

# The index of lattice Dirac operators and K-theory



Hidenori Fukaya (Osaka U.)

Shoto Aoki(U. Tokyo), HF, Mikio Furuta (U. Tokyo), Shinichiroh Matsuo(Nagoya U.), Tetsuya Onogi(Osaka U.), and Satoshi Yamaguchi (Osaka U.), [arXiv:2407.17708](https://arxiv.org/abs/2407.17708)



# What is the index of Dirac operator?

$$D\psi = 0 \quad D := \gamma^\mu (\partial_\mu + iA_\mu)$$

$$\underbrace{\text{Ind}(D)}_{n_+ - n_-} = \frac{1}{32\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \text{tr}(F_{\mu\nu} F_{\rho\sigma})$$

#sol with + chirality      #sol with - chirality

Topological charge

Index theorem

Very important both in physics (including QCD) and mathematics to understand gauge field topology, which is nonperturbative.

# Difficulty in lattice gauge theory

Both of Dirac index and topology are difficult on the lattice:

- It is difficult to define the chiral zero modes, since the standard lattice Dirac operators break the chiral symmetry.
- Lattice discretization of space time makes the topology not well-defined.

# A traditional solution = overlap Dirac operator

With the overlap Dirac operator [Neuberger 1998] satisfying the Ginsparg-Wilson relation [1982],

$$\gamma_5 D_{ov} + D_{ov} \gamma_5 = a D_{ov} \gamma_5 D_{ov}$$

a modified **chiral symmetry is exact** [Luescher 1998],

$$q \rightarrow e^{i\alpha\gamma_5(1-aD_{ov})} q, \quad \bar{q} \rightarrow \bar{q} e^{i\alpha\gamma_5}.$$

**and the index is well-defined:**

[Hasenfratz et al. 1998]

$$\text{Ind} D_{ov} = \text{Tr} \gamma_5 \left( 1 - \frac{a D_{ov}}{2} \right)$$

but this definition is so far limited to the even-dimensional flat periodic lattice.



This work = an alternative mathematical formulation of the lattice Dirac index.

In our formulation,

- No exact chiral symmetry is needed : **Wilson Dirac operator is good enough.**
- **K theory is used** to show equality to the continuum Dirac index.
- **Wider application than the overlap** Dirac operator to the systems with nontrivial boundaries in any dimensions.

# Phys-Math collaborators

## Physicists



Shoto Aoki



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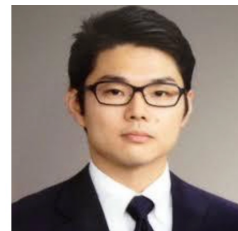


Satoshi  
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## Mathematicians



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We revisit the lattice index theorem with a K-theoretic treatment of Wilson Dirac op.

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## Nielsen-Ninomiya theorem [1981]

Nielsen-Ninomiya theorem [1981]:

If  $\gamma_5 D + D \gamma_5 = 0$ , we cannot avoid fermion doubling.

since the lattice discretization

$$p_\mu \rightarrow \frac{1}{a} \sin(p_\mu a) \quad \text{gives unphysical poles} \quad p_\mu = 0, \frac{\pi}{a}$$

$a$  :lattice spacing

## Overlap Dirac operator [Neuberger 1998]

$$D_{ov} = \frac{1}{a} (1 + \gamma_5 \text{sgn}(H_W)) \quad H_W = \gamma_5 (D_W - M) \quad M = 1/a$$

satisfies the GW relation:  $\gamma_5 D_{ov} + D_{ov} \gamma_5 = a D_{ov} \gamma_5 D_{ov}$   
and the action

$$S = \sum_x \bar{q}(x) D_{ov} q(x)$$

is invariant under the modified chiral rotation:

$$q \rightarrow e^{i\alpha\gamma_5(1-aD_{ov})} q, \quad \bar{q} \rightarrow \bar{q} e^{i\alpha\gamma_5}.$$

[Luescher 1998]

# Anomaly and index of the overlap Dirac operator

Moreover, it reproduces the anomaly.

$$q \rightarrow e^{i\alpha\gamma_5(1-aD_{ov})}q, \quad \bar{q} \rightarrow \bar{q}e^{i\alpha\gamma_5}.$$

$$Dq\bar{q} \rightarrow \exp [2i\alpha\text{Tr}(\gamma_5 + \gamma_5(1 - aD_{ov}))/2] Dq\bar{q}$$

and the index is well-defined:

$$\text{Ind}D_{ov} = \text{Tr}\gamma_5 \left( 1 - \frac{aD_{ov}}{2} \right) = n_+ - n_-$$

[Hasenfratz et al. 1998]

But  $D_{ov}$  is defined with the Wilson Dirac operator.

$$D_{ov} = \frac{1}{a} (1 + \gamma_5 \text{sgn}(H_W)) \quad H_W = \gamma_5 (D_W - M) \quad M = 1/a$$

$$\begin{aligned} \text{Ind} D_{ov} &= \text{Tr} \gamma_5 \left( 1 - \frac{a D_{ov}}{2} \right) = \underbrace{\text{Tr} \frac{\gamma_5}{2}}_{=0} - \frac{1}{2} \text{Tr} \text{sgn}(H_W) \\ &= -\frac{1}{2} \text{Tr} \text{sgn}(H_W) \end{aligned}$$

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What is this ???



