RUB

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Modern Theory of Nuclear Forces

Concepts

Chiral perturbation theory, pionless and chiral effective field theories for few-N, KSW vs Weinberg, predictive power, renormalization, …

Methods

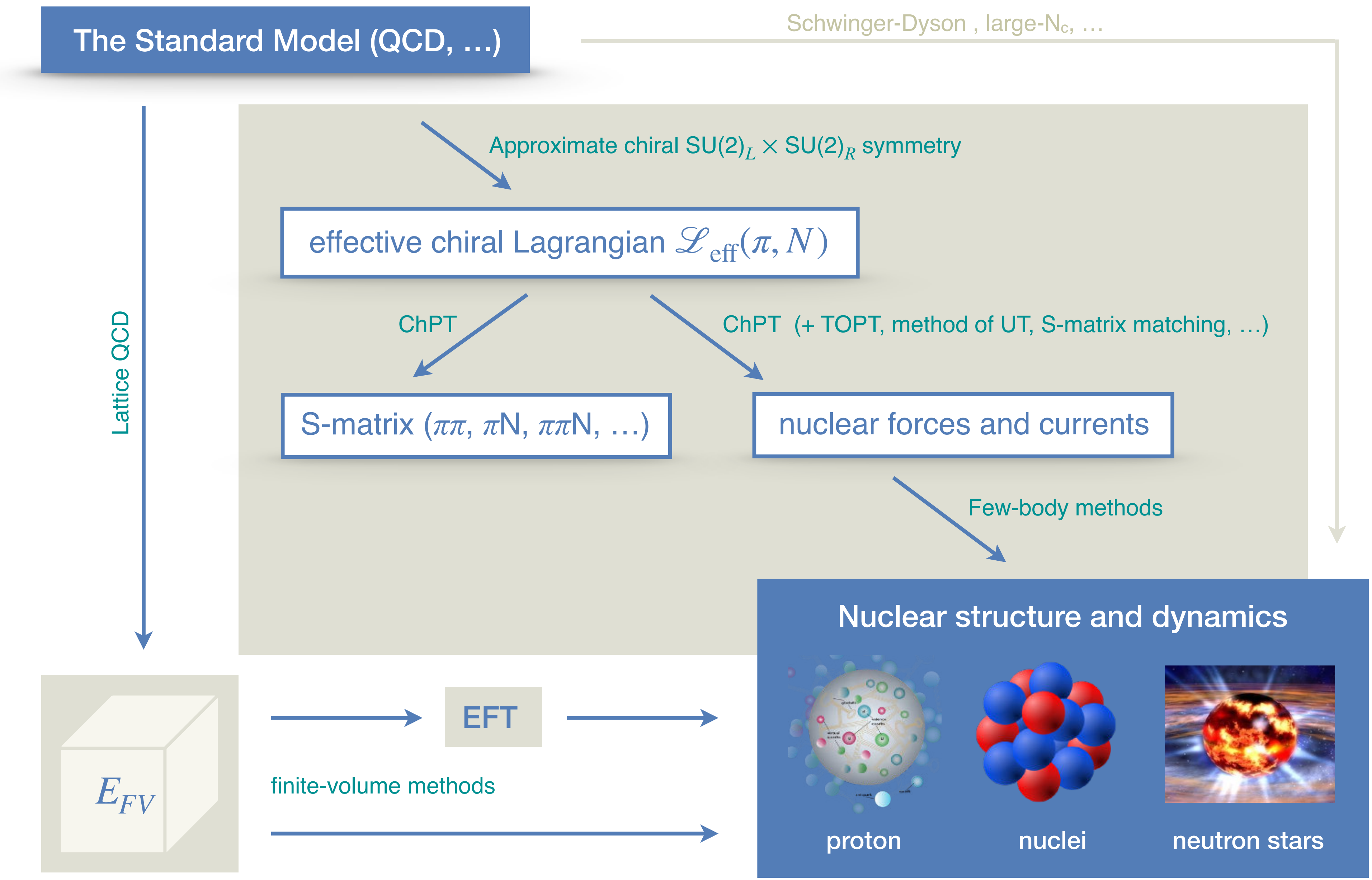
S-matrix matching, Method of Unitary Transformation, Path-integral approach …

erc European Research Council
Bundesministerium für Bildung und Forschung
DFG Deutsche Forschungsgemeinschaft
Ministerium für Kultur und Wissenschaft des Landes Nordrhein-Westfalen
ERATO Exploratory Research for Advanced Technology

YITP long-term and Nishinomiya-Yukawa memorial workshop

Hadrons and Hadron Interactions in QCD 2024 (HHIQCD 2024)

The Big Picture



I: Chiral Perturbation Theory

How to exploit the chiral symmetry of QCD to calculate low-energy reactions involving pions?

Outline

- Classical example of an effective theory
- Effective Lagrangian for pions and calculation of the S-matrix
- Inclusion of the nucleons

Selected review articles

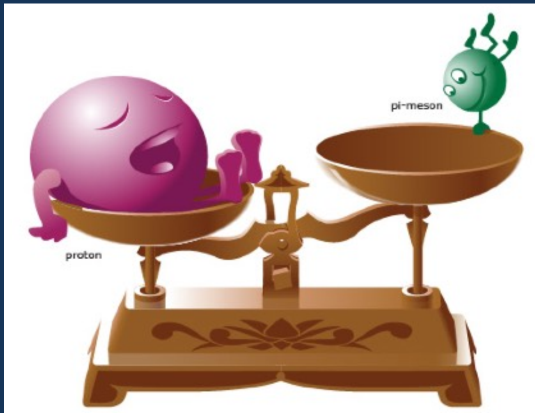
Bernard, Kaiser, Meißner, Int. J. Mod. Phys. E4 (1995) 193
Pich, Rep. Prog. Phys. 58 (1995) 563
Bernard, Prog. Part. Nucl. Phys. 60 (2007) 82
Scherer, Prog. Part. Nucl. Phys. 64 (2010) 1

Lecture notes

Scherer, Adv. Nucl. Phys. 27 (2003) 277
Gasser, Lect. Notes Phys. 629 (2004) 1

Text book

Scherer, Schindler, *A Primer for ChPT*, Lecture Notes in Physics, 2012

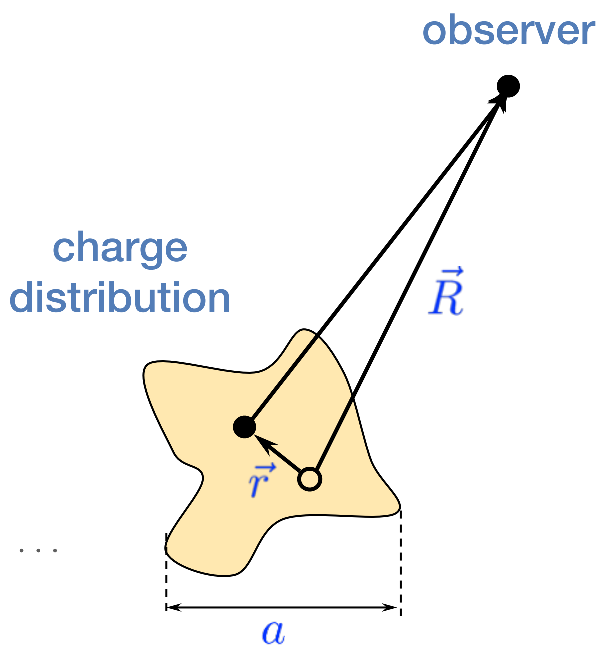


What is an effective theory?

A classical example

The goal: compute electric potential generated by a localized charge distribution $\rho(\vec{r})$

The answer is $V(\vec{R}) \propto \int d^3r \, \frac{\rho(\vec{r})}{|\vec{R}-\vec{r}|}$



An effective theory for $R \gg a$: The Top-Down approach

$$\frac{1}{|\vec{R}-\vec{r}|} = \frac{1}{R} + r_i \left[\frac{\partial}{\partial r_i}\frac{1}{|\vec{R}-\vec{r}|}\right]_{\vec{r}=0} + \frac{1}{2!} r_i r_j \left[\frac{\partial^2}{\partial r_i \partial r_j}\frac{1}{|\vec{R}-\vec{r}|}\right]_{\vec{r}=0} + \ldots$$

$$= \frac{1}{R} + \frac{R_i}{R^3} r_i + \frac{1}{2!}\frac{R_i R_j}{R^5}(3 r_i r_j - r^2 \delta_{ij}) + \ldots$$

$$\Rightarrow \boxed{V(\vec{R}) = \frac{q}{R} + \frac{R_i}{R^3} P_i + \frac{1}{2}\frac{R_i R_j}{R^5} Q_{ij} + \ldots}$$

with $q = \int d^3r \, \rho(\vec{r}), \qquad P_i = \int d^3r \, \rho(\vec{r}) \, r_i, \qquad Q_{ij} = \int d^3r \, \rho(\vec{r}) \, (3 r_i r_j - r^2 \delta_{ij})$

We have just „integrated out“ short-distance physics. For $R \gg a$, the only information needed about $\rho(\vec{r}\,)$ is hidden in the moments q, P_i, Q_{ij}, …

What is an effective theory?

An effective theory for $R \gg a$: The Bottom-Up approach

What if we cannot „integrate out" short-distance physics or don't even now $\rho(\vec{r})$, apart from the fact that it is localized in the volume $\sim a^3$?

Solution: Write down the **most general expression for V** using the **long-distance DoF** (i.e., $\vec{R}$) compatible with the **symmetry principles** (rotational invariance)

$$V(\vec{R}) = \sum \left[\begin{array}{l} \textit{rotational tensors} \\ \textit{constructed from } \vec{R} \end{array} \right] \cdot \left[\begin{array}{l} \textit{rotational tensors characterizing} \\ \textit{the system, independent of } \vec{R} \end{array} \right]$$

$$= \underbrace{\frac{1}{R}}_{[V] = \text{length}^{-1}} \text{const} + \frac{1}{R^3} R_i \underbrace{X_i}_{\sim a \text{ (NDA)}} + \frac{1}{R^5} R_i R_j \underbrace{X_{ij}}_{\sim a^2 \text{ (NDA)}} + \ldots$$

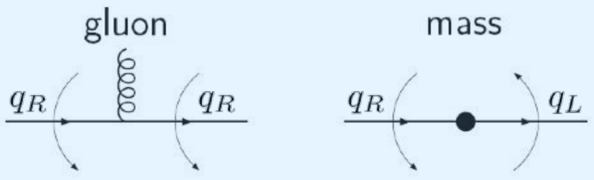
symmetric and traceless (otherwise redundant structures)

The $(2n+1)$ components of $X_{i_1 \ldots i_n}$ are called in the EFT language LECs and can be determined from experimental data.

$\Rightarrow$ systematically improvable approximation for $V(\vec{R})$ at $R \gg a$ without knowing $\rho(\vec{r})$!

Chiral perturbation theory

Chiral symmetry of QCD



$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}G_a^{\mu\nu}G_{a,\mu\nu} + \bar{q}(i\gamma^\mu D_\mu - \mathcal{M})q$$

$$= -\frac{1}{4}G_a^{\mu\nu}G_{a,\mu\nu} + \underbrace{\bar{q}_L iDq_L + \bar{q}_R iDq_R}_{} - \underbrace{q_L\mathcal{M}q_R - q_R\mathcal{M}q_L}_{}$$

q_L: $\frac{1}{2}(1-\gamma_5)q$ $\quad$ q_R: $\frac{1}{2}(1+\gamma_5)q$

$\mathrm{SU(N_f)_L \times SU(N_f)_R}$ invariant

small for $\mathrm{N_f} = 2, (3)$. Indeed: $m_u \sim 3$ MeV, $m_d \sim 5$ MeV ($\overline{\mathrm{MS}}$, $\mu = 2$ GeV)

SSB to $\mathrm{SU(N_f)_V \leq SU(N_f)_L \times SU(N_f)_R} \;\Rightarrow\; \mathrm{N_f}^2 - 1$ GBs

Chiral perturbation theory

Weinberg, Gasser, Leutwyler, Meißner, …

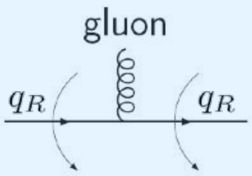
Ideal world [$m_u = m_d = 0$], zero-energy limit: non-interacting massless GBs (+ strongly interacting massive hadrons)

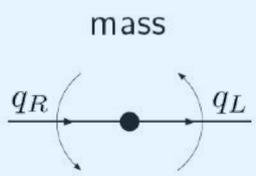
Real world [$m_u, m_d \ll \Lambda_{QCD}$], low energy: weakly interacting light GBs (pions) (+ strongly interacting massive hadrons)

$\Rightarrow$ expand about the ideal world (ChPT)

Chiral perturbation theory

Chiral symmetry of QCD





$$\mathcal{L}_{\rm QCD} = -\frac{1}{4}G_a^{\mu\nu}G_{a,\mu\nu} + \bar{q}(i\gamma^\mu D_\mu - \mathcal{M})q$$

$$= -\frac{1}{4}G_a^{\mu\nu}G_{a,\mu\nu} + \underbrace{\bar{q}_L iDq_L + \bar{q}_R iDq_R}_{\text{SU(N}_f)_L \times \text{SU(N}_f)_R \text{ invariant}} - \underbrace{q_L\mathcal{M}q_R - q_R\mathcal{M}q_L}_{\text{small}}$$

with $q_L = \frac{1}{2}(1-\gamma_5)q$, $q_R = \frac{1}{2}(1+\gamma_5)q$

$\mathrm{SU(N_f)_L \times SU(N_f)_R}$ invariant

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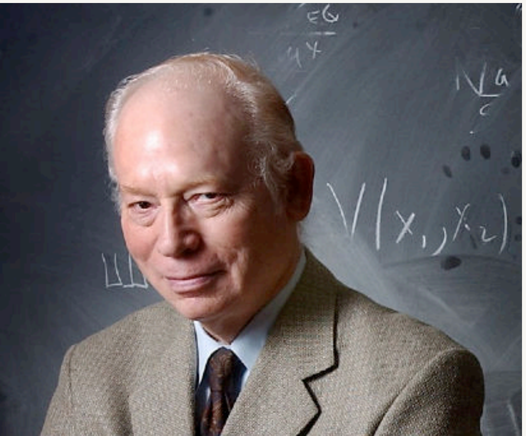
Chiral perturbation theory

Weinberg, Gasser, Leutwyler, Meißner, …



„if one writes down the most general possible Lagrangian, including all terms consistent with the assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition, and the assumed symmetry properties“

S. Weinberg, Physica 96A (1979) 327; see also H. Leutwyler, Annals Phys. (1994) 165

Effective chiral Lagrangian

Pions transform linearly under isospin (iso-triplet) but non-linearly under $SU(2)_L \times SU(2)_R$, e.g.:

$$U = \frac{1}{F}\left(\sigma\, \mathbf{1}_{2\times 2} + i\boldsymbol{\pi}\cdot\boldsymbol{\tau}\right) \xrightarrow{\text{nonlinear realization}} U = \frac{1}{F}\left(\sqrt{F^2 - \boldsymbol{\pi}^2}\,\mathbf{1}_{2\times 2} + i\boldsymbol{\pi}\cdot\boldsymbol{\tau}\right)$$

Chiral rotations: $U \longrightarrow U' = LUR^\dagger$ with $L = \exp[-i(\boldsymbol{\theta}^V - \boldsymbol{\theta}^A)\cdot\boldsymbol{\tau}/2]$, $R = \exp[-i(\boldsymbol{\theta}^V + \boldsymbol{\theta}^A)\cdot\boldsymbol{\tau}/2]$

