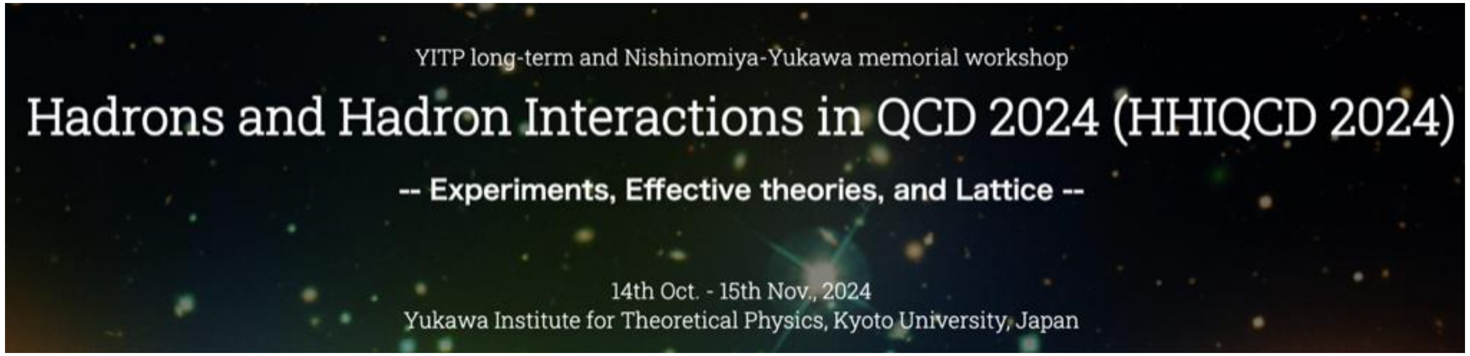


Grassmann tensor renormalization group approach to (1+1)-dimensional two-color QCD at finite density

HHIQCD 2024 @ YITP, Kyoto University

1st November 2024

Ho Pai, KWOK

Preprint available
arXiv:2410.09485v1
(HPK, Shinichiro Akiyama, Synge Todo)

Tensor network methods

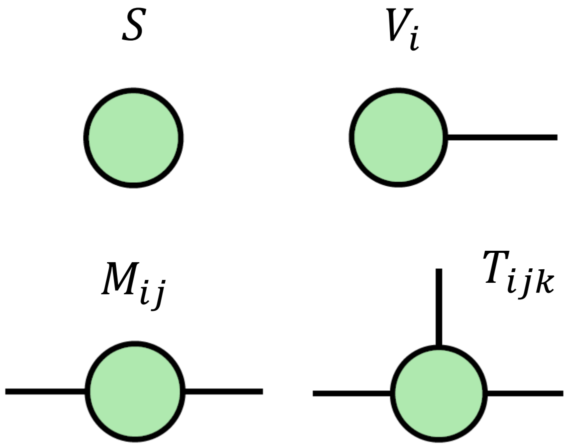
- Lattice field theories with finite density or a theta term suffer from the sign problem

$$\langle \hat{O} \rangle = \int DU\, O \boxed{D\Psi D\bar{\Psi} e^{-S}}$$

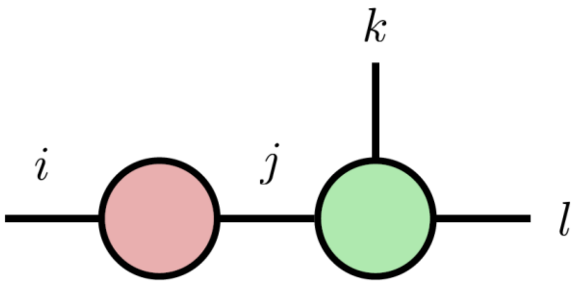
This quantity is complex (e.g., negative)
Not easy for Monte-Carlo method

- Tensor network methods are free from the sign problem (a big advantage!)

- Some terminologies:

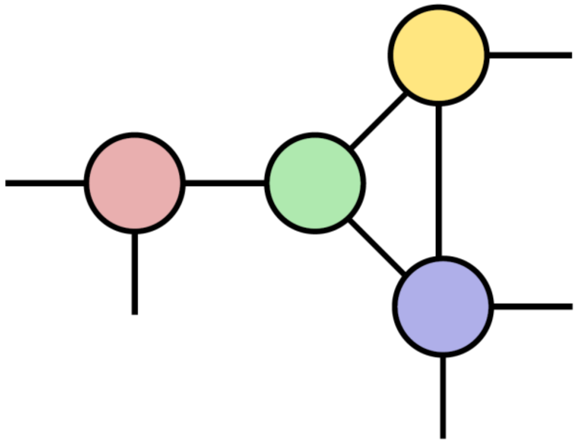


Bond dimension:
the number of values that an index can take



$$C_{ikl} = \sum_{j=1}^{\chi} A_{ij} B_{jkl}$$

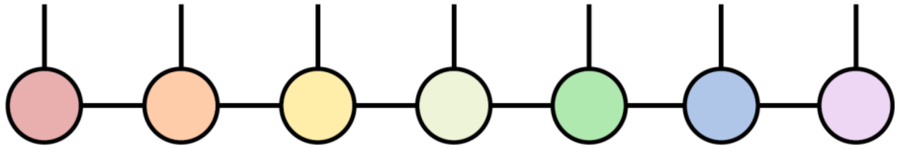
Tensor contraction



Tensor network:
a network of contracted tensors

Tensor network methods

- **Hamiltonian approach**

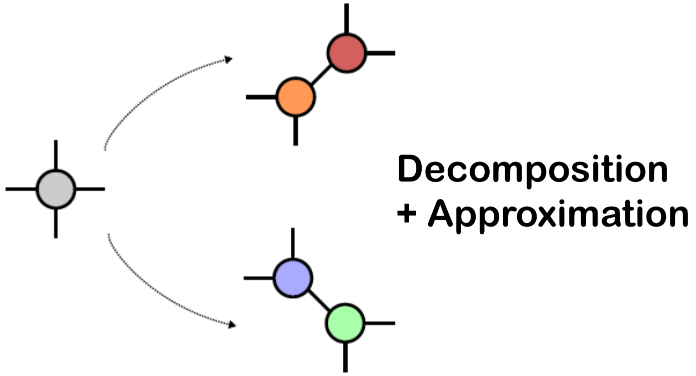


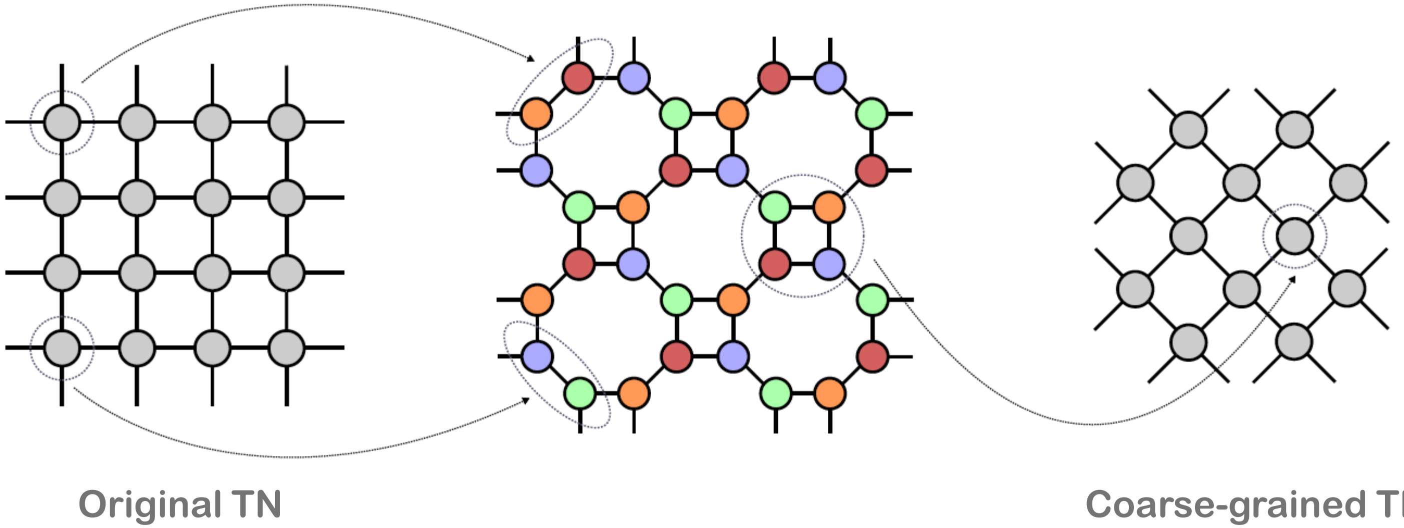
[S. Kuhn+, JHEP 07 (2015) 130] [P. Silvi+, Quantum 1 (2017) 9]
[M. C. Banuls+, PRX 7 (2017) 041046] [P. Sala+, PRD 98 (2018) 034505]
[P. Silvi+, PRD 100 (2019) 074512] [M. Rigobello+, 2308.04488]
[H. Liu+, 2312.17734] [T. Hayata+, JHEP 07 (2024) 106]

- **Lagrangian approach (Tensor Renormalization Group)**

[J. Bloch & R. Lohmayer, Nucl. Phys. B 986 (2023) 116032]
[M. Asaduzzaman+, JHEP 05 (2024) 195]

[Levin, M., & Nave, C. P. (2007). PRL, 99(12), 120601]





✓ Can achieve a large lattice efficiently
✓ Can describe fermions directly by incorporating Grassmann variables (Grassmann TRG)

What do we study?

