Novel Lattice Formulation of 2D Chiral Gauge Theory via Bosonization

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- O. Morikawa, SO, and H. Suzuki, PTEP 2024, (2024) 063B01 [arXiv:2403.03420]

Introduction

- Chiral gauge theories are very important in particle physics

  e.g. Standard model, GUT...

- Open problem: Lattice formulation of chiral gauge theories

  It is difficult to formulate lattice theories with the **same gauge anomaly structure** as in continuous theories

  Only special cases have been solved

  $U(1)$, Lüscher '98, $SU(2) \times U(1)$, Kikukawa, Nakayama '00...

Recently, **Bosonization**-based approaches are proposed in 2D $U(1)$

$\Longrightarrow$ Berkowitz, Cherman, Jacobson '23, DeMarco, Lake, Wen '23

Bosonization-based approach

- In 2D, there is a **duality** between fermions and bosons (Bosonization)
- **Instead of a fermion** itself, we can **use a boson** with the same chiral symmetry structure as chiral gauge theories

$$\textbf{Fermion} \Longleftrightarrow \textbf{Boson} \rightarrow \textbf{Lattice formulation}$$

Advantages

- Exact chiral symmetry and ultra-locality
- Simple formulation: Gauge anomalies can be calculated classically and easily from explicit breaking by gauge transf.

Continuum theory: 2D fermion action and Bosonization

In 2D, massless Dirac fermions ψ and compact boson ϕ are equivalent at a certain fixed point $R = \frac{1}{\sqrt{2}}$ Coleman '75, Mandelstam '75

According to Bosonization rule,

$$\int_{M_2} \mathrm{d}^2 x \, \bar{\psi} i \partial\!\!\!/ \psi \Longleftrightarrow \frac{1}{8\pi} \int_{M_2} \mathrm{d}^2 x \, \partial_\mu \phi \partial_\mu \phi$$

Global symmetries: $U(1)_{\text{Left}} \times U(1)_{\text{Right}} \cong U(1)_{\text{Axial}} \times U(1)_{\text{Vector}}$

Axial sym.: $\psi \to e^{i\gamma_3 \xi}\psi$, $\partial_\mu(\bar{\psi}\gamma_\mu\gamma_3\psi) = 0 \Longleftrightarrow e^{i\phi} \to e^{i\phi}e^{i\xi}$, $\partial_\mu\partial_\mu\phi = 0$

Boson counterpart of Axial sym. is shift sym. and E.o.M

Vector sym.: $\psi \to e^{i\Lambda}\psi$, $\partial_\mu(\bar{\psi}\gamma_\mu\psi) = 0$

$$\Longleftrightarrow e^{i\tilde{\phi}} \to e^{i\tilde{\phi}}e^{i\Lambda}, \ \partial_\mu\partial_\mu\tilde{\phi} = \epsilon_{\mu\nu}\partial_\mu\partial_\nu\phi = 0$$

$\tilde{\phi}$ is dual scalar $\partial_\mu\tilde{\phi} = \epsilon_{\mu\nu}\partial_\nu\phi$

Boson counterpart of Vector sym. is **dual** shift sym. and **Bianchi identity**

Breaking of Bianchi identity

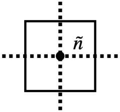
Dual vertex operator $e^{i\tilde{\phi}(x)}$ impose **breaking of Bianchi id.** at x.

In other words, $\tilde{\phi}(x)$ is a Lagrange multiplier which imposes

Bianchi id. $\mathcal{L} = \cdots + i\tilde{\phi}(x)\partial_\mu j_{V,\mu}(x), \quad j_{V,\mu}(x) \sim \epsilon_{\mu\nu}\partial_\nu\phi$

This story is almost same for lattice theories.