The leading and sub-leading Lagrangian for pions

low-energy constants

terms involving external sources

$$\mathcal{L}_\pi^{(2)} = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger + 2B(\mathcal{M}U + \mathcal{M}U^\dagger)\rangle\,,$$

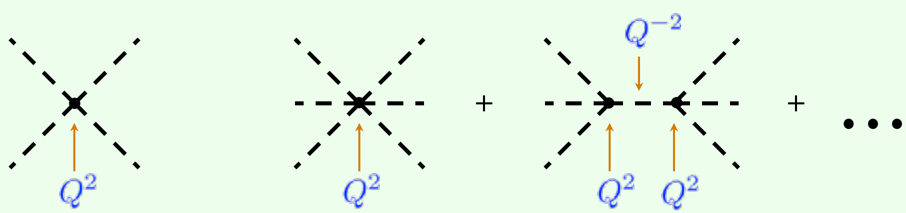
$$\mathcal{L}_\pi^{(4)} = \frac{l_1}{4}\langle \partial_\mu U \partial^\mu U^\dagger\rangle^2 + \frac{l_2}{4}\langle \partial_\mu U \partial_\nu U^\dagger\rangle\langle \partial^\mu U \partial^\nu U^\dagger\rangle + \frac{l_3}{16}\langle 2B\mathcal{M}(U+U^\dagger)\rangle^2 + \ldots$$
$$- \frac{l_7}{16}\langle 2B\mathcal{M}(U-U^\dagger)\rangle^2$$

[Gasser, Leutwyler '84]

Tree-level connected diagrams from $\mathcal{L}_\pi^{(2)}$

$2Bm_q \;\Rightarrow\; M_\pi^2 = 2Bm_q + \mathcal{O}(m_q^2)$

$$U(\boldsymbol{\pi}) = \mathbf{1}_{2\times 2} + i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{F} - \frac{\boldsymbol{\pi}^2}{2F^2} - i\alpha\frac{\boldsymbol{\pi}^2\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{F^3} + \mathcal{O}(\boldsymbol{\pi}^4) \longrightarrow \mathcal{L}_\pi^{(2)} = \frac{\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}}{2} - \frac{M^2\pi^2}{2} + \frac{(\partial_\mu\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2}{2F^2} - \frac{M^2\pi^4}{8F^2} + \ldots$$



- all diagrams scale as Q^2
- insertions from $\mathcal{L}_\pi^{(4)}$, $\mathcal{L}_\pi^{(6)}$, ... suppressed by powers of Q^2
- remarkable predictive power

Perturbative expansion of S-matrix

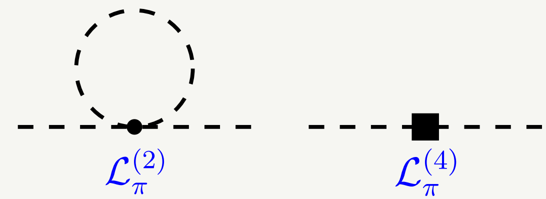
Tree-level diagrams with higher-order vertices are suppressed at low energy. In ChPT, loop contributions are also suppressed (pions as GBs) Weinberg '79

Example: pion self energy (using DimReg)

$$I = \frac{M^2}{F^2}\int \frac{d^4 l}{(2\pi)^4}\frac{i}{l^2 - M^2 + i\epsilon} \;\to\; \mu^{4-d}\frac{M^2}{F^2}\int \frac{d^d l}{(2\pi)^d}\frac{i}{l^2 - M^2 + i\epsilon}$$

$$= M^2\frac{M^2}{(4\pi F)^2}\ln\left(\frac{M^2}{\mu^2}\right) + \frac{2M^4}{F^2}L(\mu) + \ldots$$

← terms vanishing in d = 4



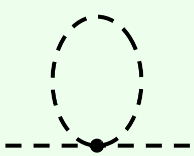
The infinity $L(\mu) = \frac{\mu^{d-4}}{16\pi^2}\left(\frac{1}{d-4} + \text{const}\right)$ is cancelled by the c.t. from $\mathcal{L}_\pi^{(4)}$: $l_i \to l_i^r(\mu) + \text{c.t.}$

The bottom line: in DimReg, all momenta flowing through loop graphs are soft, $\sim Q$

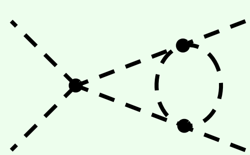
Power counting (NDA)

It is easy to show that connected diagrams scale as Q^D with $D = 2 + 2L + \sum_d N_d(d-2)$

(# of loops → L; # of vertices from $\mathscr{L}_\pi^{(d)}$ → N_d)



$D = 2 + 2 + 0 = 4$



$D = 2 + 4 + 0 = 6$

— loops are suppressed
— finite number of LECs at any order
} π's as GBs

— Λ_b at best $\sim 4\pi F_\pi$ (more realistic $\sim M_\rho$)

Inclusion of the nucleons

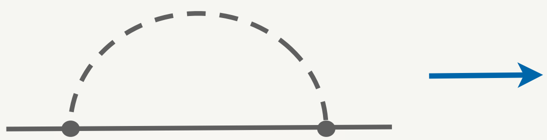
Matter fields (N, Δ,…) can be introduced via the CCWZ realization Coleman, Callan, Wess, Zumino '69

Lowest-order effective Lagrangian for a single nucleon:

known functions of the pion fields

$$\mathcal{L}_{\pi N}^{(1)} = \bar{N}\left(i\gamma^\mu D_\mu - m + \frac{g_A}{2}\gamma^\mu\gamma_5 u_\mu\right) N$$

Problem (?): new hard mass scale m ⇒ power counting ??



$$\delta m \xrightarrow{\mathcal{M}\to 0} -m\frac{3g_A^2 m^2}{(4\pi F)^2}\left[\log\frac{m}{\mu} + \mu^{d-4}\left(\frac{1}{d-4} + \text{const}\right)\right]$$

Gasser, Sainio, Svarc '88

- **Heavy-baryon ChPT ($p^\mu = mv^\mu + k^\mu, \;\; v\cdot k \ll m$)** Jenkins, Manohar '91; Bernard et al. '92; Mannel et al. '92

  Nonrelativistic expansion of $\mathscr{L}_{\pi N}$ ⇒ nucleon mass appears only in $1/m^n$—corrections

  $$\mathcal{L}_{\pi N}^{(1)} = N'^\dagger\left(iD_0 + \frac{g_A}{2}\vec{\sigma}\cdot\vec{u}\right) N' + \mathcal{O}(1/m)$$ ⟵ for $v^\mu = (1,0,0,0)$

- **Manifestly Lorentz invariant BChPT: Infrared regularization** Becher, Leutwyler '99

  Separate infrared-singular parts of the loop integrals

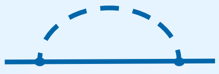
- **Manifestly Lorentz invariant BChPT: EOMS** Gegelia, Japaridze '99; Fuchs et al. '03

  Restore chiral power counting by using appropriate renormalization conditions

Chiral expansion of m_N in the HB approach

Dressed HB propagator of the nucleon: $\dfrac{i}{v \cdot k - \Sigma(k) + i\epsilon} = \dfrac{i}{v \cdot p - m - \Sigma(k) + i\epsilon}$

Up to the order Q³, the physical mass ($p =: m_N v$) is given by $m_N = m + \Sigma(0)$.

Chiral expansion of the nucleon self energy: $\Sigma(k) =$  $+$  $+ \ldots$

- $-4c_1 M^2 - \vec{k}^2/(2m)$ — Q²
- $\Sigma_{\rm loop}(k) =: -3g_A^2/(4F^2)\, I(k)$ — Q³
- $\ldots$ — Q⁴

$$I(0) = -i \int \frac{d^4 l}{(2\pi)^4} \frac{\vec{l} \cdot \vec{l}}{(l_0 + i\epsilon)(l^2 - M^2 + i\epsilon)} \xrightarrow{\text{DR}} \frac{M^3}{8\pi} \quad \Rightarrow \quad m_N = \underbrace{m - 4c_1 M^2}_{\text{not fixed by } \chi\text{-symmetry}} - \underbrace{\frac{3g_A^2}{32\pi F^2} M^3}_{\text{ChPT prediction}} + \ldots$$

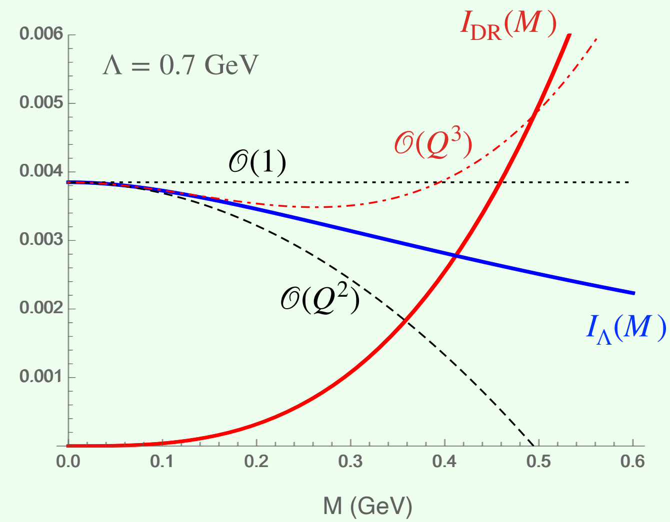
How about cutoff regularization?

Donoghue, Holstein '98

$$I_\Lambda(0) = -i \int \frac{d^4 l}{(2\pi)^4} \frac{\vec{l} \cdot \vec{l}\, e^{-\vec{l}^2/\Lambda^2}}{(l_0 + i\epsilon)(l^2 - M^2 + i\epsilon)}$$

$$= \frac{1}{16\pi^{3/2}} \left[\Lambda^3 - 2\Lambda M^2 + 2\sqrt{\pi} M^3\, e^{\frac{M^2}{\Lambda^2}}\, \text{erfc}\left(\frac{M}{\Lambda}\right) \right]$$

$$\xrightarrow{\Lambda \to \infty} \underbrace{\alpha \Lambda^3 + \beta \Lambda M^2}_{\text{can be absorbed in } m(\Lambda),\, c_1(\Lambda)} + \frac{M^3}{8\pi} + \mathcal{O}(\Lambda^{-1})$$



Pion-nucleon scattering

Effective chiral Lagrangian:

low-energy constants

$$\mathcal{L}_\pi = \mathcal{L}_\pi^{(2)} + \mathcal{L}_\pi^{(4)} + \ldots$$

$$\mathcal{L}_{\pi N} = \underbrace{\bar{N}\left(i\gamma^\mu D_\mu[\pi] - m + \frac{g_A}{2}\gamma^\mu\gamma_5 u_\mu[\pi]\right)N}_{\mathcal{L}_{\pi N}^{(1)}} + \underbrace{\sum_i \boldsymbol{c}_i \bar{N}\hat{O}_i^{(2)}[\pi]N}_{\mathcal{L}_{\pi N}^{(2)}} + \underbrace{\sum_i \boldsymbol{d}_i \bar{N}\hat{O}_i^{(3)}[\pi]N}_{\mathcal{L}_{\pi N}^{(3)}} + \underbrace{\sum_i \boldsymbol{e}_i \bar{N}\hat{O}_i^{(4)}[\pi]N}_{\mathcal{L}_{\pi N}^{(4)}} + \ldots$$

Pion-nucleon scattering amplitude for $\pi^a(q_1) + N(p_1) \to \pi^b(q_2) + N(p_2)$:

$$T_{\pi N}^{ba} = \frac{E+m}{2m}\left(\delta^{ba}\left[g^+(\omega,t) + i\vec{\sigma}\cdot\vec{q}_2\times\vec{q}_1\, h^+(\omega,t)\right] + i\epsilon^{bac}\tau^c\left[g^-(\omega,t) + i\vec{\sigma}\cdot\vec{q}_2\times\vec{q}_1\, h^-(\omega,t)\right]\right)$$

calculated within the chiral expansion

Relevant LECs (in GeV^{-n}) extracted from πN scattering

	c_1	c_2	c_3	c_4	$\bar{d}_1+\bar{d}_2$	$\bar{d}_3$	$\bar{d}_5$	$\bar{d}_{14}-\bar{d}_{15}$	$\bar{e}_{14}$	$\bar{e}_{17}$	
$[Q^4]_{\rm HB,NN}$, GW PWA	−1.13	3.69	−5.51	3.71	5.57	−5.35	0.02	−10.26	1.75	−0.58	Krebs, Gasparyan, EE, PRC85 (12) 054006
$[Q^4]_{\rm HB,NN}$, KH PWA	−0.75	3.49	−4.77	3.34	6.21	−6.83	0.78	−12.02	1.52	−0.37	
$[Q^4]_{\rm HB,NN}$, Roy-Steiner	−1.10	3.57	−5.54	4.17	6.18	−8.91	0.86	−12.18	1.18	−0.18	Hoferichter et al., PRL 115 (15) 092301
$[Q^4]_{\rm covariant}$, data	−0.82	3.56	−4.59	3.44	5.43	−4.58	−0.40	−9.94	−0.63	−0.90	Siemens et al., PRC94 (16) 014620

With the LECs taken from πN, the long-range NN force is completely fixed (parameter-free)

ChPT versus Multipole Expansion

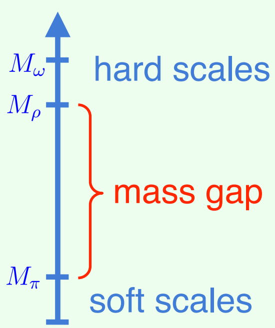
Chiral Perturbation Theory

- Most general effective Lagrangian for pions [and matter fields], chiral symmetry!

$$\mathcal{L}_\pi^{(2)} = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger + 2B(\mathcal{M}U + \mathcal{M}U^\dagger)\rangle\,,$$
$$\mathcal{L}_\pi^{(4)} = \frac{l_1}{4}\langle \partial_\mu U \partial^\mu U^\dagger\rangle^2 + \frac{l_2}{4}\langle \partial_\mu U \partial_\nu U^\dagger\rangle\langle \partial^\mu U \partial^\nu U^\dagger\rangle + \ldots$$

- The size of (ren.) LECs governed by the hard scale $\Lambda_\chi \sim 1$ GeV, LECs can be calculated (lattice-QCD) or fixed from experiment

- Separation of scales: [soft] $Q \sim M_\pi \ll \Lambda_\chi \sim M_\rho$ [hard]



- Chiral expansion of S-matrix elements (Feynman graphs, power counting, renorm.)

$$Q = \frac{\text{momenta of particles or } M_\pi \sim 140 \text{ MeV}}{\text{breakdown scale } \Lambda_b}$$

Electric potential

Most general expression for the electric potential (rotational invariance)

LECs (multipoles) governed by the size a of $\rho(\vec{r})$, they can be calculated or determined from exp.