- **(1+1)-D two-color QCD with staggered fermions on a square lattice**

What we calculate with TRG

$$\boxed{Z = \int \mathcal{D}U\mathcal{D}\chi\mathcal{D}\bar{\chi}\, \mathrm{e}^{-S}}$$

Parameters: m, β, μ, λ

$$S = S_f + S_g + S_\lambda$$

Wilson's gauge action

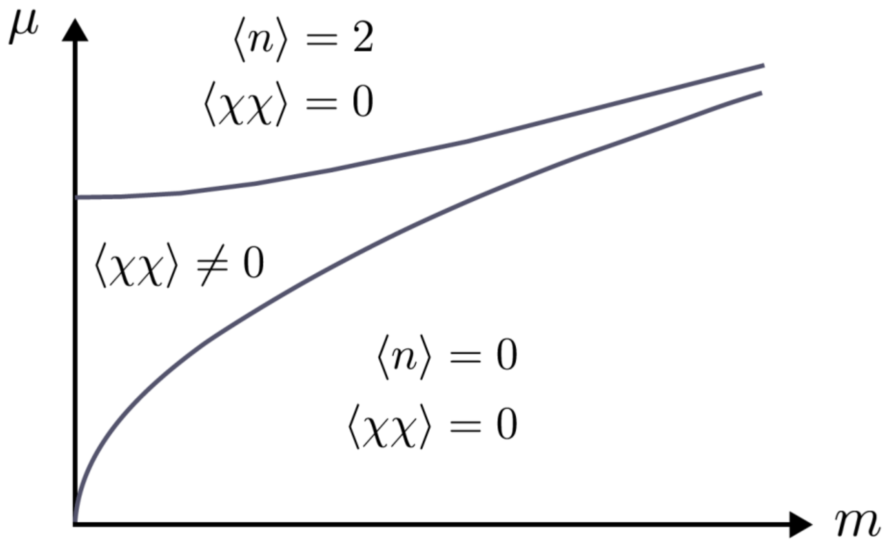
Fermion hopping term
+ mass term

Diquark source term

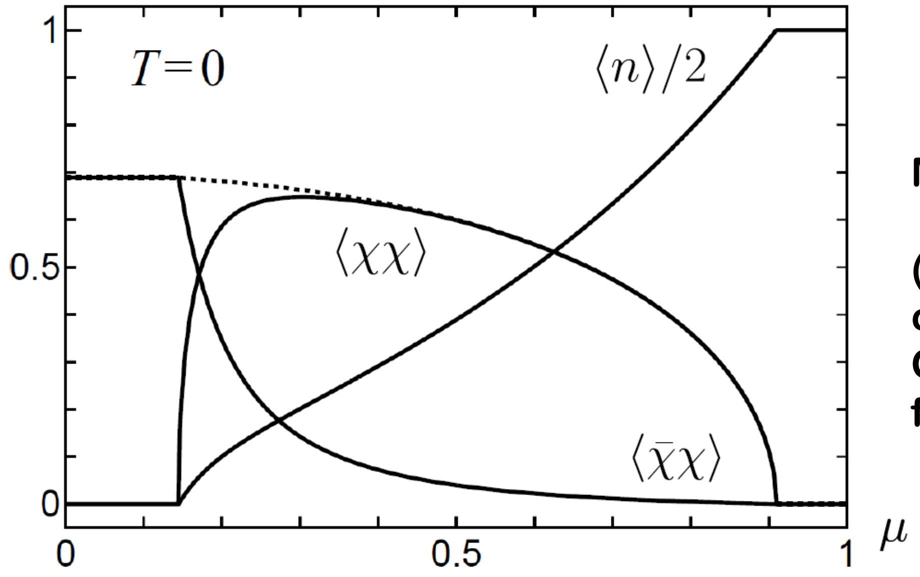
$$S_\lambda = \frac{\lambda}{2}\sum_n \left[\chi^T(n)\sigma_2\chi(n) + \bar{\chi}(n)\sigma_2\bar{\chi}^T(n)\right]$$

$$\langle\chi\chi\rangle \equiv \frac{1}{2V}\int \mathcal{D}U\mathcal{D}\chi\mathcal{D}\bar{\chi} \sum_n \left(\chi^T\sigma_2\chi + \bar{\chi}\sigma_2\bar{\chi}^T\right) \mathrm{e}^{-S}$$

- **Phase structure of the (3+1)-D theory**



[Y. Nishida+, Phys. Rept. 398 (2004) 281–300]



Mass = 0.02

(3+1)-D infinite coupling two-color QCD with staggered fermions

Our proposal

- Two-color QCD with staggered fermion has the global $U_V(1) \times U_A(1)$ symmetry at a finite μ, in the vanishing λ limit and chiral limit ($m \to 0$)

- In higher dimensions, spontaneous symmetry breaking is possible and diquark condensate $\langle \chi\chi \rangle$ may have a finite value

- However, there is NO spontaneous breaking of continuous global symmetry in two dimensions.

$$\lim_{m\to 0} \lim_{V\to\infty} \langle \bar{\chi}\chi \rangle = 0 \qquad \lim_{\lambda\to 0} \lim_{V\to\infty} \langle \chi\chi \rangle = 0$$

- Therefore, we explicitly break the $U_A(1)$ symmetry with a finite m, and the $U_V(1)$ symmetry with a finite λ

- Under this setting, we compute the expectation value of quark number density, chiral condensate, and diquark condensate with the TRG approach

1. Construction of the TN representation (Discretization of the gauge group integration)

2. The bond dimension of the tensors is inevitably large (How to handle this in practical computation)

Tensor network representation (Grassmann part)

N = no. of color

[Akiyama, S., & Kadoh, D., JHEP, 2021(10), 1-16]

- The Grassmann path integral is expressed as the trace of a Grassmann tensor network by introducing a pair of ***N***- component auxiliary Grassmann field on edges to decompose each of the hopping terms

Hopping term

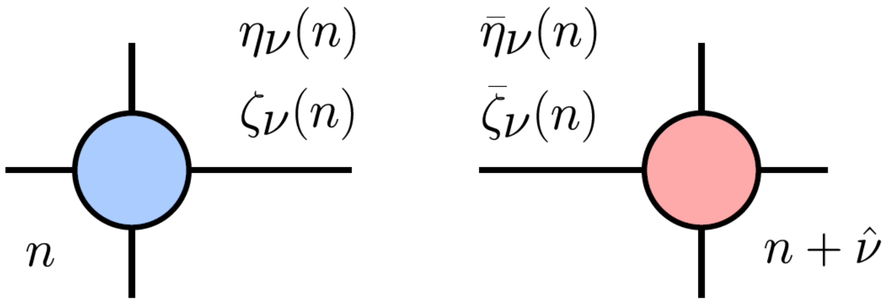
$$\exp\left[-\frac{p_\nu(n)}{2}\mathrm{e}^{\mu\delta_{\nu,2}}\bar{\chi}(n)U_\nu(n)\chi(n+\hat{\nu})\right]$$

$$\exp\left[\frac{p_\nu(n)}{2}\mathrm{e}^{-\mu\delta_{\nu,2}}\bar{\chi}(n+\hat{\nu})U_\nu^\dagger(n)\chi(n)\right]$$

