# Nuclear matter and neutron stars in the Sakai-Sugimoto model

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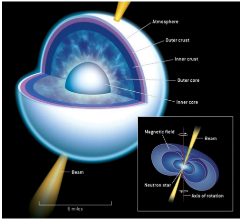
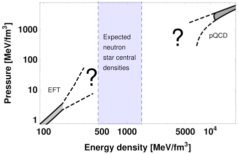
# Kaifeng, the old capital of China

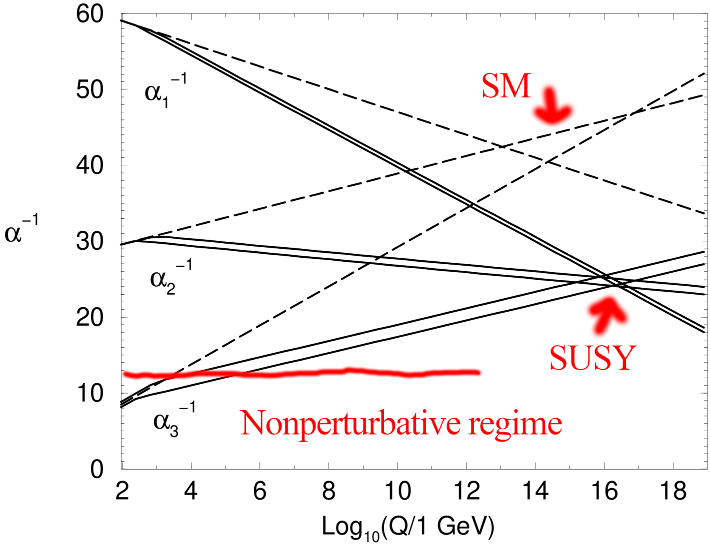
# Neutron stars – the most compact baryonic objects

- Remnants of supernovea from supergiant stars
- Remnants of scattering of massive objects that did not collapse to a BH
- Highest baryon densities in the observable universe
- Governed by the Tolman-Oppenheimer-Volkov (TOV) equations
- TOV needs the EOS as input: Nuclear physics at large density
- Neutron stars are neutron rich: Proton fraction determined by the symmetry energy (SE)
- Traditional nuclear physics and chiral perturbation theory (ChPT) predict EOS well near saturation density
- pQCD predicts EOS well at extremely large densities (energies)
- The large gap of densities in-between is where strong coupled methods have their chance



## Gauge couplings – strong coupling at low energies




[Martin, Adv.Ser.Direct.HEP21, 1-153 (2010)]

# Dualities?

- Since perturbation theory fails miserably at low energies for QCD, a duality would be the perfect candidate.
- An old example of a duality, is the EM duality

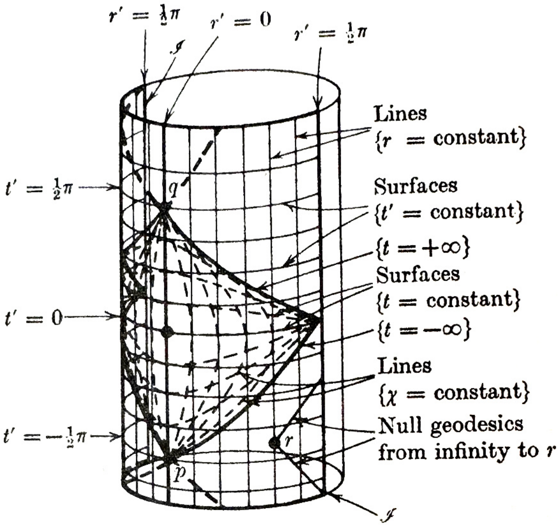
$$\begin{pmatrix} \boldsymbol{F} \\ \widetilde{\boldsymbol{F}} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \boldsymbol{F} \\ \widetilde{\boldsymbol{F}} \end{pmatrix}$$

  under which the Maxwell's equations are invariant and so is the Hamiltonian
- However, the Lagrangian is not invariant since the SO(2) rotation rotates a tensor into a pseudo-tensor

# Maldacena – AdS/CFT

- In 1997 Maldacena [Adv.Theor.Math.Phys.2,231 (1998)] proposes the AdS/CFT duality between string theory in $\mathrm{AdS}_5$ space and the conformal $\mathcal{N}=4$ super-Yang-Mills field theory on the other side
- In particular, the D3-brane in the large $N_c$ (large number of branes) is described in type IIB string theory by $\mathrm{AdS}_5 \times S^5$.
- Chiral primary operators of $\mathcal{N}=4$ super-Yang-Mills theory are mapped to Kaluza-Klein modes of type IIB supergravity on $\mathrm{AdS}_5 \times S^5$ [Witten, Adv.Theor.Math.Phys.2, 253 (1998)]
- Most importantly, the 't Hooft coupling $\lambda = g^2 N_c$ is mapped to $1/\alpha'$, so that strongly coupled gauge theory corresponds to weakly coupled string theory
- The symmetry group $\mathrm{SO}(2,d)$ of $\mathrm{AdS}_{d+1}$ acts as the conformal group on the boundary space $M_d$, which can be shown to be a compactification of $d$-Minkowski space

# AdS space




[Hawking-Ellis, 1973]

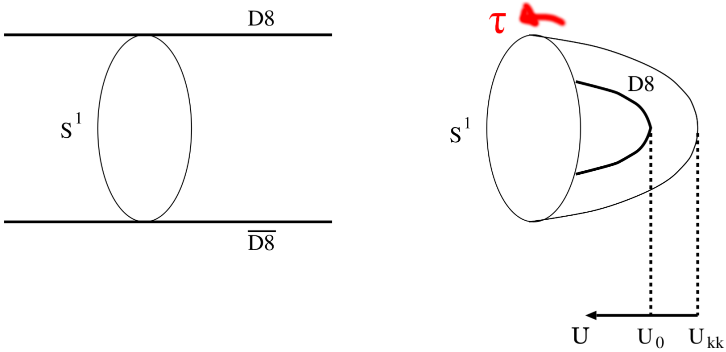
# Sakai-Sugimoto – AdS/QCD

The Sakai-Sugimoto (SS) model [Prog.Theor.Phys.113, 843 (2005)] builds on the work of Witten [Adv.Theor.Math.Phys.2, 253 (1998)] where $N_f$ D8- and $\overline{\text{D8}}$-branes are intersecting $N_c$ D4-branes

| | 0 | 1 | 2 | 3 | (4) | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| D4 | ∘ | ∘ | ∘ | ∘ | ∘ | | | | | |
| D8-$\overline{\text{D8}}$ | ∘ | ∘ | ∘ | ∘ | | ∘ | ∘ | ∘ | ∘ | ∘ |

- The SS model is in type IIA string theory, but is T-dual to a D3/D7 model by [Sugimoto-Takahashi, JHEP04, 051 (2004)], except for SUSY-breaking anti-periodic boundary conditions on the $S^1$ for the fermions on D4.
- Chiral symmetry is explicit by the two 8-branes, when they stretch
- Chiral symmetry breaking is string geometric as the 8-branes touch and merge – the low-energy supergravity geometry is that of a cigar-shaped space, which is $\text{AdS}_5$-like

# Sakai-Sugimoto – chiral symmetry breaking




[Sakai-Sugimoto, Prog.Theor.Phys.113, 843 (2005)]

