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Theta angle dependence of the vacuum energy in two-dimensional $\mathbb{C}P^{N-1}$ model

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Based on work in progress

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1. Introduction

θ term

What is a θ term ?

θ term = topological term coupled to 2π - periodic parameter θ

In 4d QCD or pure Yang-Mills theory

$$\mathscr{L}_\theta = -\frac{\theta}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}(F^{\mu\nu}F^{\rho\sigma})$$

Why do we consider θ term ?

Instanton effects on perturbative calculations via θ term

$|\theta| \lesssim 1.2 \times 10^{-10}$ (from nEDM exp.) : **Strong CP Problem** $\rightarrow$ **QCD Axion**

θ dependence of the vacuum energy = Axion potential

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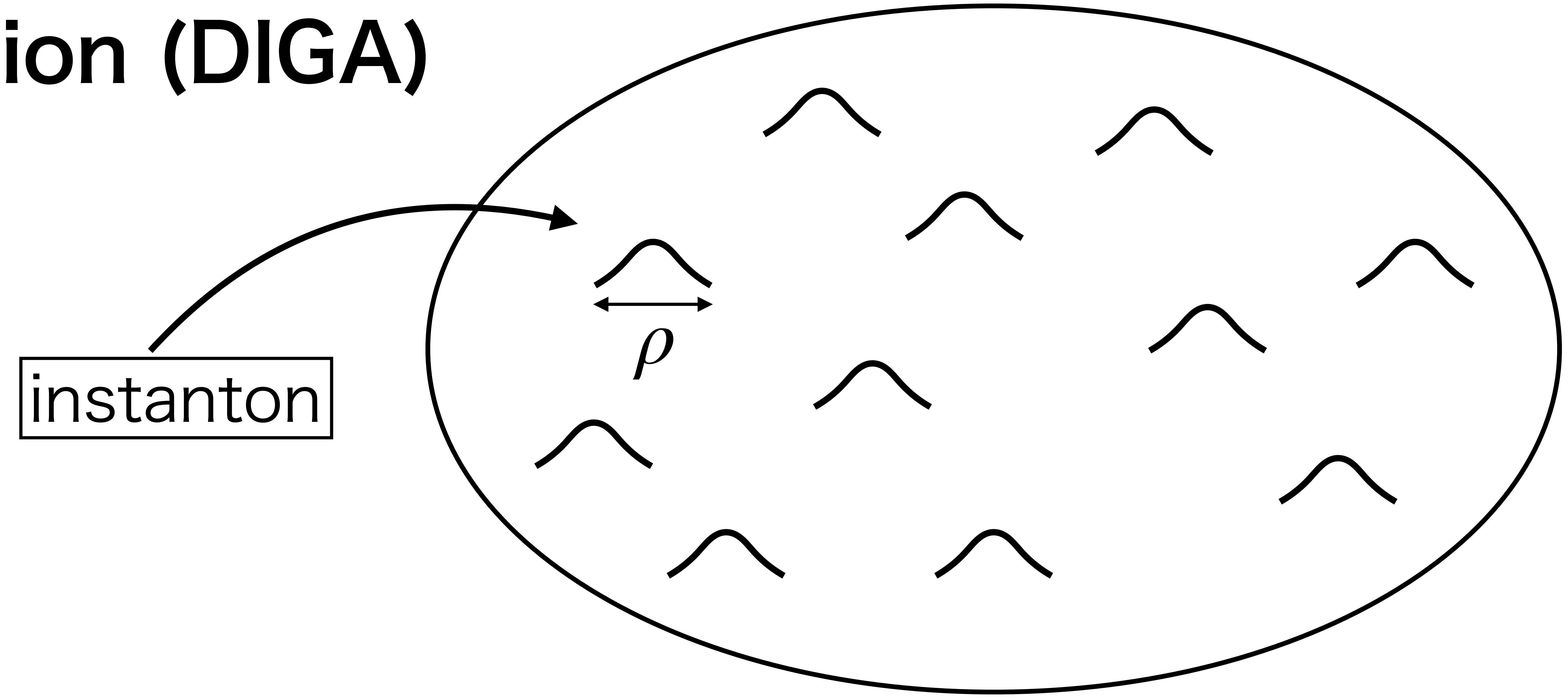
Vacuum energy in DIGA

Dilute instanton gas approximation (DIGA)

Instanton size ρ : sufficiently small

(Each instantons do not overlap)



Dilute instanton gas configuration

Vacuum energy : $E(\theta) \propto 1 - \cos\theta$

This approx. is valid at the weak coupling.