Auxiliary Grassmann field

$\eta_\nu(n)$ $\bar{\eta}_\nu(n)$

$\zeta_\nu(n)$ $\bar{\zeta}_\nu(n)$



- The bond dimension of a Grassmann tensor is $2^{\text{number of Grassmann fields on the bond}}$
  - ➢ Our construction: $\mathbf{2^{2N}}$
  - ➢ [M. Asaduzzaman+, JHEP 05 (2024) 195]: $\mathbf{2^{2N^2}}$

- In our construction, a Grassmann tensor at site n depends on the link variables $U_1(n-\hat{1})$ and $U_2(n-\hat{2})$
  - ⇨ The dependence of any link variable appears in only one local tensor
  - ⇨ At the infinite coupling limit, we can do the gauge group integration exactly for each tensor before TRG

Tensor network representation (including plaquette term)

N = no. of color

- **The gauge group integration is dicretized by a summation with group elements sampled uniformly from the group manifold**

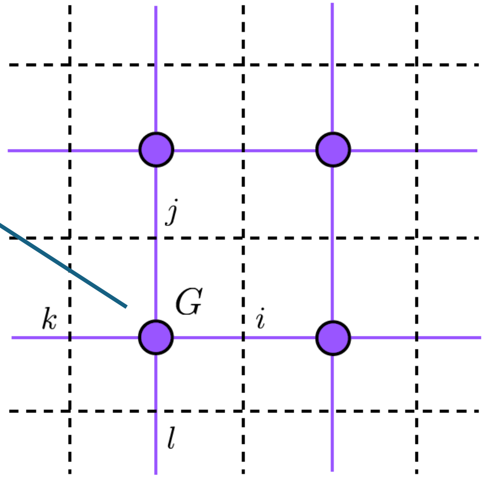
$$\int \mathrm{d}U\, f(U) \simeq \frac{1}{K}\sum_{i=1}^{K} f(U_i)$$

Sample size

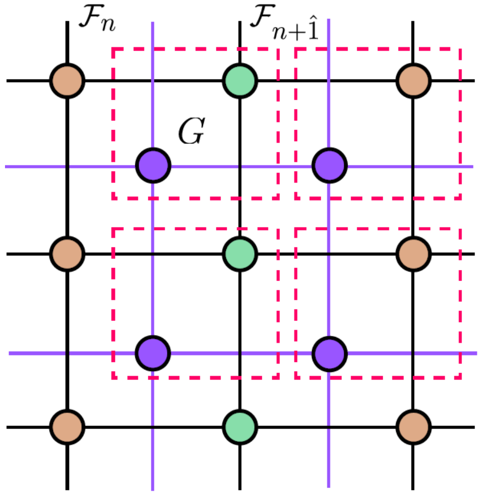
[Fukuma, M.+, PTEP, 2021(12),123B03]

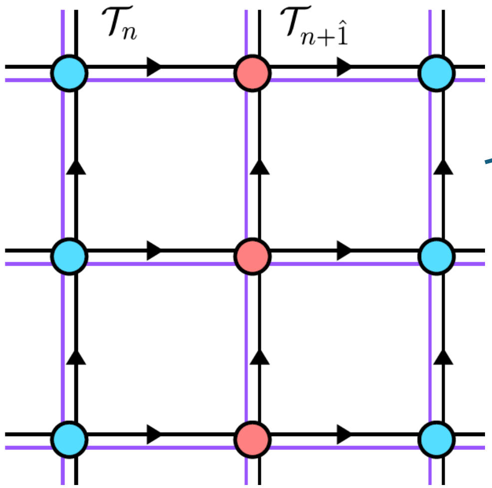
$$G_{ijkl} \equiv \frac{1}{K^2}\mathrm{e}^{(\beta/N)\,\mathrm{Re}\,\mathrm{Tr}\left(U_i U_j^\dagger U_k^\dagger U_l\right)}$$

Partition function of pure Yang-Mills on a square lattice



- **We then combine the Grassmann tensor F and the above real-valued tensor G**





Bond dimension: $2^{2N}K$

$$\boxed{Z = \int \mathcal{D}U\mathcal{D}\chi\mathcal{D}\bar{\chi}\,\mathrm{e}^{-S}}$$

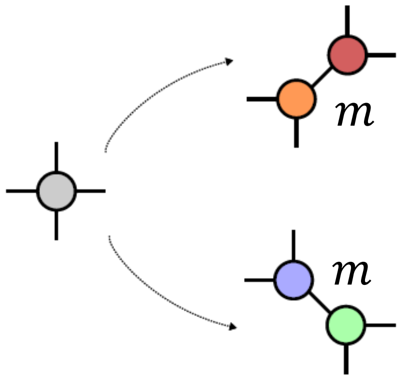
Why we need initial tensor compression

[Adachi, D.+, PRB, 105(6), L060402 (2022)]
[Akiyama, S., JHEP, 2022(11), 1-14]

- We use <u>bond-weighted tensor renormalization group</u> to coarse-grain the tensor network and reach the thermodynamic limit

- The choice of bond dimension cutoff D in TRG algorithms depends on the bond dimension of initial tensors. In our case (two-color i.e., N=2), the initial bond dimension is $16K$!

Truncated singular value decomposition (SVD)



Original bond dimension

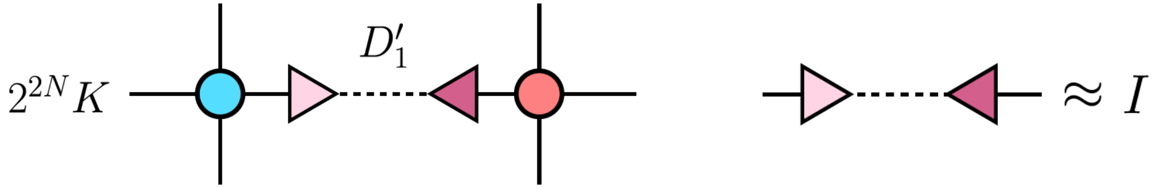
$$T_{(ij)(kl)} = \sum_{m=1}^{\chi^2} U_{(ij)m} s_m V^*_{(kl)m}$$

Bond dimension cutoff

$$\approx \sum_{m=1}^{D} U_{(ij)m} s_m V^*_{(kl)m}$$

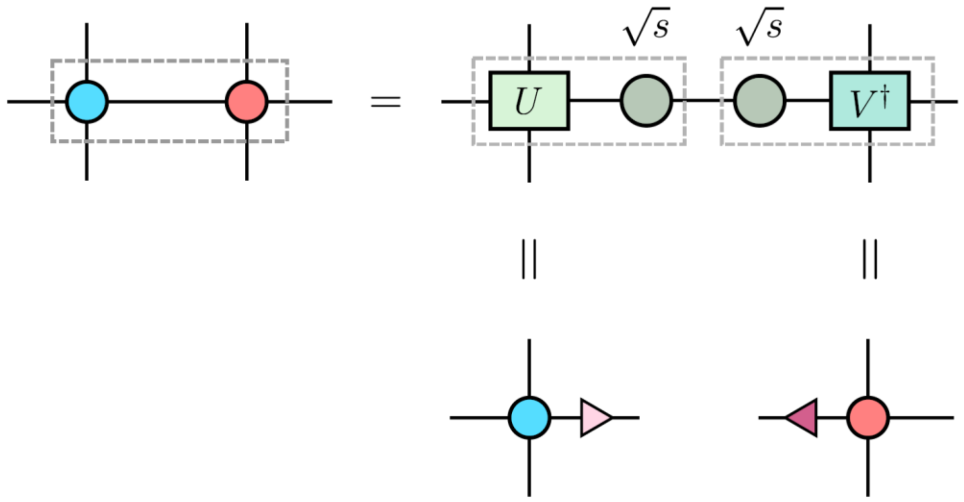
Among the χ^2 singular values s, we only keep the largest D ones and set the remaining s to zero

- Compression of the initial tensors is needed before TRG
  - ➢ insert a pair of squeezers, which acts as a good approximation of identity, on every bond of the tensor network



Construction of squeezers & Efficiency of compression

- **The insertion of squeezers is equivalent to doing a truncated SVD on the following contraction of initial tensors**



How to determine the bond dimension after compression (how many singular values are kept)?