[soft] $1/R \ll 1/a$ [hard]

Multipole expansion for $V(\vec{R})$ in powers of a/R

Take-aways of part I

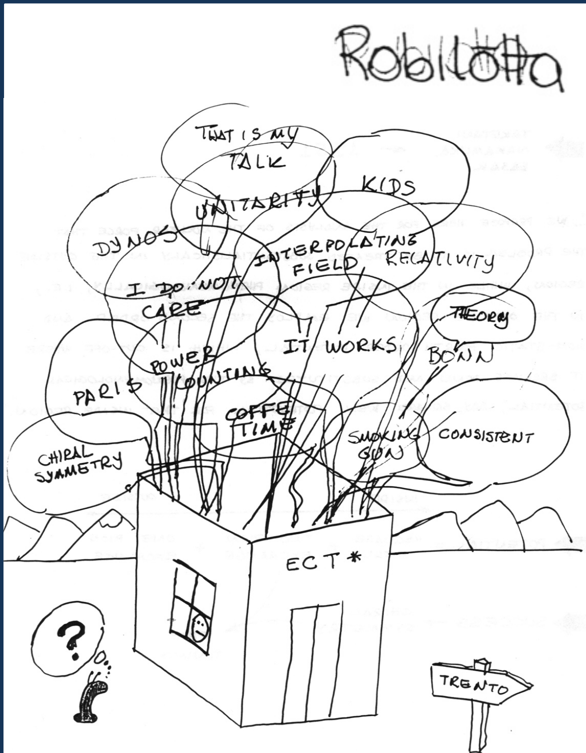
- chiral symmetry of QCD strongly constrains interactions between pions
- only irrelevant operators $\Rightarrow$ low-energy amplitudes cal-culable in perturbation theory (ChPT)
- straightforwardly generalizable to single-N processes

II: Pionless EFT for 2 nucleons

Use the simplest (analytically solvable) EFT for NN to clarify the meaning of renormalization, power counting and all that…

Outline

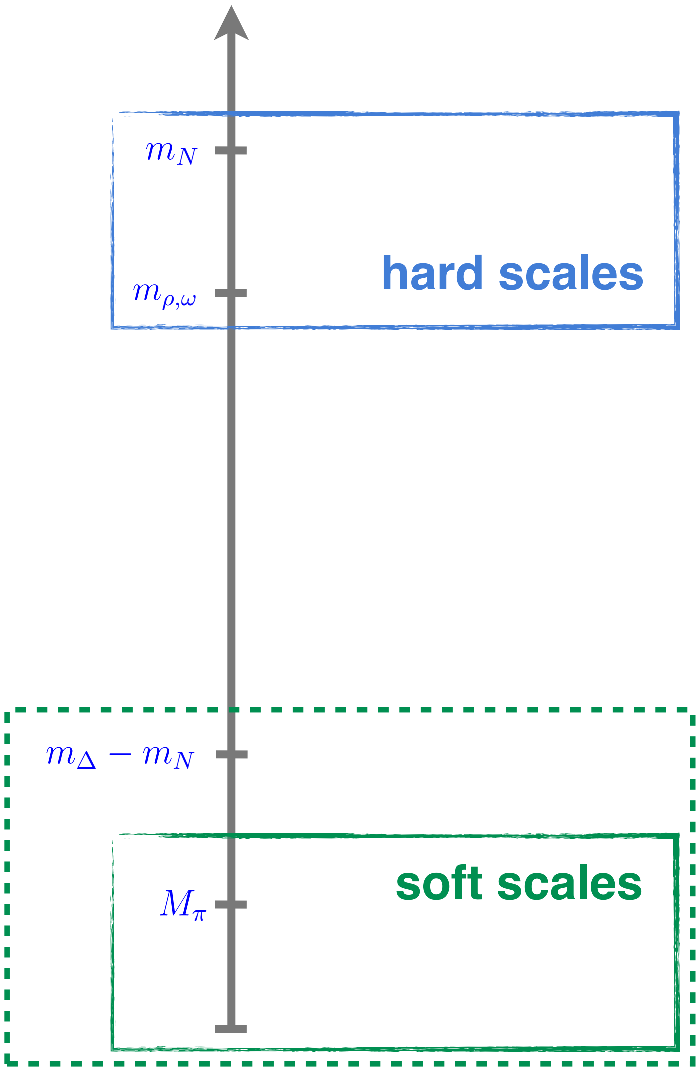
- Lippmann-Schwinger equation from Feynman diagrams
- First naive attempt and the need for fine tuning
- KSW & W power counting schemas
- Implicit renormalization



Further reading

Kaplan, Savage, Wise, NPB 478 (1996) 629
EE, Gegelia, Meißner, NPB 925 (2017) 161
EE, Gegelia, Huesmann, Meißner, FBS 62 (2021) 51

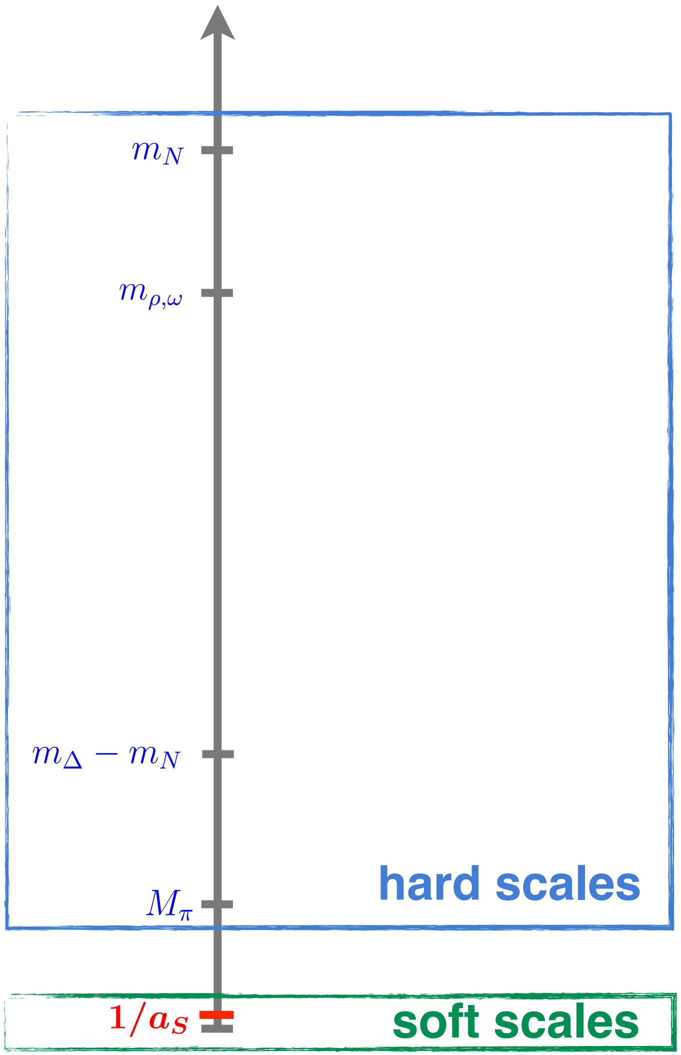
Relevant scales



- **Chiral perturbation theory** (0,1 nucleons): perturbative expansion of the amplitude in powers of

$$Q \in \left\{ \frac{M_\pi}{\Lambda}, \frac{|\vec{p}\,|}{\Lambda} \right\}, \quad \Lambda \sim m_\rho \sim 4\pi F_\pi \sim 1 \text{ GeV}$$

Relevant scales



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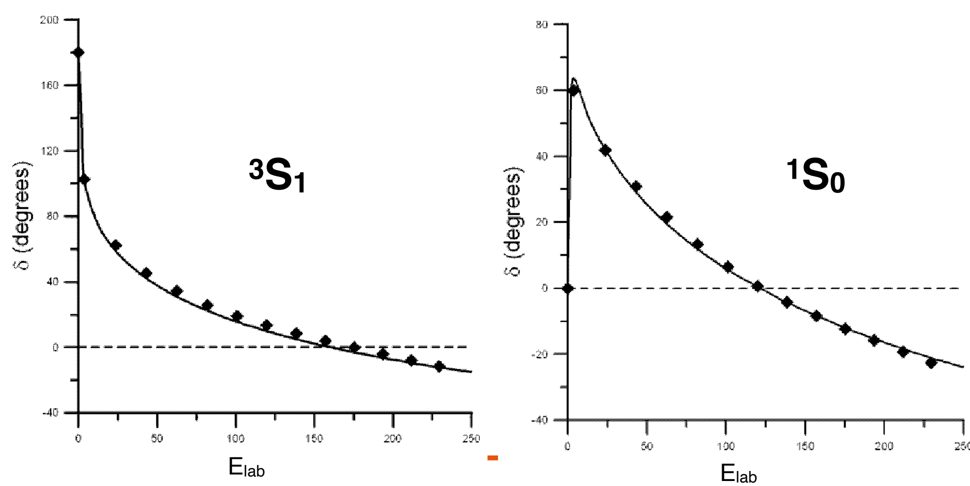
- >1 nucleons: a new very soft scale

$$1/a_S \simeq 8.5 \text{ MeV} (36 \text{ MeV})$$ in $^1S_0$ ($^3S_1$)

has to be generated dynamically ⟶ need nonperturbative resummations: **chiral EFT**



Pionless EFT

Pionless EFT Kaplan, Savage, Wise, Nucl. Phys. B478 (1996) 629

The goal: design an EFT to match ERE (no predictive power for NN beyond ERE)

DoF: nonrelativistic nucleons (use the HB formalism)

Symmetries: rotational invariance, isospin symmetry, usual discrete symmetries…

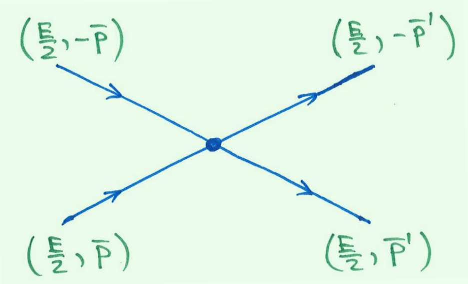
Begin with writing down the most general Lagrangian:

$$\mathcal{L} = N^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2m_N} \right) N - \frac{1}{2} C_S (N^\dagger N)^2 - \frac{1}{2} C_T (N^\dagger \vec{\sigma} N)^2 + \underbrace{\ldots}_{\text{terms with} \geq 2 \text{ derivatives}}$$

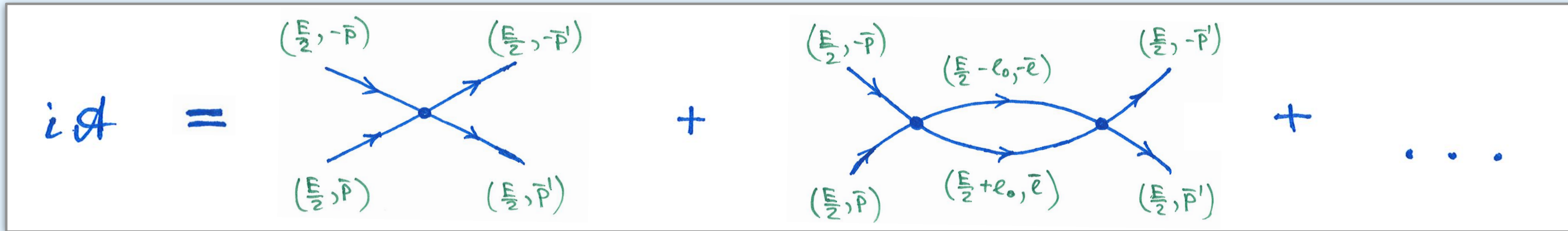
Notice: $(N^\dagger \vec{\tau} N)^2$, $(N^\dagger \vec{\tau} \vec{\sigma} N)^2$ are redundant (Pauli principle). Indeed, in there are only 2 independent s-waves ($^1S_0$ and $^3S_1$) in the isospin limit…

Feynman rule (ignore spin for the moment…):

$$i\mathcal{A}^{\text{tree}} = -i \big[\underbrace{C_0}_{\substack{\text{linear combination} \\ \text{of } C_S,\, C_T}} + C_2 (\vec{p}^{\,2} + \vec{p}^{\,\prime\,2}) + \ldots \big]$$



Pionless EFT



$$i\mathcal{A}^{1-\text{loop}} = \int \frac{d^4 l}{(2\pi)^4}(-i)\Big[C_0 + C_2(\vec{p}^{\,2} + \vec{l}^{\,2}) + \ldots\Big] \frac{i}{\frac{E}{2} + l_0 - \frac{\vec{l}^2}{2m_N} + i\epsilon} \frac{i}{\frac{E}{2} - l_0 - \frac{\vec{l}^2}{2m_N} + i\epsilon}$$

$$\times (-i)\Big[C_0 + C_2(\vec{l}^{\,2} + \vec{p}^{\,\prime\,2}) + \ldots\Big]$$

$$= (-i)\int \frac{d^3 l}{(2\pi)^3}\Big[C_0 + C_2(\vec{p}^{\,2} + \vec{l}^{\,2}) + \ldots\Big] \frac{1}{E - \frac{\vec{l}^2}{m_N} + i\epsilon}\Big[C_0 + C_2(\vec{l}^{\,2} + \vec{p}^{\,\prime\,2}) + \ldots\Big]$$

Since loop integrals factorize, the results are trivially generalizable to any number of loops. One finds for $E = p^2/m_N$:

$$\mathcal{A}(\vec{p}^{\,\prime}, \vec{p}) = V(\vec{p}^{\,\prime}, \vec{p}) - m_N \int \frac{d^3 l}{(2\pi)^3} \frac{V(\vec{p}^{\,\prime}, \vec{l})\,\mathcal{A}(\vec{l}, \vec{p})}{\vec{p}^{\,2} - \vec{l}^{\,2} + i\epsilon}$$

sign convention for V, $\mathcal{A}$

with the potential $V(\vec{p}^{\,\prime}, \vec{p}) = -(C_0 + C_2(\vec{p}^{\,2} + \vec{p}^{\,\prime 2}) + \ldots)$. As expected, the nonrelativistic treatment recovers the quantum mechanical Lippmann-Schwinger equation.

Pionless EFT

In the following, we focus on S-wave scattering. Utilizing the KSW notation,

$$i\mathcal{A}^{\text{tree}} = iV(p',p) = -i(C_0 + \underbrace{C_2(p^2+p'^2)}_{\text{terms } \sim pp' \text{ contribute to p-waves}} + \underbrace{\ldots}_{\geq 4 \text{ derivatives}})$$

The LS equation for the half-shell amplitude in the s-waves:

$$\mathcal{A}(p',p) = V(p',p) - m\int \frac{d^3l}{(2\pi)^3}\frac{V(p',l)\,\mathcal{A}(l,p)}{p^2-l^2+i\epsilon}\,, \quad \text{and} \quad S_0 = 1 + i\frac{mp}{2\pi}\mathcal{A}$$

Loop integrals are UV divergent ⇒ need regularization and renormalization…

$$m\int\frac{d^3l}{(2\pi)^3}\frac{l^{2n}}{p^2-l^2+i\epsilon} = \underbrace{-m\int\frac{d^3l}{(2\pi)^3}l^{2n-2}}_{=:\,I_{2n+1}} - \ldots - \underbrace{mp^{2n-2}\int\frac{d^3l}{(2\pi)^3}}_{=:\,p^{2n-2}I_3} + \underbrace{mp^{2n}\int\frac{d^3l}{(2\pi)^3}\frac{1}{p^2-l^2+i\epsilon}}_{=:\,p^{2n}I(p)}$$

Cutoff regularization (DimReg + PDS correspond to $\mu \to \mu\pi/2$, $\mu_i = 0$):

$$I_n \to I_n^\Lambda = \frac{-m}{2\pi^2}\int_0^\Lambda dl\, l^{n-1} = \frac{-m}{2\pi^2}\int_{\mu_n}^\Lambda dl\, l^{n-1} + \frac{-m}{2\pi^2}\int_0^{\mu_n} dl\, l^{n-1} \equiv \Delta_n(\mu_n) + I_n^R(\mu_n)$$

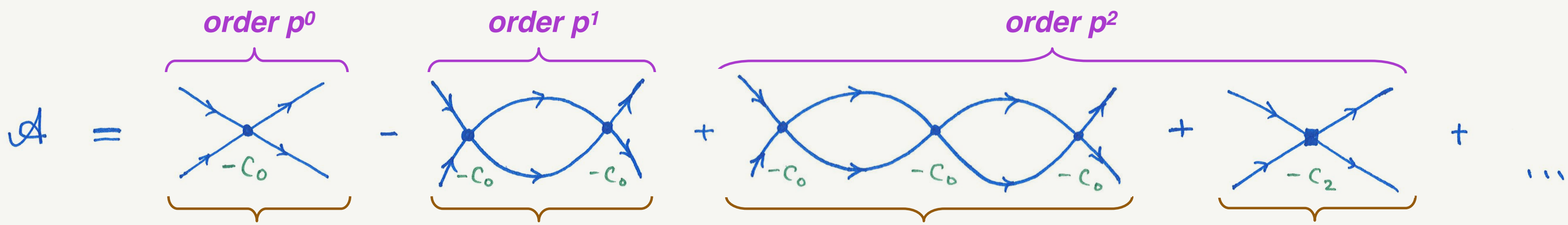
$$I(p) \to I^\Lambda(p) = \frac{m}{2\pi^2}\int_0^\Lambda dl\,\frac{l^2}{p^2-l^2+i\epsilon} = \Delta_1(\mu) + I_1^R(\mu) - \frac{i\,m\,p}{4\pi} - \frac{mp}{4\pi^2}\ln\frac{\Lambda-p}{\Lambda+p}$$

Pionless EFT

We still need to specify renormalization conditions (= choice of subtraction scales).

