 10 loops assuming 10d conformal symmetry

Hidden 10d conformal symmetry (perturbative)

- "master correlator": all half BPS operators packaged together

$$\boldsymbol{\mathcal{O}}(x,y) = \sum_{p=2}^{\infty} \frac{1}{p} \left(\frac{16\pi^4}{c}\right)^{p/4} \mathcal{O}_p(x,y)$$

- Master correlator in terms of <2222> correlator

$$\langle \boldsymbol{\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}} \rangle = \text{free} \; + \; \frac{I(x_i,y_j)}{2c} \times \sum_{l=0}^{\infty} \frac{\lambda^l}{(4\pi^2) l!} \int \frac{d^4x_5}{(-4\pi^2)} .. \frac{d^4x_{4+l}}{(-4\pi^2)} f^{(l)}(\mathbf{x}_{ij}^2)$$

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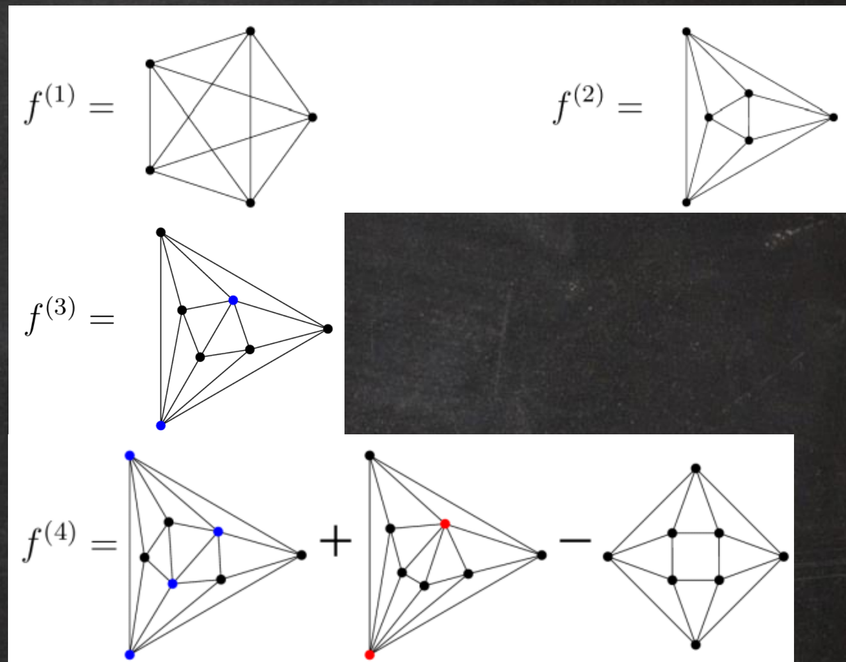
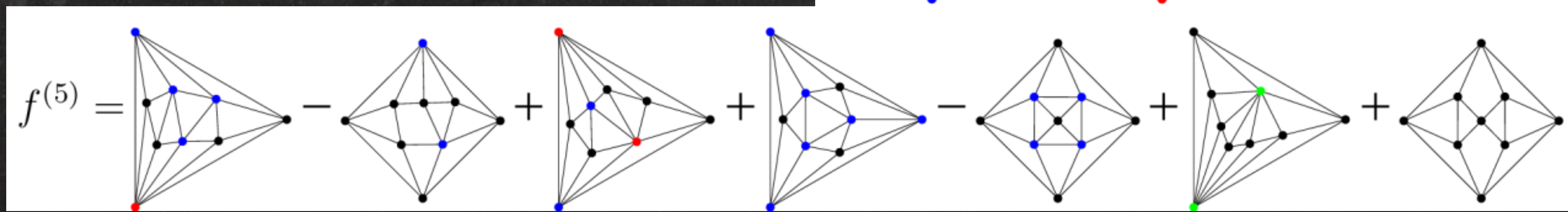
Hidden permutation symmetry (perturbative)

- <2222> correlator integrand has hidden S_{4+l} permutation symmetry mixing external and internal variables
- Sum of "f-graphs"

$$f^{(l)}(\mathbf{x}_{ij}^2) = \sum_\alpha c_\alpha^{(l)} f_\alpha^{(l)}(\mathbf{x}_{ij}^2)$$

$$f_\alpha^{(l)}(\mathbf{x}_{ij}^2) = \frac{1}{|\mathrm{aut}(\alpha)|} \sum_{\sigma_{l+4}} \prod_{i,j=1}^{4+l} \frac{1}{(\mathbf{x}_{\sigma_i \sigma_j}^2)^{e_{ij}^\alpha}}$$





4 point correlators @ strong coupling

Tree-level SUGRA

- 1998-> : Similar progress in parallel at stong coupling (large λ)
- 1999: <2222> leading order in $1/\lambda = \alpha'^2$ from Tree-level SUGRA
- 2000-2016: <pqrs> for many values of p,q,r,s from SUGRA
- 2016: <pqrs> all values, bootstrap [Rastelli, Zhou]
- 2018: 10d conformal symmetry [Caron-Huot, Trinh]
- => <pqrs> generating function from <2222>

master correlator

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle_{\frac{1}{c},\lambda\to\infty} = \text{free} - \frac{I(x_i,y_i)}{4c}\frac{D_{2422}(\mathbf{x}_i)}{\mathbf{x}_{13}^2\mathbf{x}_{14}^2\mathbf{x}_{34}^2}$$

<2222> correlator

(NB 10d conformal symmetry here for the correlator. Perturbative 10d conformal symmetry for the integrand only (broken by 4d integration).