Bond dimension after compression

$$\frac{\sum_{y=1}^{D'} s_y^2}{\sum_{y=1}^{2^{2N}K} s_y^2} \geq r$$

Ratio parameter

Initial bond dimension

- **Efficiency of our compression scheme**

$m = 0.1 \quad \beta = 1.6 \quad \mu = 0.4 \quad \lambda = 0 \quad K = 14$

r	D_1'	D_2'	D_3'	D_4'	compression rate
1	224	224	224	224	100%
0.99999	148	148	143	143	17.8%
0.99995	122	122	118	118	8.23%
0.9999	110	110	105	105	5.30%
0.9995	80	80	79	79	1.59%
0.999	70	70	67	67	0.874%
0.99	35	35	33	33	0.0530%

$m = 0.1 \quad \beta = 0.8 \quad \mu = 0.4 \quad \lambda = 0 \quad K = 14$

r	D_1'	D_2'	D_3'	D_4'	compression rate
1	224	224	224	224	100%
0.99999	86	86	84	84	2.07%
0.99995	68	68	66	66	0.800%
0.9999	61	61	59	59	0.514%
0.9995	46	46	43	43	0.155%
0.999	39	39	37	37	0.0827%
0.99	19	19	19	19	0.00518%

Calculation of observables

Free energy density:

$$f = \ln Z / V$$

What we calculate directly with TRG

Quark number density:

$$\langle n \rangle = \frac{\partial f}{\partial \mu} \simeq \frac{f(\mu + \Delta\mu) - f(\mu)}{\Delta\mu}$$

$$\Delta\mu = 0.04 \text{ for } m = 0.1$$

$$\Delta\mu = 0.02 \text{ for } m = 1$$

Chiral condensate:

$$\langle \bar{\chi}\chi \rangle = \frac{\partial f}{\partial m} \simeq \frac{f(m + \Delta m) - f(m)}{\Delta m}$$

$$\Delta m = 10^{-4}$$

Diquark condensate:

$$\langle \chi\chi \rangle = \frac{\partial f}{\partial \lambda} \simeq \frac{f(\lambda + \Delta\lambda) - f(\lambda)}{\Delta\lambda}$$

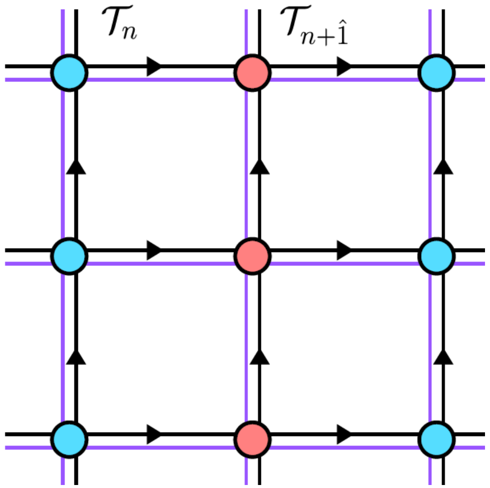
$$\lambda = \Delta\lambda = 10^{-4}$$

- Another way to evaluate expectation values: **Impurity tensor method**

$$\langle O \rangle \equiv Z^{-1} \boxed{\int \mathcal{D}U \mathcal{D}\chi \mathcal{D}\bar{\chi}\, O\, \mathrm{e}^{-S}}$$

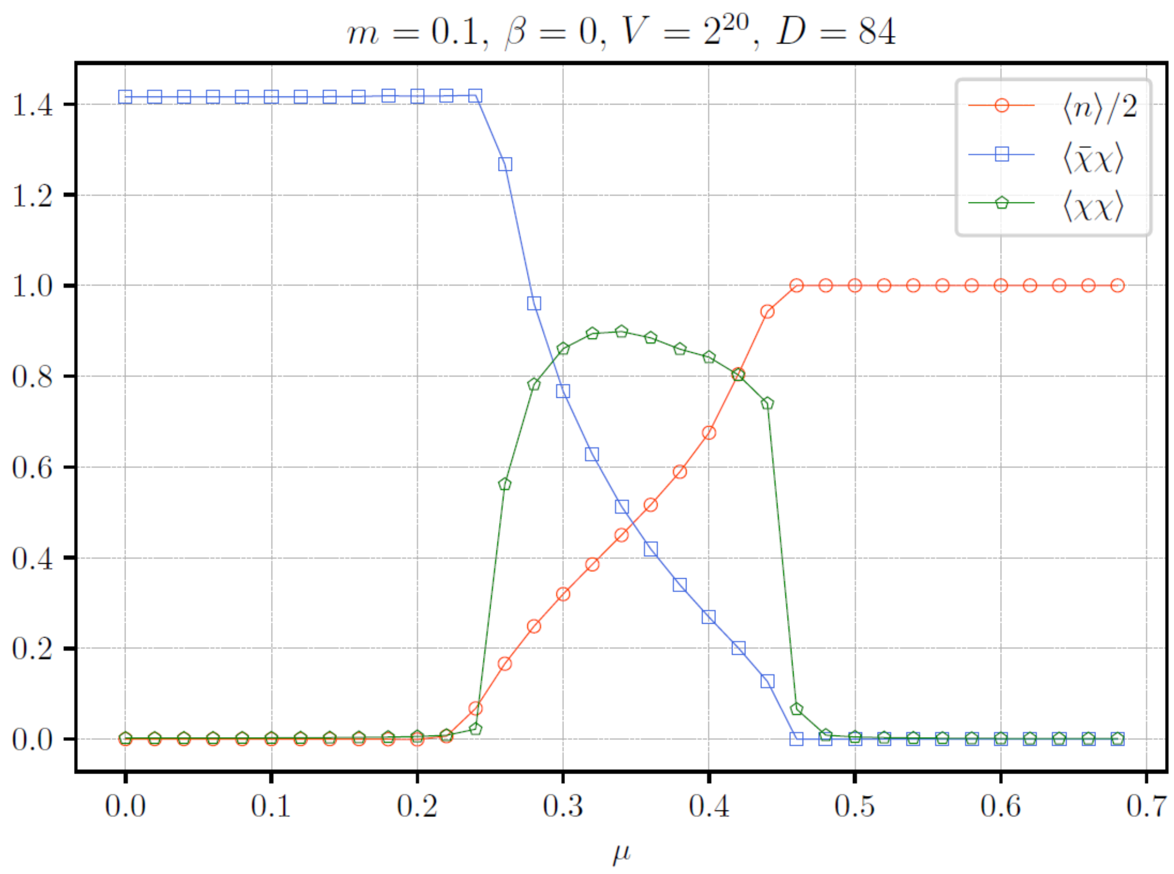
Trace of TN composed of uniform tensors, and some impurity tensors

