# Instanton Density Operator in Lattice QCD from Higher Category Theory

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Long term problem:

*Define instanton on lattice in a natural manner.*

Conclusion:

*Must refine Wilson's lattice gauge field at a definition level.*
*This can be—and has to be—done by higher category theory.*

This leads to a systematic rethinking of what "lattice QFT" really is.

motivation, principle & math: [2406.06673]

explicitly: with Peng Zhang [2411.07195]

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*I will NOT: introduce category theory itself*

*in fact, lattice theory helped me learn higher category theory*

*The story can be highly formal, but it can also be very physically intuitive.*

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*The Big Picture*

*Refined Lattice Yang-Mills Path Integral*

- *Plaquette d.o.f. and weight*
- *Cube d.o.f. and weight; Chern-Simons saddle*
- *Hypercube d.o.f. and weight; instanton*

*Further Remarks*

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*Further Remarks*

*Problem:* continuum $\mathcal{I} := \frac{1}{2}\,\mathbf{tr}\left[\frac{F}{2\pi} \wedge \frac{F}{2\pi}\right], \qquad I := \int_{\mathcal{M}} \mathcal{I} \in \mathbb{Z}$

Wilson's lattice gauge theory: *G* d.o.f. on each link

$$\prod_{\text{links } l} G_l \longrightarrow \mathbb{Z}$$

config space instanton number

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| | |
|---|---|
| config space | instanton number |
| *connected* | *discrete* |
| *any two configs can continuously deform to each other* | either single-valued or discontinuous map |

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*current methods*
restricted config space,
cooling/flow,
fermion index

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same issue for group cohomology lattice models for topological order, when the group becomes continuous

*Very general problem:*

topological operators lost

when putting continuous-valued d.o.f. on lattice

$$\prod_{\text{local}} X \longrightarrow \text{topological number}$$

*connected* *discrete*

either single-valued
or discontinuous

topological operators lost

when putting continuous-valued d.o.f. on lattice

- $S^1$ nlsm (XY): winding, vortex
- *U(1)* gauge: Dirac quantization, monopole, abelian CS, abelian instanton

} $\pi_1$ physics

- $S^2$ nlsm: Berry phase, skyrmion, hedgehog

} $\pi_2$ physics

- $S^3$ nlsm: WZW, skyrmion, hedgehog
- *SU(N)* gauge: CS, instanton, Yang monopole

… whatever …

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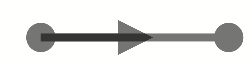
} $\pi_3$ (or higher) physics
groups/fibre bundles fail

higher category theory necessary

$$Z = \int D\Phi \; e^{i \int \mathcal{L}(\Phi, \partial\Phi, \cdots)}$$

$$\partial\Phi \;\sim\; \Phi_{v'} - \Phi_v$$



becomes finite dimensional

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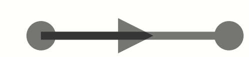
becomes finite dimensional