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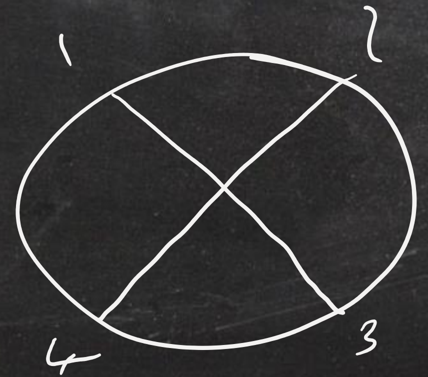
4 point correlators @ strong coupling
Tree-level String corrections

- 2014-> : $1/\lambda = \alpha'^2$ corrections [Goncalves; Binder, Chester, Pufu, Wang; Alday,Bissi; Drummond, Nandan,Paul, Rigatos; Aprile, Drummond, Paul Stangata; Abl, Lipstein, PH]
- 10d conformal symmetry broken but ...
- Master correlator from single 10d scalar effective action on $AdS_5 x S^5$ [Abl, Lipstein, PH]
- EG First string correction, α'^3 from scalar ϕ^4:

$$S_{\alpha'^3} = \frac{1}{8.4!}\left(\frac{\alpha'}{2}\right)^3 \times 2\zeta_3 \times \int_{\mathrm{AdS}\times\mathrm{S}} d^{10}\mathbf{z}\, \phi(\mathbf{z})^4$$



$AdS_5 x S^5$ Witten daigram

$AdS_5 x S^5$ Bulk to Boundary propagator $\frac{\mathcal{C}_\Delta}{(-2)^\Delta}(\mathbf{z}.\mathbf{x})^{-\Delta}$

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle|_{c^{-1}\lambda^{-3/2}} \sim \frac{2\zeta_3}{8.4!}\left(\frac{\alpha'}{2}\right)^3 \frac{(\mathcal{C}_4)^4}{(-2)^{16}}\, \mathcal{I}(x_i, y_i) \int_{\mathrm{AdS}\times\mathrm{S}} \frac{d^{10}\mathbf{z}}{(\mathbf{z}.\mathbf{x}_1)^4(\mathbf{z}.\mathbf{x}_2)^4(\mathbf{z}.\mathbf{x}_3)^4(\mathbf{z}.\mathbf{x}_4)^4}$$

4 point correlators @ strong coupling

Tree-level String corrections

- α'^5: four-derivative interactions

$$S_{\alpha'^5} = \frac{\zeta_5}{8.4!}\left(\frac{\alpha'}{2}\right)^5 \int_{\mathrm{AdS}\times\mathrm{S}} d^{10}\mathbf{z}\Big(3(\nabla\phi.\nabla\phi)(\nabla\phi.\nabla\phi) - 9\nabla^2\nabla_\mu\phi\nabla^\mu\phi\phi^2 - 30\phi^4\Big)$$

- α'^6: six-derivative interactions (fixed)
- α'^7: eight-derivative interactions not completely fixed:

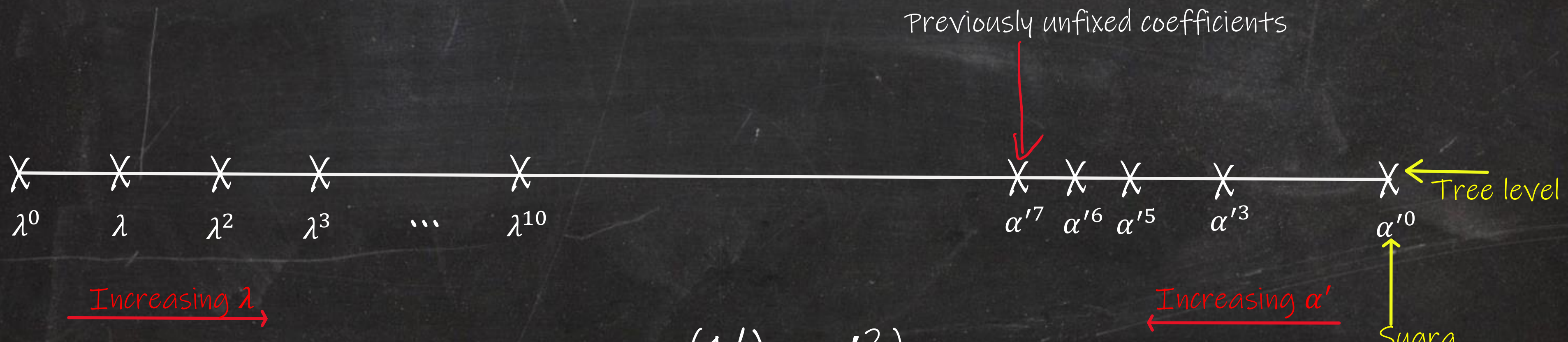
$$S_{\alpha'^7} = \frac{1}{8}\left(\frac{\alpha'}{2}\right)^7 \Big(\tfrac{1}{2}\zeta_7 S^{\mathrm{main}}_{\alpha'^7} + G_{1;0}S^{\mathrm{amb}_1}_{\alpha'^7} + G_{2;0}S^{\mathrm{amb}_2}_{\alpha'^7} + G_{3;0}S^{\mathrm{amb}_3}_{\alpha'^7} + G_{4;0}S^{\mathrm{amb}_4}_{\alpha'^7} + G_{5;0}S^{\mathrm{amb}_5}_{\alpha'^7}$$
$$+ D_1 S^{\mathrm{main}}_{\alpha'^6} + E_1 S^{\mathrm{amb}}_{\alpha'^6} + B_2 S^{\mathrm{main}}_{\alpha'^5} + C_2 S^{\mathrm{amb}}_{\alpha'^5} + A_4 S^{\mathrm{main}}_{\alpha'^3}\Big) ,$$