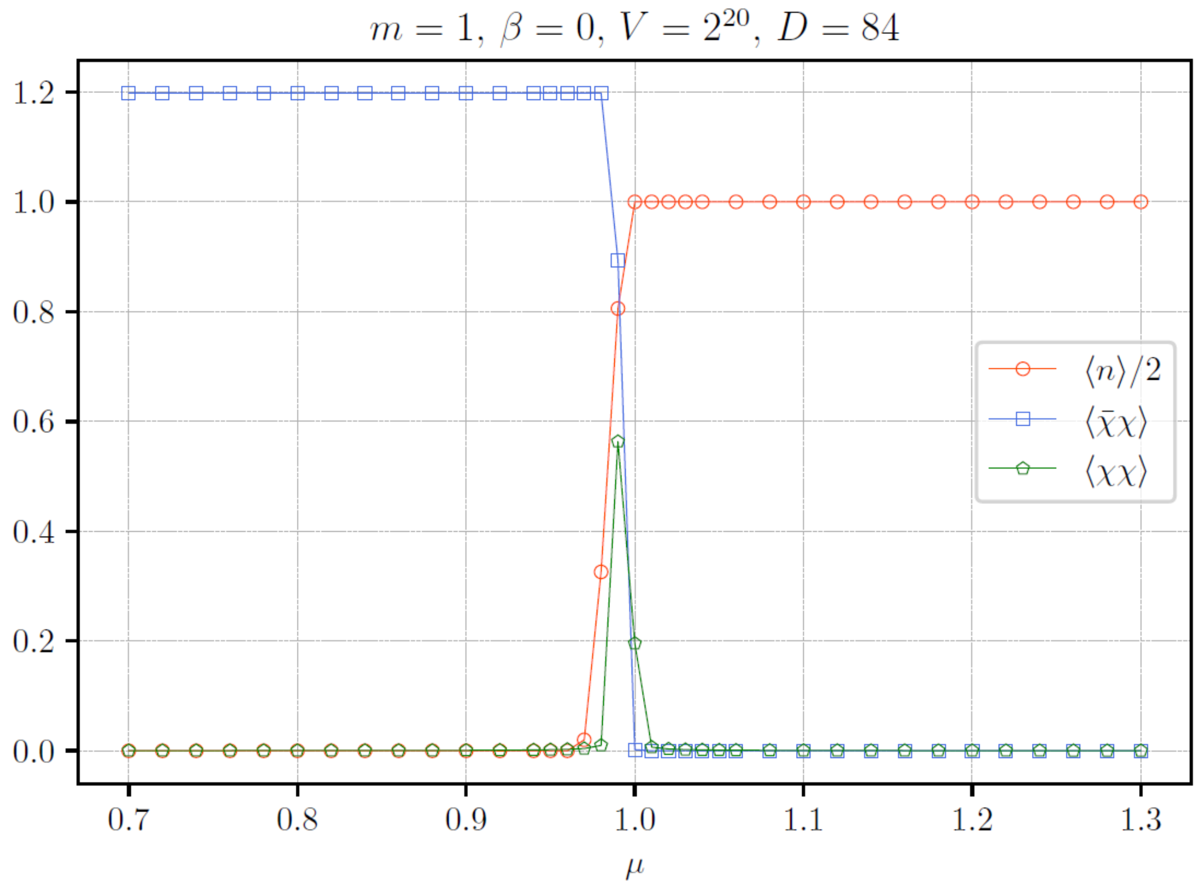
Trace of TN composed of uniform tensors



Numerical results: infinite coupling limit

The bond dimension is 16 only, so no initial tensor compression here





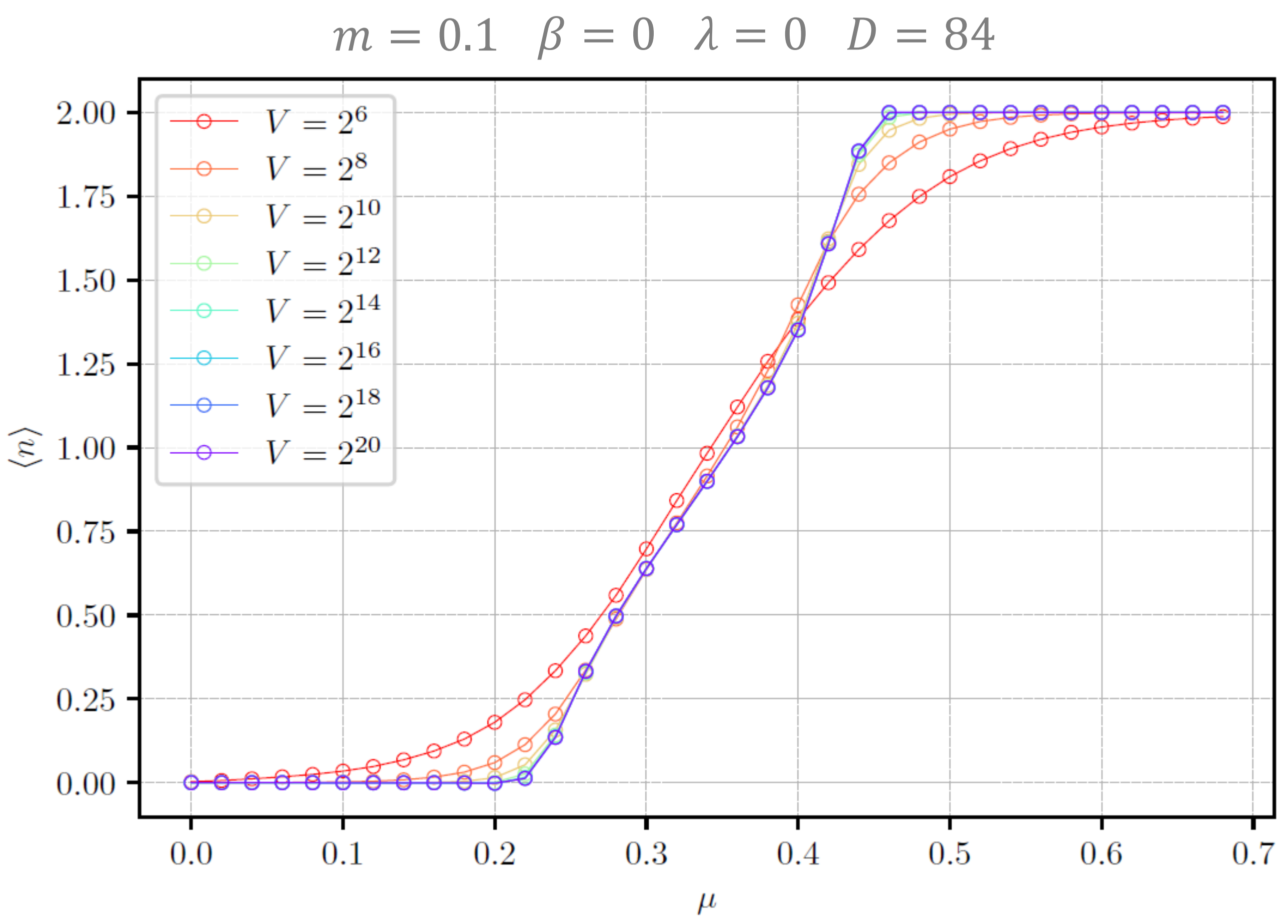
At m = 0.1: an intermediate phase is observed in a finite region of μ

At m = 1: a sharp transition is seen, and the intermediate phase becomes a very narrow region in μ

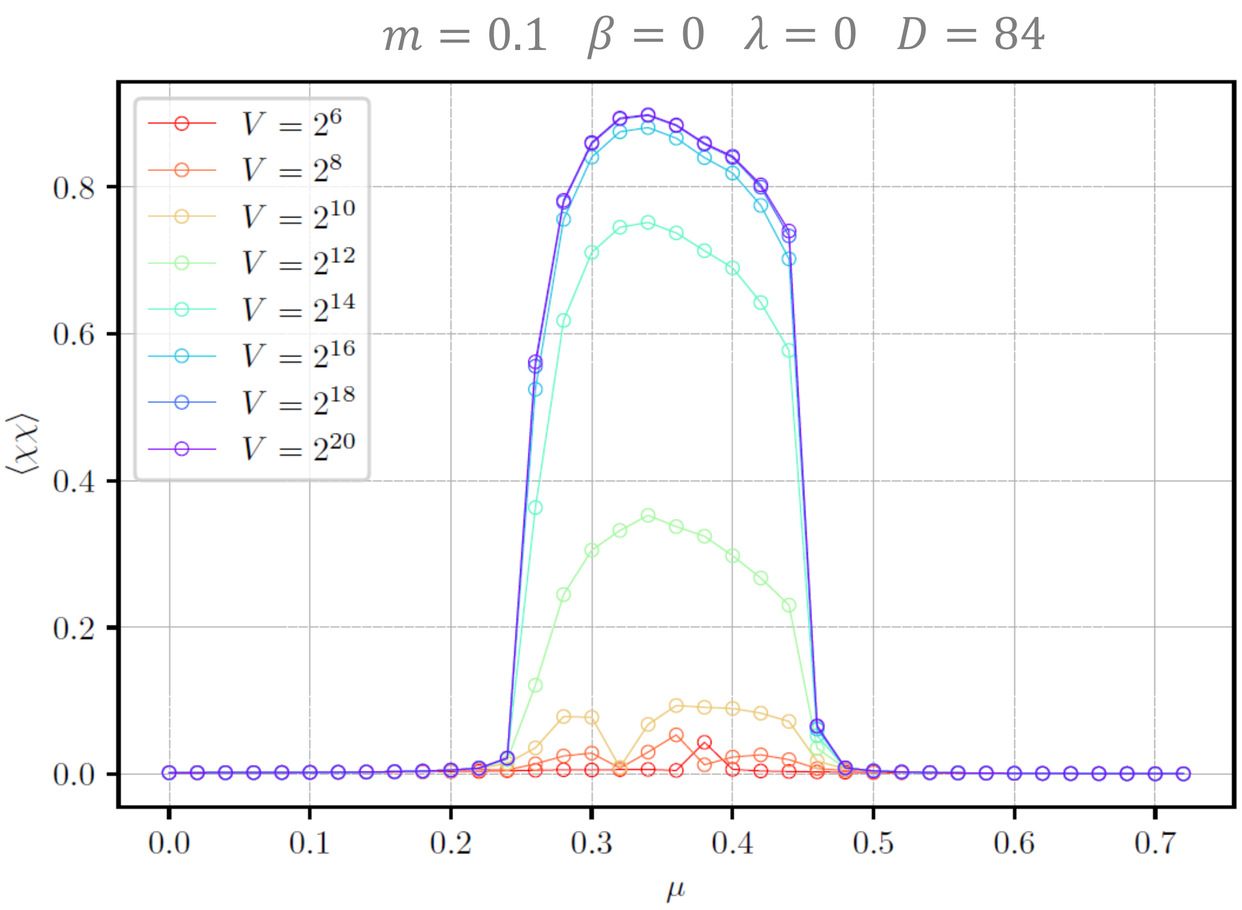
- The qualitative behavior of the observables at finite m and/or λ is similar to that exhibited in a mean-field study of the (3+1)-D theory, where spontaneous symmetry breaking exists [Y. Nishida+, Phys. Rept. 398 (2004) 281–300]