The vacuum energy is nontrivial in the strong coupling regime

But it is difficult to analyze at strong coupling⋯

Vacuum energy in non-DIGA

Expectation : Vacuum energy has multi-branch structure

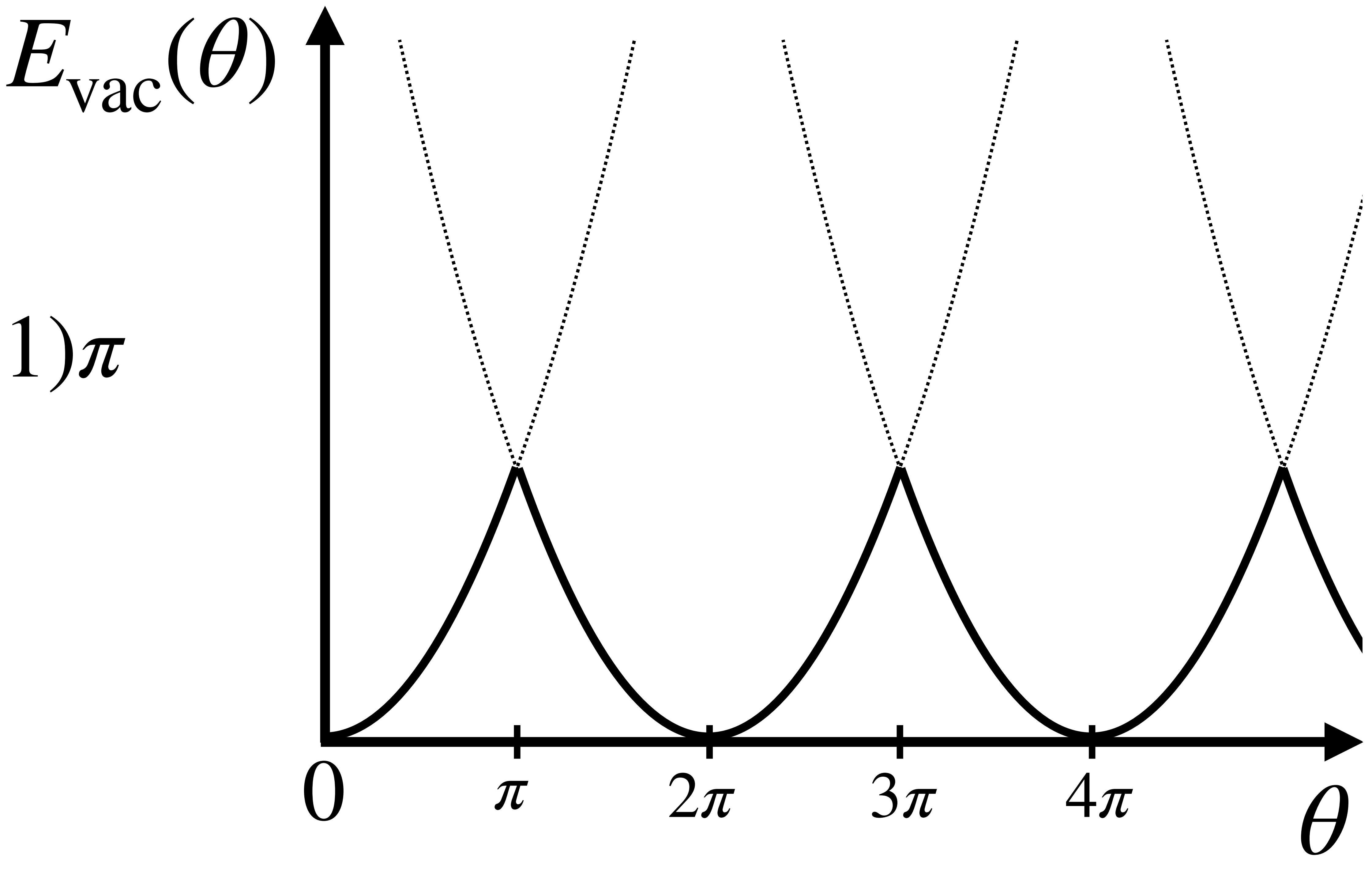
Vacua (all branch) = True vacuum (1 branch) + Metastable vacua (other branches)

Natural candidate

- Trivially gapped at $\theta \notin (2\mathbb{Z}+1)\pi$
- CP symmetry breaking at $\theta \in (2\mathbb{Z}+1)\pi$

At $\theta = (2\mathbb{Z}+1)\pi$, true vacuum $\rightarrow$ metastable vacuum, one of metastable vacua $\rightarrow$ true vacuum

Evidence : Large N [Witten, '80], Holographic QCD [Witten, '98], etc⋯



— ⋯ true vacuum

······· ⋯ metastable vacuum

One branch of vacuum energy

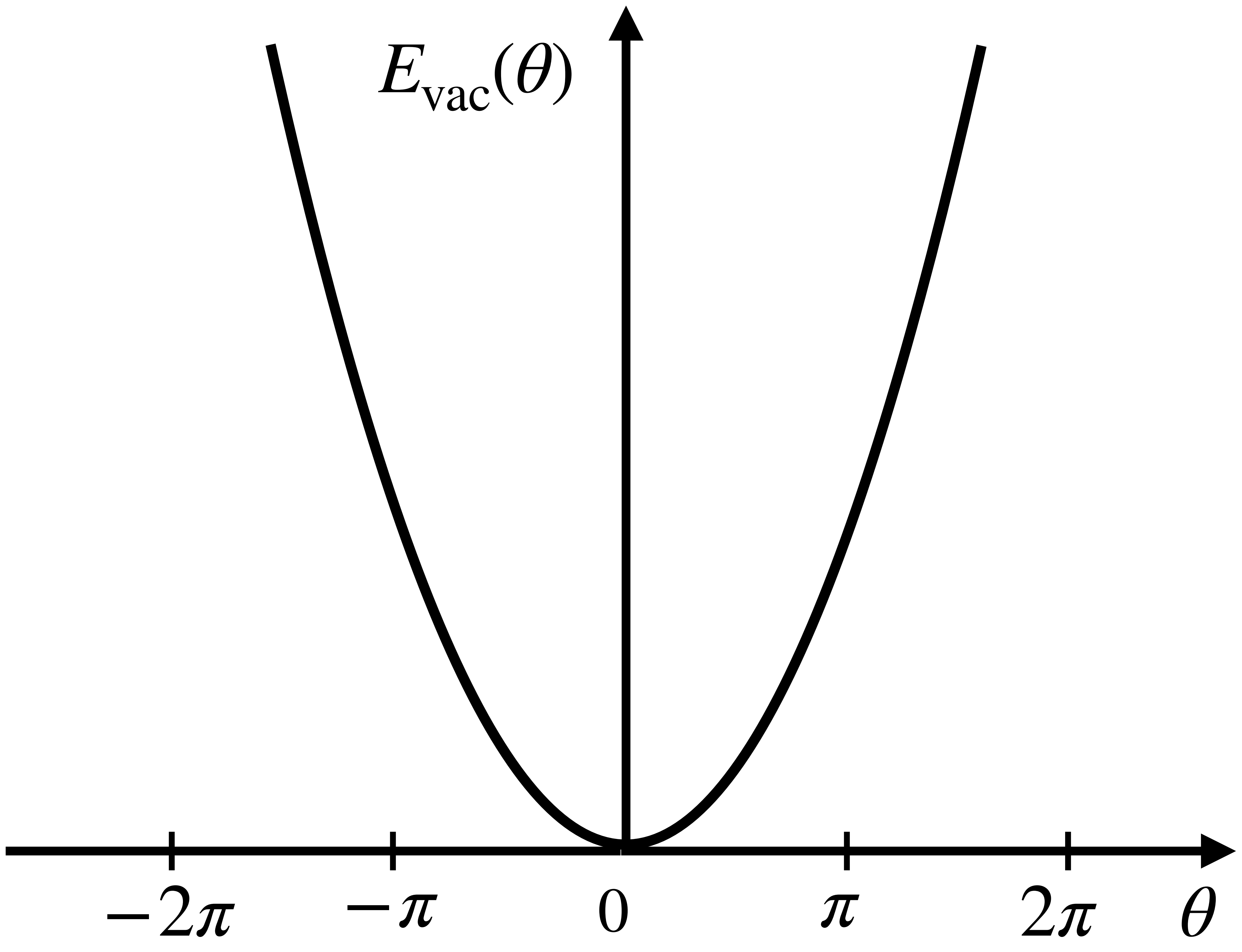
One branch of vacuum energy = axion potential