(return to this shortly)

NB 10d master correlator but not 10d conformal symmetry

Summary: large N known correlators (all charges)

X = result known (integrand) for master correlator



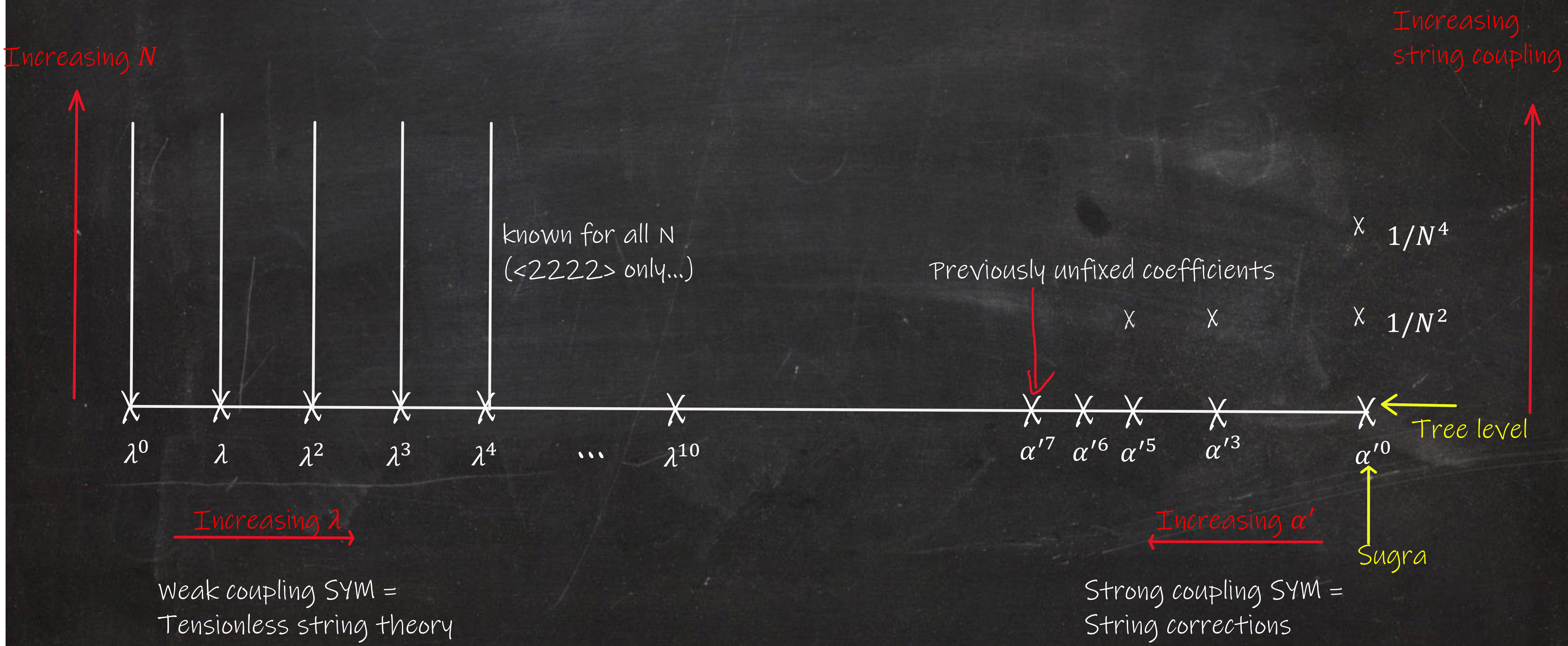
$(1/\lambda = \alpha'^2)$

Weak coupling SYM =
Tensionless string theory

Strong coupling SYM =
String corrections

Summary: beyond large N

- No results for all charges beyond large N (gravity loop corrections)
- Some results for small charges [MANY HERE, ... Fernando Alday, Shai Chester, Tobias Hansen, Zhongjie Huang, Michele Santagata, Bo Wang, Ellis Ye Yuan, Xinan Zhou, PH...]