Numerical results: Volume dependence

Quark number density:



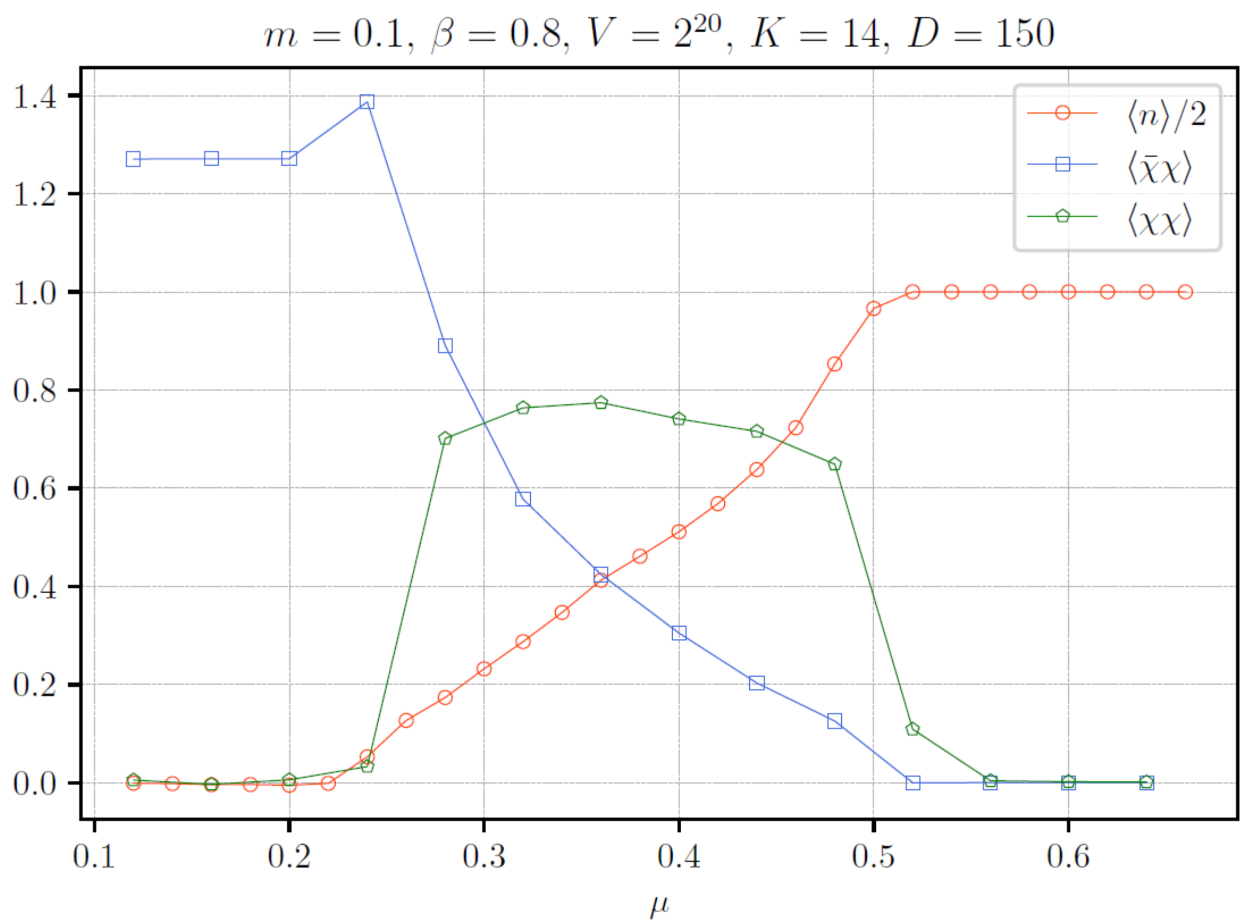
Diquark condensate:

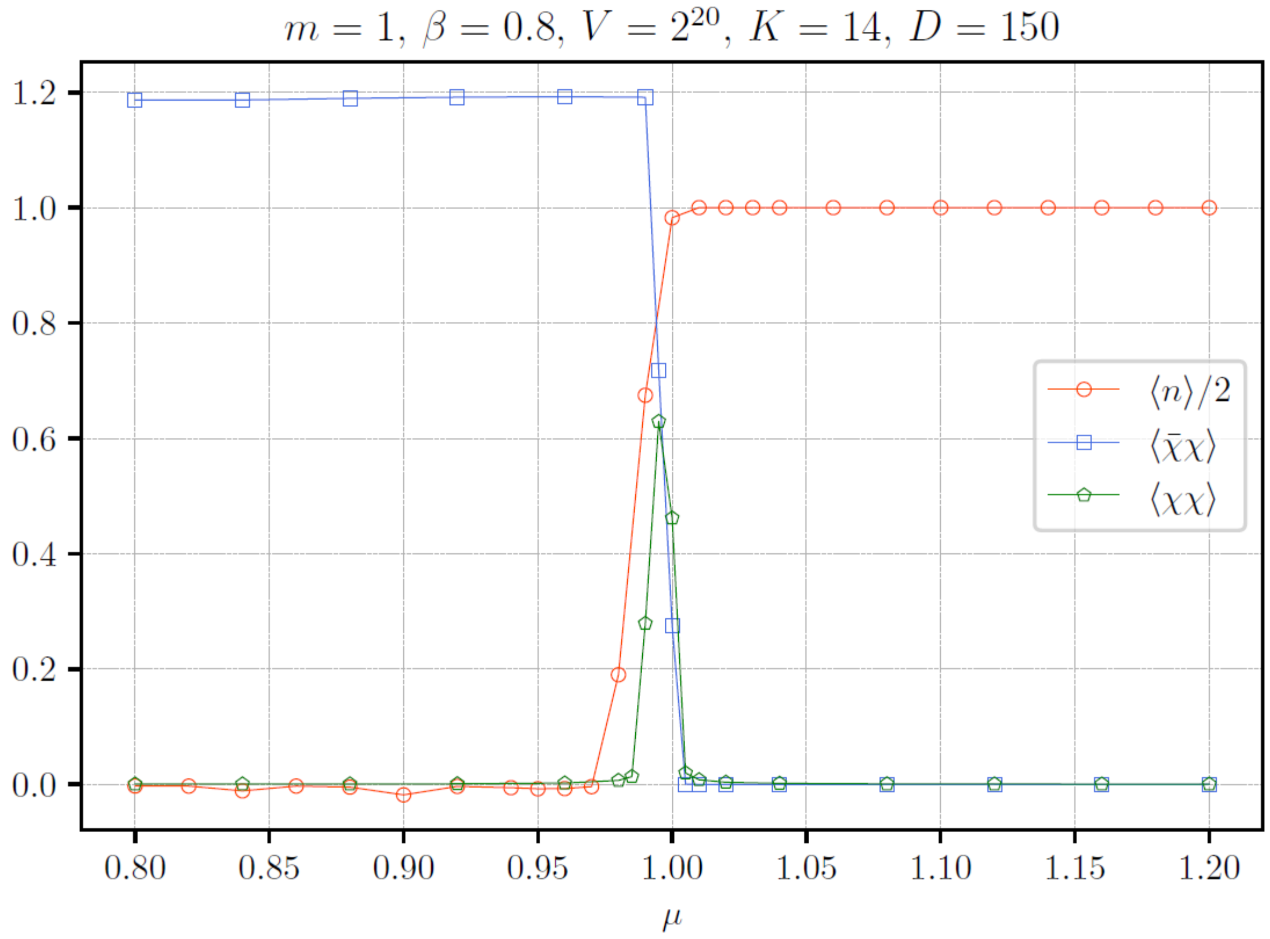


- The thermodynamic limit is reached when $V = 2^{20}$

Numerical results: $\beta = 0.8$

K = 14

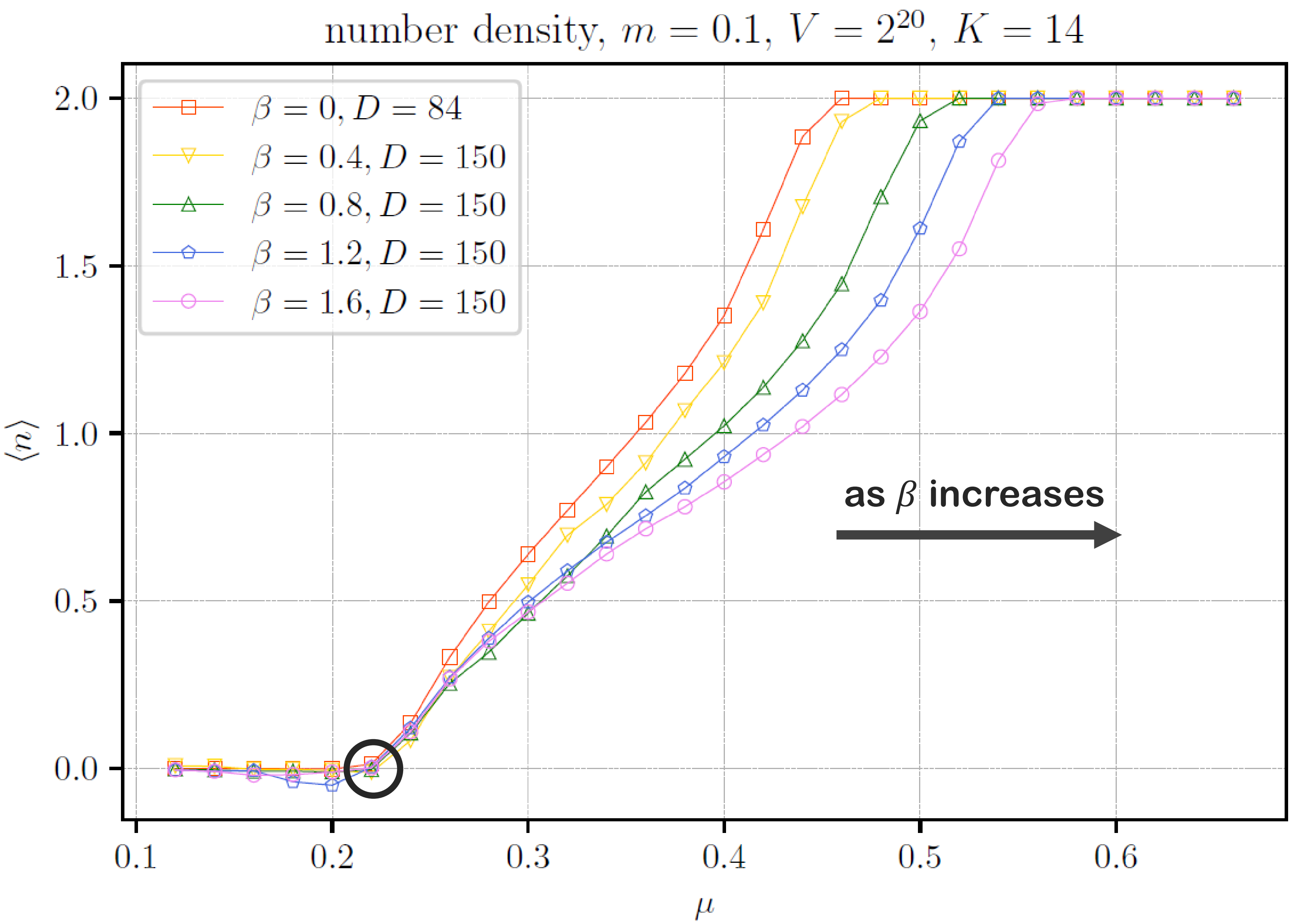




- The behavior at finite coupling is similar to that at infinite coupling
- As β becomes nonzero, the intermediate phase becomes broader at m = 0.1

$\beta = 0 \quad 0.22 \le \mu \le 0.46$ $\qquad$ $\beta = 0.8 \quad 0.22 \le \mu \le 0.52$

β dependence of transition points



- **The first transition point (the one at a smaller μ) seems to be robust against β**
- **The second transition point locates at larger chemical potential as β increases**
- **$\langle n \rangle$ does not saturate in regions of larger chemical potential as the gauge interaction is weakened, approaching the continuum limit**

Summary

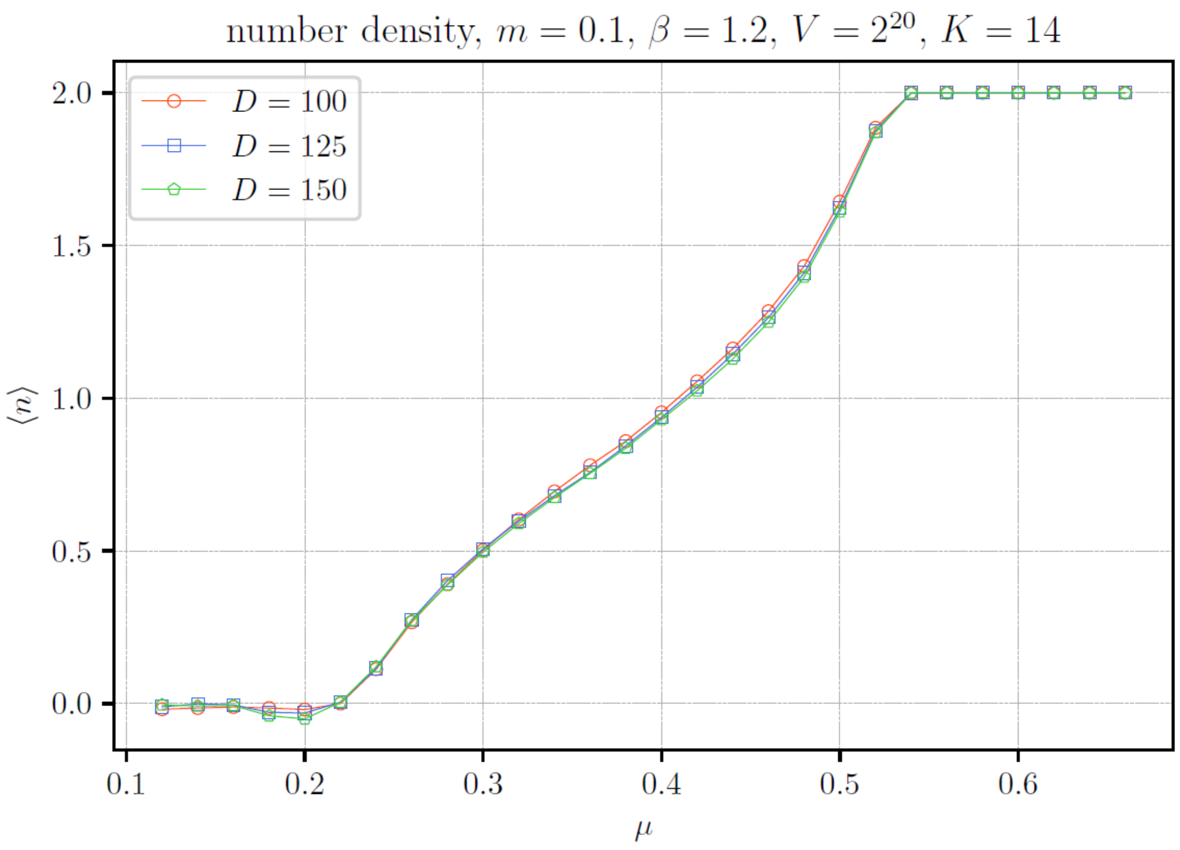
- This is a TRG study on non-Abelian gauge theory coupled with standard staggered fermions at finite density and finite coupling
- Tensor network calculation for this kind of theories is computationally challenging because of the very large initial bond dimension
- We introduce an efficient initial tensor compression scheme to deal with this issue
- TRG enables the calculation of important physical quantities at the infinite coupling limit and finite β regime
- Future directions:
  1) Improved construction of tensor which allows a larger sample size K for the discretization of gauge group
  2) Chiral limit and vanishing λ limit in higher dimensions
  3) Extension to the SU(3) gauge group
  4) Investigation of inhomogeneous phase [T. Kojo, Nucl. Phys. A 877(2012) 70-94] [T. Hayata+, JHEP 07 (2024) 106]

Thank you for listening!