- The energy of true vacuum is not smooth and have 2π - periodicity.

$\rightarrow$ θ is 2π periodic parameter.

- Each branches of vacuum energy are analytic functions which are not 2π - periodic.

$\rightarrow$ θ is not 2π periodic parameter.



One branch of vacuum energy

I would like to find whole behavior of one branch

However it is difficult to simulate θ angle because of sign problem.

I will use 2d $\mathbb{C}P^{N-1}$ model instead of 4d pure Yang-Mills theory.

2. 2d $\mathbb{C}P^{N-1}$ model

2d $\mathbb{C}P^{N-1}$ model

$$\mathscr{L} = (D_\mu z)^\dagger D^\mu z + D\left(z^\dagger z - N/g_0^2\right) + \frac{i\theta}{2\pi}E$$

z : N complex scalar field, A_μ : aux. $\mathrm{U(1)}$ - gauge field, D : aux. scalar field,

g_0 : 't Hooft coupling, $E := F_{12}$: Euclidean electric field

Similarities with 4d pure Yang-Mills theory

- Strong coupling theory
- Topological θ term, instanton
- Vacuum energy has multi-branch structure (from large N analysis)

Vacuum energy in large N limit

I will focus on the branch that is true vacuum around $\theta = 0$

One branch of vacuum energy [D'adda, Lüscher, '78]

$$E_{\mathrm{vac}}(\theta) \simeq \frac{3N\Lambda^2}{2\pi}\left(\frac{\theta}{N}\right)^2 \quad \text{at } \theta \sim O(1)$$

However, if $\theta/N \sim O(1)$ ($\theta \sim O(N)$), higher order terms of θ/N contribute to vacuum energy.

→ At large θ, the vacuum energy is still nontrivial even in large N limit.

Goal : Find behavior of vacuum energy at large θ in large N limit

3. Method & Result

Summary of method

Basically our method is a **saddle point approximation** (Large N)

Step 1 : Calculate the effective Lagrangian $\mathscr{L}_{\mathrm{eff}}(E, D)$ (by performing the path integral over $z, z^\dagger$)

Step 2 : Determine an analytic form of $\mathscr{L}_{\mathrm{eff}}(E, D)$

Step 3 : Find a saddle point $(E(\theta), D(\theta))$ (such that $(E(0), D(0)) = (0, \Lambda^2)$) & Evaluate the eff. Lagrangian on the saddle point

Vacuum energy : $E_{\mathrm{vac}} = \mathrm{Re}\, \mathscr{L}_{\mathrm{eff}} |_{\mathrm{saddle}}$

Decay rate : $\Gamma = -2\, \mathrm{Im} \mathscr{L}_{\mathrm{eff}} |_{\mathrm{saddle}}$

Step 1. Eval. of eff. Lagrangian

Performing path integral over $z, z^\dagger$

$$S_{\text{eff}} \supset N\mathrm{Tr}\log(-D_\mu D^\mu + D) = -\int \mathrm{d}^2x \int_\varepsilon^\infty \frac{\mathrm{d}t}{t}\mathrm{Tr}e^{-t[(\partial_\mu + iA_\mu)^2 + D]}$$

Choice of a constant configuration

$$A_\mu = -\epsilon_{\mu\nu}x_\nu E/2,\ E = \text{const.},\ D = \text{const.}$$

$\mathrm{Tr}e^{-t[(\partial_\mu + iA_\mu)^2 + D]}$ = Trace of Boltzmann factor in Hilbert space of Landau level

$$\int_\varepsilon^\infty \frac{\mathrm{d}t}{t}\mathrm{Tr}e^{-t[(\partial_\mu + iA_\mu)^2 + D]} = \int \mathrm{d}^2x \int_\varepsilon^\infty \frac{\mathrm{d}t}{4\pi t}\frac{Ee^{-tD}}{\sinh Et}$$