2) Integrated correlators

- Very few quantites known for all values of coupling in QFT
- 2019: Integrated <2222> and <22pp> found exactly as $f(N, \lambda)$ [Binder, Chester, Pufu, Wang; + Dorigoni, Wen, Green; Paul, Perlmutter, Raj]
- Integrated correlator <-> partition function of $N=2^*$ theory on S^4
- Partition function computed by SU(N) matrix model via SUSY localisation [Pestun]
- But: Only for <22pp>....
- ....What can we say about other charges <pqrs> ?
- Systematic study of all integrated half BPS correlators....

Integrated Correlators beyond <22pp>

- Q0: What precise functions to integrate?
- Measure = natural conformally invariant integral over x_1, x_2, x_3, x_4 [Wen, Zhang] (same as periods)

$$\int d\mu \ldots = -\frac{1}{\pi^2} \int \frac{d^4x_1 \ldots d^4x_4}{\mathrm{vol}(SO(2,4))} \ldots .$$

- Only makes sense to integrate objects with weight 4 at each point
- The coefficient functions $F_{\{b_{ij}\}}$ have this property:

$$F^{(l)}_{\{b_{ij}\}}(x_1, .., x_4) = \frac{\xi^{(4)}}{l!} \int \frac{d^4x_5}{(-4\pi^2)} .. \frac{d^4x_{4+l}}{(-4\pi^2)} f^{(l)}_{\{b_{ij}\}}(x_1, .., x_{4+l})$$

- Agrees with the <22pp> integrated correlator from localisation: b34=p-2, bij=0 otherwise.

Integrated master correlator

- Define "Integrated master correlator", generator of all integrated correlators
- Recall master correlator

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle = \text{free} + \frac{I(x_i,y_j)}{2c} \times \sum_{l=0}^{\infty} \frac{\lambda^l}{(4\pi^2)l!} \int \frac{d^4x_5}{(-4\pi^2)} .. \frac{d^4x_{4+l}}{(-4\pi^2)} f^{(l)}(\mathbf{x}_{ij}^2)$$

$$\mathbf{x}_{ij}^2 := x_{ij}^2 - y_{ij}^2 = x_{ij}^2(1-g_{ij})$$

- Integrated master correlator:

$$\mathcal{C}(\lambda; g_{ij}) := -\sum_{\ell=1}^{\infty} \left(\frac{\lambda}{4\pi^2}\right)^{\ell} \int \frac{d^4x_1 \dots d^4x_{4+\ell}}{\text{vol}(SO(2,4))} \frac{f^{(\ell)}(x_{ij}^2(1-g_{ij}))}{\pi^2 \ell!(-4\pi^2)^{\ell}}$$

NB treat x_{ij}^2, g_{ij} as independent quantities (rather than x_{ij}^2, y_{ij}^2)

- generates all integrated half BPS correlators
- g_{ij} keeps track of all possible scalar contractions AND ensures correct conformal weight 4 to be integrated

Permutation symmetry for integrated correlators

- S_{4+l} symmetry => only one integrated corelator per f-graph !!
- (S_{4+l} symmetric measure)
- All Feynman diagrams from the same f-graph contribute equally
- So the x_{ij}^2, $1-g_{ij}$ contributions for each f-graph factorise:

$$f^{(l)}(\mathbf{x}_{ij}^2) = \sum_\alpha c_\alpha^{(l)} f_\alpha^{(l)}(\mathbf{x}_{ij}^2)$$

=>

$$\mathcal{C}(\lambda; g_{ij}) = -\sum_{\ell=1}^{\infty} \left(\frac{\lambda}{4\pi^2}\right)^\ell \frac{1}{\ell!(-4)^{\ell+1}} \sum_\alpha c_\alpha^{(\ell)} \mathcal{P}_{f_\alpha^{(\ell)}} f_\alpha^{(\ell)}(1-g_{ij})$$

$$\mathbf{x}_{ij}^2 = x_{ij}^2 - y_{ij}^2 = x_{ij}^2(1-g_{ij})$$

$$f_\alpha^{(l)}(\mathbf{x}_{ij}^2) = \frac{1}{|\mathrm{aut}(\alpha)|} \sum_{\sigma_{l+4}} \prod_{i,j=1}^{4+l} \frac{1}{(\mathbf{x}_{\sigma_i\sigma_j}^2)^{e_{ij}^\alpha}}$$

- Where P is the "period" of the corresponding f-graph

$$\mathcal{P}_{f_\alpha^{(\ell)}} = \frac{1}{(\pi^2)^{\ell+1}} \int \frac{d^4x_1 \dots d^4x_{4+\ell}}{\mathrm{vol}(SO(2,4))} f_\alpha^{(\ell)}(x_{ij}^2)$$

- Periods extensively studied by mathematicians [Broadhurst, Kreimer; Brown, Schnetz, Panzer] HypExp/Hyperlog procedures [Schnetz, Panzer]

Integrated master correlators: examples

- 1 loop:

$$f^{(1)}(x_{ij}^2) = \frac{1}{\prod_{1\leq i<j\leq 5} x_{ij}^2}$$

=> integrated correlator: $-\frac{1}{1!(-4)^1}\frac{\mathcal{P}_{f^{(1)}}}{\prod_{1<i<j<4}(1-g_{ij})}$