Conventional wisdom suggests: $\mu,\ \mu_i \sim$ soft scale $\sim p \ll M_\pi$ [i.e., all loop momenta are of the order of the soft scale after renormalization…]

The amplitude at 2 loops assuming NDA scaling of LECs:



$$-C_0 \qquad \hbar\,(-C_0)^2\,I(p) \qquad \hbar^2\,(-C_0)^3\,\big(I(p)\big)^2 \qquad (-C_2)2p^2$$

Recall: $$I(p) = \underbrace{\lim_{\Lambda\to\infty}\left(\frac{m}{2\pi^2}(\mu-\Lambda)\right)}_{=\,\Delta(\mu)} \underbrace{-\ \frac{m}{4\pi}\left(ip+\frac{2}{\pi}\mu\right)}_{=\,I^{\rm R}(\mu,p)}$$

Renormalization: $$C_0 = C_0^{\rm R}(\mu) + \hbar\,\underbrace{\delta C_{0,1}}_{-(C_0^{\rm R})^2\,\Delta} + \hbar^2\,\underbrace{\delta C_{0,2}}_{(C_0^{\rm R})^3\,\Delta^2} + \mathcal{O}(\hbar^3)$$

Thus, finally: $$\boxed{\mathcal{A} = -C_0^{\rm R}(\mu) - \big(C_0^{\rm R}(\mu)\big)^2 I^R(\mu,p) - \big(C_0^{\rm R}(\mu)\big)^3\big(I^R(\mu,p)\big)^2 - 2C_2p^2 + \ldots}$$

[If all c.t. are included, renormalization amounts to replacing $C_i \to C_i^{\rm R}(\mu)$, $I_{[n]} \to I_{[n]}^{\rm R}(\mu)$.]

Pionless EFT

To determine LECs, we have to match the amplitude to the ERE:

$$\mathcal{A} = \frac{4\pi}{m}\frac{1}{p\cot\delta - ip} = \frac{4\pi}{m}\frac{1}{\left[-\frac{1}{a} + \frac{1}{2}rp^2 + v_2p^4 + \ldots\right] - ip}$$

Such matching is possible provided $a, r, v_i \sim \mathcal{O}(1)$. One then has:

$$\mathcal{A} = \frac{4\pi}{m}\left(-a + ia^2p + a^3p^2 - \frac{a^2r}{2}p^2 + \ldots\right) \overset{!}{=} -C_0^{\rm R} - C_0^{\rm R\,2}\underbrace{I^R(\mu,p)}_{-m/(4\pi)(2\mu/\pi + ip)} - C_0^{\rm R\,3}\left(I^R(\mu,p)\right)^2 - 2C_2p^2 + \ldots$$

$$\rightarrow \begin{cases} C_0^{\rm R} = \dfrac{4\pi a}{m}\left[1 + \mathcal{O}(a\mu)\right] \\ C_2 = \dfrac{\pi a^2}{m} r \end{cases}$$

[Choosing $\mu = 0$, one reproduces exactly the first four terms in the expansion of $\mathcal{A}$ …]

However, in reality, the scattering lengths are large:

$$a_{^1S_0} = -23.714 \text{ fm} \sim -16.6\, M_\pi^{-1}, \qquad a_{^3S_1} = 5.42 \text{ fm} \sim 3.8\, M_\pi^{-1}$$

Thus, it seems more appropriate to count $a \sim p^{-1}$. This leads to the expansion:

$$\mathcal{A} = -\frac{4\pi}{m}\Bigg[\underbrace{\frac{1}{a^{-1} + ip}}_{\text{order } p^{-1}} + \underbrace{\frac{rp^2}{2(a^{-1} + ip)^2}}_{\text{order } p^{0}} + \underbrace{\frac{r^2p^4}{4(a^{-1} + ip)^3}}_{\text{order } p} + \ldots\Bigg]$$

Pionless EFT

The large scattering length signals non-perturbative physics. In order to accommodate for it, some fine tuning must be built in to the EFT.

The resulting power counting depends on the choice of renormalization conditions!
[for details see EE, Gegelia, Meißner, Nucl. Phys. B925 (2017) 161]

Consider a general expansion for the potential: $V = V^{\rm LO} + V^{\rm NLO} + V^{\rm N^2LO} + \ldots$

Want to assign powers of p to match: $\mathcal{A} = \mathcal{A}^{(-1)} + \mathcal{A}^{(0)} + \mathcal{A}^{(1)} + \ldots$

LS equation: $\hat{\mathcal{A}}^{(-1)} = \hat{V}^{\rm LO} - \hat{V}^{\rm LO}\hat{G}_0\hat{\mathcal{A}}^{(-1)} \;\longrightarrow\; 1 + \hat{V}^{\rm LO}\hat{G}_0 = \hat{V}^{\rm LO}\,[\hat{\mathcal{A}}^{(-1)}]^{-1}$

Let: $\hat{V}^{\rm LO} \sim \mathcal{O}(p^x) \;\longrightarrow\; 1 + \hat{V}^{\rm LO}\hat{G}_0 \sim \mathcal{O}(p^{1+x}) \;\longrightarrow\; \begin{cases} \hat{G}_0 \sim \mathcal{O}(p), & x \le -1 \\ \hat{G}_0 \sim \mathcal{O}(p^{-x}), & x \ge -1 \end{cases}$

A desired scaling of $\hat{G}_0$ can be realized by choosing the renormalization conditions:

- **Weinberg:** $\mu \sim \mathcal{O}(1),\;\; \mu_i \sim \mathcal{O}(p) \;\longrightarrow\; x = 0 \;\longrightarrow\; V^{\rm LO}_{\rm Weinberg} \sim \mathcal{O}(1)$
- **KSW:** $\mu, \mu_i \sim \mathcal{O}(p) \;\longrightarrow\; x = -1 \;\longrightarrow\; V^{\rm LO}_{\rm KSW} \sim \mathcal{O}(p^{-1})$

Scaling of $V^{\rm NLO}$ can be read off from:

$$\hat{\mathcal{A}}^{(0)} = \hat{V}^{\rm NLO} - \hat{V}^{\rm NLO}\hat{G}_0\hat{\mathcal{A}}^{(-1)} - \hat{\mathcal{A}}^{(-1)}\hat{G}_0\hat{V}^{\rm NLO} + \hat{\mathcal{A}}^{(-1)}\hat{G}_0\hat{V}^{\rm NLO}\hat{G}_0\hat{\mathcal{A}}^{(-1)} \longrightarrow \begin{cases} V^{\rm NLO}_{\rm Weinberg} \sim \mathcal{O}(p^2) \\ V^{\rm NLO}_{\rm KSW} \sim \mathcal{O}(1) \end{cases}$$

Pionless EFT

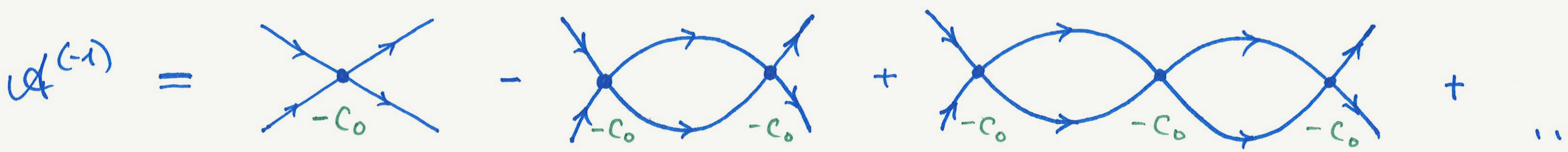
Both choices of the renormalization conditions

— lead to self-consistent approaches

— are equivalent for pionless EFT but lead to different EFTs with pions

— involve some fine tuning beyond NDA [see: EE, Gegelia, Meißner, NPB 925 (2017) 161]

Leading order (p^{-1}):



$$\mathcal{A}^{(-1)} = -C_0 - C_0^2 I(p) - C_0^3 (I(p))^2 + \ldots = -\frac{1}{C_0^{-1} - I(p)} = -\frac{1}{(C_0^{\mathrm{R}}(\mu))^{-1} - I^{\mathrm{R}}(\mu, p)}$$

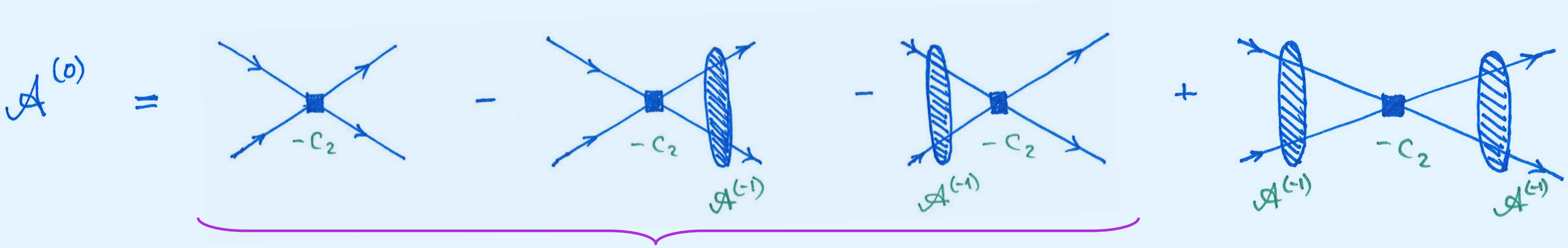
$$= -\frac{4\pi}{m}\frac{1}{\frac{4\pi}{m}(C_0^{\mathrm{R}}(\mu))^{-1} + \frac{2}{\pi}\mu + ip} \overset{!}{=} -\frac{4\pi}{m}\frac{1}{a^{-1} + ip} \quad \longrightarrow \quad C_0^{\mathrm{R}} = \frac{4\pi}{m}\frac{1}{a^{-1} - \frac{2}{\pi}\mu}$$

One recovers the Weinberg/KSW scaling of C_0^{R} depending on the choice of μ :

$C_0^{\mathrm{R}} \sim \mathcal{O}(1)$ for $\mu \sim \mathcal{O}(1)$; $\quad C_0^{\mathrm{R}} \sim \mathcal{O}(p^{-1})$ for $\mu \sim \mathcal{O}(p)$.

Pionless EFT

Subleading order (p^0):



The subleading amplitude (including terms suppressed in W. scheme) reads:

$$\mathcal{A}^{(0)} = -2C_2p^2 - 2\Big(-C_2(p^2I(p) + \underbrace{J_1(p)}_{=-I_3+p^2I(p)})\Big)\mathcal{A}^{(-1)} - 2C_2J_1(p)I(p)\Big(\mathcal{A}^{(-1)}\Big)^2$$

For the sake of simplicity, choose $\mu_3 = 0$ (so that $I_3^{\rm R} = 0$).

After renormalization ($I(p) \to I^{\rm R}(\mu, p), \;\; C_2 \to C_2^{\rm R}(\mu)$), one finds:

$$\mathcal{A}^{(0)} = -2\,C_2^{\rm R}\,p^2\,\frac{\left(a^{-1} - \frac{2}{\pi}\mu\right)^2}{(a^{-1}+ip)^2} \overset{!}{=} -\frac{4\pi}{m}\,r\,p^2\,\frac{1}{2(a^{-1}+ip)^2} \;\longrightarrow\; C_2^{\rm R} = \frac{\pi}{m}\,\frac{r}{\left(a^{-1} - \frac{2}{\pi}\mu\right)^2}$$

Again, we recover: $C_2^{\rm R} \sim \mathcal{O}(1)$ for $\mu \sim \mathcal{O}(1)$; $\quad C_2^{\rm R} \sim \mathcal{O}(p^{-2})$ for $\mu \sim \mathcal{O}(p)$.

Pionless EFT: Implicit renormalization

In EFT with non-perturbative pions, the amplitude cannot be calculated analytically
$\Rightarrow$ renormalization has to be performed implicitly.

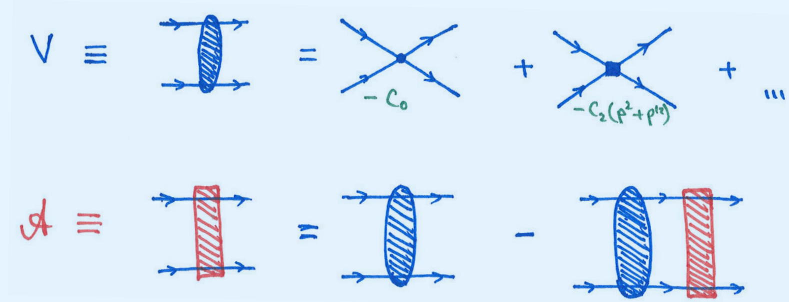
„The theory is fully specified by the values of the bare constants once a suitable regularization procedure is chosen. In principle, the renormalization program is straightforward: one calculates quantities of physical interest in terms of the bare parameters at given, large value of (ultraviolet cutoff) Λ. Once a sufficient number of physical quantities have been determined as functions of the bare parameters one inverts the result and expresses the bare parameters in terms of physical quantities, always working at some given, large value of Λ. Finally, one uses these expressions to eliminate the bare parameters in all other quantities of physical interest.“

Gasser, Leutwyler, Phys. Rep. **87** (1982) 77

Let’s see how this works in pionless EFT…

Pionless EFT: Implicit renormalization

Define the contact potential; introduce a UV cutoff $\Lambda \sim \Lambda_b \sim M_\pi$; solve the LS equation; tune **bare** LECs $C_0(\Lambda)$, $C_2(\Lambda)$ to a, r.