$\eta$  invariant of the massive Wilson Dirac operator

$$-\frac{1}{2} \text{Tr} \, \text{sgn}(H_W) = -\frac{1}{2} \sum_{\lambda_{H_W}} \text{sgn}(\lambda_{H_W}) = -\frac{1}{2} \eta(H_W)$$

$$H_W = \gamma_5(D_W - M) \quad M = 1/a$$

This quantity is known as **the Atiyah-Patodi-Singer  $\eta$  invariant** (of the massive Wilson Dirac operator).

[Atiyah, Patodi and Singer, 1975]

# The Wilson Dirac operator and K-theory

$$\text{Ind}D_{ov} = -\frac{1}{2}\eta(H_W) \qquad H_W = \gamma_5(D_W - M)$$
$$M = 1/a$$

In this talk, we try to show a deeper mathematical meaning of the right-hand side of the equality,

and try to convince you that the massive Wilson Dirac operator is an equally good or even better object than  $D_{ov}$  to describe the gauge field topology in terms of K-theory [Atiyah-Hilzebruch 1959, Karoubi 1978...]

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# What is fiber bundle?

A united manifold of spacetime (= base manifold) and field (fiber= vector space)

$$\phi(x) \rightarrow (x, \phi) \in X \times F$$

Spacetime                      Field space  
= base space                      = fiber space

The product structure is realized only locally.  
In general, it is “twisted” by gauge fields (connections).

In mathematics, the (isomorphism class of) total space is denoted by  $E$

# Analogy found on the web

$X$  base space (space-time)

= your head

$F$  fiber (field)

= your hair

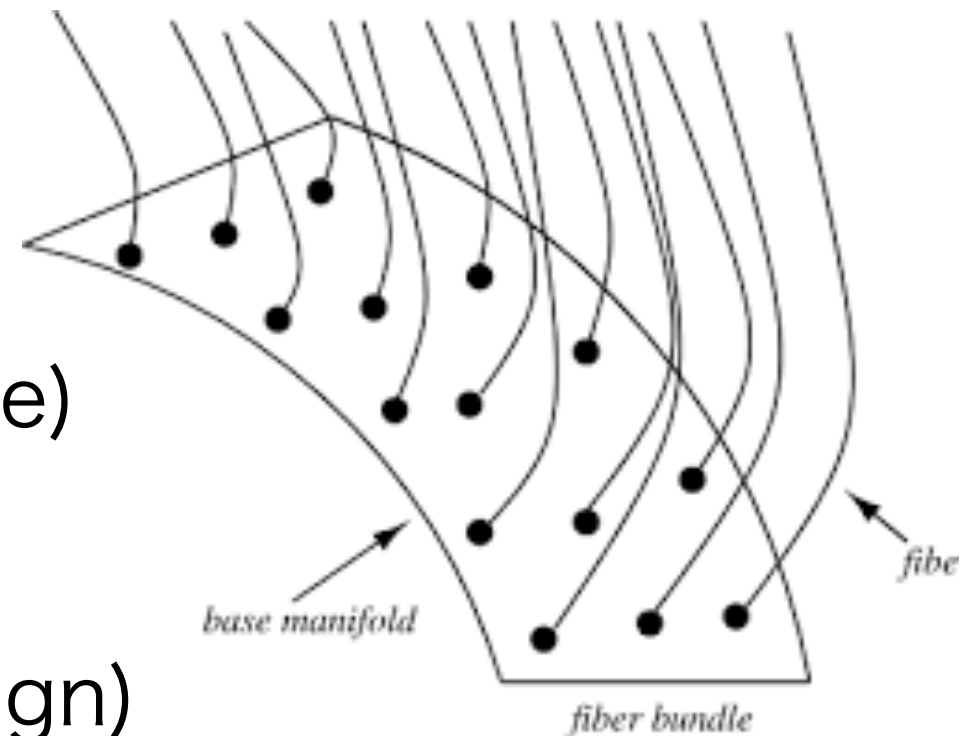
$E$  (= locally  $X \times F$ ) (total space)

= your hair style

Gauge field (connection)

= hair wax (local hair design)

Figure from Wolfram Math world



# What is K-theory?

- A mathematical theory which **classifies the fiber (vector) bundles** [or more general additive categories].
- One of generalized cohomology theories (**stronger** than ordinary cohomology) : without the dimension axiom:  
$$H^{n>0}(\text{point}) = \{0\}$$
- It is **weaker** than homotopy theory but **easier to “compute”**.

# $K^0(X)$ group

The element of  $K^0(X)$  group is given by  $[E_1, E_2]$   
[ ] denotes the equivalence class.

Equivalently, we can consider an operator and its conjugate,

$$D_{12} : E_1 \rightarrow E_2 \quad D_{12}^\dagger : E_2 \rightarrow E_1$$

to represent the same element by  $[E, D, \gamma]$   
where

$$E = E_1 \oplus E_2, \quad D = \begin{pmatrix} & D_{12} \\ D_{12}^\dagger & \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

\*  $K^0$  group classifies the Dirac operator which anticommutes with chirality operator.

# K-theory pushforward (Gysin map)

When you are interested in global structure only,  
You can forget about details of the base manifold  $X$  by taking  
“one-point compactification” by the K-theory pushforward :

$$\begin{aligned} G : K^0(X) &\rightarrow K^0(\text{point}) && \text{The map just forgets all} \\ [E, D, \gamma] &\rightarrow [H_E, D, \gamma] && \text{but the chiral symmetry.} \end{aligned}$$

$H_E$  : The whole Hilbert space on which  $D$  acts.