$\tilde{\phi}(x) \to \tilde{\phi}(\tilde{n})$ ($\tilde{n}$: dual lattice) e.g. Berkowitz, Cherman, Jacobson '23



In this method, breaking of Bianchi id. is imposed into a plaquette

to introduce a vector charged object $\partial_\mu j_{V,\mu}(x) \sim \partial_\mu(\epsilon_{\mu\nu}\partial_\nu\phi) \neq 0$

We propose yet another formulation that respects **"smoothness"**

Lattice field contents

Let us consider a 2D compact boson on the lattice $e^{i\phi(n)}$

"Derivative" respecting the compactness

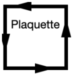
$\partial\phi(n,\mu) \equiv \frac{1}{i}\ln[e^{-i\phi(n)}e^{i\phi(n+\hat{\mu})}] = \Delta_\mu\phi(n) + 2\pi\ell_\mu(n),\ \ell_\mu(n) \in \mathbb{Z}$

$-\pi \leq \frac{1}{i}\ln e^{i\phi(n)} < \pi,\ \phi(n) = \frac{1}{i}\ln e^{i\phi(n)}$

$\epsilon_{\mu\nu}\partial\phi(n,\nu)$ are lattice counterparts of $j_{V,\mu}$

Under the **admissibility condition** $\sup_{n,\mu}|\partial\phi(n,\mu)| < \epsilon < \frac{\pi}{2}$,

$\Longrightarrow \epsilon_{\mu\nu}\Delta_\mu\partial\phi(n,\nu) = 0$ (Bianchi identity) e.g. Fujiwara, Suzuki, Wu, '00



$= 0$

Admissibility imposes a conservation law corresponding to **vector symmetry** on the lattice

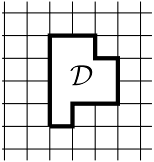
Excision method Abe, SO, Morikawa, Suzuki, Tanizaki '23

We propose a definition of vector charge (Noether charge) that keeps admissibility (in other word, without breaking of $\epsilon_{\mu\nu}\Delta_\mu \partial\phi(n,\nu) = 0$)$\Longrightarrow$ **Excision method (a charged object corresponding to vector symmetry is a "hole")**

Excision method

We can introduce **conserved and non-zero vector charge** $m \equiv \frac{1}{2\pi}\sum_{(n,\mu)\in\partial\mathcal{D}} \partial\phi(n,\mu) \in \mathbb{Z}$

When the holes are sufficiently large, m can be non-zero value.



Short proof

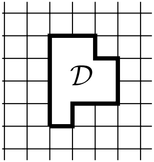
According to admissibility condition,

$|\Delta_1 \ell_2(n) - \Delta_2 \ell_1(n)| = \frac{1}{2\pi}|\Delta_1 \partial\phi(n,2) - \Delta_2 \partial\phi(n,1)| < \frac{\epsilon}{2\pi} \times 4 < 1$

$\Longrightarrow \underline{\epsilon_{\mu\nu}\Delta_\mu \partial\phi(n,\nu) = 0}$ Here, the 4 is # of links in a plaquette

In the presence of "hole" $\mathcal{D}$ and considering $\sum_{(n,\mu)\in\partial\mathcal{D}} \partial\phi(n,\mu)$

we can avoid the bound: $\frac{\epsilon}{2\pi}|\partial\mathcal{D}| > 1 \Longrightarrow \underline{\sum_{(n,\mu)\in\partial\mathcal{D}} \partial\phi(n,\mu) \neq 0}$



Excision method

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Chiral gauge theories in continuum

Introducing flavor d.o.f. and gauging **both** $U(1)_{\text{Axial},\alpha} \times U(1)_{\text{Vector},\alpha}$

$$\int_{M_2} \mathrm{d}^2 x \sum_{\alpha:\text{flavor}} \bar{\psi}_\alpha \{ i\partial\!\!\!/ + q_{V,\alpha} A\!\!\!/ + \gamma_3 q_{A,\alpha} A\!\!\!/ \} \psi_\alpha$$

Bosonized-counterpart:

$$\int_{M_2} \mathrm{d}^2 x \sum_{\alpha:\text{flavor}} \left[\frac{R^2}{4\pi} (\partial_\mu \phi_\alpha + 2 q_{A,\alpha} A_\mu)^2 + \frac{i q_{V,\alpha}}{2\pi} A_\mu \epsilon_{\mu\nu} (\partial_\nu \phi_\alpha + 2 q_{A,\alpha} A_\nu) \right]$$

Under the gauge transf. (Λ is a gauge transf. parameter)

$$(\text{gauge anomaly}) \propto \left(\sum_\alpha q_{A,\alpha} q_{V,\alpha} \right) \int_{M_2} \mathrm{d}^2 x \, \Lambda F_{12}$$

Our lattice formulation reproduces this structure!