Step 2. Analytic form of eff. Lagrangian

For $E > 0$

$$\int_{\varepsilon}^{\infty} \frac{\mathrm{d}t}{4\pi t} \frac{E e^{-tD}}{\sinh Et} = D\left[\frac{2E}{D}\log\Gamma\left(\frac{1}{2}+\frac{D}{2E}\right) + \log 2\varepsilon E - \frac{E}{D}\log 2\pi + \gamma + \frac{1}{\varepsilon D}\right] + O(\varepsilon)$$

$g_0 \longrightarrow g$: renormalization

$$\mathscr{L}_{\mathrm{eff}} = \frac{\mathrm{i}\theta}{2\pi}E - \frac{N}{4\pi}\left[2E\log\Gamma\left(\frac{1}{2}+\frac{D}{2E}\right) + D\log\frac{2E}{\Lambda^2} - E\log 2\pi\right]$$

up to higher derivative terms of (E, D)

Note : We get the analytic form for $E < 0$ by $E \to -E$

Step 3. Find a saddle point

Ansatz : There is a saddle point for constant E and D

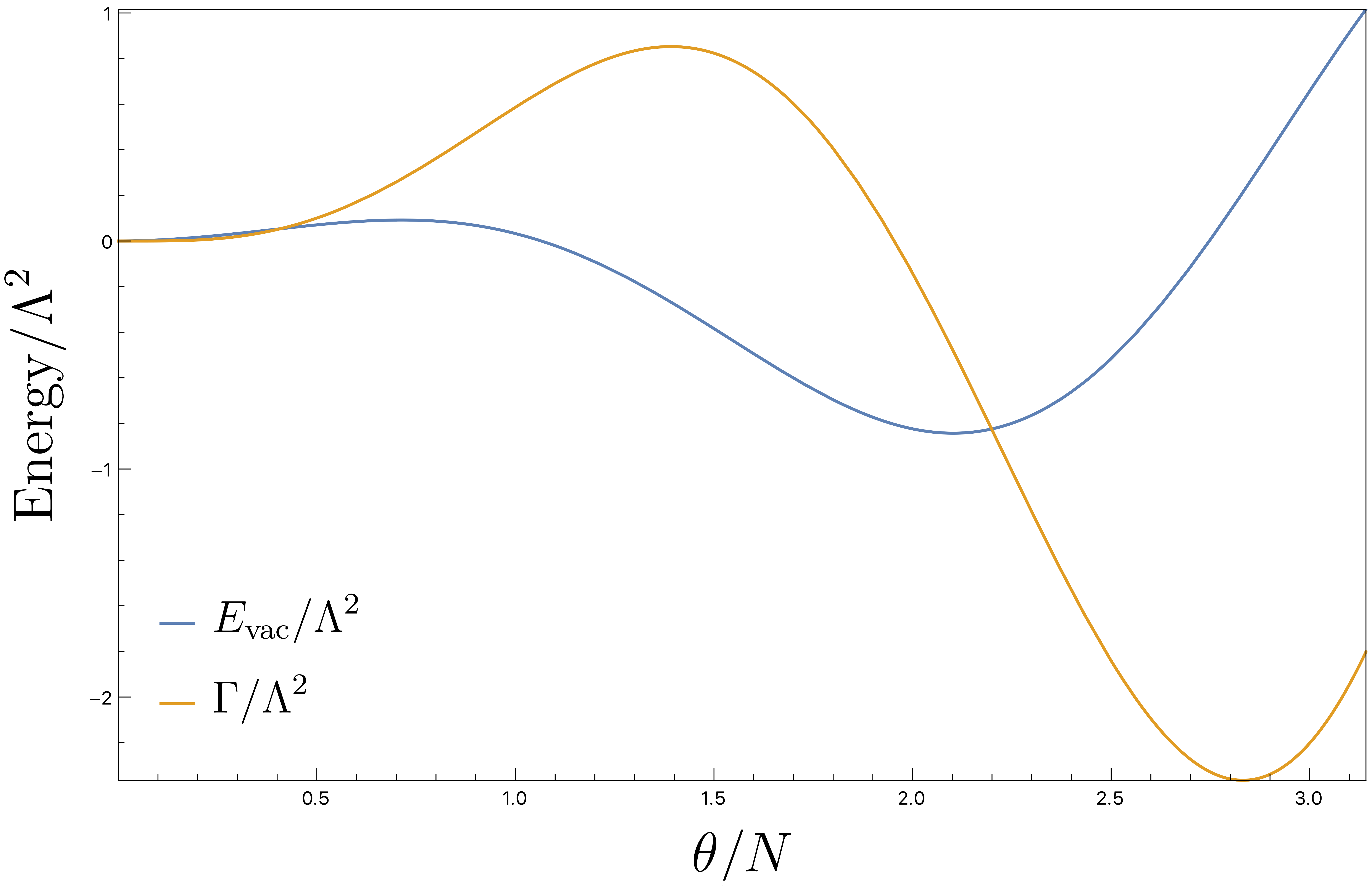
The higher derivative terms in $\mathscr{L}_{\text{eff}}$ are neglected.

$$\frac{\partial \mathscr{L}_{\text{eff}}}{\partial E} = \frac{\partial \mathscr{L}_{\text{eff}}}{\partial D} = 0$$

We find a saddle point such that $(E(0), D(0)) = (0, \Lambda^2)$ by numerical calculation. (Solve the saddle point condition)

Result

$E_{vac}(\theta = 0)$ is set to zero.