- Notice that the confined geometry ends at $U_{\text{KK}}$

# Sakai-Sugimoto – AdS/QCD

- The 't Hooft limit is considered $N_c \gg N_f$, so that the 8-branes can be considered in the probe branes embedded in the D4-background (color d.o.f.)

$$\mathrm{d}s^2 = \left(\frac{u}{R}\right)^{3/2}(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + f(u)\mathrm{d}\tau^2) + \left(\frac{R}{u}\right)^{3/2}\left(\frac{\mathrm{d}u^2}{f(u)} + u^2\mathrm{d}\Omega_4^2\right)$$

$$e^\phi = g_s\left(\frac{u}{R}\right)^{3/4}, \qquad F_4 = \mathrm{d}C_3 = \frac{2\pi N_c}{V_4}\epsilon_4, \qquad f(u) = 1 - \frac{u_{\mathrm{KK}}^3}{u^3}.$$

- The flavor d.o.f. are described by the DBI action and the Chern-Simons term at level $N_c$ – both scale as $N_c$
- The leading order approximation to the DBI action is the 5-dimensional Yang-Mills term, hence the Sakai-Sugimoto model:

$$S = -\kappa\,\mathrm{tr}\int_{\mathrm{AdS}_5}\mathcal{F}\wedge *\mathcal{F} + N_c\int_{\mathrm{AdS}_5}\omega_5,$$

with YM coefficient $\kappa = \frac{\lambda N_c}{216\pi^3}$, and 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N_c$ (fixed),

$$g = h(z)k(z)\mathrm{d}x^\mu \mathrm{d}x_\mu + h^2(z)\mathrm{d}z^2, \qquad k(z) = h^{-3}(z) = 1 + z^2.$$

# Sakai-Sugimoto – Mesons and baryons

- The scale of the theory (glueballs) is

$$R_\tau = \frac{4\pi}{3}\frac{R^{3/2}}{u_{\rm KK}^{1/2}}, \qquad M_{\rm KK} = \frac{3}{2}\frac{u_{\rm KK}^{1/2}}{R^{3/2}},$$

- And in terms of string theory

$$R^3 = \frac{1}{2}\frac{g_{\rm YM}^2 N_c l_s^2}{M_{\rm KK}}, \qquad u_{\rm KK} = \frac{2}{9} g_{\rm YM}^2 N_c M_{\rm KK} l_s^2, \qquad g_s = \frac{1}{2\pi}\frac{g_{\rm YM}^2}{M_{\rm KK} l_s}.$$

- The flavor fields can be expanded as

$$\mathcal{A}_\mu = \sum_n v_\mu^{2n-1}(x)\psi_{2n-1}(z) + \sum_n a_\mu^{2n}(x)\psi_{2n}(z), \qquad \mathcal{A}_z = \Pi(x)\phi_0(z),$$

  with profile functions

$$\text{pions} : \phi_0(z) = \frac{1}{\sqrt{\pi\kappa}}\frac{1}{k(z)}, \qquad \text{vectors} : -h^{-1}(z)\partial_z(k(z)\partial_z\psi_n) = \lambda_n\psi_n,$$

  with vector meson masses $M_n \sim \sqrt{\lambda_n}$
- Fitting the pion decay constant and the rho meson mass, one obtains

$$M_{\rm KK} = 949\,\text{MeV}, \qquad \lambda = 16.63$$

# Sakai-Sugimoto – Mesons and baryons

- If we truncate to the pions, one gets the Skyrme model

$$S = \widetilde{T}(2\pi\alpha')^2 \int \mathrm{d}^4x \;\mathrm{tr}\left[AL_\mu^2 + B[L_\mu, L_\nu]^2\right],$$

with the left-invariant chiral current $L_\mu = U^{-1}\partial_\mu U$, and the constants determined by string theory

$$A = \frac{9u_{\rm KK}}{4\pi}, \qquad B = \frac{R^3 b}{2\pi^4}, \qquad b \sim 15.25$$

- Notice that the Skyrme coupling is determined by the model

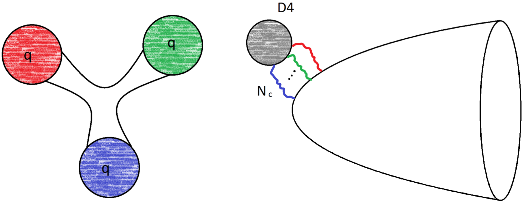
$$e^2 = \frac{27\pi^7}{2b}\frac{1}{\lambda N_c} \sim (7.32\cdots)^2$$

which can be compared to [Adkins-Nappi-Witten, NPB228, 552 (1983)], where they find $e = 5.45$ by fitting to the masses of the nucleon and Delta resonance

# Sakai-Sugimoto – Where is the baryon?

The baryon is:

- the coupling of $N_c$ strings from the D4-branes to the 8-branes




- the instanton in an $(x^1, x^2, x^2, z)$ slice of the AdS$_5$-like geometry
- the Skyrmion in the pion effective theory, which is a soliton of 1 dimension less than the instanton, but same $S^3$ target space

$$\pi_3(S^3) = \mathbb{Z} \ni k = B$$

- The 3rd homotopy group is due to the mappings being from $\sim \partial\mathbb{R}^4 \simeq S^3$ in the instanton case, and from $\mathbb{R}^3 \cup \{\infty\} \simeq S^3$ in the Skyrmion case
- The precise mathematical relation is given by the instanton holonomy of [Atiyah-Manton, PLB, 438 (1989)]

# The instanton in SS

- In the large-$\lambda$ limit, the curved-space instanton is well approximated by the flat-space BPST instanton solution in the non-Abelian fields [Hata-Sakai-Sugimoto-Yamato, Prog.Theor.Phys.117, 1157 (2007)]

$$A_M = -\mathrm{i} f(\xi) g \partial_M g^{-1}, \qquad f(\xi) = \frac{\xi^2}{\xi^2+\rho^2},$$
$$\xi^2 = (x-X)^2 + (z-Z)^2, \qquad g(x) = \frac{(z-Z) - \mathrm{i}(\mathbf{x}-\mathbf{X})\cdot\boldsymbol{\tau}}{\xi}.$$

- The Abelian electric field is new, this field acts as a size stabilization against gravitational collapse

$$\widehat{A}_0 = \frac{N_c}{8\pi^2\kappa}\frac{1}{\xi^2}\left(1 - \frac{\rho^4}{(\xi^2+\rho^2)^2}\right).$$

- Minimization of the pseudo-moduli $Z, \rho$ gives

$$Z = 0, \qquad \rho^2 = \frac{N_c}{8\pi^2\kappa}\sqrt{\frac{6}{5}}.$$

- Rotation of the instanton gives rise to spin and isospin quantum numbers

# BPST Instantons

The Belavin-Polyakov-Schwartz-Tyupkin [PLB59, 85 (1975)] instanton can easiest be understood as the solution to the self-dual equation

$$\begin{aligned} S &= \frac{1}{2e^2}\int \mathrm{d}^4x \;\operatorname{tr} F_{\mu\nu}F_{\mu\nu} \\ &= \frac{1}{4e^2}\int \mathrm{d}^4x \left[\operatorname{tr}\left(F_{\mu\nu} \mp \widetilde{F}_{\mu\nu}\right)^2 \pm 2\operatorname{tr} F_{\mu\nu}\widetilde{F}_{\mu\nu}\right] \\ &= \frac{1}{4e^2}\int \mathrm{d}^4x \left[\operatorname{tr}\left(F_{\mu\nu} \mp \widetilde{F}_{\mu\nu}\right)^2 \pm \epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\partial_\mu\left(A_\nu F_{\rho\sigma} + \frac{\mathrm{i}2}{3}A_\nu A_\rho A_\sigma\right)\right], \end{aligned}$$

with $\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. The self-dual equation implies the full second-order equation of motion

$$D_\mu F^{\mu\nu} = \pm D_\mu \widetilde{F}^{\mu\nu} = 0.$$

The first equality holds because of the selfdual equation (BPS equation), whereas the latter vanishes due to the Bianchi identity.