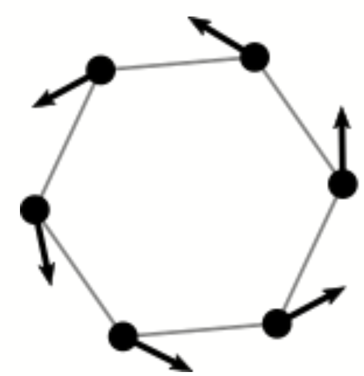
but not in the naive way

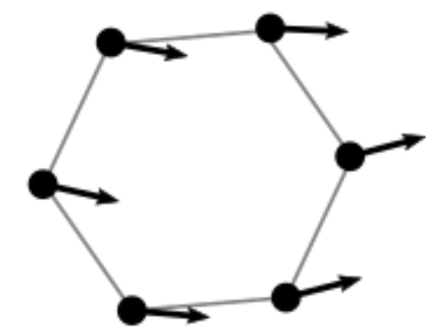
$\partial\Phi \sim \Phi_{v'} - \Phi_v$



conceptually refine to capture
homotopy / interpolation

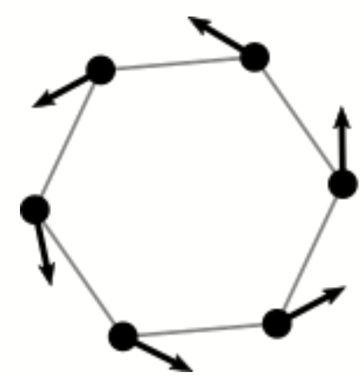
$S^1$ nlsm (XY): Villainization

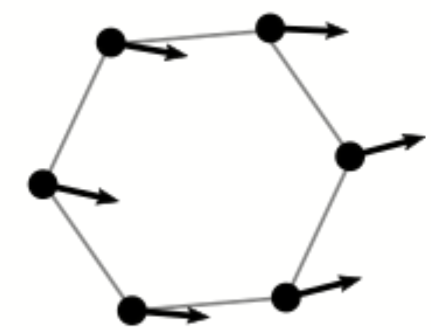




we feel: *w=1* *w=0*

## $S^1$ nlsm (XY): Villainization





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Continuous deformable to each other!
(only on lattice, not in continuum)

## $S^1$ nlsm (XY): Villainization

If link embedded in continuum,
how does $\theta$ interpolate?

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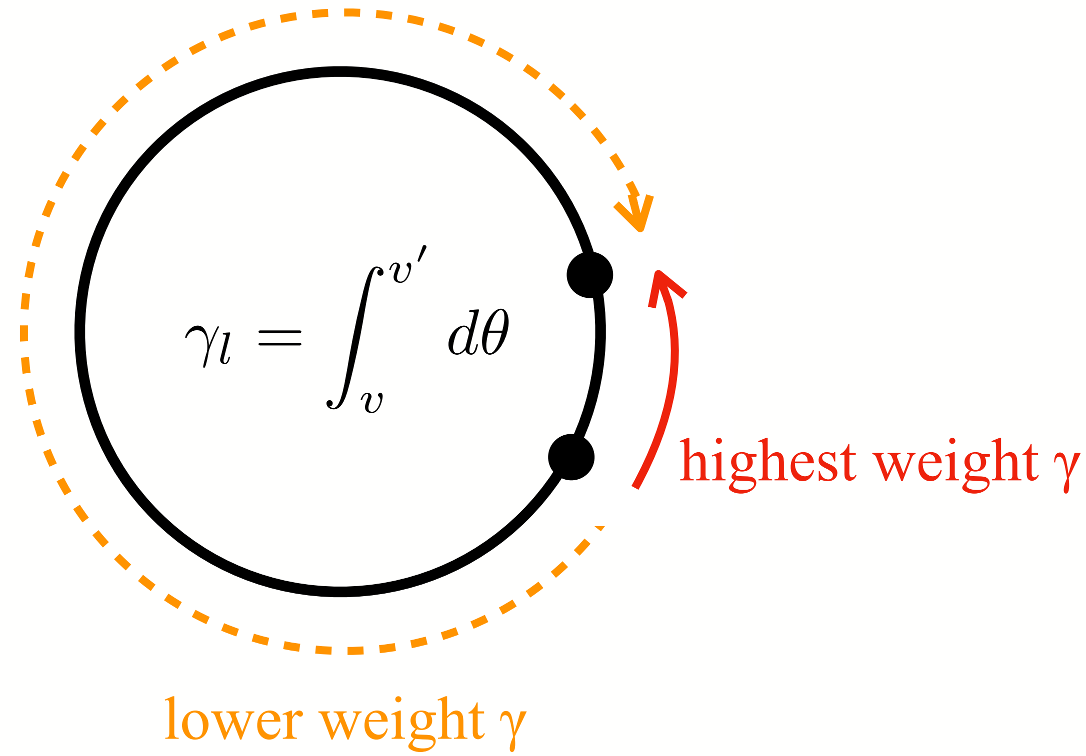