($x_{ij}^2 \to x_{ij}^2(1-g_{ij})$, $g_{i5}=0$)

$$\mathcal{P}_{f^{(1)}} = 6\zeta(3)$$

- 2 loops:

$$f^{(2)}(x_{ij}^2) = \frac{1}{48}\sum_{\sigma\in S_6}\frac{x_{\sigma_1\sigma_2}^2 x_{\sigma_3\sigma_4}^2 x_{\sigma_5\sigma_6}^2}{\prod_{1\leq i<j\leq 6} x_{ij}^2}$$

=>

Non-trivial function of g_{ij}

$$-\frac{\mathcal{P}_{f^{(2)}}}{2!(-4)^2}\frac{g_{12}g_{34}+g_{13}g_{24}+g_{14}g_{23}-3\sum_{1\leq i<j\leq 4} g_{ij}+15}{\prod_{1<i<j<4}(1-g_{ij})}$$

$$\mathcal{P}_{f^{(2)}} = 20\zeta(5)$$

- eg Extract <22pp> correlators by taking g_{34}^{p-2} coefficient

$$-\frac{\mathcal{P}_{f^{(2)}}}{2!(-4)^2}\left(\frac{12}{1-g_{34}}+3\right) = \begin{cases} -75\zeta(5)/8 & \text{for } \langle 2222\rangle \\ -15\zeta(5)/2 & \text{for } \langle 22pp\rangle \end{cases}$$

5 loop integrated correlator

- All periods known to 5 loops (HYPEXP + <22pp> correlator gave new period [Wen, Zhang])
- Therefore complete integrated master correlator to 5 loops known
- At 5 loops multi-zetas (and products of zetas) appear
- EG <4444> 5 loops: $C_{4,4,4,4}$ =

$$\frac{g_{12}^2 g_{34}^2}{40320}\Big[7560\zeta(5,3,3)+14685615\zeta(11)+56700\pi^2\zeta(9) +252\pi^4\zeta(7)+31500\zeta(5)^2+6300\zeta(3)^2\zeta(5)-20\pi^6\zeta(5)\Big]$$
$$-\frac{g_{12}g_{23}g_{34}g_{14}}{40320}\Big[7560\zeta(5,3,3)+569205\zeta(11)+56700\pi^2\zeta(9) +252\pi^4\zeta(7)+31500\zeta(5)^2+6300\zeta(3)^2\zeta(5)-20\pi^6\zeta(5)\Big]$$
$$+\ 3 \text{ terms of crossing}\,.$$

- Localisation computation predicts only zetas
- Indeed for <22pp> all multi-zetas cancel and only ζ(11) remains

Sum over channels

- But notice: sum over <4444> channels (g_{ij}->1) the multi-zetas cancel
- Same happens for all higher charges!
- Only $\zeta(11)$ remains when summing over channels....
- Let C_{pqrs} be integrated <pqrs> summed over channels:

Localisation =>

$$\mathcal{C}_{2,2,p,p}(\lambda) = \sum_{s=2}^{\infty} \frac{4(-1)^s \pi^{1-2s}(s+1)\Gamma\left(s+\frac{1}{2}\right)^2}{(2s-1)\Gamma(s-1)\Gamma(s+2)} \left(1 + \frac{(-1)^p\Gamma(s-1)\Gamma(s+1)}{\Gamma(s-p)\Gamma(p+s)}\right) \lambda^{s-1}$$

- Examine data for other families <33pp> etc
- Following formula is consistent with <33pp> to 5 loops:

$$\mathcal{C}_{3,3,p,p}(\lambda) = \sum_{s=2}^{\infty} \frac{16(-1)^s \pi^{1-2s}\lambda^{s-1}\Gamma\left(s+\frac{1}{2}\right)^2}{\Gamma(s-1)\Gamma(s+3)} \left(\frac{s+7}{2s-1} + \frac{(-1)^p\left(s-2p^2\right)\Gamma(s-1)\Gamma(s+3)}{(2s-1)\Gamma(-p+s+1)\Gamma(p+s+1)}\right)$$

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C44pp

And the following for the <44pp> family

$$C_{4,4,p,p} = \zeta(2s-1)\Bigg(\frac{4\pi^{1-2s}(-1)^{p+s}\left(17p^4+6p^2s^2-54p^2s-17p^2+s^4-2s^3+11s^2+14s\right)\lambda^{s-1}\Gamma\left(s-\frac{1}{2}\right)\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(-p+s+2)\Gamma(p+s+2)} + \frac{(-1)^s\pi^{1-2s}(2s-1)\left(-\frac{1120}{s+2}+\frac{1260}{s+3}-\frac{504}{s+4}+\frac{348}{s+1}+2\right)\lambda^{s-1}\Gamma\left(s-\frac{1}{2}\right)^2}{\Gamma(s-1)\Gamma(s+1)}\Bigg)$$

- Only used data to 5 loops, but, all <pqrs> so a lot of data
- Incredibly though it sums up -> agreement with strong coupling (return shortly)
- Simplify / structure?