- The vacuum becomes no longer metastable for $\theta/N \gtrsim 0.4$ because of $E_{vac} \sim \Gamma$. (If vacuum is metastable, $E_{vac} \gg \Gamma$)
- $E_{vac} < 0$ for $1.1 \lesssim \theta/N \lesssim 2.7$

Puzzle ?

Physically negative vacuum energy is strange because

- Negative vacuum energy is smaller than the true vacuum energy ($E_{\mathrm{vac}} \sim 0$ for true vacuum).
- Saddle point which gives $E_{\mathrm{vac}} < 0$ have large $\exp(-S_{\mathrm{eff}})$, contribution of this saddle to path integral is larger than true vacuum.
- $E_{\mathrm{vac}} < 0$ exist in each branches. (Each branches are labeled by integers n. By $\theta \to \theta - 2\pi k$, we get branch $n = k$ from branch $n = 0$. If k/N is subleading, branch $n = k$ is almost same as branch $n = 0$.).

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We have not understood the solution for this puzzle.

I will show you possibilities of the solution.

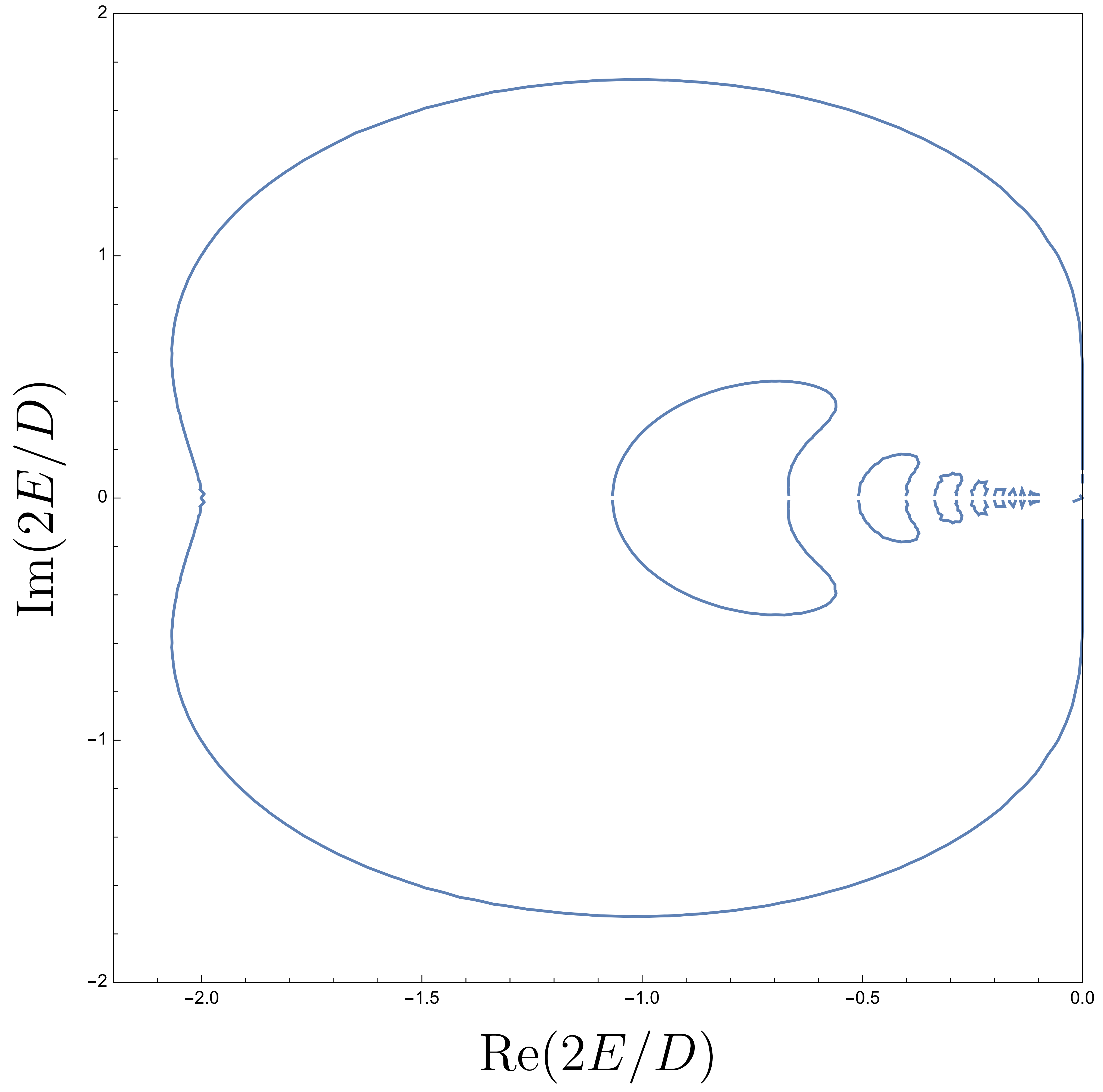
Possibilities of solution for the puzzle

1. The saddle point in previous slide does not contribute, other saddle points contribute (There are infinitely saddle points.).