Cf. forgetful map  $f : \mathbb{C} \rightarrow \mathbb{R}^2, \quad a + ib \rightarrow (a, b)$

Many information is lost but one (the Dirac operator index) remains.



# Suspension isomorphism

“point” can be suspended to an interval:

There is an isomorphism between 

$$K^0(\text{point}) \cong K^1(I, \partial I)$$

$$[H_E, D, \gamma] \leftrightarrow [H_E \times I, D_t]$$

where “1” denotes removal of the chirality operator.

Instead, the Dirac operator must become one-to-one (no zero mode) at the two endpoints :  $\partial I$

Physical meaning of the isomorphism will be given soon later .

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# Atiyah-Singer index

$$\overbrace{n_+ - n_-}^{\text{Ind}(D)} = \frac{1}{32\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \text{tr}(F_{\mu\nu}F_{\rho\sigma})$$

#sol with + chirality      #sol with - chirality

Index theorem

In the standard formulation, we need a massless Dirac operator and its zero modes with definite chirality :  $[H_E, D, \gamma] \in K^0(\text{point})$

But we will show that it is isomorphic to

$$[H_E \times I, \gamma(D + m)] \in K^1(I, \partial I)$$

# Eigenvalues of continuum massive Dirac operator

$$H(m) = \gamma_5(D_{\text{cont.}} + m) \quad \begin{array}{l} \text{on Euclidean even-dimensional manifold.} \\ \text{Gauge group is U(1) or SU(N)} \end{array}$$

$$\text{For } D_{\text{cont.}}\phi = 0, \quad H(m)\phi = \gamma_5 m\phi = \underbrace{\pm}_{\text{chirality}} m\phi.$$

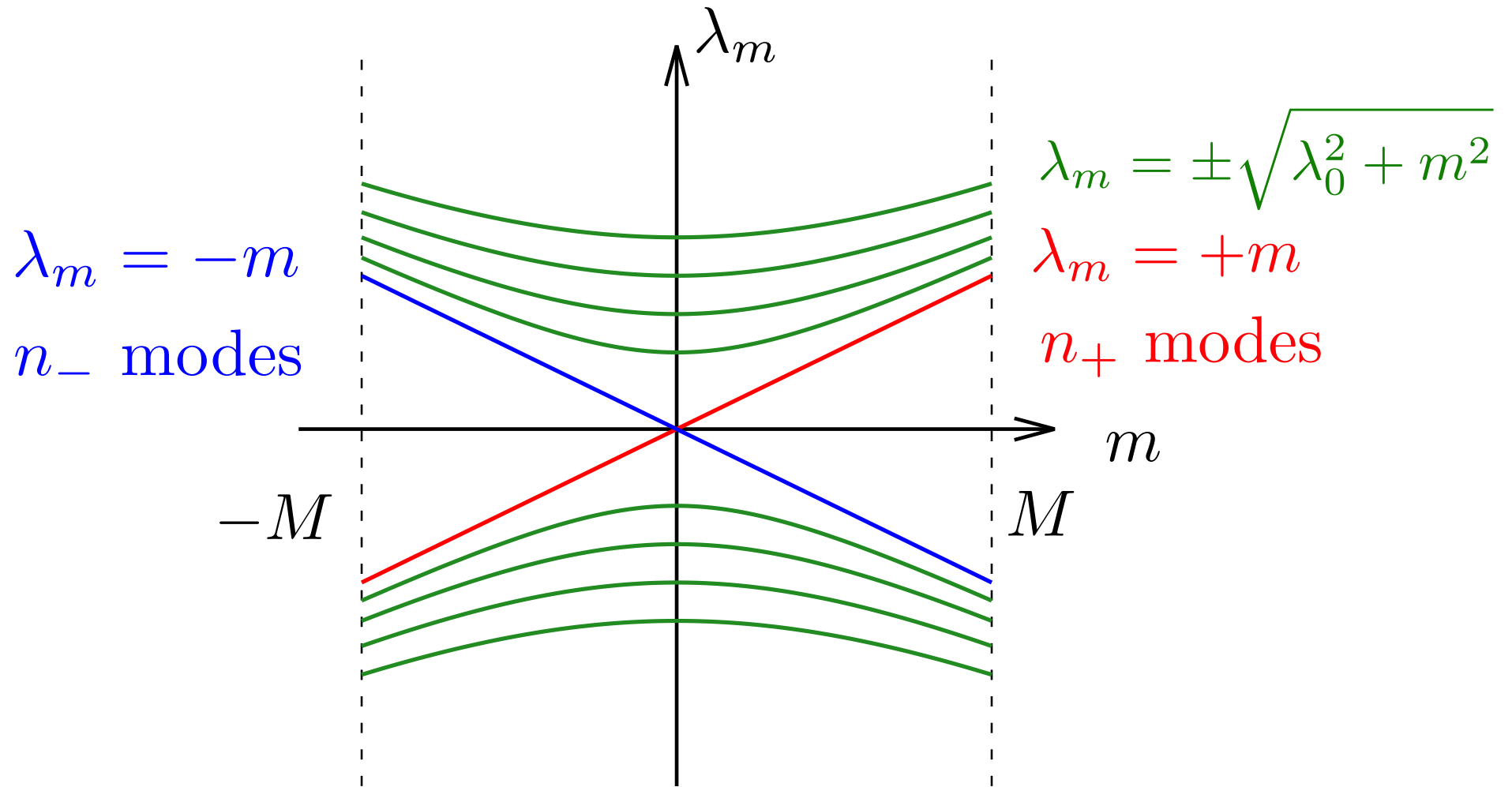
$$\text{For } D_{\text{cont.}}\phi \neq 0, \quad \{H(m), D_{\text{cont.}}\} = 0.$$

$$\text{The eigenvalues are paired: } H(m)\phi_{\lambda_m} = \lambda_m\phi_{\lambda_m}$$

$$H(m)D_{\text{cont.}}\phi_{\lambda_m} = -\lambda_m D_{\text{cont.}}\phi_{\lambda_m}$$

$$\text{As } H(m)^2 = -D_{\text{cont.}}^2 + m^2, \text{ we can write them } \lambda_m = \pm\sqrt{\lambda_0^2 + m^2}$$