For a sharp cutoff, one finds at NLO:

$$mC_0 = \frac{6\pi^2\left(\beta - 6\sqrt{3}\sqrt{\alpha(\pi - 2a\Lambda)^2}\right)}{5\alpha\Lambda}, \qquad mC_2 = \frac{6\pi^2\left(\sqrt{3}\sqrt{\alpha(\pi - 2a\Lambda)^2} - \alpha\right)}{\alpha\Lambda^3}$$

where I have introduced:

$$\alpha \equiv 16a^2\Lambda^2 - \pi a\Lambda\left(a\Lambda^2 r + 12\right) + 3\pi^2, \qquad \beta \equiv 64a^2\Lambda^2 - \pi a\Lambda\left(3a\Lambda^2 r + 62\right) + 18\pi^2$$

Implicitly renormalized expression for the inverse amplitude:

$$\frac{4\pi}{m}\frac{1}{\mathcal{A}(p)} = \left[-\frac{1}{a} + \frac{1}{2}rp^2 + \frac{\pi\left(8 - 3a\Lambda^2 r(\pi\Lambda r - 8)\right) - 64a\Lambda}{12\pi\Lambda^3(\pi - 2a\Lambda)}p^4 + \mathcal{O}(p^6)\right] - ip$$

⇒ Well-defined and correct (up to higher-order terms) result for $\Lambda \sim r^{-1} \sim M_\pi$. $\Delta(\mu)$
However, things may (and, in general, would) go wrong if choosing $\Lambda \gg r^{-1}$
(complex C_i's, Wigner bound, peratization…)

Pionless EFT: Implicit renormalization

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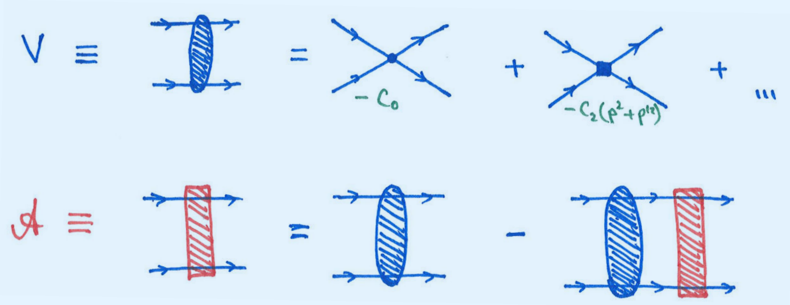

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To summarize:

- Contrary to the previous cases, not all c.t. needed to remove UV divergences are included $\Rightarrow$ it is not legitimate to take the limit $\Lambda \to \infty$.
- Higher-order terms are indeed small provided Λ ~ hard scale (NDA…).
- Implicit renormalization (i.e. no explicit splitting of C_i into $C_i^{\mathrm{R}}(\mu)$ and $\Delta(\mu)$).
- Bare LECs $C_i(\Lambda)$ must be re-fitted at every order.

Take-aways of part II

— the appearance of shallow NN states signals fine tuning (non-perturbative physics) that must be built in to an EFT

— different choices of renormalization conditions in pionless EFT lead to different power countings (KSW vs W)

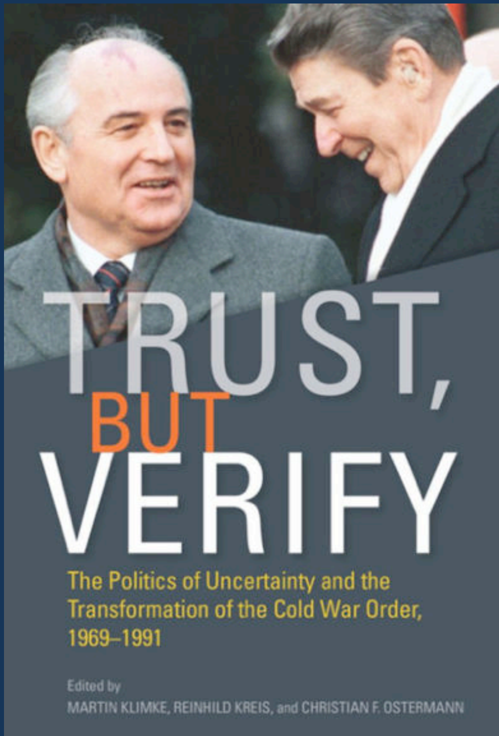
— renormalization can also be carried out implicitly by tuning bare LECs to experimental data

III: Inclusion of pions

Are pions perturbative? How to test the long-range dynamics?
What is the predictive power of chiral EFT for few N's?

Outline

- Low-energy theorems (LETs) and the modified ERE
- KSW with perturbative pions
- Non-perturbative inclusion of pions



Further reading

van Haeringen, Kok, PRA 26 (1982) 1218

Lepage, *How to renormalize the Schrödinger equation*, nucl-th/9607029

Kaplan, Savage, Wise, NPB 534 (1998) 329

Cohen, Hansen, PRC 59 (1999) 13, 3047

Fleming, Mehen, Stewart, NPA 677 (2000) 313

EE, Gegelia, EPJA 41 (2009) 341

EE, *Nuclear forces from chiral EFT: A primer*, arXiv:1001.3229 [nucl-th]

Modified Effective Range Expansion (MERE)

LETs and the MERE

What are the low-energy theorems?

Two-range potential: $V(r) = V_L(r) + V_S(r)$

with $M_L^{-1} \gg M_H^{-1}$

- $F_l(k^2)$ is meromorphic in $|k| < M_L/2$

- $$F_l^M(k^2) \equiv M_l^L(k) + \frac{k^{2l+1}}{|f_l^L(k)|^2} \cot\left[\delta_l(k) - \delta_l^L(k)\right]$$

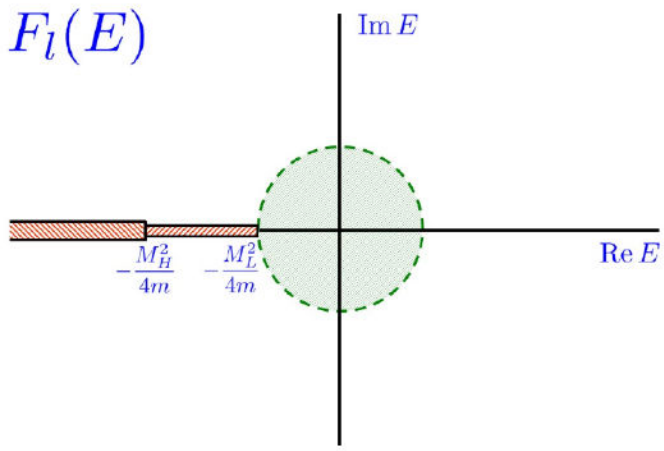
← modified effective range function
van Haeringen, Kok '82

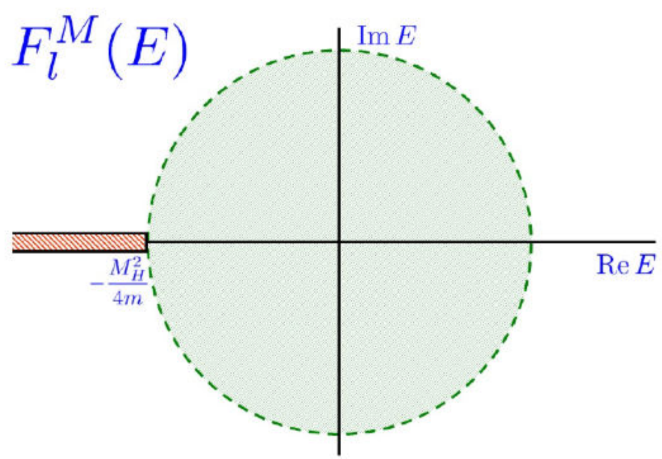
$$f_l^L(k) = \lim_{r\to 0}\left(\frac{l!}{(2l)!}(-2ikr)^l f_l^L(k,r)\right)$$

Jost function for $V_L(r)$ — Jost solution for $V_L(r)$

$$M_l^L(k) = Re\left[\frac{(-ik/2)^l}{l!}\lim_{r\to 0}\left(\frac{d^{2l+1}}{dr^{2l+1}}\frac{r^l f_l^L(k,r)}{f_l^L(k)}\right)\right]$$

Per construction, F_l^M reduces to F_l for $V_L = 0$ and is meromorphic in $|k| < M_H/2$





MERE and low-energy theorems

Example: proton-proton scattering

$$F_C(k^2) = C_0^2(\eta)\, k\, \cot[\delta(k) - \delta^C(k)] + 2k\,\eta\, h(\eta) = -\frac{1}{a^M} + \frac{1}{2} r^M k^2 + v_2^M k^4 + \ldots$$

where $\underbrace{\delta^C \equiv \arg \Gamma(1 + i\eta)}_{\text{Coulomb phase shift}}$, $\eta = \frac{m}{2k}\alpha$, $\underbrace{C_0^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta} - 1}}_{\text{Sommerfeld factor}}$, $h(\eta) = \text{Re}\Big[\underbrace{\Psi(i\eta)}_{\text{Digamma function}}\Big] - \ln(\eta)$, $\Psi(z) \equiv \Gamma'(z)/\Gamma(z)$

MERE and low-energy theorems

Long-range forces impose correlations between the ER coefficients (low-energy theorems)

Cohen, Hansen '99; Steele, Furnstahl '00

The emergence of the LETs can be understood in the framework of MERE:

$$F_l^M(k^2) \equiv M_l^L(k) + \frac{k^{2l+1}}{|f_l^L(k)|^2} \cot\left[\delta_l(k) - \delta_l^L(k)\right]$$

$F_l^M(k^2)$: meromorphic for $k^2 < (M_H/2)^2$

$M_l^L(k)$, $|f_l^L(k)|^2$, $\delta_l^L(k)$: can be computed if the long-range force is known

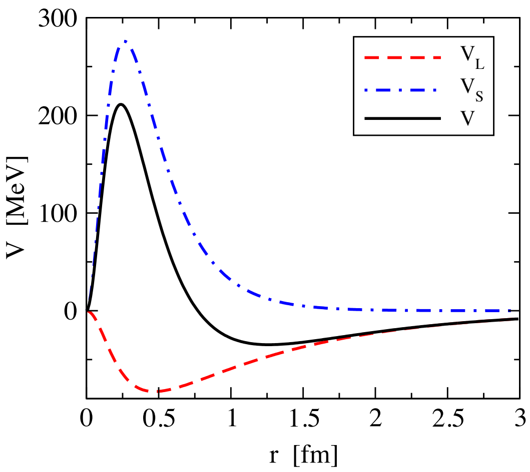
- approximate $F_l^M(k^2)$ by first 1,2,3,… terms in the Taylor expansion in k^2
- calculate all "soft" quantities
- reconstruct $\delta_l^L(k)$ and predict all coefficients in the ERE

Toy model: Low-energy theorems

$$V(r) = \underbrace{v_L e^{-M_L r} f(r)}_{V_L} + \underbrace{v_H e^{-M_H r} f(r)}_{V_H}$$

where $f(r) = \frac{(M_H r)^2}{1+(M_H r)^2}$

and $M_L = 1.0\,,\; v_L = -0.875\,,\; M_H = 3.75\,,\; v_H = 7.5$ (all in fm^{-1})



ERE and MERE

	a	r	v_2	v_3	v_4
F_0 [fm^n]	5.458	2.432	0.113	0.515	−0.993
F_0^M [M_S^{-n}]	1.710	−1.063	−0.434	−0.680	2.624

for an analytic example, see EE, Gegelia, EPJ A41 (2009) 341

Low-Energy Theorems

	LO	NLO	NNLO	"Exp"
r				2.432197161
v_2				0.112815751
v_3				0.51529
v_4				−0.9928

Toy model: Low-energy theorems

$$V(r) = \underbrace{v_L e^{-M_L r} f(r)}_{V_L} +$$

where $f(r) = \frac{(M_H r)^2}{1 + (M_H r)^2}$

and $M_L = 1.0\,,\; v_L = -0.875\,,$

ERE and MERE

	a	r	v_2	v_3	v_4
F_0 [fm^n]	5.458	2.432	0.113	0.515	-0.993
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for an analytic example, see EE, Gegelia, EPJ A41 (2009) 341

Low-Energy Theorems

	LO	NLO	NNLO	"Exp"
r	2.447(38)			2.432197161
v_2	0.12(11)			0.112815751
v_3	0.61(12)			0.51529
v_4	$-0.95(5)$			-0.9928

Toy model: Low-energy theorems

$$V(r) = \underbrace{v_L e^{-M_L r} f(r)}_{V_L} \;+$$

where $f(r) = \frac{(M_H r)^2}{1 + (M_H r)^2}$

and $M_L = 1.0\,,\; v_L = -0.875\,,$

ERE and MERE

	a	r	v_2	v_3	v_4
F_0 [fm^n]	5.458	2.432	0.113	0.515	−0.993
F_0^M [M_S^{-n}]	1.710	−1.063	−0.434	−0.680	2.624

for an analytic example, see EE, Gegelia, EPJ A41 (2009) 341

Low-Energy Theorems

	LO	NLO	NNLO	"Exp"
r	2.447(38)	2.432197161		2.432197161
v_2	0.12(11)	0.1132(29)		0.112815751
v_3	0.61(12)	0.517(16)		0.51529
v_4	−0.95(5)	−0.991(14)		−0.9928

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v_3	0.61(12)	0.517(16)	0.51533(20)	0.51529
v_4	−0.95(5)	−0.991(14)	−0.9925(11)	−0.9928

Chiral EFT for NN scattering

KSW with perturbative pions

Recall the differences between the W and KSW counting schemes:

- **Weinberg:** $\mu \sim \mathcal{O}(1), \quad \mu_i \sim \mathcal{O}(p)$ → $V^{\rm LO}_{\rm Weinberg} \sim \mathcal{O}(1), \quad V^{\rm NLO}_{\rm Weinberg} \sim \mathcal{O}(p^2)$

  [i.e. scaling of C_{2n} according to NDA (~ O(1))]

- **KSW:** $\mu, \mu_i \sim \mathcal{O}(p)$ → $V^{\rm LO}_{\rm KSW} \sim \mathcal{O}(p^{-1}), \quad V^{\rm NLO}_{\rm KSW} \sim \mathcal{O}(1)$

  [i.e. scaling of C_{2n} as $C_{2n} \sim O(p^{-1-n})$]

While the two schemes are equivalent for pionless theory, they suggest different scenarios for pionful (chiral) EFT:

$$V_{1\pi} = -\left(\frac{g_A}{2F_\pi}\right)^2 \frac{\vec{\sigma}_1 \cdot \vec{q}\, \vec{\sigma}_2 \cdot \vec{q}}{q^2 + M_\pi^2}\, \vec{\tau}_1 \cdot \vec{\tau}_2 \sim \mathcal{O}(1)$$

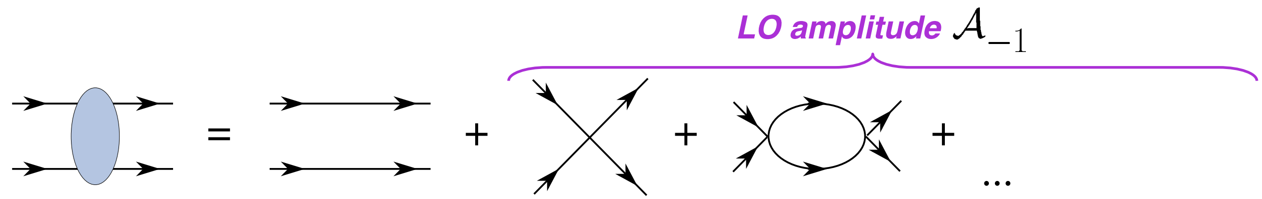
OPE is expected to be:

— LO contribution (nonperturbative) in the Weinberg scheme,

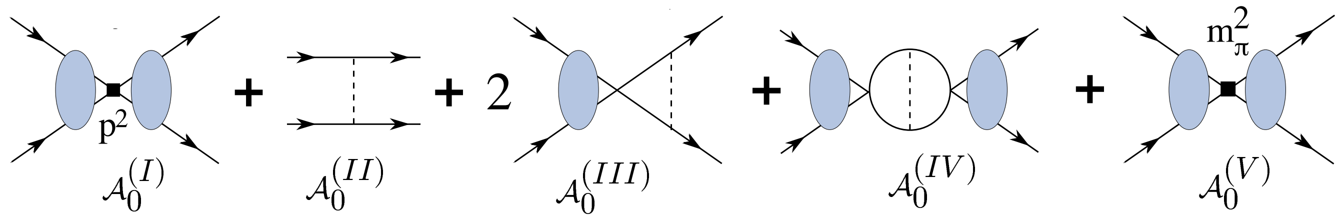
— NLO contribution (perturbative) in the KSW scheme.