2. Lefschetz thimble method is invalid (so we cannot use the saddle point approximation.).

Other saddles

The plot of saddles as functions of θ angle



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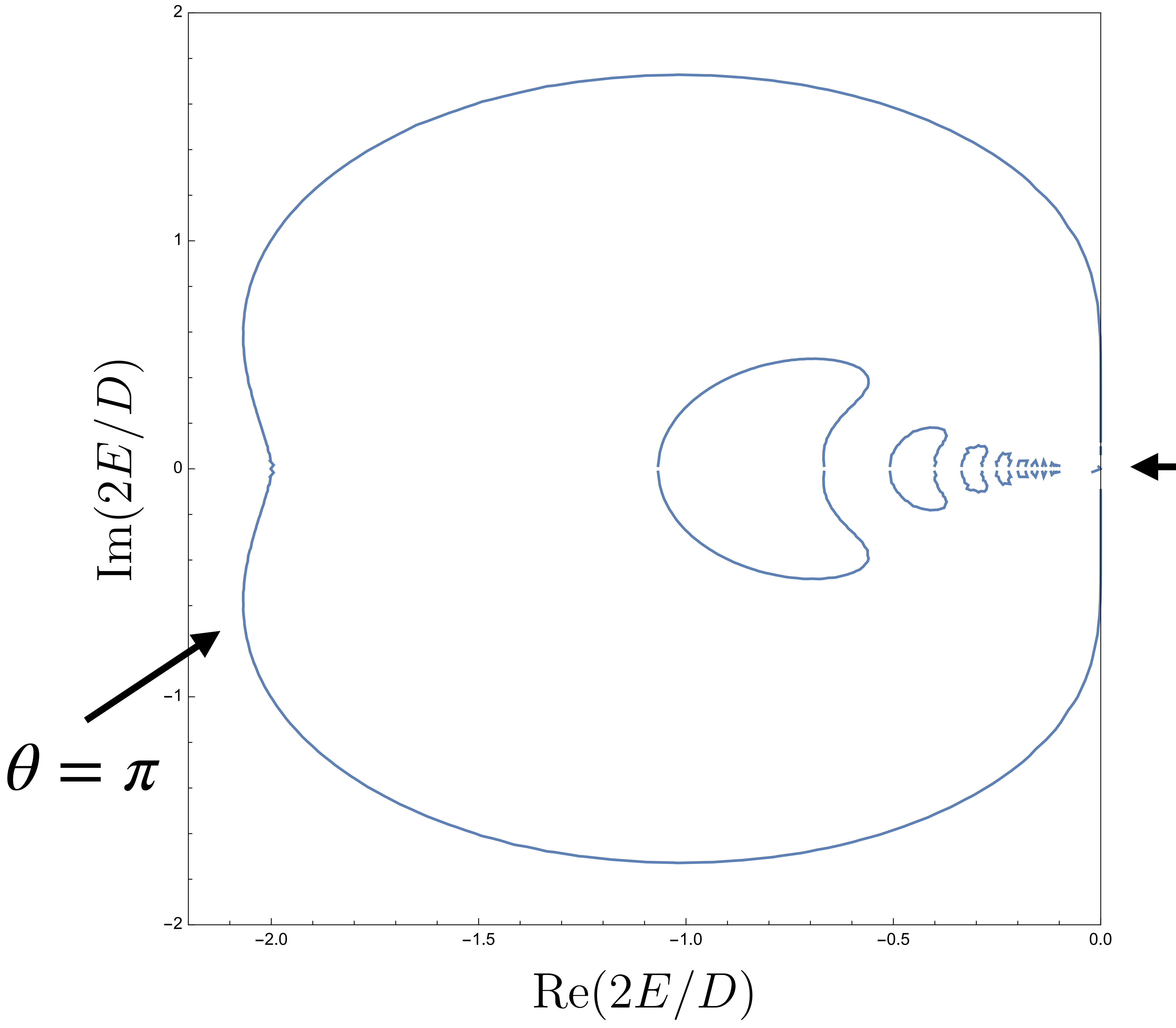
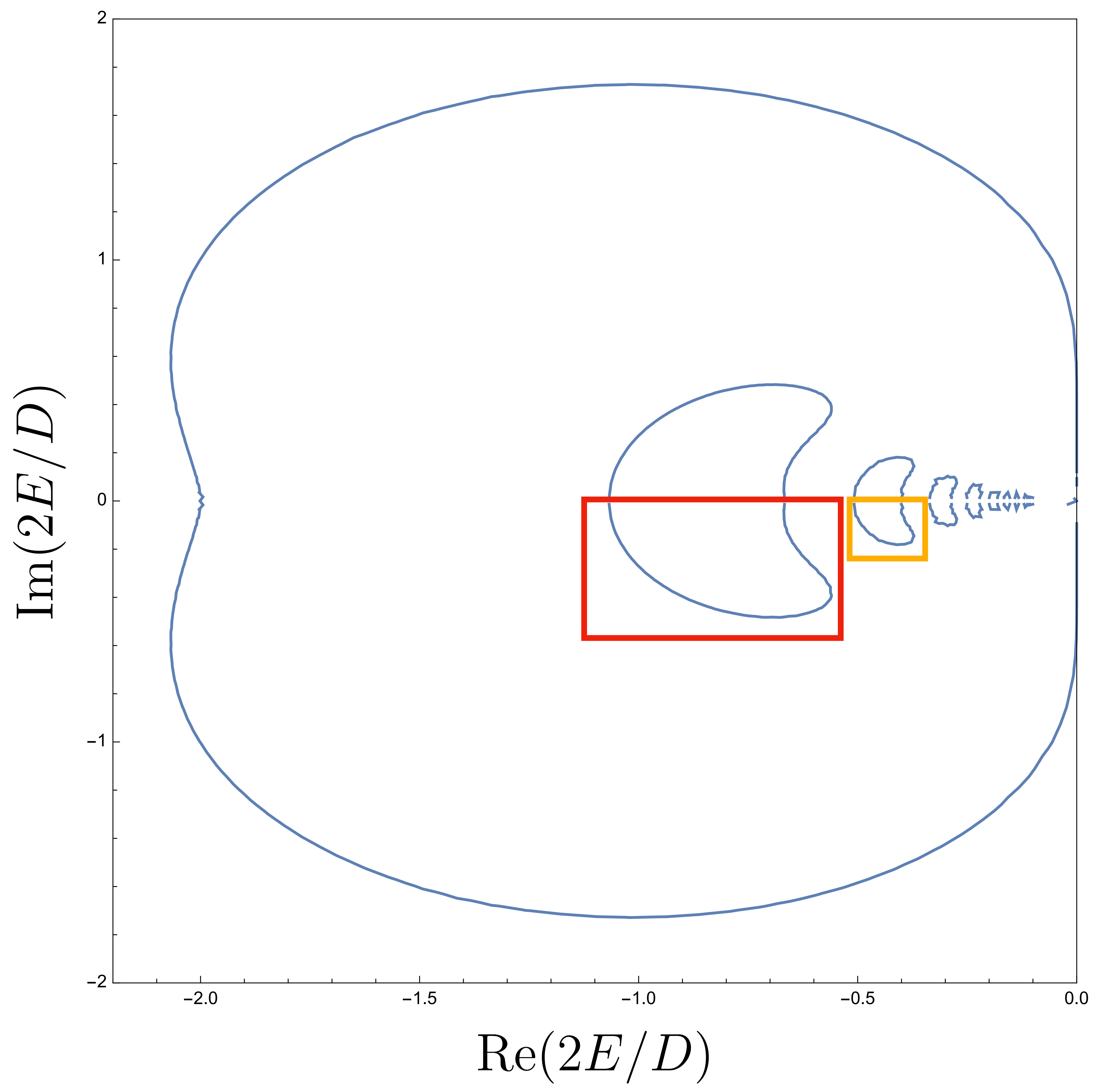