Master correlator summed over channels

- Sum over channels at level of master correlator?
- Easy, just let $g_{ij} \to \gamma_i \gamma_j$
- Then all channels contribute equally as required
- $(g_{12})^2 (g_{34})^2 \to (\gamma_1\gamma_2\gamma_3\gamma_4)^2, \quad g_{12}g_{23}g_{34}g_{14} \to (\gamma_1\gamma_2\gamma_3\gamma_4)^2$
- <4444> -> ζ(11)
- Define integrated master correlator summed over channels:

$$\mathcal{C}(\lambda; \gamma_i \gamma_j)$$

- Integrated <p1p2p3p4> correlator summed over channels is the appropriate γ coefficient:

$$\mathcal{C}_{p_1p_2p_3p_4}(\lambda) := \mathcal{C}(\lambda; \gamma_i\gamma_j)\Big|_{\gamma_1^{p_1-2}\gamma_2^{p_2-2}\gamma_3^{p_3-2}\gamma_4^{p_4-2}}$$

Crossing and Schur polynomials

- Crossing symmetry => Cpqrs = Cqprs ETC.
- [For unintegrated correlator x_1<->x_2 and y_1 <-> y_2 but these are integrated over (symmetrically) and summed over]
- Therefore $\mathcal{C}(\lambda;\gamma_i\gamma_j)$ is a symmetric function of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$
- Natural basis for symmetric polynomials:

Schur polynomials: $$\mathcal{S}_{\nu_1,\nu_2,\nu_3,\nu_4}(\gamma_i) = \frac{\det\left(\gamma_i^{4+\nu_j-j}\right)_{i,j=1,2,3,4}}{\prod_{1\le i<j\le 4}(\gamma_i-\gamma_j)}$$

- Rewrite the <22pp>, <33pp>,<44pp> families in terms of SCHURS

- Big simplicity: Results imply only $\mathcal{S}_{\nu,\nu,0,0}$, $\mathcal{S}_{\nu,\nu,1,1}$ survive!

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All orders, all charges formula

- Examining the coefficients, previous families simplifies hugely

- All data to 5 loops consistent with the simple formula:

$$\mathcal{C}(\lambda;\gamma_i\gamma_j) = \sum_{\ell=1}^{\infty} \lambda^{\ell} \sum_{\nu=2}^{\infty} \frac{4(-1)^{\nu+\ell+1}\Gamma\left(\ell+\frac{3}{2}\right)^2 \zeta(2\ell+1)}{\pi^{2\ell+1}\Gamma(\ell+2-\nu)\Gamma(\ell+\nu+1)} F_{\nu}(\gamma_i)$$

$$F_{\nu}(\gamma_i) = \frac{\mathcal{S}_{\nu-2,\nu-2,0,0}(\gamma_i) - \mathcal{S}_{\nu-2,\nu-2,1,1}(\gamma_i)}{\prod_{1\le i<j\le 4}(1-\gamma_i\gamma_j)}$$

- Gives all integrated correlators <pqrs> summed over channels, to all orders in λ

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Summed up formula

- The formula can be resummed through a modified Borel transform
- Replace $\zeta(2l+1)$ with: $\zeta(n)\,\Gamma(n+1) = 2^{n-1}\int_0^\infty dw \frac{w^n}{\sinh^2(w)}$ and resum [Russo; Hatsuda, Okuyama]

An analytic formula for all integrated correlators:

$$\mathcal{C}(\lambda;\gamma_i\gamma_j) = \int_0^\infty \frac{w\,dw}{\sinh^2(w)} \sum_{\nu=2}^{\infty} \left(J_{\nu-1}(u)^2 - J_\nu(u)^2\right) F_\nu(\gamma_i)$$

$$u = \frac{w\sqrt{\lambda}}{\pi}$$

$J_\nu(u)$ Bessel functions

- This reproduces <22pp> result from localisation [Binder, Chester, Pufu, Wang]

$$\mathcal{C}_{2,2,p,p}(\lambda) = \int_0^\infty \frac{w\,dw}{\sinh^2(w)} \left(J_1(u)^2 - J_p(u)^2\right)$$

- For <33pp> predicts:

$$\mathcal{C}_{3,3,p,p}(\lambda) = \int_0^\infty \frac{w\,dw}{\sinh^2(w)} \Big(3J_1(u)^2 + 4J_2(u)^2 + J_3(u)^2 - 2J_{p-1}(u)^2 - 4J_p(u)^2 - 2J_{p+1}(u)^2\Big)$$

Summary: large N known correlators (all charges)

X = result known (integrand) for master correlator

Weak coupling SYM =
Tensionless string theory

Strong coupling SYM =
String corrections

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Summary: large N known integrated correlators (all charges)

Known as analytic function of λ

λ^0 λ λ^2 λ^3 $\cdots$ λ^{10} α'^7 α'^6 α'^5 α'^3 α'^0

Increasing λ →

← Increasing α'

Weak coupling SYM =
Tensionless string theory

Strong coupling SYM =
String corrections

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Checks and Predictions: Strong coupling

- Re-expand in $\frac{1}{\lambda}$ strong coupling by using Mellin-Barnes representation of products of Bessel functions [Binder, Chester, Pufu; Wang]

$$\mathcal{C}(\lambda;\gamma_i\gamma_j)\Big|_{\text{strong}} = \sum_{\nu=2}^{\infty}\Big(\frac{1}{2\nu(\nu-1)} + \sum_{n=1}^{\infty}\frac{4n(-1)^n\,\Gamma(n+\frac{1}{2})\,\Gamma(\nu+n-\frac{1}{2})\,\zeta(2n+1)}{\lambda^{n+\frac{1}{2}}\sqrt{\pi}\Gamma(n)\Gamma(\nu-n+\frac{1}{2})}\Big)F_\nu(\gamma_i)$$