Chiral EFT for NN: The KSW approach

- Leading order:



- NLO: $\mathcal{A}_0$ =



$$\mathcal{A}_0^{(I)} = -C_2^{(^1S_0)} p^2 \left[\frac{\mathcal{A}_{-1}}{C_0^{(^1S_0)}}\right]^2, \qquad \mathcal{A}_0^{(II)} = \left(\frac{g_A^2}{2f^2}\right)\left(-1 + \frac{m_\pi^2}{4p^2}\ln\left(1+\frac{4p^2}{m_\pi^2}\right)\right)$$

$f \to \sqrt{2}F_\pi$

$$\mathcal{A}_0^{(III)} = \frac{g_A^2}{f^2}\left(\frac{m_\pi M \mathcal{A}_{-1}}{4\pi}\right)\left(-\frac{(\mu+ip)}{m_\pi} + \frac{m_\pi}{2p}\left[\tan^{-1}\left(\frac{2p}{m_\pi}\right) + \frac{i}{2}\ln\left(1+\frac{4p^2}{m_\pi^2}\right)\right]\right)$$

$$\mathcal{A}_0^{(IV)} = \frac{g_A^2}{2f^2}\left(\frac{m_\pi M \mathcal{A}_{-1}}{4\pi}\right)^2\left(-\left(\frac{\mu+ip}{m_\pi}\right)^2 + \left[i\tan^{-1}\left(\frac{2p}{m_\pi}\right) - \frac{1}{2}\ln\left(\frac{m_\pi^2+4p^2}{\mu^2}\right)+1\right]\right)$$

$$\mathcal{A}_0^{(V)} = -D_2^{(^1S_0)} m_\pi^2 \left[\frac{\mathcal{A}_{-1}}{C_0^{(^1S_0)}}\right]^2$$

For more details see:
Kaplan, Savage, Wise, Nucl. Phys. B534 (1998) 329

LETs for S-waves: KSW approach

Use these results to test the LETs for S-waves: [Cohen, Hansen, PRC 59 (1999) 13]

$$p\cot\delta_0(p) = \frac{4\pi}{m}\left[\frac{1}{\mathcal{A}_{-1}} - \frac{\mathcal{A}_0}{(\mathcal{A}_{-1})^2} + \ldots\right] + ip \overset{!}{=} -\frac{1}{a} + \frac{1}{2} r p^2 + v_2 p^4 + v_3 p^6 + v_4 p^8 + \ldots$$

Express the LECs C_0, C_2, in terms of a and r to predict the shape parameters, e.g.:

$$v_2 = \frac{g_A^2 m}{16\pi F_\pi^2}\left(-\frac{16}{3a^2 M_\pi^4} + \frac{32}{5a M_\pi^3} - \frac{2}{M_\pi^2}\right), \qquad v_3 = \frac{g_A^2 m}{16\pi F_\pi^2}\left(\frac{16}{a^2 M_\pi^6} - \frac{128}{7a M_\pi^5} + \frac{16}{3M_\pi^4}\right), \ldots$$

$^1S_0$ partial wave	a [fm]	r [fm]	v_2 [fm^3]	v_3 [fm^5]	v_4 [fm^7]
NLO KSW Cohen, Hansen '99	fit	fit	-3.3	18	-108
Nijmegen PWA	-23.7	2.67	-0.5	4.0	-20

$^3S_1$ partial wave	a [fm]	r [fm]	v_2 [fm^3]	v_3 [fm^5]	v_4 [fm^7]
NLO KSW Cohen, Hansen '99	fit	fit	-0.95	4.6	-25
Nijmegen PWA	5.42	1.75	0.04	0.67	-4.0

→ large deviations suggest that pions should be treated nonperturbatively…

[even stronger evidence comes from phase shifts at N²LO, see: Fleming, Mehen, Stewart, NPA 677 (2000) 313]

Nonperturbative inclusion of pions

LO scattering amplitude:

$$T(\vec p\,', \vec p\,) = \left[V_{\rm cont}(\vec p\,', \vec p\,) + V_{1\pi}(\vec p\,', \vec p\,)\right] + m \int \frac{d^3 l}{(2\pi)^3} \frac{\left[V_{\rm cont}(\vec p\,', \vec l\,) + V_{1\pi}(\vec p\,', \vec l\,)\right] T(\vec l, \vec p\,)}{p^2 - l^2 + i\epsilon}$$

Complications (as compared to pionless theory):

— $V_{1\pi}$ is not separable, no analytic results beyond 2 loops are available,

— $1/r^3$ singularity of $V_{1\pi}$

Static OPEP in coordinate space:

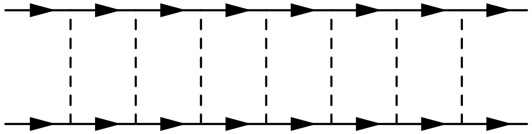
tensor operator: $S_{12} = 3\,\vec\sigma_1 \cdot \hat r\, \vec\sigma_2 \cdot \hat r - \vec\sigma_1 \cdot \vec\sigma_2$

$$V_{1\pi}(\vec r\,) = \left(\frac{g_A}{2F_\pi}\right)^2 \tau_1 \cdot \tau_2 \left[M_\pi^2 \frac{e^{-M_\pi r}}{12\pi r}\left(S_{12}(\hat r)\left(1 + \frac{3}{M_\pi r} + \frac{3}{(M_\pi r)^2}\right) + \vec\sigma_1 \cdot \vec\sigma_2\right) - \frac{1}{3}\vec\sigma_1 \cdot \vec\sigma_2\, \delta^3(r)\right]$$

singular potential in all S=1 channels
(solutions to the Schröd./LS equation still exist in repulsive cases)

⇒ Need ∞ many c.t.'s in all spin-triplet channel to remove UV divergences from iterations…

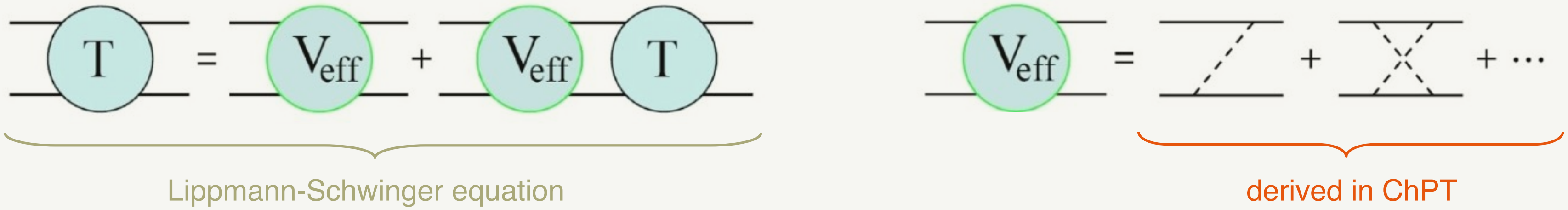
E.g.:



$$\propto \frac{1}{d-4}\, \vec p^{\,6}\, m_N^6 \quad \text{(spin-triplet)}$$

Chiral EFT for nuclear systems

Ladder graphs are responsible for the failure of perturbation theory and must be re-summed:

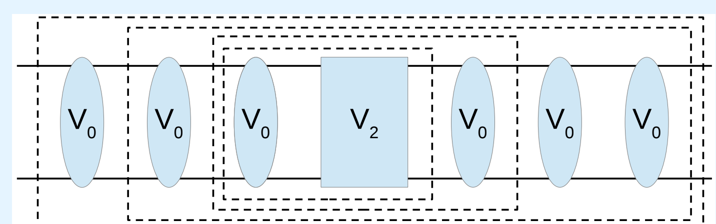


Nuclear forces and currents = irreducible parts of the amplitude (scheme-dependent)

Divergent integrals in the Lippmann-Schwinger equation are regularized using a cutoff Λ:

- the „RG invariant“ approach with $\Lambda \gg \Lambda_b$: $T \sim 1 + \Lambda + \Lambda^2 + \ldots = (1-\Lambda)^{-1}$ van Kolck, Long, Yang, …
  - — criticized in EE, Gegelia, EPJA 41 (09) 341; EE, Gasparyan, Gegelia, Meißner, EPJA 54 (18) 186
  - — not cutoff-independent RG-invariant beyond LO Gasparyan, EE, PRC 107 (23) 034001

- finite-Λ EFT with $\Lambda \lesssim \Lambda_b \sim 600$ MeV Lepage, EE, Gegelia, Meißner, Reinert, Entem, Machleidt, …
  - — phenomenologically successful; approximate Λ-independence verified a posteriori
  - — renormalizability (in the EFT sense) has been rigorously proven to NLO using the BPHZ subtraction method (forest formula) Gasparyan, EE, PRC 105 (22) 024001; PRC 107 (23) 044002



LETs for S-waves: KSW vs Weinberg

LETs for neutron-proton scattering: nonperturbative vs perturbative OPEP

	a [fm]	r [fm]	v_2 [fm^3]	v_3 [fm^5]	v_4 [fm^7]
$^1S_0$ partial wave					
LO EE, Gegelia, PLB617 (12) 338	fit	1.50	−1.9	8.6(8)	−37(10)
NLO EE et al., EPJA51 (15) 71	fit	fit	$-0.61\ldots-0.55$	$5.1\ldots5.5$	$-30.8\ldots-29.6$
NLO KSW Cohen, Hansen '98	fit	fit	−3.3	18	−108
Empirical values	−23.7	2.67	−0.5	4.0	−20

	a [fm]	r [fm]	v_2 [fm^3]	v_3 [fm^5]	v_4 [fm^7]
$^3S_1$ partial wave					
LO EE, Gegelia, PLB617 (12) 338	fit	1.60	−0.05	0.82	−5.0
NLO Baru et al., PRC94 (16) 014001	fit	fit	0.06	0.70	−4.0
NLO KSW Cohen, Hansen '98	fit	fit	−0.95	4.6	−25
Empirical values	5.42	1.75	0.04	0.67	−4.0

- perturbative inclusion of pions (KSW approach) fails
- $^1S_0$ channel: limited predictive power of the LETs due to the weakness of the OPEP; taking into account the range correction (NLO) leads to improvement
- $^3S_1$ channel: LETs work as advertised (strong tensor part of the OPEP)

Take-aways of part III

— long-range interactions govern energy dependence of the amplitude and lead to correlations between coefficients in the ERE (LETs) that can be tested

— the failure of the LETs in the KSW approach suggests that pion exchange should be treated non-perturbatively

— iterations of the 1π-exchange are non-renormalizable (in the usual sense) $\Rightarrow$ finite-cutoff formulation of chiral EFT

IV: From $\mathscr{L}_{\rm eff}$ to nuclear forces

How to derive nuclear forces from the effective Lagrangian?
What is the current state-of-the-art?

Outline

- Methods: S-matrix matching, TOPT, MUT, a path integral approach
- Example: chiral expansion of the 2π-exchange 3N force
- State-of-the-art for nuclear forces

Further reading

EE, PPNP 57 (2006) 654

EE, Hammer, Meißner, RMP 81 (2009) 1773

Entem, Machleidt, Phys. Rept. 503 (2011) 1

EE, Krebs, Reinert, Front. In Phys. 8 (2020) 98

Krebs, EE, PRC 110 (2024) 044003

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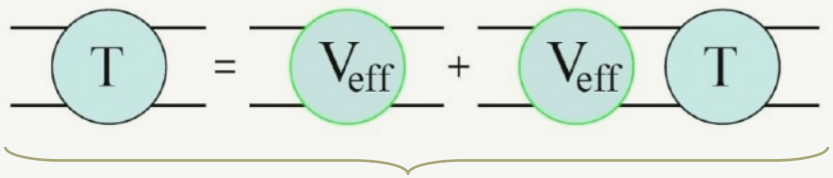
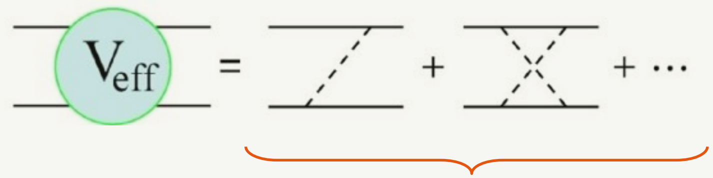


Lippmann-Schwinger equation



derived in ChPT

Nuclear forces and currents = irreducible parts of the amplitude (scheme-dependent)

They can be derived using a variety of methods including [In all cases, utilize a perturba-tive expansion within ChPT]:

— S-matrix matching Kaiser et al.

— time-ordered perturbation theory Pastore, Baroni, Schiavilla et al.

— method of unitary transformations (UTs) EE, Glöckle, Meißner, Krebs, Kölling

— path integral approach Krebs, EE

More demanding than just calculating Feynman diagrams:

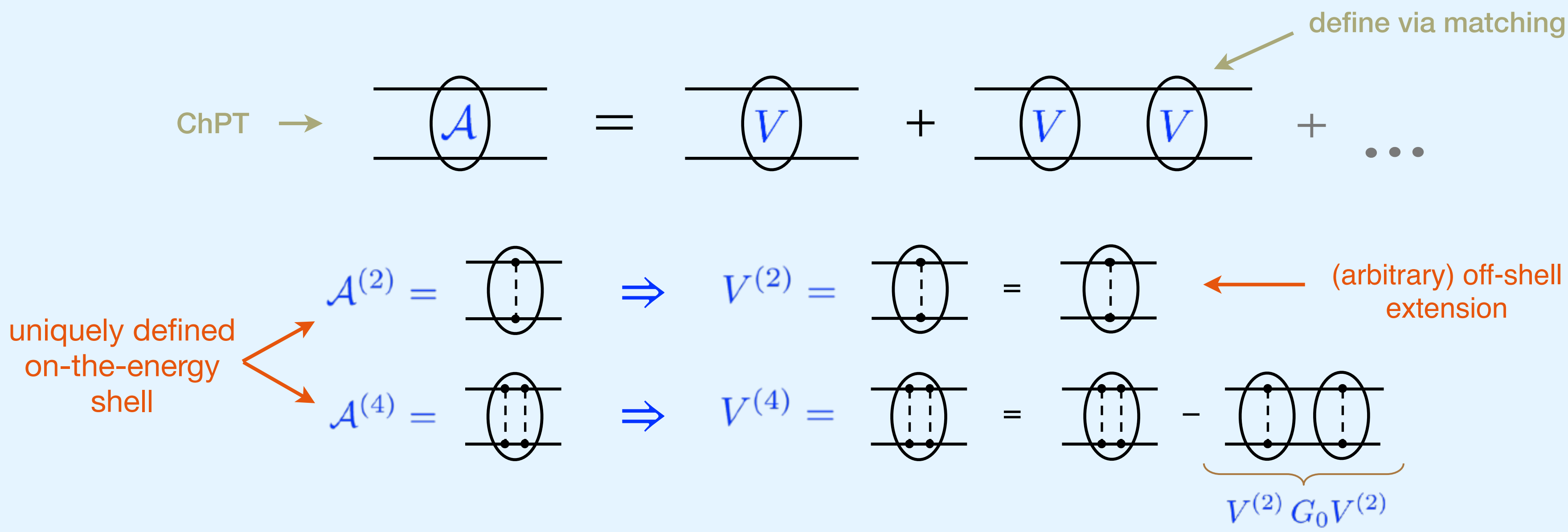
— need to subtract reducible pieces in order to avoid double counting

— have to deal with non-uniqueness of nuclear potentials

— maintaining renormalizability non-trivial…

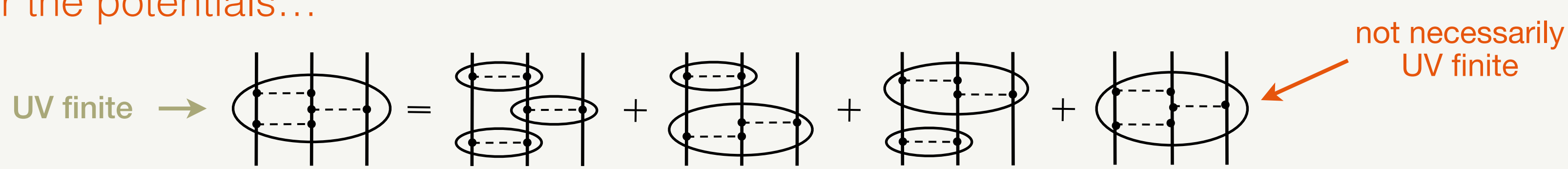
S-matrix matching

Matching to the amplitude Kaiser et al.