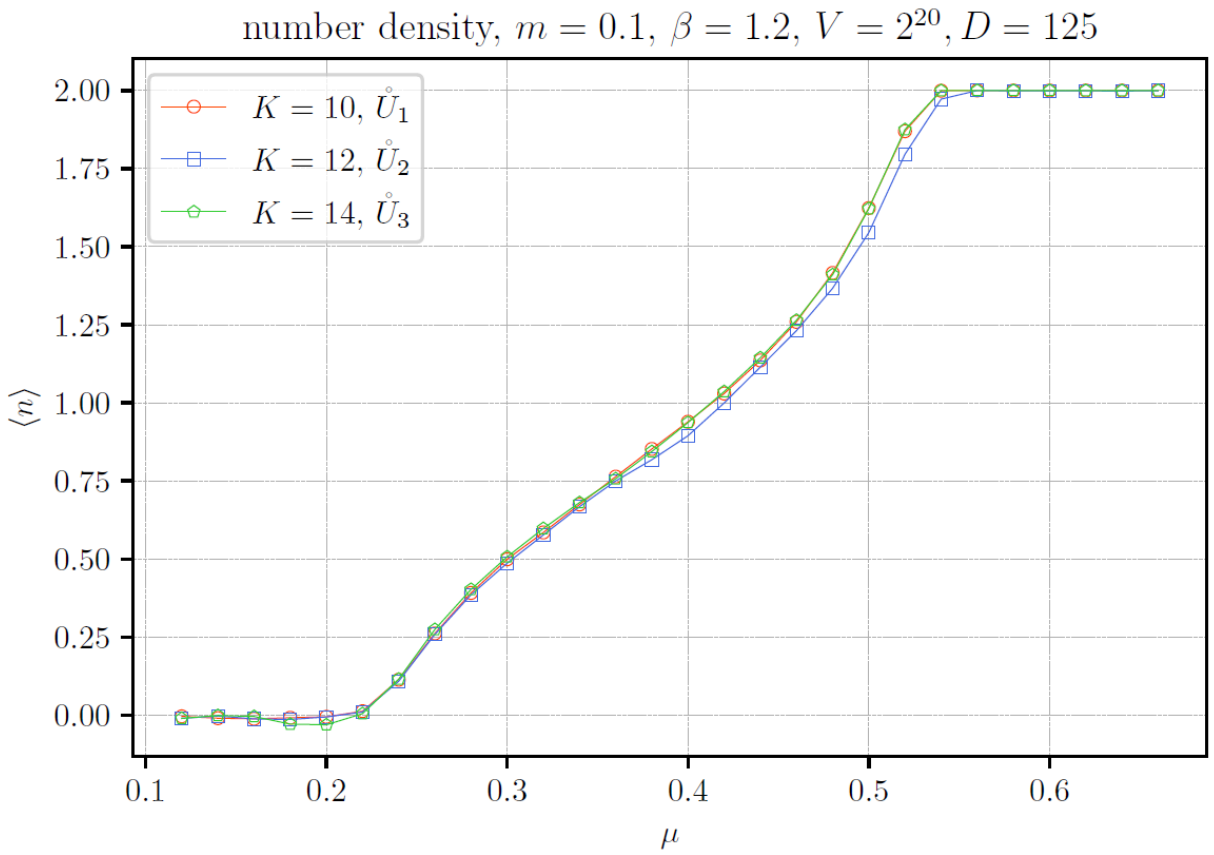
Backup slides

Algorithmic parameter dependence

***D* dependence:**



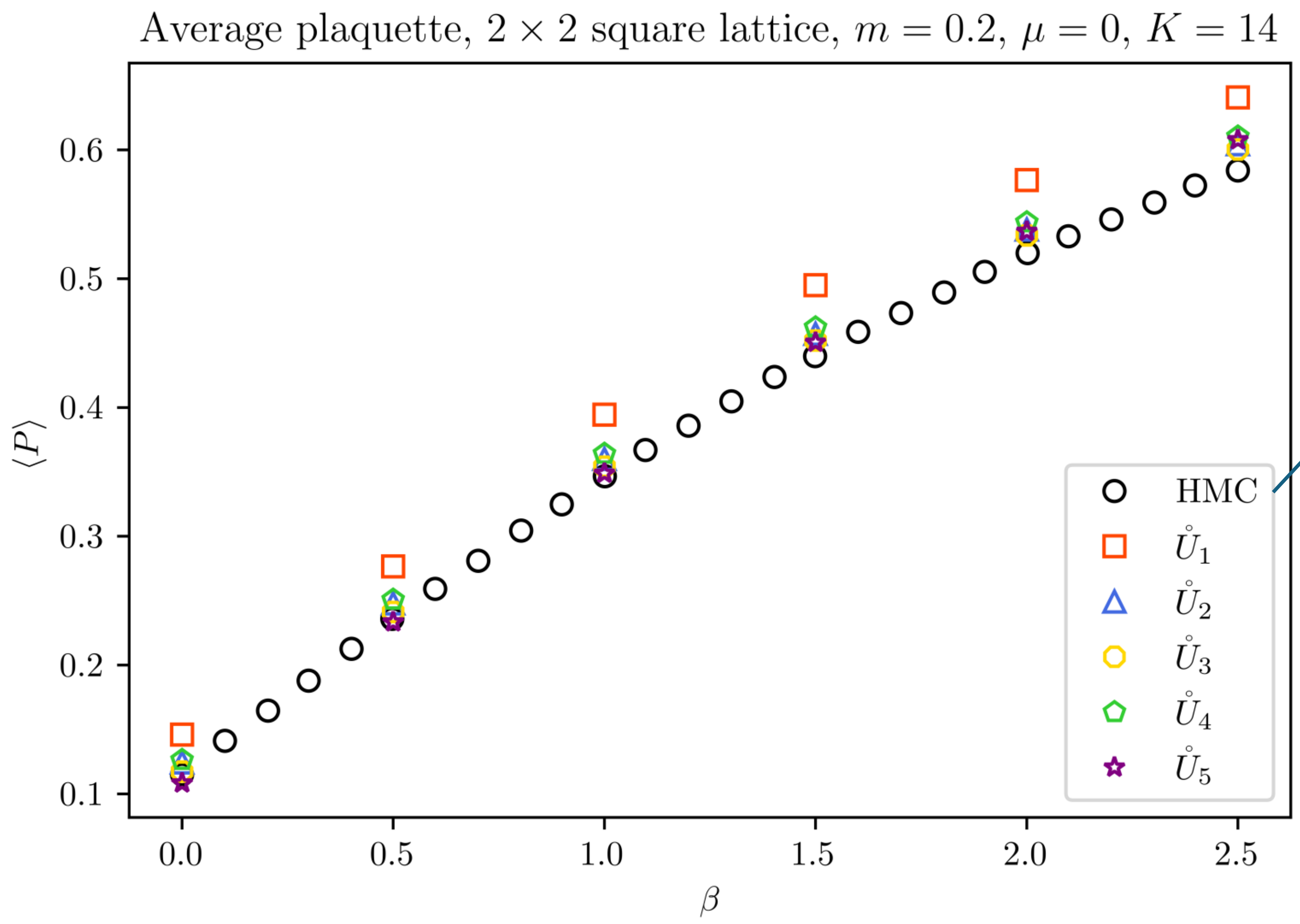
***K* dependence:**



Results of a 2×2 lattice by exact contraction

K = 14

$$\langle P \rangle = \frac{\partial f}{\partial \beta}$$



[G. Gagliardi & W. Unger, PRD 101, 034509 (2020)]

Scheme of squeezer construction