Spectrum of  $H(m) = \gamma_5(D_{\text{cont.}} + m)$



# Spectral flow = Atiyah-Singer index = $\eta$ invariant

$n_+$  = # of zero-crossing eigenvalues from - to +      $H(m) = \gamma_5(D_{\text{cont.}} + m)$

$n_-$  = # of zero-crossing eigenvalues from + to -

$n_+ - n_- =:$  **spectral flow** of  $H(m)$     $m \in [-M, M]$

Equivalent to the eta invariant: whenever an eigenvalue crosses zero,

$\eta(H(m))$  jumps by two.

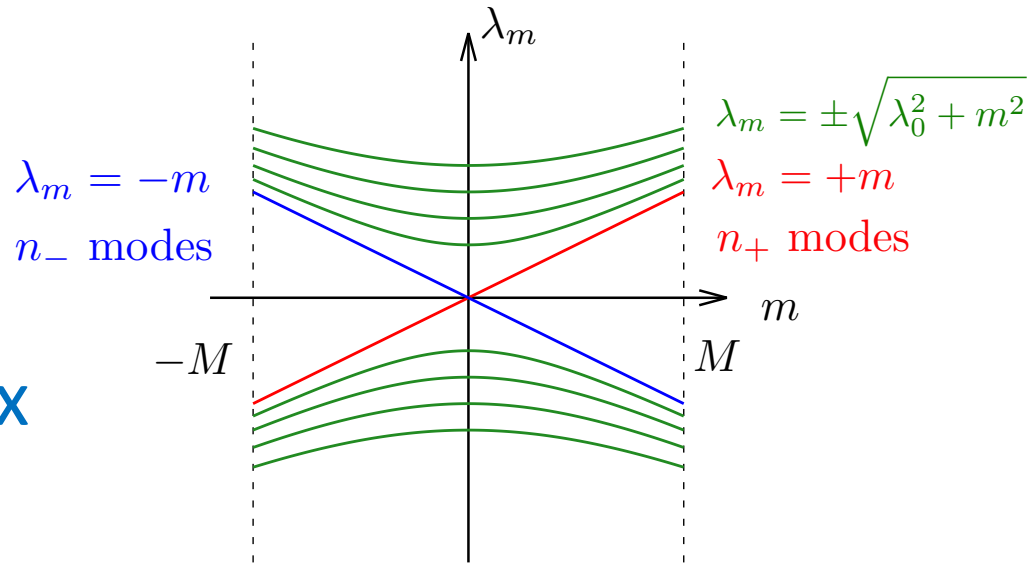
$$\eta(H) = \sum_{\lambda \geq 0}^{\text{reg}} - \sum_{\lambda < 0}^{\text{reg}}$$