Couple to gauge fields on the lattice

"Covariant derivatives"

$D\phi(n,\mu) \equiv \frac{1}{i} \ln \left[e^{-i\phi_\alpha(n)} U(n,\mu)^{2q_{A,\alpha}} e^{i\phi_\alpha(n+\hat{\mu})} \right]$

$F_{\mu\nu}(n) \equiv \frac{1}{i} \ln \Box = \Delta_\mu A_\nu(n) - \Delta_\nu A_\mu(n) + 2\pi N_{\mu\nu}(n)$

Admissibility condition: $\sup_{n,\mu} |D\phi_\alpha(n,\mu)| < \epsilon$, $\sup_{n,\mu,\nu} |2q_{A,\alpha} F_{\mu\nu}(n)| < \delta$

Analogue of Bianchi id.: $\Delta_\mu D\phi_\alpha(n,\nu) - \Delta_\nu D\phi_\alpha(n,\mu) = 2q_{A,\alpha} F_{\mu\nu}(n)$

Excision method works in gauge theory as well

$\sum_{(n,\mu)\in\partial D} D\phi_\alpha(n,\mu) = m_\alpha + 2q_{A,\alpha} F(\partial D)$

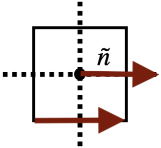
$F(\partial\mathcal{D}) \equiv \frac{1}{i} \ln \prod_{(n,\mu)\in\partial\mathcal{D}} U(n,\mu)$

Gauged action

Our lattice action:

$$S_{\mathrm{B}} = \sum_{\text{flavor}\,:\,\alpha} \sum_{n \in M_2} \left[\frac{R^2}{4\pi} D\phi_\alpha(n,\mu) D\phi_\alpha(n,\mu) + \frac{i}{2\pi} q_{V,\alpha} \epsilon_{\mu\nu} A_\mu(\tilde{n}) D\phi_\alpha(n+\hat{\mu},\nu) + \frac{i}{2} q_{V,\alpha} \epsilon_{\mu\nu} N_{\mu\nu}(\tilde{n}) \phi_\alpha(n+\hat{\mu}+\hat{\nu}) \right]$$

For technical reasons, we put copies on dual lattice $U(n,\mu) = U(\tilde{n},\mu)$



Gauge anomaly

Under the gauge transformation,

$$e^{\phi_\alpha(n)} \to e^{\phi_\alpha(n)} e^{-2q_{A,\alpha} i\Lambda(n)},$$

$$U(n,\mu) \to e^{-i\Lambda(n)} U(n,\mu) e^{i\Lambda(n+\hat{\mu})},$$

$$U(\tilde{n},\mu) \to e^{-i\Lambda(\tilde{n})} U(\tilde{n},\mu) e^{i\Lambda(\tilde{n}+\hat{\mu})}$$

Then, we can check

$\Delta S_B = (\text{gauge anomaly}) \propto (\sum_\alpha q_{A,\alpha} q_{V,\alpha})(\sum_{n\in M_2} \Lambda(\tilde{n}) F(n) + \cdots)$

Anomaly cancellation condition : $\sum_\alpha q_{A,\alpha} q_{V,\alpha} = 0$

$\Longrightarrow$ The gauge field can be dynamical.