# Instanton solutions

The instanton solution can be found on Euclidean $\mathbb{R}^4$ for $G = \mathrm{SU}(2)$ using the Ansatz

$$A_\mu = \frac{\mathrm{i}}{2}\sigma_{\mu\nu}\partial_\nu \log\rho,$$

with the 't Hooft symbols

$$\sigma_{i4} = \sigma_i, \qquad \sigma_{ij} = \epsilon_{ijk}\sigma_k, \qquad i,j,k = 1,2,3,$$

and the 't Hooft Ansatz

$$\rho = 1 + \sum_{I=1}^{N} \frac{\lambda_I^2}{|x - a_I|^2},$$

which encodes 5 parameters or moduli per instanton (in total $N$ instantons). The $N = 1$ solution is the BPST instanton solution.

# Instanton moduli

For $N = 1$ (a single instanton), this is the complete number of moduli according to the Atiyah-Hitchin-Singer [Proc.Natl.Acad.Sci.USA 74, 2662 (1977)] index theorem

$$\dim \mathcal{M}_{N,\mathrm{SU}(2)} = 8N.$$

which agrees with the BPST solution, since $a_1$ are spatial translations in $\mathbb{R}^4$, $\lambda_1^2$ is the size of the instanton and 3 global rotations in $\mathrm{SU}(2)$ correspond to the instanton's orientation in the gauge group

$$A_\mu \to g A_\mu g^{-1}.$$

# Instanton moduli

For $N = 2$, $5 + 5 < 16$ but Jackiw-Nohl-Rebbi [PRD15, 1642 (1980)] found that

$$\rho = \sum_{I=0}^{2} \frac{\lambda_I^2}{|x - a_I|^2},$$

is the complete number of parameters for $N = 2$ instantons, since $3 \times 5 + 3 = 18$, but only the ratio of sizes is physical due to the derivative of the logarithm and a further parameter is simply a gauge transformation.

For $N > 2$, the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction is the only way to parametrize the entire $8N$ moduli. The reason that this construction is possible is due to the integrability properties of Yang-Mills theory.

# Instantons in string theory

- The instanton is in fact expected on general grounds from string theory
- The simplest description is the D0-D4-brane system in type IIA or its T-dual D(-1)-D3-brane system in type IIB [Polchinsky II, 1998]
- The orthogonality of the branes means there are no forces between the D0-branes – they are BPS objects
- The Higgs branch condition in the D4-branes gives rise to the self-dual equation

$$F = \pm * F,$$

- The intersecting, but orthogonal D-branes break exactly 1/2 of supersymmetry – the instantons are 1/2-BPS objects, just like YM instantons
- Performing T-duality, we can easily arrive at the D4-D8-branes system in type IIB – the Sakai-Sugimoto soliton

# Hashimoto-Sakai-Sugimoto

[Hashimoto-Sakai-Sugimoto, Prog.Theor.Phys.120, 1093 (2008)] computed baryon observables and showed general improvements over the Skyrme model computations by [Adkins-Nappi-Witten, NPB228, 552 (1983)]

| | our model | Skyrmion[14] | experiment |
|---|---|---|---|
| $\langle r^2 \rangle^{1/2}_{I=0}$ | 0.742 fm | 0.59 fm | 0.806 fm |
| $\langle r^2 \rangle^{1/2}_{M,\,I=0}$ | 0.742 fm | 0.92 fm | 0.814 fm |
| $\langle r^2 \rangle_{E,\mathrm{p}}$ | $(0.742\ \mathrm{fm})^2$ | $\infty$ | $(0.875\ \mathrm{fm})^2$ |
| $\langle r^2 \rangle_{E,\mathrm{n}}$ | 0 | $-\infty$ | $-0.116\ \mathrm{fm}^2$ |
| $\langle r^2 \rangle_{M,\mathrm{p}}$ | $(0.742\ \mathrm{fm})^2$ | $\infty$ | $(0.855\ \mathrm{fm})^2$ |
| $\langle r^2 \rangle_{M,\mathrm{n}}$ | $(0.742\ \mathrm{fm})^2$ | $\infty$ | $(0.873\ \mathrm{fm})^2$ |
| $\langle r^2 \rangle^{1/2}_{A}$ | 0.537 fm | – | 0.674 fm |
| $\mu_p$ | 2.18 | 1.87 | 2.79 |
| $\mu_n$ | $-1.34$ | $-1.31$ | $-1.91$ |
| $\left\lvert \frac{\mu_p}{\mu_n} \right\rvert$ | 1.63 | 1.43 | 1.46 |
| $g_A$ | 0.734 | 0.61 | 1.27 |
| $g_{\pi NN}$ | 7.46 | 8.9 | 13.2 |
| $g_{\rho NN}$ | 5.80 | – | $4.2 \sim 6.5$ |

# Approximations made

A lot of approximations has been made under the way of constructing this version of holographic QCD (many are similar in other HQCD models):

- Large-$N_c$: Nature is only $N_c = 3$
- Large-$\lambda$, this is necessary for using the BPST solution and corresponding analytic basis of $\psi_n$
- QCD is not supersymmetric: in SS SUSY is broken at $\sim M_{\text{KK}}$
- The SS model is not asymptotically free – it is only applicable as a model for low-energy QCD
- The SS has no quark masses (see next)
- The single instanton is simple, but multi-instantons are also quite complicated! How to get large nuclei or neutron star equations of state (EOS)

# Quark and pion masses in SS

- The issue with the quark mass in SS, is that it involves both the left- and right-handed fermions

$$m_q(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L),$$

which must be nonlocal in the bulk!

- [Aharony-Kutasov, PRD78, 026005 (2008)] solved this problem with a Wilson line

$$S_{\mathrm{AK}} = c\int \mathrm{d}^4x\ \mathrm{tr}\,P\left[M\left(e^{\mathrm{i}\varphi}+e^{-\mathrm{i}\varphi}-2\mathbf{1}\right)\right], \qquad \varphi = -\int \mathrm{d}z\,A_z, \tag{1}$$

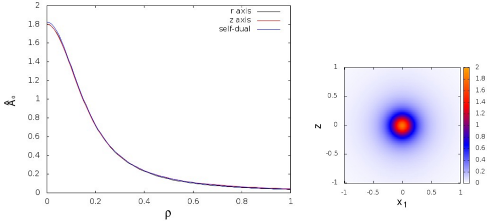
- It can be interpreted also as an effect from world-sheet instantons
- Inclusion of the AK action (quark masses) deforms the size of the instanton and the mass of the instanton
- It also induces a pion mass ($k_0$ of $\phi_0$ becomes nonvanishing) and in turn yields the Gell-Man-Oakes-Renner relation

$$4mc = f_\pi^2 m_\pi^2.$$

i.e. the pion mass squared is proportional to the quark mass

# The instanton tail

- Although most of the instanton's energy is situated near $z = 0$ (for $Z = 0$), the tail of the flat instanton is incorrect at long distances, leading to contradicting and erroneous results in the literature: e.g. exponentially suppressed EM form factors
- This problem was addressed in [Bolognesi-Sutcliffe, JHEP01, 078 (2014)], where the non-commutativity of $\lambda$ and large radius is discussed in detail