$$\frac{1}{2}\eta(H(M)) - \frac{1}{2}\eta(H(-M)) = n_+ - n_-.$$

Pauli-Villars subtraction

# Suspension isomorphism in K theory

Massless=  
counting index  
by points



Massive=  
counting  
index by lines

$$K^0(\text{point}) \cong K^{-1}(I, \partial I)$$

point

line=interval

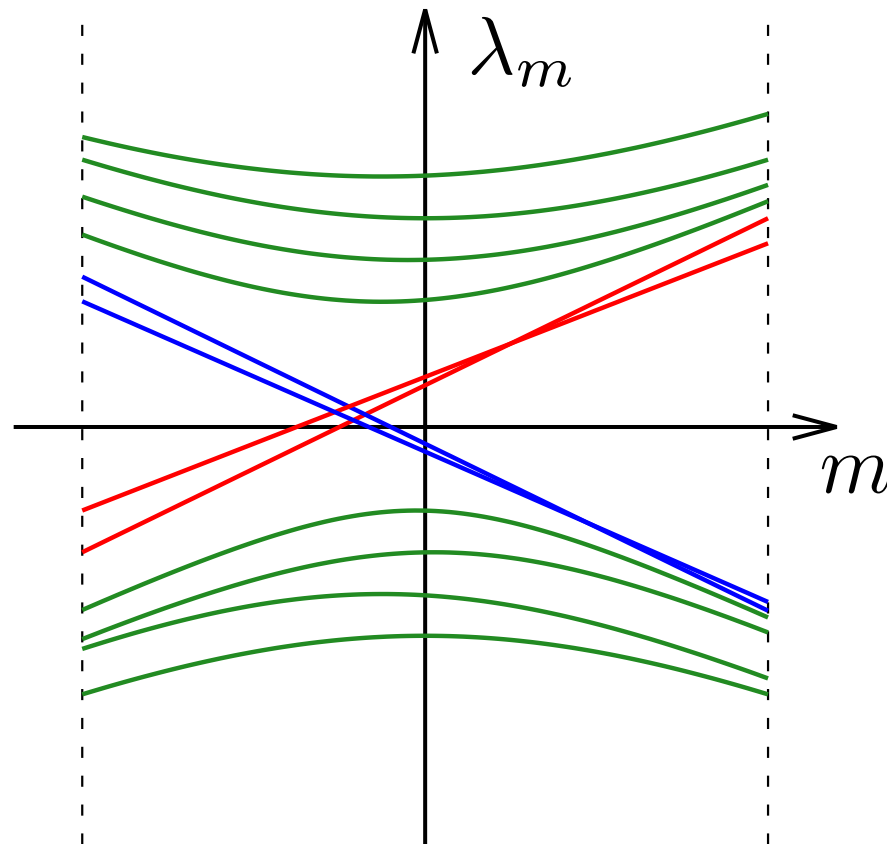
With chirality operator

Without chirality operator

$\Rightarrow$  The two definitions of the index agree.

With chiral symmetry breaking regularization (on a lattice), counting points (**massless**) is difficult but counting lines (**massive**) still works.

Standard definition:  
Where is  $m=0$ ?  
What are zero modes?



Eta invariant:  
If  $m = \pm M$  points are gapped, we can still count the crossing lines.

Note) this fact is known even before overlap Dirac by Itoh-Iwasaki-Yoshie 1982 and other literature, but its mathematical meaning was not discussed. See also Adams, Kikukawa-Yamada, Luescher, Fujikawa, and Suzuki



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# Dirac operator in continuum theory

E : Complex vector bundle

Base manifold M: **2n-dimensional flat torus**  $T^{2n}$

Fiber F : vector space of rank r with a Hermitian metric

Connection : Parallel transport with **gauge field**  $A_i$

D : Dirac operator on sections of E

$$D_{\text{cont.}} = \gamma_i (\partial_i + A_i)$$

Chirality ( $Z_2$  grading) operator:  $\gamma = i^n \prod_i \gamma_i$

$$\{\gamma, D\} = 0, \{\gamma, \gamma_i\} = 0.$$

# Wilson Dirac operator on a lattice

We regularize  $T^{2n}$  is by a **square lattice with lattice spacing**  $a$

(The fiber is still continuous.)

We denote the bundle by  $E^a$  and

link variables :

$$U_k(\mathbf{x}) = P \exp \left[ i \int_0^a A_k(\mathbf{x}') dl \right],$$

$$D_W = \sum_i \left[ \gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{a}{2} \nabla_i^f \nabla_i^b \right]$$

**Wilson term**

$$a \nabla_i^f \psi(\mathbf{x}) = U_i(\mathbf{x}) \psi(\mathbf{x} + \mathbf{e}_i) - \psi(\mathbf{x})$$

$$a \nabla_i^b \psi(\mathbf{x}) = \psi(\mathbf{x}) - U_i^\dagger(\mathbf{x} - \mathbf{e}_i) \psi(\mathbf{x} - \mathbf{e}_i)$$

Note: In our paper, we consider "generalized link variables" to determine the gauge fields both in continuum and on a lattice simultaneously. But the standard Wilson line works, too.

## Definition of $K^1(I, \partial I)$ group

Let us consider a Hilbert bundle with

Base space  $I = \text{range of mass } [-M, M]$

boundary  $\partial I = \pm M$  points

Fiber space  $\mathcal{H} = \text{Hilbert space to which } D \text{ acts}$

$D_m$  : one-parameter family labeled by  $m$ .

We assume that  $D_{\pm M}$  has no zero mode.

The group element is given by equivalence classes of the pairs:

$[ (\mathcal{H}, D_m) ]$  **having the same spectral flow.**

Note:  $K^1$  group does **NOT** require any chirality operator.

## Definition of $K^1(I, \partial I)$ group

Group operation:  $[(\mathcal{H}^1, D_m^1)] \pm [(\mathcal{H}^2, D_m^2)] = [(\mathcal{H}^1 \oplus \mathcal{H}^2, \begin{pmatrix} D_m^1 & \\ & \pm D_m^2 \end{pmatrix})]$

Identity element:  $[(\mathcal{H}, D_m)]|_{\text{Spec.flow}=0}$

We compare  $[(\mathcal{H}_{\text{cont.}}, \gamma(D_{\text{cont.}} + m))]$  and  $[(\mathcal{H}_{\text{lat.}}, \gamma(D_W + m))]$

taking their difference, and confirm if **the lattice-continuum combined Dirac operator**

$$\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & f_a \\ f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

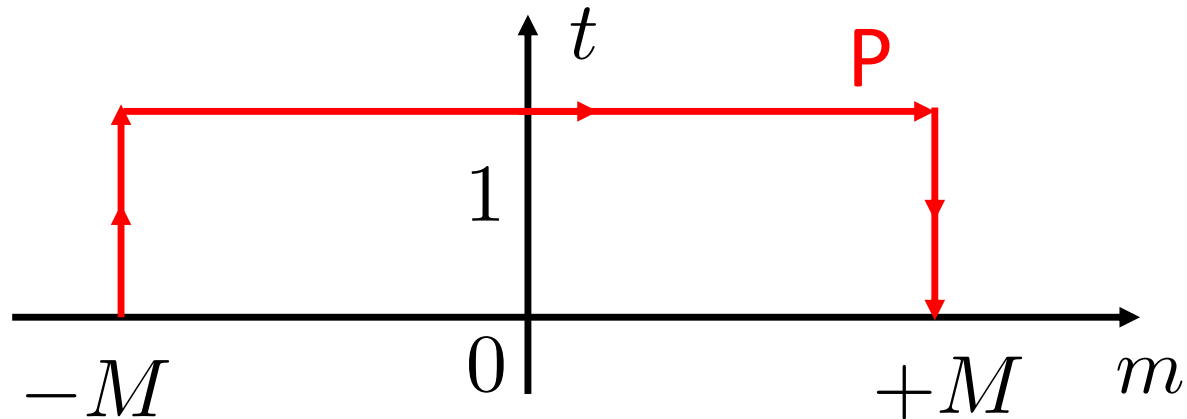
has Spectral flow =0 where  $f_a^*$   $f_a$  are “**mixing mass term**” with some “nice” mathematical properties (see our paper for the details).