The saddle point in previous slide

$\theta = 0$

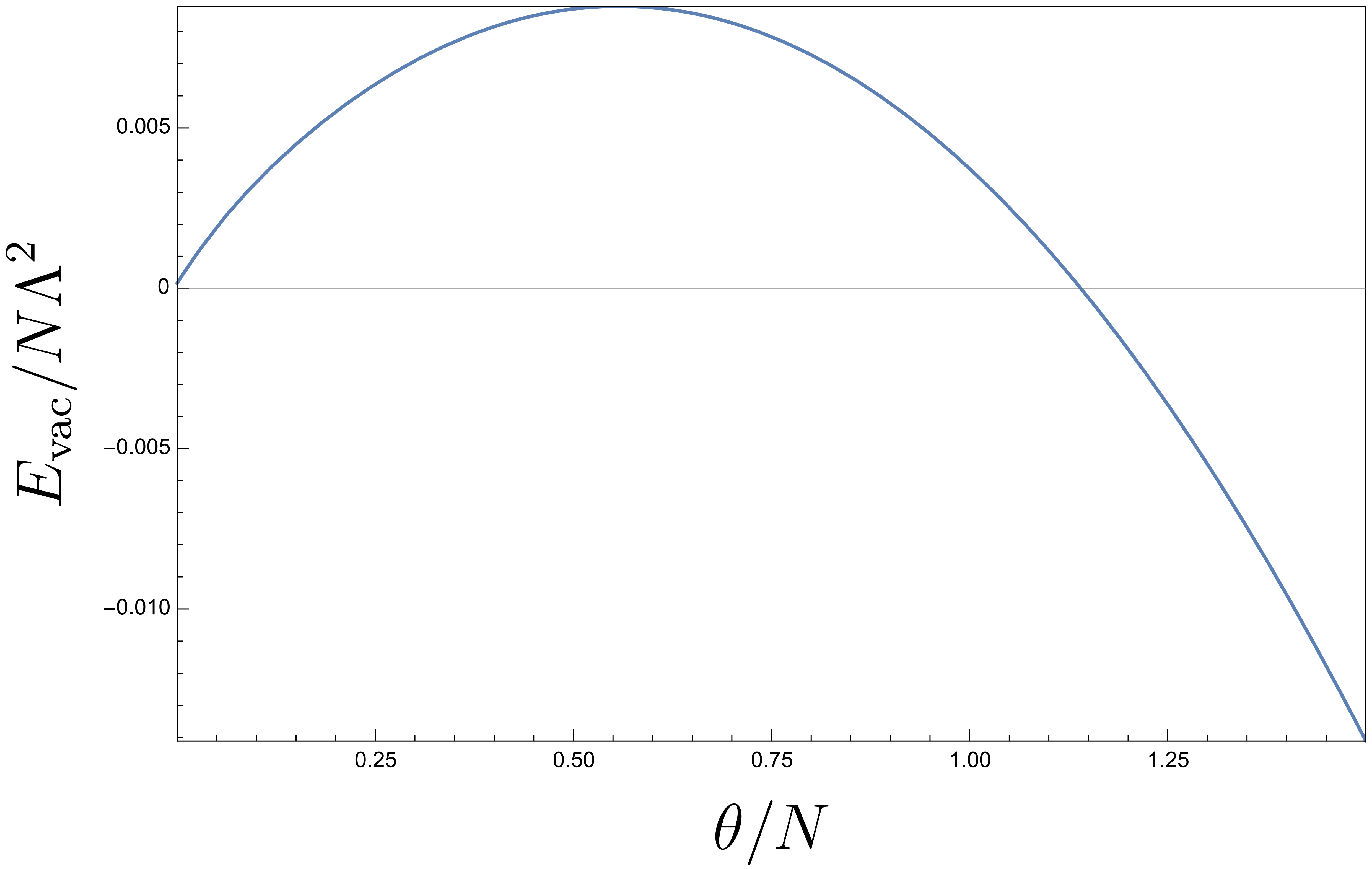
Other saddles

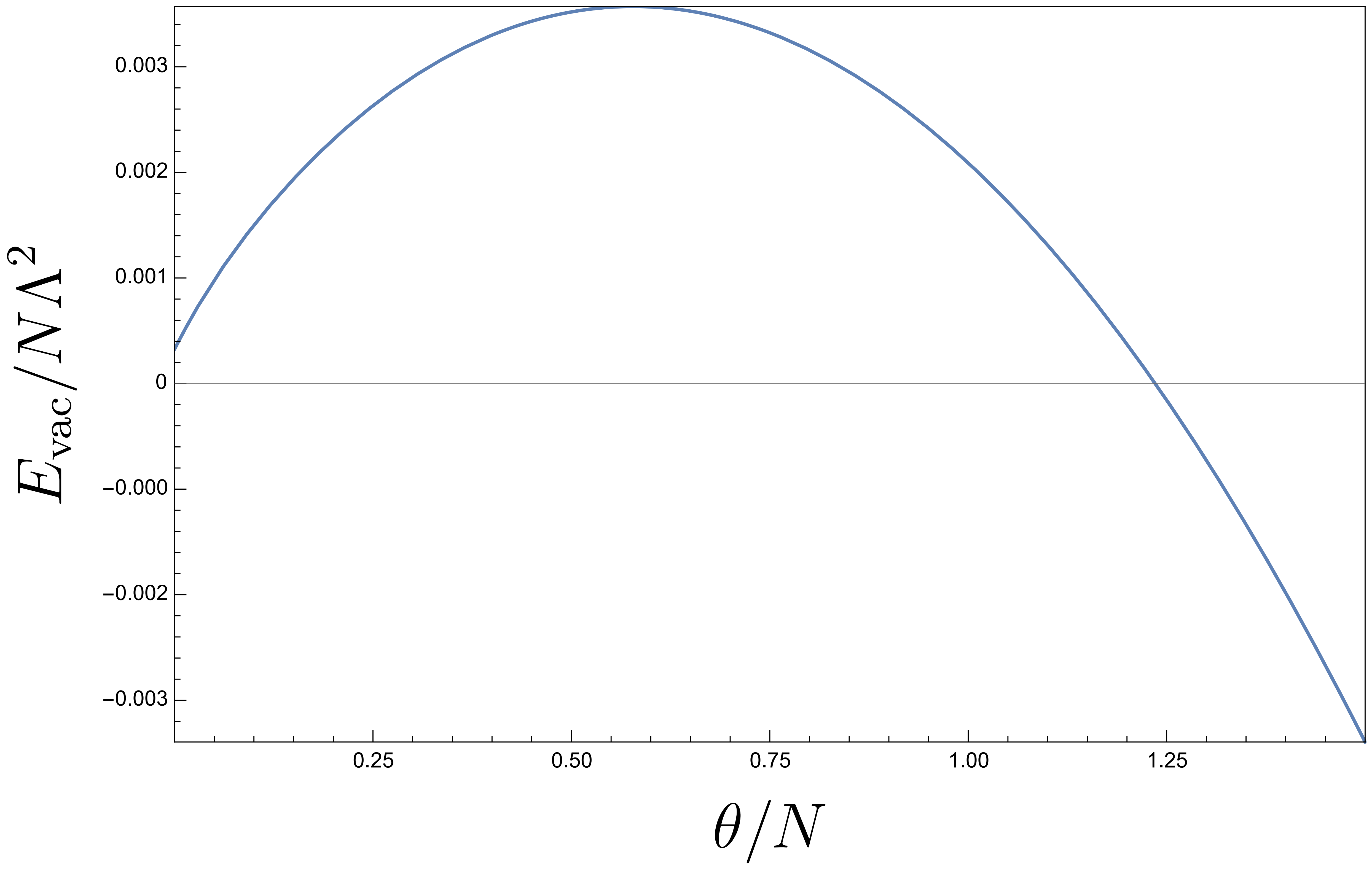
I will show you some example of vacuum energy at other saddles.



I will focus on two cases.

Other saddles





Left case $E_{\rm vac} < 0$ for $\theta > 1.15N$

Right case $E_{\rm vac} < 0$ for $\theta > 1.25N$

They have large $\exp(-S_{\rm eff})$ near θ which gives $E_{\rm vac} < 0$ for first saddle.

Other saddles

As long as we research, 10 saddles have large $\exp(-S_{\text{eff}})$ for large θ.

It seems that the puzzle is not solved by this method.

But we are not sure because there are many other saddle points.

Lefschetz thimble method

We would like to deform original path integral contour $(-\infty, \infty)$ by Cauchy theorem (Lefschetz thimble method)

$$\mathscr{L}_{\text{eff}} \supset -\int_{\varepsilon}^{\infty} \frac{\mathrm{d}t}{4\pi t} \frac{Ee^{-tD}}{\sinh Et} \sim \sum_{k \in \mathbb{Z}} (2k)! E^{2k} \text{ : asymptotic expansion}$$

→ The effective Lagrangian is not analytic at $E = 0$

We cannot deform the original contour directly because $E = 0$ is on the original contour.

It may not be possible to use the (standard) saddle point approximation in this case.

Summary

- We studied one branch of the vacuum energy beyond $\theta \sim O(1)$ of large N 2d $\mathbb{C}P^{N-1}$ model. (Motivation : axion potential, 't Hooft anomaly)
- At large θ, there are saddle points which have larger contributions to the path integral than the true vacuum.
- It is not obvious whether we can use the saddle point approximation or not.