- The resolution is the discovery of a new scale, after which numerical solutions or nonlinear analysis is required

$$\begin{array}{rcll} 0 < & \rho & \lesssim L/\sqrt{\Lambda}, & \text{flat and nonlinear} \\ L/\sqrt{\Lambda} \lesssim & \rho & \lesssim L, & \text{flat and linear} \\ L \lesssim & \rho & \lesssim L\log\Lambda, & \text{curved and linear} \\ L\log\Lambda \lesssim & \rho & & \text{curved and nonlinear.} \end{array}$$

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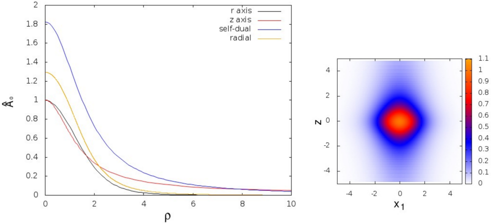


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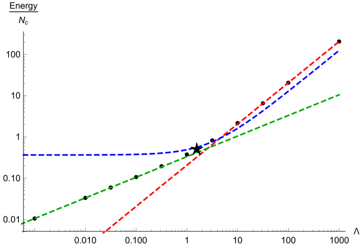
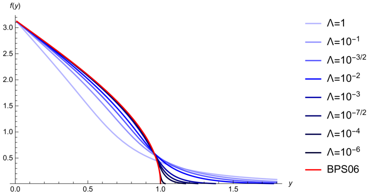
# Large and small ’t Hooft coupling

- [Bartolini-Bolognesi-Proto, PRD97, 014024 (2018)] studied the behavior of the SS model in the limits of small and large ’t Hooft coupling
- The large-$\lambda$ limit essentially converges to the results given by the BPST instanton. In the case of taking into account the correction for the quark mass, the energy is given by [Hashimoto-Hirayama-Hong, PRD81, 045016 (2010)] – in this limit the instantons are point-like
- The small $\lambda$-limit corresponds to the so-called BPS-Skyrme model, which is a field theory incarnation of a liquid-drop model



# Large and small 't Hooft coupling

- The Sakai-Sugimoto model with the tower of vector mesons integrated out is

$$S = \frac{\lambda N_c}{216\pi^4}\int \mathrm{d}^4x \ \mathrm{tr}\, L_\mu^2 + \frac{\lambda N_c}{216\pi^3}\int \mathrm{d}^4x \ \mathrm{tr}[L_\mu, L_\nu]^2 + \frac{51 N_c}{8960\lambda}\int \mathrm{d}^4x \ (\mathrm{tr}\, L_\mu L_\nu L_\rho \epsilon^{\mu\nu\rho\sigma})^2 + 4mc\int \mathrm{d}^4x \ (\sigma - 1).$$

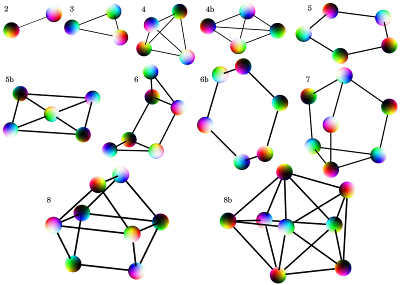
- The large-$\lambda$ limit gives the Skyrme model, as expected (to leading order)
- The small-$\lambda$ limit makes the volume-preserving diffeomorphism invariant sextic term the dominant one
- Upon scaling the size of the soliton, one obtains the sextic term + mass term, which is exactly the BPS-Skyrme model of [Adam-Sanchez-Guillen-Wereszczynski, PLB691, 105 (2010)]

# Nucleon-nucleon potential

- [Baldino-Bolognesi-Gudnason-Koksal, PRD96, 034008 (2017)] and [Baldino-Bartolini-Bolognesi-Gudnason, PRD103, 126015 (2021)] obtain the nucleon-nucleon potential from the SS model

$$V(r, B^\dagger C) = \frac{4\pi N_c}{\Lambda}\left(\sum_{n=1}^{\infty}\left(\frac{1}{c_{2n-1}}\frac{e^{-k_{2n-1}r}}{r} + \frac{6}{5}\frac{1}{c_{2n-1}}M_{ij}(B^\dagger C)P_{ij}(r_i, k_{2n-1})\frac{e^{-k_{2n-1}r}}{r^3}\right.\right.$$
$$\left.\left. - \frac{6}{5}\frac{1}{d_{2n}}\frac{e^{-k_{2n}r}}{r^3}M_{ij}(B^\dagger C)P_{ij}(r_i, k_{2n})\right) - \frac{6}{5\pi}\frac{e^{-k_0 r}}{r^3}M_{ij}(B^\dagger C)P_{ij}(r_i, k_0)\right).$$

- Point-like nucleons in the following configurations minimize the 2-body potential



# Hashimoto – matrix model for nuclei

- Instead of studying the instanton in the SS model as the baryon, one can write down the matrix model corresponding to description of the zeromodes of the D8-branes wrapping the $S^4$
- The matrix model is very similar to the ADHM construction of instantons
- However, with the addition of a 1-dimensional Chern-Simons term and an important mass deformation – not present in ADHM
- The symmetry of the fields is $\mathrm{U}(k) \times \mathrm{SU}(N_f) \times \mathrm{SO}(3)$
- The model is much simpler than solving ODEs in the bulk (or PDEs)
- A question whether the (lifted) zeromodes on the 8-branes is enough for capturing all phenomena in nuclear physics

# The Sutcliffe model

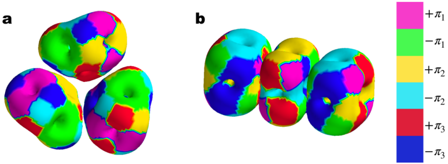
- The Sutcliffe model is a flat-holography model of YM on $\mathbb{R}^5$
- The kink function is introduced for the pion profile

$$\psi_+(z) = \frac{1}{\sqrt{2}\pi^{\frac{1}{4}}} \int_{-\infty}^{z} \psi_0(\xi)\mathrm{d}\xi = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{z}{\sqrt{2}}\right),$$

and the gauge fields are as usual expanded as

$$A_i = -\partial_i U U^{-1}\psi_+(z) + \sum_{n=0}^{K-1} V_i^n(\mathbf{x})\psi_n(z),$$

- Notice that no electric field or CS is present, since there is no curvature to fight against
- The KK scale appears due to truncation of the infinite modes to $K$ modes