$(\alpha`)^{2n+1}$

- Asymptotic expansion: resurgence anlaysis a la [Dorigoni, Green, Wen; Hatsuda, Okuyama] give world-sheet instanton-like corrections

$$\Delta\mathcal{C}(\lambda;\gamma_i\gamma_j) = \pm\frac{i}{2}\sum_{\nu=2}^{\infty}(-1)^\nu(2\nu-1)^2\Big(\frac{8\,\mathrm{Li}_0(z)}{(2\nu-1)^2} + \frac{2\,\mathrm{Li}_1(z)}{\lambda^{1/2}} + \frac{(4\nu^2-4\nu+5)\mathrm{Li}_2(z)}{4\,\lambda} + \dots\Big)F_\nu(\gamma_i)$$

$$z = e^{-2\sqrt{\lambda}} = e^{-2L^2/\alpha'}$$

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Checks and Predictions: Strong coupling

- Integrate strong coupling correlators directly from Mellin amplitude expressions [Chester, Pufu]
- AdS_5xS^5 Effective action -> Mellin amplitude -> Integrated correlator (master correlator)
- Agrees with above prediction, all charges!
- Fixes all previously unfixed coefficients at α'^7 (with additional <2222> data from [Alday, Hansen])

α'^7 Coefficients

$$A_4 = -\frac{1575\zeta(7)}{4}\,, \quad C_2 = \frac{641\zeta(7)}{16}\,, \quad D_1 = 0\,,$$
$$E_1 = 0\,, \quad F_0 = \frac{\zeta(7)}{2}\,, \quad G_{1;0} = -\frac{11\zeta(7)}{64}$$
$$G_{2;0} = \frac{71\zeta(7)}{64}\,, \quad G_{3;0} = \frac{141\zeta(7)}{256}\,, \quad G_{5;0} = -\frac{51\zeta(7)}{64}$$
$$B_2 = 20G_{4;0} + \frac{259\zeta(7)}{4}$$

- Bootstrap "rank constraints" [Aprile, Drummond, Paul, Santagata]

$$B_2 = \frac{170839\zeta(7)}{1664}$$
$$G_{4;0} = \frac{12619\zeta(7)}{6656}$$

- All <pqrs> correlators (unintegrated) fixed to α'^7

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Checks and Predictions: Weak coupling

- Data to 5 loops used to obtain all orders result
- Consistency / Predictions at 6 loops?
- 6 loop result given in terms of 26 non vanishing f-graphs
- HypExp gives periods of 16 of these
- Our formula:
  - Is consistent with these
  - Predicts 9 of the 10 remaining periods (beyond HypExp [hyperlogprocedures] [Panzer, Schnetz] techniques) giving 9 new 7-loop periods*

Eg

$$\mathcal{P}_{f_2^{(6)}} = \frac{1}{(\pi^2)^7}\int \frac{d^4x_1\ldots d^4x_{10}}{\mathrm{vol}(SO(2,4))}\,\frac{x_{18}^2 x_{2\,10}^2 x_{46}^2 x_{47}^2}{x_{12}^2 x_{13}^2 x_{14}^2 x_{15}^2 x_{16}^2 x_{23}^2 x_{26}^2 x_{27}^2 x_{28}^2} \frac{1}{x_{34}^2 x_{38}^2 x_{45}^2 x_{48}^2 x_{49}^2 x_{4\,10}^2 x_{56}^2 x_{5\,10}^2 x_{67}^2 x_{6\,10}^2 x_{78}^2 x_{79}^2 x_{7\,10}^2 x_{89}^2 x_{9\,10}^2}\,.$$

$$\begin{aligned}\mathcal{P}_{f_2^{(6)}} &= 2880\zeta(5)\zeta(5,3) + 1440\zeta(5,5,3) - 720\zeta(7,3,3) \\ &\quad - 106632\zeta(13) + 8580\pi^2\zeta(11) + 176\pi^4\zeta(9) \\ &\quad + 2800\zeta(5)\zeta(7) + 1680\zeta(3)^2\zeta(7) + \frac{64\pi^6\zeta(7)}{21}\,.\end{aligned}$$

*6 loop integrated correlator = 7 loop period

Checks and Predictions: Octagon

- 10d lightlike limit of master correlator = new regularised amplitude^2 = octagon^2 [Caron-Huot, Coronado]

$$\lim_{\mathbf{x}^2_{i,i+1}\to 0} \frac{\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle}{\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle_{\text{free}}} = M^2$$

$$M = \mathbb{O}/\mathbb{O}_{\text{free}}$$

= massive 4-particle amplitude on the Coulomb branch

- Valid for perturbative integrands

- Both sides finite… Therefore can be integrated

- Octagon given (by integrability) in terms of sums of products of ladders [Belitsky, Korchemsky]