# Main theorem

Consider a continuum-lattice combined Dirac operator

$$\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & t f_a \\ t f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

on the path  $P$  :



## Main theorem

There exists a finite lattice spacing  $a_0$  such that for any  $a < a_0$

$$\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & tf_a \\ tf_a^* & -\gamma(D_W + m) \end{pmatrix}$$

is invertible (having no zero mode) on the staple-shaped path P

[which is a sufficient condition for Spec.flow=0]

$\Rightarrow \gamma(D_{\text{cont.}} + m), \gamma(D_W + m)$  have the same spec.flow

$$\Rightarrow \frac{1}{2}\eta(\gamma(D - M))^{\text{PV reg.}} = \frac{1}{2}\eta(\gamma(D_W - M))$$

The continuum and lattice indices agree.

## Proof (by contradiction)

Assume  $\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & t f_a \\ t f_a^* & -\gamma(D_W + m) \end{pmatrix}$

has zero mode(s) at arbitrarily small lattice spacing.

$\Rightarrow$  For a decreasing series of  $\{a_j\}$

$$\begin{pmatrix} \gamma(D_{\text{cont.}} + m_j) & t_j f_{a_j} \\ t_j f_{a_j}^* & -\gamma(D_W^{a_j} + m_j) \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix} = 0$$

is kept.

$\Rightarrow$  In the continuum limit,  $m_\infty = t_\infty = 0$ .

**Contradiction with  $m^2 + t^2 > 0$  along the path P.**



# What not shown in this talk

Because of time limitation, we cannot explain the following details.

- The map  $f_a, f_a^*$  between lattice and continuum Hilbert spaces
- Convergence of  $f_a f_a^* \rightarrow 1, f_a^* f_a \rightarrow 1$ .
- Convergence of  $f_a^* D_W f_a \rightarrow D_{\text{cont}}$ .
- Elliptic estimate for the Wilson Dirac operator
- Relich theorem

Please see our paper [S. Aoki, HF, M. Furuta, S. Matsuo, T. Onogi, S. Yamaguchi, [arXiv:2407.17708](https://arxiv.org/abs/2407.17708)] or invite us to your (online) seminar.

[For other mathematical approaches, see Kubota 2020 and Yamashita 2021]

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The proof is given by lattice-continuum combined Dirac operator, which is gapped.

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Wilson Dirac operator is **equally good** as  $D_{ov}$  to describe the index.

$$\text{Ind}D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\text{cont.}} - M)) = \text{Ind}D_{\text{cont.}}$$

By  $K^1(I, \partial I)$   
for sufficiently small  
lattice spacings

Suspension  
isomorphism

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$$\text{Ind}D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\text{cont.}} - M)) = \text{Ind}D_{\text{cont.}}$$

↑  
By  $K^1(I, \partial I)$   
for sufficiently small  
lattice spacings

↑  
Suspension  
isomorphism

Or even better?

# Application to the manifolds with boundaries

Periodic b.c.

$$\text{Ind}D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\text{cont.}} - M)) = \text{Ind}D_{\text{cont.}}$$

With free b.c. (Shamir domain-wall fermion) we can show

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = -\frac{1}{2}\eta(\gamma_5(D_{DW}^{\text{cont.}})) = \text{Ind}_{\text{APS}}D^{\text{cont.}}$$

[perturbative equality F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019].

[F, Furuta, Matuso, Onogi, Yamaguchi, Yamashita 2019].

But the **overlap Dirac is missing** because Ginsparg-Wilson relation is broken by the boundary [Luescher 2006].

# Real Dirac operators and the mod-two index

For general complex Dirac operators,

$$K^1(I, \partial I) \rightarrow -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D - M))$$

For real Dirac operators, for example, in SU(2) gauge theory in 5D (origin of Witten anomaly), we will be able to show

$$\begin{aligned} KO^0(I, \partial I) &\rightarrow -\frac{1}{2} \left[ 1 - \text{sgn det} \left( \frac{D_W - M}{D_W + M} \right) \right] = -\frac{1}{2} \left[ 1 - \text{sgn det} \left( \frac{D_{\text{cont.}} - M}{D_{\text{cont.}} + M} \right) \right] \\ &= \text{Ind}_{\text{mod-two}} D_{\text{cont.}} \end{aligned}$$

But there is no overlap Dirac counterpart.