# Homogeneous Ansatz

- The homogeneous Ansatz is an approximation where the single instantons are sitting so densely, that they lose their individual identity
- Unfortunately, it was proved in 2007 by [Rozali-Shieh-Van Raamsdonk-Wu, JHEP01, 053 (2008)] that it is impossible for a continuous gauge field configuration to have a nonvanishing instanton number in the homogeneous Ansatz
- However, it can be done by introducing a discontinuous gauge field, which acts as a source of baryons – usually put at $z = 0$ [Li-Schmitt-Wang, PRD92, 026006 (2015)]
- The homogeneous Ansatz for SS (for $\mathrm{SU}(2)$) is

$$\mathcal{A}_0 = \frac{1}{2}\widehat{a}_0(z), \qquad \mathcal{A}_i = -\frac{1}{2}H(z)\tau^i, \qquad \mathcal{A}_z = 0,$$

with the discontinuous BCs:

$$\widehat{a}_0'(0) = 0, \qquad H(0^\pm) = \pm(4\pi^2 d)^{\frac{1}{3}}.$$

# Finite isospin density

- Following [Adkins-Nappi-Witten, NPB228, 552 (1983)], we can turn on isospin, by rotating the baryons in $\mathrm{SU}(2)$ [Bartolini-Gudnason, 2209.14309]

$$\mathcal{A}_0 = G(z) a \boldsymbol{\chi} \cdot \boldsymbol{\tau} a^{-1} + \frac{1}{2}\widehat{a}_0(z), \qquad \mathcal{A}_i = -\frac{1}{2}\left(H(z) a \tau^i a^{-1} + L(z)\chi^i\right),$$
$$\mathcal{A}_z = 0,$$

with boundary angular isospin velocity $a\boldsymbol{\chi} \cdot \boldsymbol{\tau} a^{-1} = -2i\dot{a}a^{-1}$ and the discontinuous BCs:

$$\widehat{a}_0'(0) = 0, \qquad H(0^\pm) = \pm(4\pi^2 d)^{\frac{1}{3}}.$$

- What are the suitable BCs for $G$ and $L$?
- Guessing suitable boundary conditions based on parity, one would get

$$G'(0) = 0, \qquad L(0) = 0. \tag{2}$$

- Unfortunately, scrutinizing a bit, it turns out that $G'(0)$ does not minimize the entire action!

# Taking a closer look at the Chern-Simons term

- SS Model

$$S = -\kappa \,\mathrm{tr} \int_{\mathrm{AdS}_5} \mathcal{F} \wedge * \mathcal{F} + N_c \int_{\mathrm{AdS}_5} \omega_5, \qquad \kappa = \frac{\lambda N_c}{216\pi^3},$$

  or in more details

$$S = -\frac{\kappa}{2} \,\mathrm{tr} \int \mathrm{d}^5 x \left[h(z)\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + 2k(z)\mathcal{F}_{\mu z}\mathcal{F}^{\mu z}\right] + \frac{N_c}{24\pi^2} \,\mathrm{tr} \int \left(\mathcal{A} \wedge \mathcal{F}^2 - \frac{\mathrm{i}}{2}\mathcal{A}^3 \wedge \mathcal{F} - \frac{1}{10}\mathcal{A}^5\right)$$

- The CS term can be integrated by parts:

$$+\frac{N_c}{24\pi^2} \,\mathrm{tr} \int \left[3\widehat{A} \wedge F^2 + \widehat{A} \wedge \widehat{F}^2 + \mathrm{d}\left(\widehat{A} \wedge \left(2F \wedge A - \frac{\mathrm{i}}{2}A^3\right)\right)\right].$$

  with $\mathcal{F} = F^a \frac{\tau^a}{2} + \widehat{F}\frac{\mathbf{1}}{2}$.

- After integrating by part, the entire CS term is proportional to $\widehat{A}$, but this depends on what is chosen as the boundary term (ambiguity problem)

# Discarding the total derivative of CS

- Let's discard the total derivative in the CS
- and use the homogeneous Ansatz:

$$\mathcal{A}_0 = \frac{1}{2}\widehat{a}_0(z), \qquad \mathcal{A}_i = -\frac{1}{2}H(z)\tau^i, \qquad \mathcal{A}_z = 0,$$

- The full variation of the action:

$$\delta S = \int \mathrm{d}^4x\mathrm{d}z \left[(\mathrm{E.o.M.})_H\, \delta H + (\mathrm{E.o.M.})_{\widehat{a}_0}\, \delta\widehat{a}_0\right] + \delta S_{\mathrm{boundary}},$$

$$\delta S_{\mathrm{boundary}} = 2\kappa \int \mathrm{d}^4x \left[k(z)\widehat{a}_0'\delta\widehat{a}_0 - 3\left(k(z)H' + \frac{N_c}{16\pi^2\kappa}\widehat{a}_0 H^2\right)\delta H\right]_{z=0}^{z=\infty}.$$

- We arrive at the conditions:

$$\widehat{a}_0'(0)\delta\widehat{a}_0(0) = 0,$$

$$\left(H'(0) + \frac{N_c}{16\pi^2\kappa}\widehat{a}_0(0)H^2(0)\right)\delta H(0) = 0.$$

- and the solution for $\widehat{a}_0$:

$$\widehat{a}_0' = -\frac{N_c}{16\pi^2\kappa}\frac{1}{k(z)}\left(H^3(z) - H^3(0)\right),$$

# Discarding the total derivative of CS

- Looking at the baryon current at the conformal boundary

$$d_B = \frac{2}{N_c}\kappa\left[k(z)\widehat{a}_0'\right]_{-\infty}^{\infty} = \frac{4}{N_c}\kappa\left[k(z)\widehat{a}_0'\right]_{z=\infty} = d,$$

- we find the same baryon density $d$ as given by the topological integral

$$\begin{aligned} d &= \frac{1}{32\pi^2}\int \mathrm{d}z\,\epsilon^{MNPQ}\,\mathrm{tr}\,\boldsymbol{F}_{MN}\boldsymbol{F}_{PQ} \\ &= -\frac{1}{8\pi^2}\int \mathrm{d}z\,\partial_z\left(H^3\right) \\ &= -\frac{1}{8\pi^2}\left[H^3\right]_{z=0^+}^{z=+\infty} - \frac{1}{8\pi^2}\left[H^3\right]_{z=-\infty}^{z=0^-} \end{aligned}$$

which is consistent.

- For the chemical potential

$$\mu_B d_B = -\frac{N_c}{8\pi^2}\mu\int_0^{\infty}\mathrm{d}z\;\partial_z(H^3) = \frac{N_c}{2}\mu d \qquad \Rightarrow \qquad \mu_B = \frac{N_c}{2}\mu.$$

# Including the total derivative of CS

- Let's now include the total derivative in the CS
- The full variation of the action now gives the conditions:

$$\left(\widehat{a}_0'(0) + \frac{3N_c}{64\pi^2\kappa}H^3(0)\right)\delta\widehat{a}_0(0) = 0,$$
$$\left(H'(0) + \frac{N_c}{16\pi^2\kappa}\widehat{a}_0(0)H^2(0) - \frac{3N_c}{64\pi^2\kappa}\widehat{a}_0(0)H^2(0)\right)\delta H(0) = 0.$$

- and the solution for $\widehat{a}_0$:

$$\widehat{a}_0' = -\frac{1}{k(z)}\frac{N_c}{16\pi^2\kappa}\left(H^3(z) - \frac{H^3(0)}{4}\right).$$

- Looking at the baryon current at the conformal boundary, we now find:

$$d_B = \frac{2}{N_c}\kappa\left[k(z)\widehat{a}_0'\right]_{-\infty}^{\infty} = \frac{4}{N_c}\kappa\left[k(z)\widehat{a}_0'\right]_{z=\infty} = \frac{d}{4},$$

  which differs from the topological density by a factor of $1/4$.
- The chemical potential is unchanged.