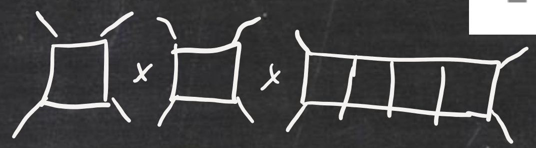
Checks and Predictions: Octagon

- Everything is beautifully consistent with our proposal to 6 loops
- Relates the remaining unfixed period to products of ladders

$$\mathcal{P}_{f_1^{(6)}} = \frac{1}{(\pi^2)^7}\int \frac{d^4x_1\ldots d^4x_{10}}{\mathrm{vol}(SO(2,4))}\, \frac{x_{19}^2 x_{2\,10}^2 x_{36}^2 x_{58}^2}{x_{12}^2 x_{13}^2 x_{14}^2 x_{15}^2 x_{16}^2 x_{23}^2 x_{26}^2 x_{27}^2 x_{28}^2} \frac{1}{x_{34}^2 x_{38}^2 x_{39}^2 x_{45}^2 x_{49}^2 x_{56}^2 x_{59}^2 x_{5\,10}^2 x_{67}^2 x_{6\,10}^2 x_{78}^2 x_{7\,10}^2 x_{89}^2 x_{8\,10}^2 x_{9\,10}^2}\,.$$

$$\begin{aligned}\mathcal{P}_{f_1^{(6)}} ={}& 6L_{1,1,4} - L_{1,2,3} - 6880\zeta(5)\zeta(5,3) + 1536\zeta(5,3,3)\\ &- 608\zeta(5,5,3) - 640\zeta(7,3,3) + 408252\zeta(13) - 35640\pi^2\zeta(11)\\ &- 124064\zeta(11) - 5760\zeta(3)\zeta(9) - 368\pi^4\zeta(9) + 11520\pi^2\zeta(9)\\ &- 13120\zeta(5)\zeta(7) - 4480\zeta(3)^2\zeta(7) + \frac{512\pi^6\zeta(7)}{189} + \frac{256\pi^4\zeta(7)}{5}\\ &- 7200\zeta(3)\zeta(5)^2 + 3200\zeta(5)^2 + 1280\zeta(3)^2\zeta(5) - \frac{256\pi^6\zeta(5)}{63}\,,\end{aligned}$$

$$L_{1,1,4} = \int \frac{d^4x_1 \ldots d^4x_4}{SO(2,4)}$$



- $L_{1,1,4}$ and $L_{1,2,3}$ are computable by Hyperlog procedures yielding the remaining unknown 7 loop period:

$$\begin{aligned}\mathcal{P}_{f_1^{(6)}} ={}& -1760\zeta(5)\zeta(5,3) + 768\zeta(5,3,3) - 1696\zeta(5,5,3)\\ &+ 1120\zeta(7,3,3) + 28220\zeta(13) - 880\pi^2\zeta(11) - 62032\zeta(11)\\ &- 2880\zeta(3)\zeta(9) - \frac{368\pi^4\zeta(9)}{3} + 5760\pi^2\zeta(9) + 4640\zeta(5)\zeta(7)\\ &- 1120\zeta(3)^2\zeta(7) - \frac{128\pi^6\zeta(7)}{27} + \frac{128\pi^4\zeta(7)}{5} + 2400\zeta(3)\zeta(5)^2\\ &+ 1600\zeta(5)^2 + 640\zeta(3)^2\zeta(5) - \frac{128\pi^6\zeta(5)}{63}\,.\end{aligned}$$

Conclusions

- Why does this work? Localisation? Apparently not....
- Non-planar formula? Modular invariance suggests replacing power series terms with non-holomorphic Eisenstein series and instanton terms by functions $D_N(s;\tau,\bar{\tau})$ introduced by [Dorigoni,Green,Wen,Xie;Luo, Wang]

$$E(s;\tau,\bar{\tau}) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{\pi^s |m+n\tau|^{2s}}$$

$$D_N(s;\tau,\bar{\tau}) = \sum_{(m,n)\neq(0,0)} e^{-4\sqrt{N\pi}\frac{|m+n\tau|}{\sqrt{\tau_2}}} \frac{\tau_2^s}{\pi^s |m+n\tau|^{2s}}$$

$\tau = \theta/2\pi + 4\pi i N/\lambda$

$$\begin{aligned}\mathcal{C}(\tau,\bar{\tau};\gamma_i\gamma_j) = \sum_{\nu=2}^{\infty} \Big[& \frac{1}{2(\nu-1)\nu} - \frac{2\nu-1}{2^4 N^{\frac{3}{2}}} E(3/2;\tau,\bar{\tau}) \\ & + \frac{3(2\nu-3)(4\nu^2-1)}{2^8 N^{\frac{5}{2}}} E(5/2;\tau,\bar{\tau}) + \dots \\ & \pm 2i(-1)^\nu D_N(0;\tau,\bar{\tau}) + \dots \Big] F_\nu(\gamma_i)\,,\end{aligned}$$

Conclusions

- More general integrated correlators ie beyond summing over channels?
  (Summing over channels the natural analgue of integrating over spacetime?)

- Integrated correlator with different measure (box insertion also obtained by localisation for <2222>) for more general correlators (no permutation symmetry => much more complicated!)

- Higher points?