# Application to curved manifolds

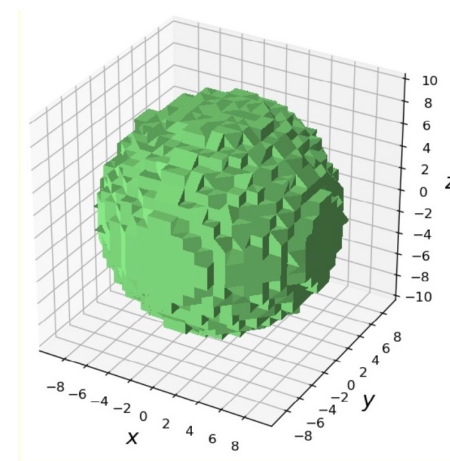
Nontrivial gravitational background can be formulated with curved domain-wall fermions.

This topic will be presented in the 5<sup>th</sup> week by

Shoto Aoki : Nov. 11(Mon) 11:30

Naoto Kan : Nov. 11(Mon) 12:00

Stay tuned!



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✓ 6. Comparison with the overlap Dirac index

we expect wider applications (to APS boundary and real case) than the overlap index.

7. Summary and discussion



## Summary

$$\text{Ind} \cancel{D_{ov}} = -\frac{1}{2}\eta(H_W) = \overset{K^1(I, \partial I)}{\downarrow} -\frac{1}{2}\eta(\gamma_5(D_{\text{cont.}} - M)) = \text{Ind}D_{\text{cont.}}$$
$$H_W = \gamma_5(D_W - M)$$

The chiral symmetry is **NOT essential** to describe gauge field topology: **massive Wilson Dirac op. is good enough** to be identified as a  $K^1(I, \partial I)$  group element.

Or even better : we expect wider applications to the systems with boundaries or odd dimensions.

Backup slides

# The overlap Dirac operator index

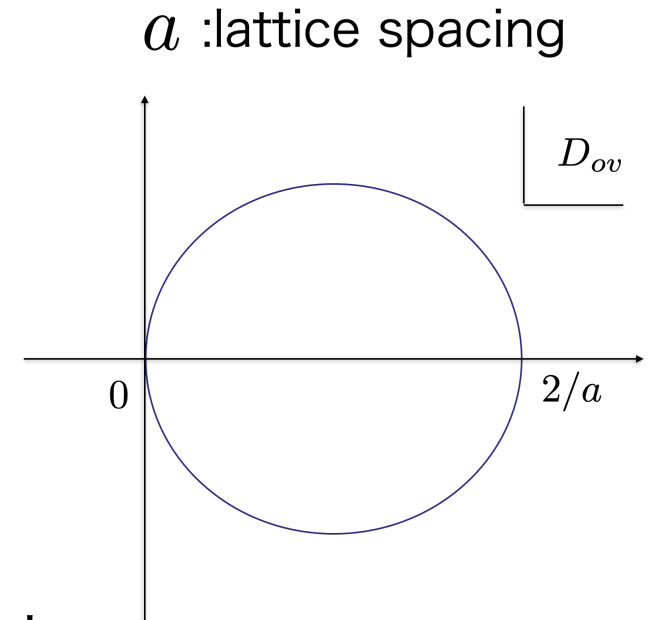
Overlap Dirac spectrum lies on a circle with radius  $1/a$

For complex eigenmodes  $D_{ov}\psi_\lambda = \lambda\psi_\lambda$

$$\psi_\lambda^\dagger \gamma_5 \left( 1 - \frac{aD_{ov}}{2} \right) \psi_\lambda = 0.$$

(therefore, no contribution to the trace).

The real  $2/a$  (doubler poles) do not contribute.



$$\text{Tr} \gamma_5 \left( 1 - \frac{aD_{ov}}{2} \right) = \text{Tr}_{\text{zero-modes}} \gamma_5 = n_+ - n_-$$

# Physicist-friendly index project in continuum

- Physicist-friendly Atiyah-Patodi-Singer (APS) index on a flat space [F, Onogi, Yamaguchi 2017]
- Mathematical proof for the physicist-friendly APS index on general curved manifold [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita 2019]
- Mod-two APS index [F, Furuta, Matsuki, Matsuo, Onogi, Yamaguchi, Yamashita 2020]

Q. How physicist-friendly?

A. We do not need to take care of chiral symmetry and boundary conditions in our formulation.

# Classification of **vector bundles**

Let us consider the case  $F =$  some vector space.

Compare two vector bundles  $E_1$  and  $E_2$ .

It was proved that the **homotopy theory** can completely classify the vector bundles. **But concrete computation is very difficult.**

**K-theory** can classify the vector bundles **when their rank is large enough**, detecting some topological invariants to characterize the bundles with sophisticated computational techniques (more powerful than **the standard** (de Rham) **cohomology** theory with respect to characteristic classes).

# Bott periodicity theorem

Interestingly, we have another isomorphism  
(Bott periodicity theorem) :

$$K^1(X, Y) \cong K^{-1}(X, Y)$$

”+1” adds a Clifford generator.

In the following, we simply denote it by  $K^1$  .

In this talk,  $K^1(I, \partial I)$  is the most important.

## Continuum limit

Multiplying  $\begin{pmatrix} 1 \\ f_{a_j} \end{pmatrix}$  and taking the continuum limit

$$\begin{pmatrix} \gamma(D_{\text{cont.}} + m_\infty) & t_\infty \\ t_\infty & -\gamma(D_{\text{cont.}} + m_\infty) \end{pmatrix} \begin{pmatrix} u_\infty \\ v_\infty \end{pmatrix} = 0$$

is obtained.