# Proposal: Matching the currents and the topological data

- We propose to choose which total-derivative term to discard in the CS action, by matching the physical currents with the topological degrees calculated in the bulk.
- This fixes the CS term to:

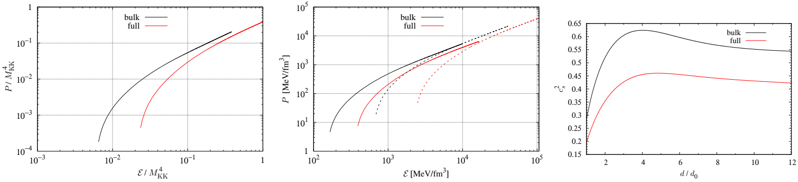
$$S_{\text{CS}} = \frac{N_c}{24\pi^2} \operatorname{tr} \int \left[ 3\widehat{A} \wedge F^2 + \widehat{A} \wedge \widehat{F}^2 \right].$$

- Using this CS term, the variation of the fields $G, L$:

$$\left[ G'(0) + \frac{N_c}{32\pi^2 \kappa} H^2(0) L(0) \right] \delta G(0) = 0,$$
$$L'(0) \delta L(0) = 0.$$

- Luckily, this same choice of CS term yields consistency also between the isospin quantum number and its current

# Effects on observables




$$d_0^{\rm bulk} = 0.436 \left(\frac{M_{\rm KK}}{949\,{\rm MeV}}\right)^3 {\rm fm}^{-3}, \qquad d_0^{\rm bulk} = 0.15\,{\rm fm}^{-3} \quad \Rightarrow \quad M_{\rm KK} = 665.0\,{\rm MeV},$$

$$d_0^{{\rm bulk}+\partial} = 0.601 \left(\frac{M_{\rm KK}}{949\,{\rm MeV}}\right)^3 {\rm fm}^{-3}, \qquad d_0^{{\rm bulk}+\partial} = 0.15\,{\rm fm}^{-3} \quad \Rightarrow \quad M_{\rm KK} = 597.6\,{\rm MeV},$$

# Thermodynamic equilibrium condition

- Thermodynamic equilibrium condition can be found from

$$\widehat{a}_0(z) = \mu - \int_z^\infty \mathrm{d}z'\, \widehat{a}_0'(z'), \qquad \widehat{a}_0' = -\frac{N_c}{16\pi^2\kappa}\frac{1}{k(z)}\left(H^3(z) - H^3(0)\right),$$

Solving for $\mu$:

$$\mu(d) = -\frac{4\kappa}{N_c}\left(\frac{4\pi^2}{d^2}\right)^{\frac{1}{3}} H'(0) + \frac{N_c d}{4\kappa}\int_0^\infty \mathrm{d}z\,\frac{1}{k(z)}\left(1 - \frac{H^3(z)}{H^3(0)}\right),$$

- The trick is that $H(z)$ can be solved without knowing $\mu$, which it is simply an overall shift in $\widehat{a}_0$, that the EOMs do not observe.
- Hence, fixing $d$, we can calculate the corresponding chemical potential $\mu$, to within numerical precision.

# Equivalence between quantization and chemical potential for isospin

- Start with Ansatz:

$$A_0 = Ga\boldsymbol{\chi}\cdot\boldsymbol{\tau}a^{-1} \quad A_i = -\frac{H}{2}a\tau^i a^{-1}, \quad A_z = 0.$$

- Perform a gauge transformation:

$$\begin{aligned}
A_0 &\to \widetilde{A}_0 = Gba\boldsymbol{\chi}\cdot\boldsymbol{\tau}a^{-1}b^{-1} - \mathrm{i}b\partial_0 b^{-1},\\
A_i &\to \widetilde{A}_i = -\frac{H}{2}ba\tau^i a^{-1}b^{-1},\\
A_z &\to \widetilde{A}_z = 0.
\end{aligned}$$

- Choose $b = a^{-1}$ rotating the fields $A_i$ back to the standard orientation, while modifying the field $A_0$ with an additional term:

$$\widetilde{A}_0 = G\boldsymbol{\chi}\cdot\boldsymbol{\tau} - \mathrm{i}a^{-1}\dot{a} \quad \widetilde{A}_i = -\frac{H}{2}\tau^i, \quad \widetilde{A}_z = 0.$$

- Using identity $-\mathrm{i}a^{-1}\dot{a} = \frac{1}{2}\boldsymbol{\chi}\cdot\boldsymbol{\tau}$, we get

$$\widetilde{A}_0 = \left(G + \frac{1}{2}\right)\boldsymbol{\chi}\cdot\boldsymbol{\tau}.$$

# Equivalence between quantization and chemical potential for isospin

- Since $G(\infty)$ vanishes, we have

$$\widetilde{A}_0(z \to z_{\mathrm{UV}}) = \frac{1}{2}\boldsymbol{\chi} \cdot \boldsymbol{\tau}.$$

  The boundary value of the field $A_0$ is dual to an isospin chemical potential.

- Choosing $\boldsymbol{\chi} = (0, 0, \mu_I)$ and defining:

$$\widetilde{G}(z) = \left( G(z) + \frac{1}{2} \right),$$

- we obtain the familiar expressions for the gauge field and its boundary condition

$$\widetilde{A}_0 = \widetilde{G}\tau^3 \mu_I, \qquad \widetilde{A}_0(z \to z_{\mathrm{UV}}) = \frac{1}{2}\mu_I \tau^3.$$

# Symmetry energy

- Using the Ansatz

$$\mathcal{A}_0 = Ga\boldsymbol{\chi}\cdot\boldsymbol{\tau}a^{-1} + \tfrac{1}{2}\widehat{a}_0, \quad \mathcal{A}_i = -\tfrac{1}{2}\left(Ha\tau^i a^{-1} + L\chi^i\right), \quad \mathcal{A}_z = 0,$$

- The action is

$$S_{\mathrm{YM}} = -\kappa\int \mathrm{d}^4x\int \mathrm{d}z\Big[-8hH^2\left(G+\frac{1}{2}\right)^2\boldsymbol{\chi}\cdot\boldsymbol{\chi} + 3hH^4$$
$$+k\left[(L')^2 - 4(G')^2 + 8(KH)^2\right]\boldsymbol{\chi}\cdot\boldsymbol{\chi} + 3k(H')^2 - k(\hat{a}_0')^2\Big],$$
$$S_{\mathrm{CS}} = -\frac{N_c}{8\pi^2}\int \mathrm{d}^4x\int \mathrm{d}z\;\hat{a}_0H'H^2 + \frac{N_c}{4\pi^2}\int \mathrm{d}^4x\int \mathrm{d}z\;(LH' - L'GH)\,H\boldsymbol{\chi}\cdot\boldsymbol{\chi},$$

# Symmetry energy

- Quantizing the isospin Hamiltonian

$$\begin{aligned} H &= \frac{1}{2} V \Lambda \boldsymbol{\chi} \cdot \boldsymbol{\chi} + VU \\ &= 2V\Lambda \dot{a}_m^2 + VU \\ &= \frac{\pi_m^2}{8V\Lambda} + VU \\ &= \frac{I(I+1)}{2V\Lambda} + VU, \end{aligned}$$

with $\pi_m = \frac{\partial H}{\partial \dot{a}_m} = 4V\Lambda \dot{a}_m$

- The inertia and potential energy are

$$\Lambda = 2\kappa \int \mathrm{d}z \left[ 2hH^2(2G+1)^2 + k((L')^2 + 4(G')^2) \right],$$