$\gamma_l \in \mathbb{R}$

$$e^{i\gamma_l} = e^{i(\theta_{v'} - \theta_v)}$$

$2\pi\mathbb{Z}$ part unfixed $\gamma_l = \theta_{v'} - \theta_v + 2\pi m_l$
sum over all possibilities with suitable weights

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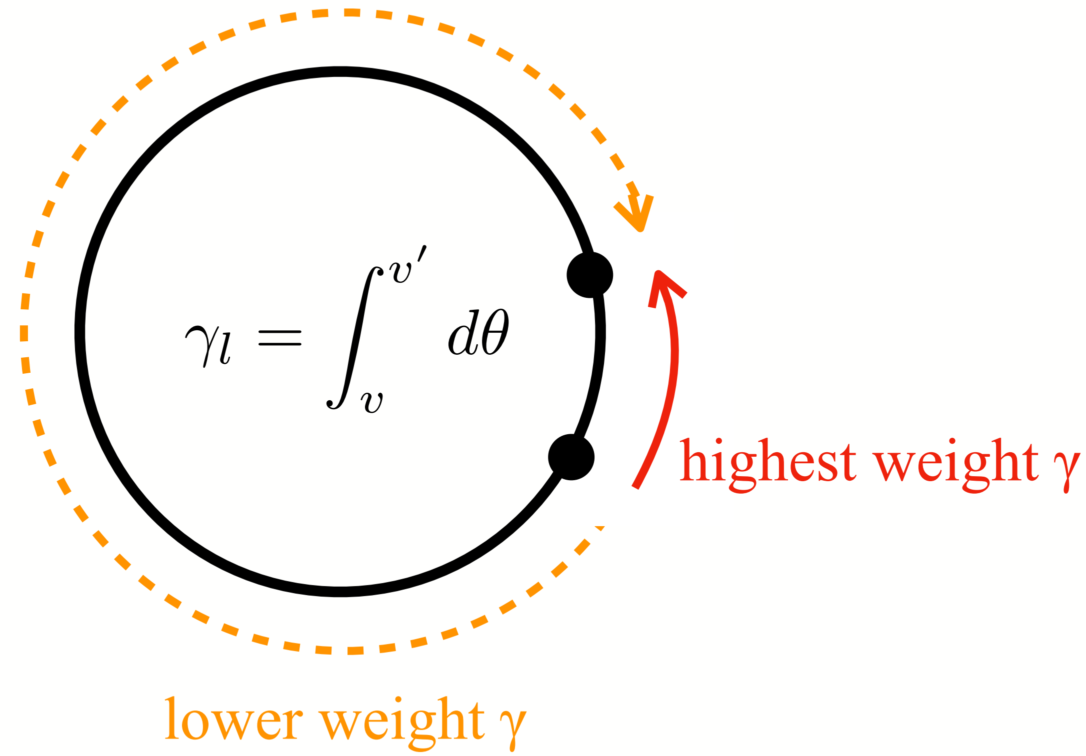
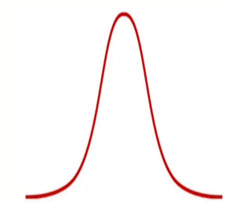