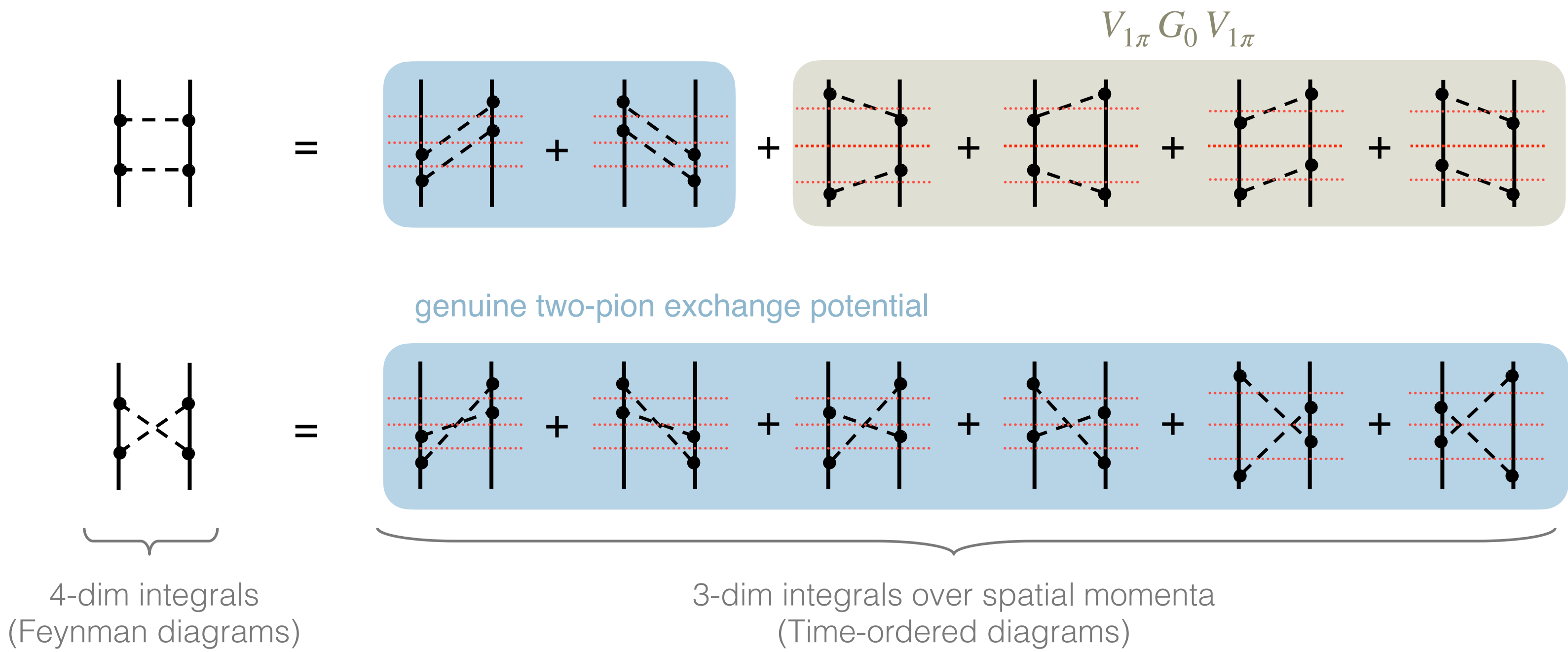
$\Rightarrow$ higher-order terms in the Hamiltonian „know" about the choice made for the off-shell extension (consistency...)

S-matrix in ChPT is renormalizable (in the EFT sense). But this should not be taken for granted for the potentials…



Renormalization can be enforced by systematically exploiting unitary ambiguities…

Time-ordered perturbation theory



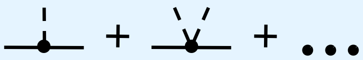
— has been used by Weinberg in his original publications

— leads to energy-dependent potentials which are inconvenient for many-body calculations (the energy dependence can be eliminated)

— changes the normalization of few-nucleon states

Method of Unitary Transformation

Taketani, Mashida, Ohnuma'52; Okubo '54; EE, Glöckle, Meißner, Krebs, Kölling, ...

- Canonical transformation and quantization: $\mathcal{L}_{\pi N} \longrightarrow \mathcal{H}_{\pi N} =$ 

nucleonic states $|N\rangle,\ |NN\rangle,\ \ldots$

EOM: $$\begin{pmatrix} \eta H \eta & \eta H \lambda \\ \lambda H \eta & \lambda H \lambda \end{pmatrix} \begin{pmatrix} |\phi\rangle \\ |\psi\rangle \end{pmatrix} = E \begin{pmatrix} |\phi\rangle \\ |\psi\rangle \end{pmatrix}$$ ⟵ can not solve (infinite-dimensional eq.)

projectors *states with mesons* $|N\pi\rangle,\ |N\pi\pi\rangle,\ \ldots$

- Decouple pions via a suitable UT: $$\tilde{H} \equiv U^\dagger \begin{pmatrix} \eta H \eta & \eta H \lambda \\ \lambda H \eta & \lambda H \lambda \end{pmatrix} U = \begin{pmatrix} \eta \tilde{H} \eta & 0 \\ 0 & \lambda \tilde{H} \lambda \end{pmatrix}$$

Minimal parametrization of the UT: $$U = \begin{pmatrix} \eta(1 + A^\dagger A)^{-1/2} & -A^\dagger(1 + AA^\dagger)^{-1/2} \\ A(1 + A^\dagger A)^{-1/2} & \lambda(1 + AA^\dagger)^{-1/2} \end{pmatrix}, \quad A = \lambda A \eta$$
Okubo '54

Require: $\eta \tilde{H} \lambda = \lambda \tilde{H} \eta = 0 \quad \Rightarrow \quad \boxed{\lambda\left(H - [A,\ H] - AHA\right)\eta = 0}$

The solution of the nonlinear decoupling equation, calculation of the UT and of the nuclear potentials is carried out in perturbation theory (chiral expansion) EE, EPJA 34 (2007) 197

Notice: Similar methods are widely used in nuclear and many-body physics (Lee-Suzuki)

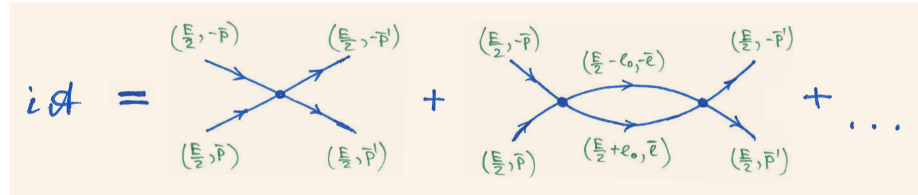
Path-integral approach

Krebs, EE, PRC 110 (2024) 044003

Pion-less EFT:

$$\mathcal{L} = N^\dagger \left[i\partial_0 + \frac{\vec{\nabla}^2}{2m_N} \right] N - \frac{C_S}{2} (N^\dagger N)^2 + \ldots$$

$$\Rightarrow \quad \mathcal{A}_{\text{tree}} = \left[C_0 + C_2 (\vec{p}^{\,2} + \vec{p}^{\,\prime 2}) + \ldots \right]$$

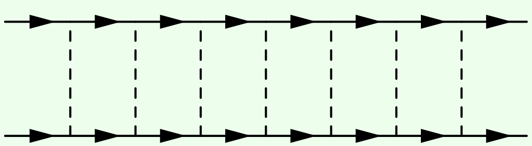


Scattering amplitude to 1 loop:

$$-i\mathcal{A}_{1-\text{loop}} = \int \frac{d^4 l}{(2\pi)^4} \left[C_0 + C_2 (\vec{p}^{\,2} + \vec{l}^{\,2}) + \ldots \right] \frac{1}{\left(\frac{E}{2} + l_0 - \frac{\vec{l}^{\,2}}{2m_N} + i\epsilon \right) \left(\frac{E}{2} - l_0 - \frac{\vec{l}^{\,2}}{2m_N} + i\epsilon \right)} \left[C_0 + \ldots \right]$$

$$= -i \int \frac{d^3 l}{(2\pi)^3} \left[C_0 + C_2 (\vec{p}^{\,2} + \vec{l}^{\,2}) + \ldots \right] \frac{1}{E - \frac{\vec{l}^{\,2}}{m_N} + i\epsilon} \left[C_0 + (\vec{l}^{\,2} + \vec{p}^{\,\prime 2}) \ldots \right]$$

All l_0-integrals factorize $\Rightarrow$ Lippmann-Schwinger eq. $\mathcal{A} = \mathcal{V} + \mathcal{V}\, G_0\, \mathcal{A}$ with $\mathcal{V} = -\mathcal{L}_{\text{int}}$



But l_0-integrals do not factorize for pions due to l_0-dependence of π-propagators…

Idea: $$Z[\eta^\dagger, \eta] = A \int \mathcal{D}N^\dagger\, \mathcal{D}N\, \mathcal{D}\pi \exp\left(iS^{\Lambda}_{\text{eff}} + i \int d^4x [\eta^\dagger N + N^\dagger \eta] \right)$$

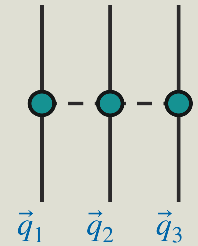
Hermann Krebs, EE, 2311.10893

nonlocal redefinitions of $N, N^\dagger$ → loops from functional determinant

$$A \int \mathcal{D}\tilde{N}^\dagger\, \mathcal{D}\tilde{N} \exp\left(iS^{\Lambda}_{\text{eff, N}} + i \int d^4x [\eta^\dagger \tilde{N} + \tilde{N}^\dagger \eta] \right)$$

instantaneous ($S^{\Lambda}_{\text{eff, N}}$)

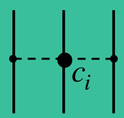
Example: 2π-exchange 3NF



$$V_{3N} = \frac{\vec{\sigma}_1 \cdot \vec{q}_1 \, \vec{\sigma}_3 \cdot \vec{q}_3}{(q_1^2 + M_\pi^2)\,(q_3^2 + M_\pi^2)} \Big[\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_3 \, \mathcal{A}(q_2) + \boldsymbol{\tau}_1 \times \boldsymbol{\tau}_3 \cdot \boldsymbol{\tau}_2 \, \vec{q}_1 \times \vec{q}_3 \cdot \vec{\sigma}_2 \, \mathcal{B}(q_2) \Big]$$

$$+ \quad \text{short-range terms} \; + \; \text{permutations}$$

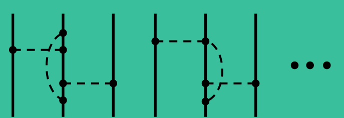
N²LO (Q³)



$$\mathcal{A}^{(3)} = \frac{g_A^2}{8F_\pi^4} \Big[(2c_3 - 4c_1) M_\pi^2 + c_3 q_2^2 \Big], \qquad \mathcal{B}^{(3)} = \frac{g_A^2 c_4}{8F_\pi^4}$$

N³LO (Q⁴)

Bernard, EE, Krebs, Meißner '08



$$\mathcal{A}^{(4)} = \frac{g_A^4}{256\pi F_\pi^6} \Big[\left(4g_A^2 + 1\right) M_\pi^3 + 2\left(g_A^2 + 1\right) M_\pi q_2^2 + A(q_2)\left(2M_\pi^4 + 5M_\pi^2 q_2^2 + 2q_2^4\right) \Big]$$

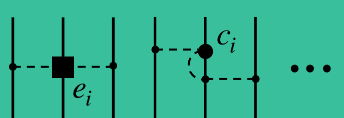
$A(q_2) = \frac{1}{2q_2} \arctan \frac{q_2}{2M_\pi}$

$$\mathcal{B}^{(4)} = -\frac{g_A^4}{256\pi F_\pi^6} \Big[A(q_2)\left(4M_\pi^2 + q_2^2\right) + (2g_A^2 + 1) M_\pi \Big]$$

calculated using DimReg

N⁴LO (Q⁵)

Krebs, Gasparyan, EE '12



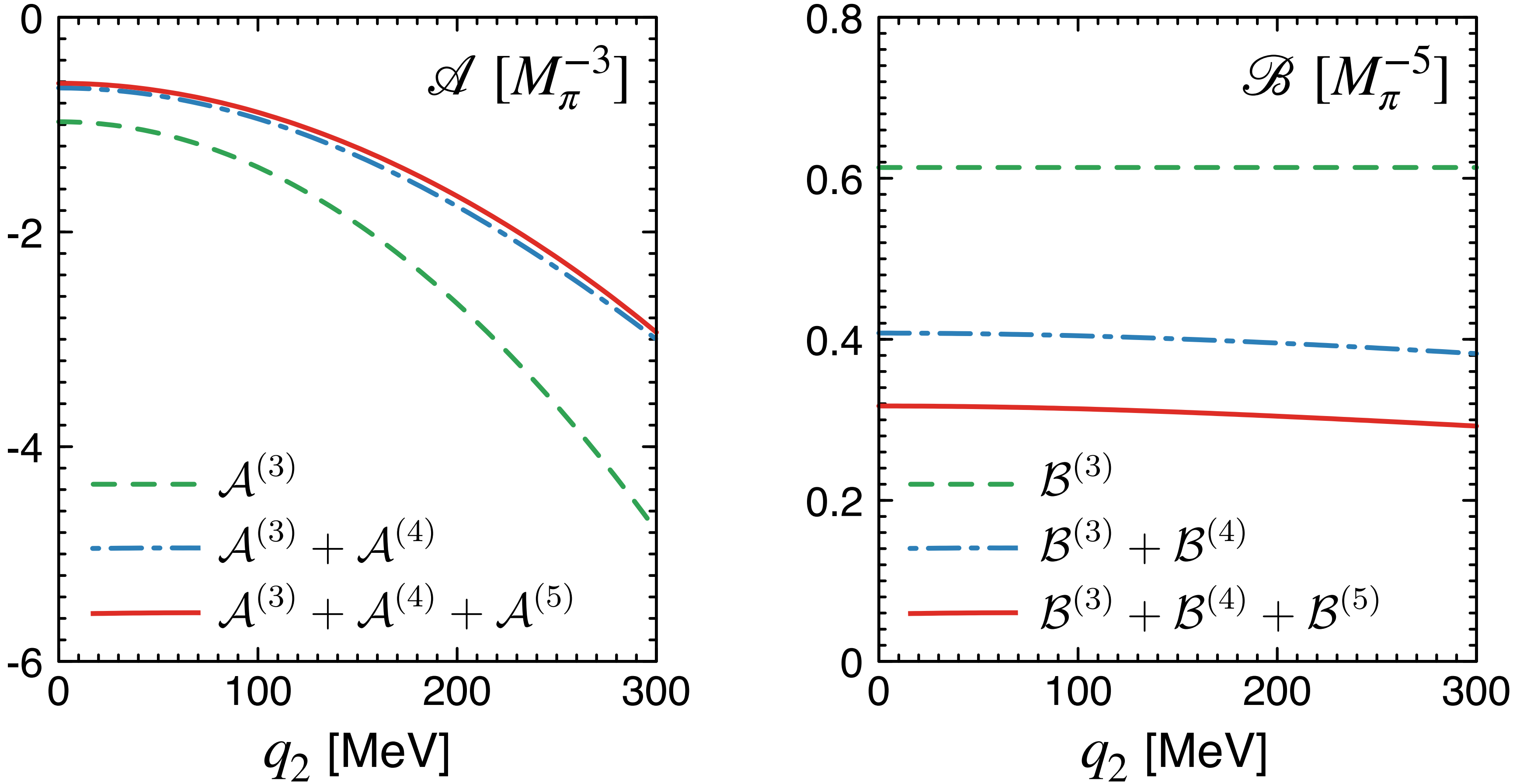
$$\mathcal{A}^{(5)} = \frac{g_A^2 \left(M_\pi^2 + 2q_2^2\right)}{4608\pi^2 F_\pi^6} \Big\{ \big[6c_1 - 2c_2 - 3c_3 - 2(6c_1 - c_2 - 3c_3) L(q_2) \big] 12 M_\pi^2 - q_2^2 \big[5c_2 + 18c_3 - 6L(q_2)(c_2 + 6c_3) \big] \Big\} + \frac{g_A^2 \bar{e}_{14}}{2F_\pi^4} \left(2M_\pi^2 + q_2^2\right)^2$$