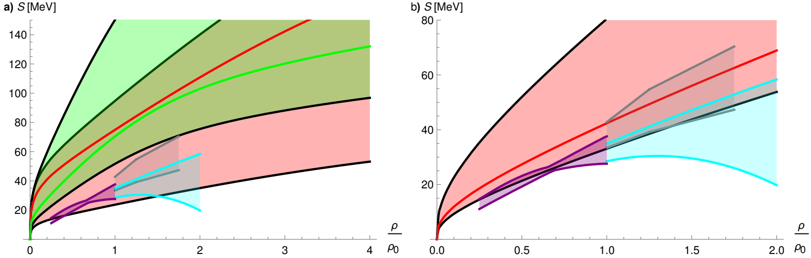
$$U = \kappa \int \mathrm{d}z \left[ 3hH^4 + 3k(H')^2 + k(\widehat{a}_0')^2 \right],$$

- The symmetry energy can be read off:

$$\frac{H}{A} = \frac{U}{d} + S(d)\beta^2 + \mathcal{O}(V^{-1}),$$

$$S(d) = \frac{d}{8\Lambda},$$

# Symmetry energy with homogeneous Ansatz in WSS (and HW)

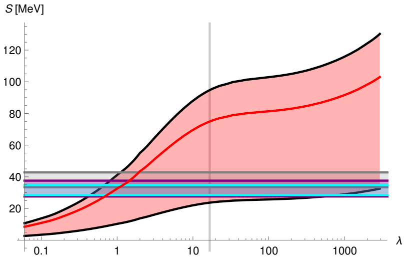


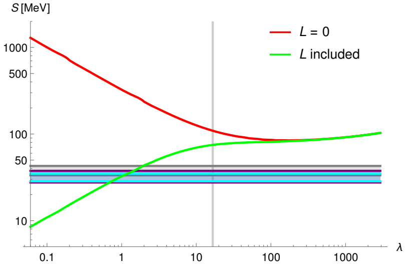

a) WSS: $M_{\mathrm{KK}} \in [300, 1200]$ MeV, HW: $\mathsf{L}^{-1} \in [110, 320]$ MeV,
b) WSS: $M_{\mathrm{KK}} \in [370, 949]$ MeV

$$S(d) = \frac{d}{8\Lambda}, \qquad \Lambda = 2\kappa \int \mathrm{d}z \left[2hH^2(2G+1)^2 + k((L')^2 + 4(G')^2)\right].$$

[Bartolini-Gudnason, SciPost Phys.16, 156 (2024)]

# Symmetry energy with homogeneous Ansatz in WSS




[Kovensky-Schmitt, SciPost Phys.11, 209 (2021)] did not include the Abelian field $L$, leading to large inconsistencies for finite $\lambda$.

# Holographic neutron stars

Some phenomenological bounds:

- Radius:
  1. 11.4 to 13.7 km for stars of about 2.1 $M_\odot$ (PSR J0740+6620)
  2. 12.2 to 16.3 km for stars of about 2.1 $M_\odot$ (PSR J0740+6620)
  3. 11.5 to 13.9 km for stars of about 1.4 $M_\odot$ (J0030+0451)
  4. 12.0 to 14.3 km for stars of about 1.4 $M_\odot$ (J0030+0451)
- Mass: maximum measured is $2.35\pm0.17$ $M_\odot$ (PSR J0952-0607)
- Tidal deformability: 70 to 620 for stars of about 1.4 $M_\odot$ (GW170817)

Some holographic works on neutron stars:

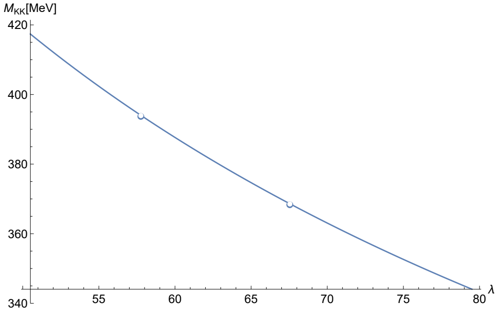
- D3-D7 [Hoyos-Fernández-Jokela-Vuorinen] [Fadafan-Rojas-Evans]
- D4-D8 (WSS) [Kovensky-Poole-Schmitt] [Bartolini-Gudnason]
- VQCD [Järvinen-Jokela-Remes]
- Hard-Wall [Bartolini-Gudnason-Leutgeb-Rebhan]

Merger events simulations from holography:

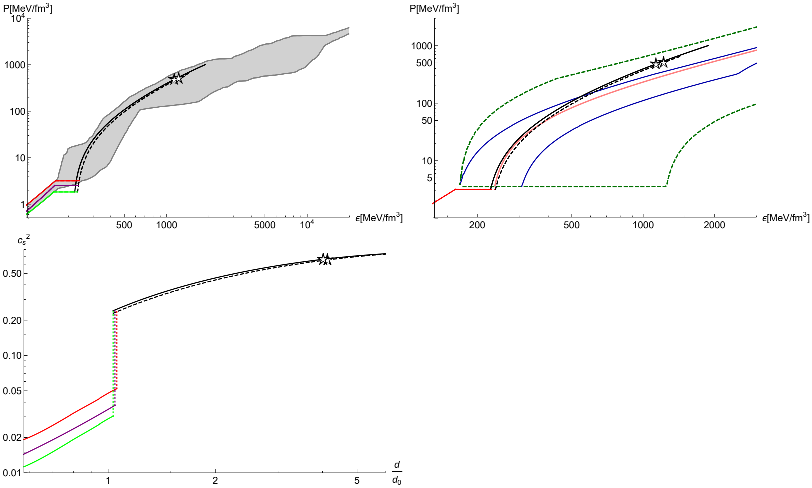
- VQCD [Ecker-Järvinen-Nijs-van der Schee]
- Hard-Wall [Bartolini-Gudnason-Leutgeb-Rebhan]

# Our fit in WSS

- WSS and homogeneous Ansatz for the core
- Solve EOM: free parameters $\lambda$, $d$, $n_I$, $M_{\mathrm{KK}}$.
- Impose $\beta$-equilibrium and charge neutrality: find $n_I(d)$.
- Impose phenomenological saturation density, $d_0 = 0.16\,\mathrm{fm}^{-3}$: find $M_{\mathrm{KK}}(\lambda)$.
- Impose phenomenological SE, $S(d_0) = (30.6 \div 32.8)\,\mathrm{MeV}$: find $\lambda$.

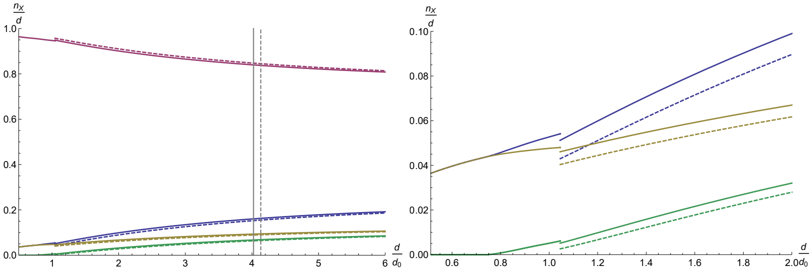


# Holographic EOS in our fit



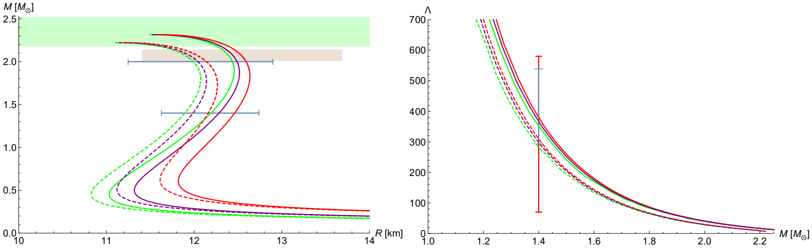

3 different ChPT EFT EOSs are glued with the holographic WSS EOS

# Proton fractions




- $\beta$-equilibrium: $\mu_\ell = \mu_N - \mu_P = -\mu_I$
- charge neutrality: $\frac{1}{2}d + n_I = \sum_\ell d_\ell$, with $\ell = e, \mu, \ldots$
- proton and neutron densities: $d_{P,N} = \frac{1}{2}d \pm n_I$
- isospin chemical potential: $\mu_I = \Lambda^{-1} n_I$