$\Longrightarrow$ We can construct **anomaly-free lattice chiral gauge theory!**

Selection rule I

In the presence of the "hole" (labeled by $\tilde{I}$),

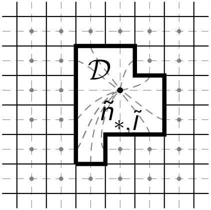
ΔS_B gets additional term: $\Delta S_B = -i\sum_{\tilde{I},\alpha} q_{V,\alpha} m_{\tilde{I},\alpha} \Lambda(\tilde{n}_{*,\tilde{I}})$

$e^{-\mathbf{i}\sum_{\tilde{I},\alpha} \mathbf{q}_{\mathbf{V},\alpha} \mathbf{m}_{\tilde{I},\alpha} \mathbf{\Lambda}(\tilde{\mathbf{n}}_{*,\tilde{I}})}$ **shows vector gauge transf. of vector charged objects**

By definition from $m_{\tilde{I},\alpha}$,

$\sum_{\tilde{I}} m_{\tilde{I},\alpha} = -\frac{2q_{A,\alpha}}{2\pi}\left(\sum_{p\in M_2-\sum_{\tilde{I}}\mathcal{D}_{\tilde{I}}} F_{12}(p) + \sum_{\tilde{I}} F(\partial\mathcal{D}_{\tilde{I}})\right) = -2q_{A,\alpha}Q$

Vector charges saturate 1st Chern number Q $\Longrightarrow$ **index theorem**!



Selection rule II

The case of axial charged objects?

Axial charged objects are vertex operator: $V_{\{n_\alpha\}}(n) \equiv e^{i\sum_\alpha n_\alpha \phi_\alpha(n)}$

For a non-zero correlation function

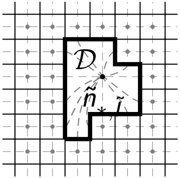
(under global shift: $\phi_\alpha(n) \to \phi_\alpha(n) + \xi_\alpha$),

$\sum_I n_{I,\alpha} = \frac{q_{V,\alpha}}{2\pi} \sum_{\tilde{p}\in \tilde{M}_2} F_{12}(\tilde{p}) = q_{V,\alpha}\tilde{Q}$ (I labels vertex operators)

Axial charges saturate 1st Chern number $\tilde{Q}$ **on dual lattice**

In the presence of holes, lattice and dual lattice **do not** have 1 to 1

correspondence $\Longrightarrow$ Generally $Q \neq \tilde{Q}$



Assuming sufficiently "strict" admissibility (δ is small enough),

$|Q - \tilde{Q}| < 1 \Longrightarrow Q = \tilde{Q}$

Selection rule III

Considering **Weyl fermion operators**

$U(1)_{\mathrm{Axial}} \times U(1)_{\mathrm{Vector}} \cong U(1)_{\mathrm{Left}} \times U(1)_{\mathrm{Right}}$

$q_R = q_V + q_A$, $q_L = q_V - q_A$

When $Q = \tilde{Q}$, results of "vector case" and "axial case" can be combined

As a result, consistent with **the fermion number anomaly**

$\int_{M_2} \mathrm{d}^2 x\, \partial_\mu J_\mu^{L,R}(x) = \mp q_{L,R} Q \quad (Q = \frac{1}{2\pi} \int_{M_2} F)$

Future directions

- **Lattice formulation via non-abelian bosonization** cf. Witten '84

Our admissibility-respecting formulation may be compatible with topologies of lattice gauge theories, including non-abelian cases

cf. Lüscher '82

$$\frac{-i}{2\pi}\oint \mathrm{Tr}\{g^{-1}\mathrm{d}g\} \overset{\text{discretize}}{\sim} \frac{-i}{2\pi}\sum \log\det\{g(n+\hat{\mu})g(n)^{-1}\} = 0 \text{ (w/ admissibility)}$$

If $\sum \to \sum_{\text{around hole}}$, then winding $\# \neq 0$

- **Generalization of Excision method to higher dimensions**

In 4D $U(1)$ lattice Maxwell theory, 't Hooft line can be defined by using Excision method (hole $\cong S^2 \times S^1$)

$\Longrightarrow$ Witten effect and dyon's statistics can be observed on lattice (SO, arXiv:2411.xxxxx)

Summary

- We construct 2D anomaly-free U(1) chiral gauge theory on the lattice with exact gauge symmetry
- In paticular, the vector symmetry is exactly realized by Excision method (Vector charged objects are “hole”) and admissibility