$u_\infty, v_\infty$  are  $L_1^2$  weakly convergent

$L^2$  strongly convergent

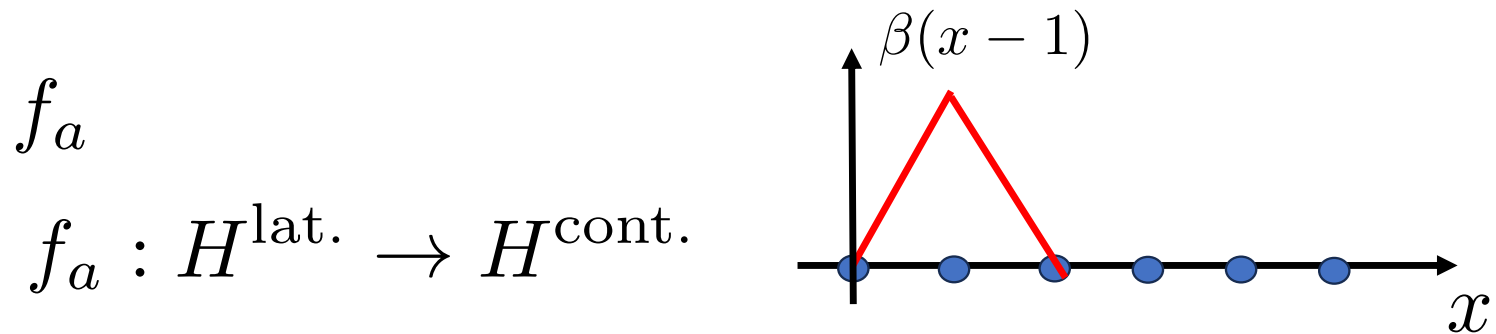
(Rellich's theorem)

$$\hat{D}_\infty^2 = D_{\text{cont.}}^2 + m_\infty^2 + t_\infty^2$$

requires

$$m_\infty = t_\infty = 0.$$

Contradiction with  $m^2 + t^2 > 0$  along the path P.



From **finite-dimensional** vector bundle on a discrete lattice we need to make **infinite-dimensional** vector bundle on continuous  $x$  :

$$f_a \phi^{\text{lat.}}(x) = \sum_{l \in C_x} \beta(x-l) P(x-l) \phi^{\text{lat.}}(l)$$

$C_x$  : a hyper cube containing  $x$  .  $l$  : lattice sites

$$P(x-l) = P \exp \left[ i \int_l^x dx'^i A_i(x') \right] \quad \text{Wilson line.}$$

$\beta(x-l)$  : linear partition of unity s.t.

$$\beta(0) = 1, \beta(\pm a e_\mu) = 0, \quad \sum_{l \in C_x} \beta_l(x) = 1.$$



$f_a^*$ 

$$f_a^* : H^{\text{cont.}} \rightarrow H^{\text{lat.}}$$

Is defined by

$$f_a^* \phi^{\text{cont.}}(l) = \int_{y \in C_l} dy \beta(l-y) P(l-y) \phi^{\text{cont.}}(y)$$

Note)  $f_a^* f_a$  is not the identity but smeared to nearest-neighbor sites. (The gauge invariance is maintained by the Wilson lines.)

## Continuum limit of $f_a^* f_a$

1. For arbitrary  $\phi^{\text{lat.}}$

$\lim_{a \rightarrow 0} f_a \phi^{\text{lat.}}$  weakly converges to a  $\phi_0^{\text{cont.}} \in L_1^2$

where  $L_1^2$  is the square-integrable subspace of  $H^{\text{cont.}}$

to the first derivatives.

2.  $\lim_{a \rightarrow 0} f_a \gamma(D_W + m) \phi^{\text{lat.}}$  weakly converges to

$\gamma(D + m) \phi_0^{\text{cont.}} \in L^2$

3. There exists  $c$  s.t.  $\|f_a^* f_a \phi^{\text{lat.}} - \phi^{\text{lat.}}\|_{L^2}^2 < ca^2 \|\phi^{\text{lat.}}\|_{L_1^2}^2$

4. For any  $\phi^{\text{cont.}} \in L_1^2$ ,  $\lim_{a \rightarrow 0} f_a f_a^* \phi^{\text{cont.}}$

converges to  $\phi_0^{\text{cont.}} \in L_1^2$  and  $\lim_{a \rightarrow 0} f_a f_a^* \phi_0^{\text{cont.}} = \phi_0^{\text{cont.}}$

# Elliptic estimate

In continuum theory, For any  $\phi \in \Gamma(E)$  and  $i$ , a constant  $c$  exists such that

$$||D_i \phi||^2 \leq c(||\phi||^2 + ||D\phi||^2)$$

When a covariant derivative is large,  $D$  is also large.  
This property is nontrivial on a lattice.

$$||\nabla_i^f \phi||^2 \leq c(||\phi||^2 + ||D_W \phi||^2)$$

Doubler modes have small Dirac eigenvalue with large wave number.

-> Wilson term is mathematically important, too!

# What are the weak convergence and strong convergence?

The sequence  $v_j$  weakly converges to  $v_\infty$  when for arbitrary  $w$

$$\lim_{j \rightarrow \infty} \langle (v_j - v_\infty), w \rangle = 0.$$

Note)  $\lim_{j \rightarrow \infty} (v_j - v_\infty)(x) \rightarrow \lim_{k \rightarrow \infty} e^{ikx}$  is weakly convergent.

Strong convergence means  $\lim_{j \rightarrow \infty} \|v_j - v_\infty\|^2 = 0$ .

Rellich's theorem:

$$L_1^2 \text{ weak convergence} = L^2 \text{ convergence}$$