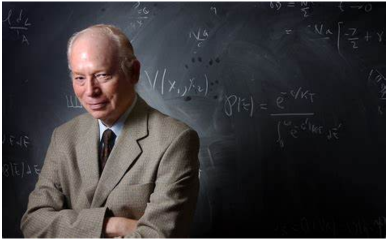
# Mass-radius and tidal deformabilities




- The mass radius plot is a curve of solutions, which each point representing a different star
- $R$ is determined as the radius with vanishing pressure
- $P(r = 0)$ is the initial condition of the TOV equations, leading to different solutions
- Tidal deformability measures the stiffness of the star

# Which is the correct (dual) description of low-energy QCD?

- Weinberg says: if you get the symmetries right, then the theory is the right theory



- In that sense, the chiral Lagrangian is correct
- But this quickly becomes insufficient, unless we reliably can determine the LECs

# Which is the correct (dual) description of low-energy QCD?

- In [Bijnens-Gudnason-Yu-Zhang, JHEP05, 061 (2023)] we have determined the pure pion terms up to operator dimension 16 (and also for any other spacetime dimension $\leq 12$ as well as other $\mathrm{O}(N)$ groups)

| $n_d$ | # EFT terms (in 3+1 dim) | | | |
|---|---|---|---|---|
| 2 | 1 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 |
| 4 | 2 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 |
| 6 | 5 | 0 | 1 | 0 |
| 7 | 0 | 0 | 0 | 0 |
| 8 | 16 | 2 | 4 | 0 |
| 9 | 0 | 0 | 0 | 0 |
| 10 | 55 | 14 | 27 | 11 |
| 11 | 0 | 0 | 0 | 0 |
| 12 | 253 | 115 | 160 | 99 |
| 13 | 0 | 0 | 0 | 0 |
| 14 | 1260 | 806 | 1021 | 779 |
| 15 | 0 | 0 | 0 | 0 |
| 16 | 7140 | 5564 | 6379 | 5426 |

# Which is the correct (dual) description of low-energy QCD?

- So the problem is to determine the many many LECs
- One way is to assume hidden local symmetry, where

$$SU(N_f)_V \subset SU(N_f)_L \times SU(N_f)_R \tag{3}$$

  becomes a local (gauged) group
- This is essentially already built-in in the SS model
- However, the symmetry group of SS is larger and hence gives numerically slightly different results, with respect to HLS
- One could contemplate that a different background gives rise to HLS
- Notice that SS, HW, HLS and the Sutcliffe model all determine all the LECs (up to the scale and 't Hooft coupling and their analogues in the other models), but all give different LECs
- Determining which background is correct amounts to determining with high precision the correct values of the LECs

# WSS and Hard-wall on the same footing

- If we fold the SS model [Gorsky-Gudnason-Krikun, PRD91, 126008 (2015)], then we can multiply the action by 2 and integrate over $z \in [0, \infty)$

$$S \to 2 \int_0^\infty \mathrm{d}z \int \mathrm{d}^4x \ \operatorname{tr} \mathcal{F} \wedge *\mathcal{F} + \ldots$$

- In the hard-wall model in the baryonic phase, where the scalar only determines the onset of the baryons, we can identify

$$L_i = -R_i = -\frac{1}{2} H(z) \tau^i, \qquad L_0 = R_0 = \frac{1}{2} \widehat{a}_0(z).$$

- Up to a coordinate transformation, the hard-wall model is formally identical to the WSS model with a different warp factor!

# Chern says physics is geometry

物理几何是一家
共同携手到天涯
黑洞單極窮奧秘
纤維連絡織錦霞
進化方程孤立異
对偶曲率瞬息空
疇算竟有天人用
拈花一笑不言中

Photography by Peg Skorpinski.

Shiing Shen Chern

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Photography by Peg Skorpinski.

Shiing Shen Chern

# Chern says physics is geometry

Physics and geometry are one family.
Together and holding hands they roam
to the limits of outer space.
Black hole and monopole exhaust
the secret of myths;
Fiber and connections weave to interlace
the roseate clouds.
Evolution equations describe solitons;
Dual curvatures defines instantons.
Surprisingly, Math. has earned
its rightful place
for man and in the sky;
Fondling flowers with a smile – just wish
nothing is said!
– Shiing-Shen Chern

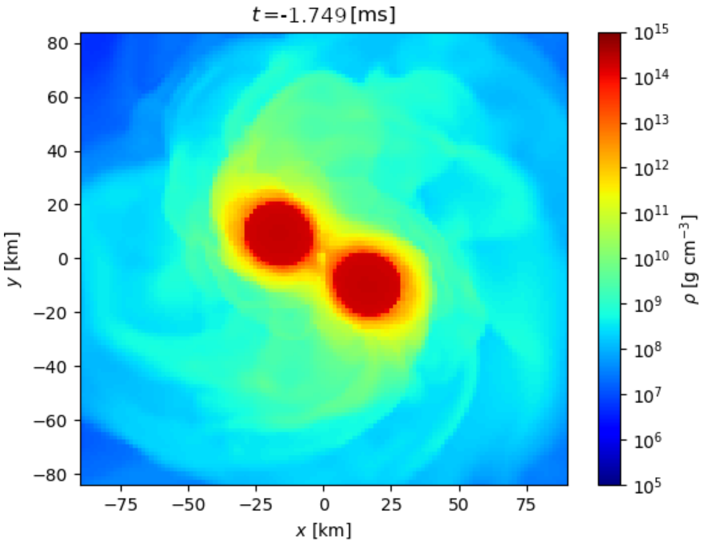
A. Jackson and D. Kotschick, Notices of the AMS, 45 7 (1998)

Perhaps QCD is just geometry?!

# Outlook/open problems

- How to systematically study and converge to the correct bulk geometry describing real world QCD?
- Using deep learning? [Hashimoto-Sugishita-Tanaka-Tomiya, PRD98, 046019 (2018)]
- How to systematically improve holographic QCD?
  - $1/N_c$ corrections?
  - $1/\lambda$ corrections? (notice that the liquid-drop-like theory term appears as $1/\lambda$ after integrating out the vectors)
  - Beyond the 8-branes being probe branes?
  - $\alpha'$ corrections?
- How to improve the finite/large density description, instead of using a homogeneous Ansatz?
- Is the Chern-Simons term the right one? [Lau-Sugimoto, PRD95, 126007 (2017)]
- For a list of known problems with many holographic QCD models, see [Aoki-Hashimoto-Iizuka, Rept.Prog.Phys.76, 104301 (2013)], section 2.2
- Is the lack of asymptotic freedom going to be a problem for the low-energy effective theory?

# Full hydro simulation of merger



ありがとうございます!
Mange tak!
Grazie mille!
非常感谢!

# Backup slide

# Solution at $\lambda = 16.63$