$L(q_2) = \frac{\sqrt{q_2^2 + 4M_\pi^2}}{q_2} \log \frac{\sqrt{q_2^2 + 4M_\pi^2} + q_2}{2M_\pi}$

$$\mathcal{B}^{(5)} = \frac{g_A^2 \bar{e}_{17}}{2F_\pi^4} \left(2M_\pi^2 + q_2^2\right) - \frac{g_A^2 c_4}{2304\pi^2 F_\pi^6} \Big\{ q_2^2 \big[5 - 6L(q_2) \big] + 12 M_\pi^2 \big[2 + 9g_A^2 - 2L(q_2) \big] \Big\}$$

calculated using DimReg

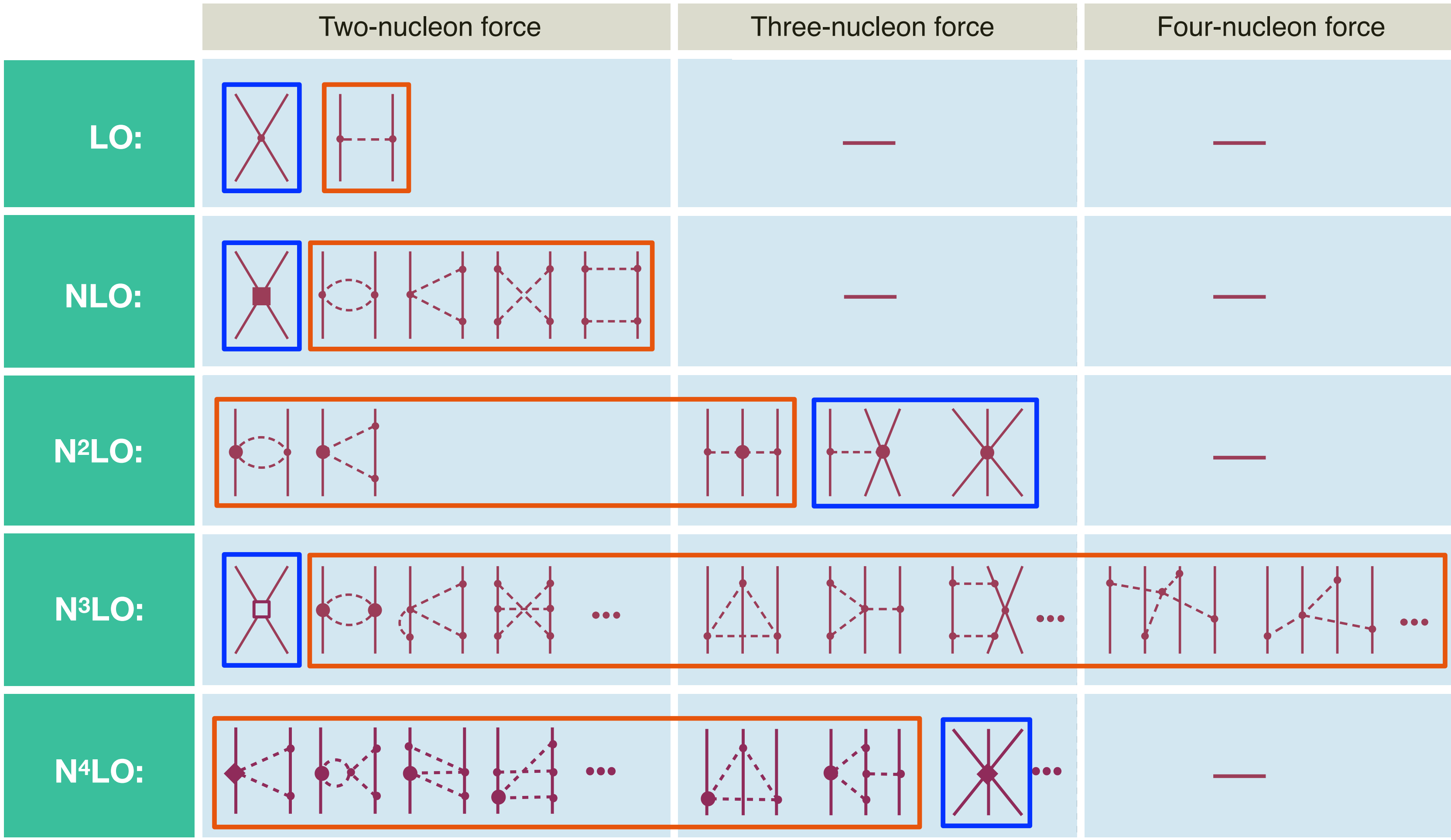
Example: 2π-exchange 3NF

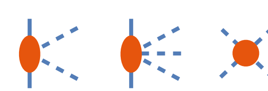


— all LECs c_i and $\bar{e}_i$ are known from the Roy-Steiner-equation analysis of the πN system

— the results are only meaningful (converged) at small momenta ⇒ cutoff needed

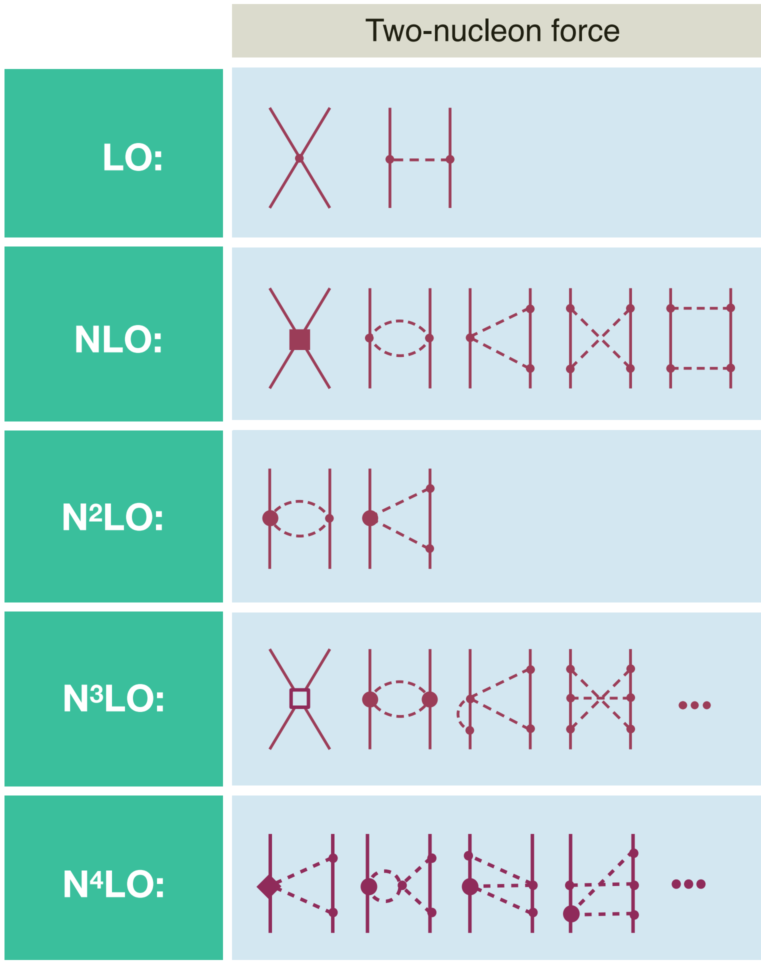
Chiral expansion of nuclear forces



Chiral dynamics: Long-range interactions are predicted in terms of on-shell amplitudes 

Short-range few-N interactions are tuned to experimental data

Chiral expansion of nuclear forces



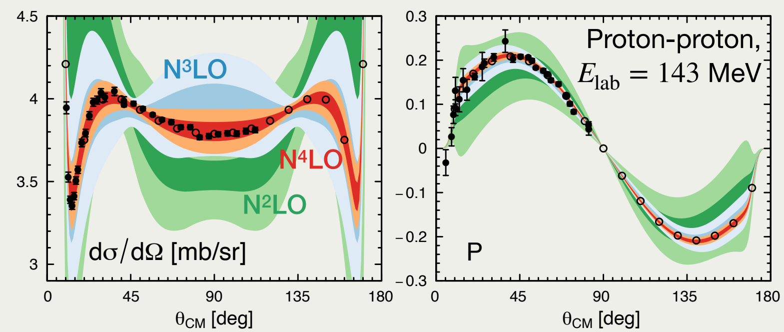
χEFT as a precision tool in the 2N sector

— N⁴LO⁺: currently most accurate and precise NN interactions on the market

— clear evidence of the TPEP from NN data

— almost no residual cutoff dependence

— Bayesian truncation-error estimation



— Precision calculations for 2 nucleons:

$g_{\pi NN} = 13.24 \pm 0.04$ Reinert, Krebs, EE '20

$r_{\rm str}^{^2\rm H} = 1.9729^{+0.0015}_{-0.0012}$ fm Filin et al., '21

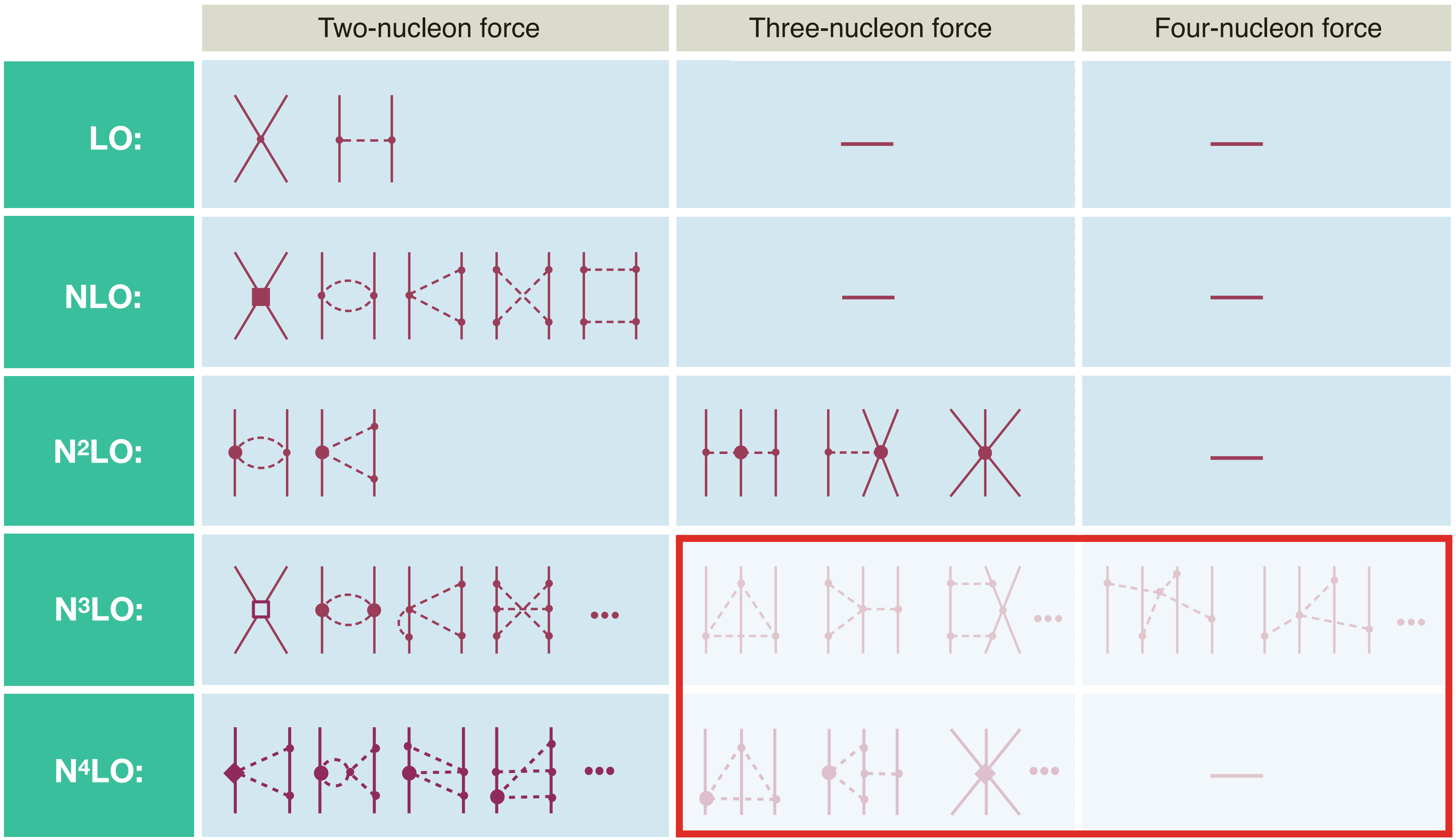
Semi-local regularization in momentum space Reinert, Krebs, EE, EPJA 54 (2018) 86; PRL 126 (2021) 092501

$$V_{1\pi}(q) = \frac{\alpha}{\vec{q}^{\,2} + M_\pi^2}\, e^{-\frac{\vec{q}^{\,2}+M_\pi^2}{\Lambda^2}} + \text{subtraction},$$

$$V_{2\pi}(q) = \frac{2}{\pi}\int_{2M_\pi}^{\infty} d\mu\, \mu \frac{\rho(\mu)}{\vec{q}^{\,2}+\mu^2}\, e^{-\frac{\vec{q}^{\,2}+\mu^2}{2\Lambda^2}} + \text{subtractions}$$

+ nonlocal (Gaussian) cutoff for contacts

Chiral expansion of nuclear forces



have been worked out using dimensional regularization

mixing DimReg with Cutoff violates χ-symmetry (also for current operators)

$\Rightarrow$ need to be re-derived using invariant cutoff regulator

Krebs, EE, PRC 110 (2024) 044004

Take-aways of part IV

— nuclear forces can be derived from the effective Lagrangian using a variety of methods

— important to maintain consistency (nuclear potentials are scheme dependent)

— regularization of 3NFs and currents beyond tree level is nontrivial