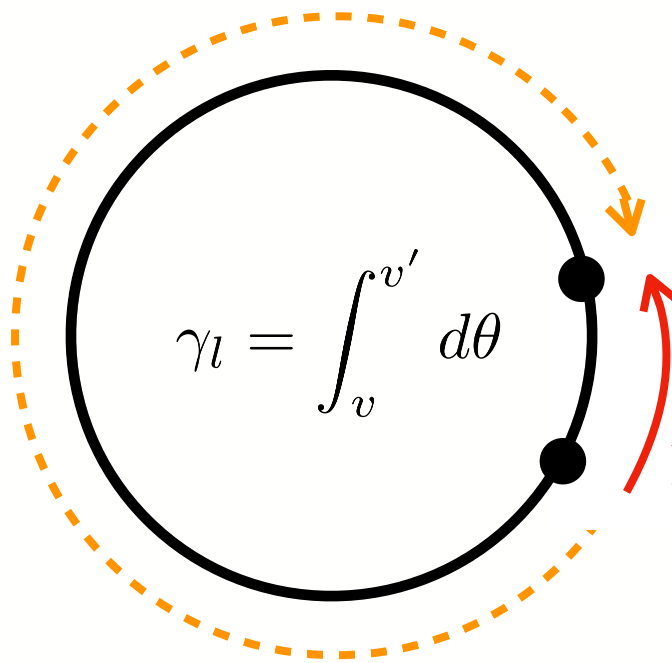

$$w := \sum_{l \text{ on loop}} \gamma_l / 2\pi = \sum_l m_l$$

winding well-defined

# $S^1$ nlsm (XY): Villainization

$$Z = \left[\prod_{v'} \int_{-\pi}^{\pi} \frac{d\theta_{v'}}{2\pi}\right] \left[\prod_{l'} \sum_{m_{l'} \in \mathbb{Z}}\right] \prod_{l} W_1(\gamma_l)$$



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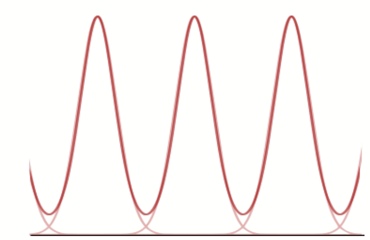

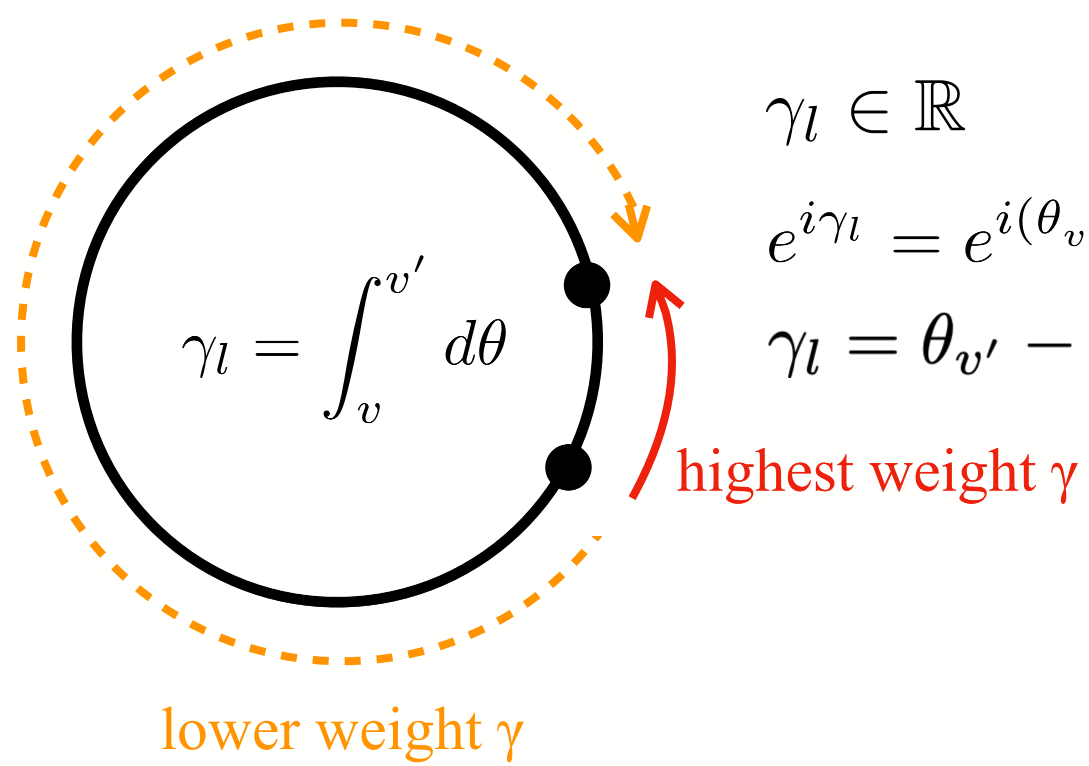


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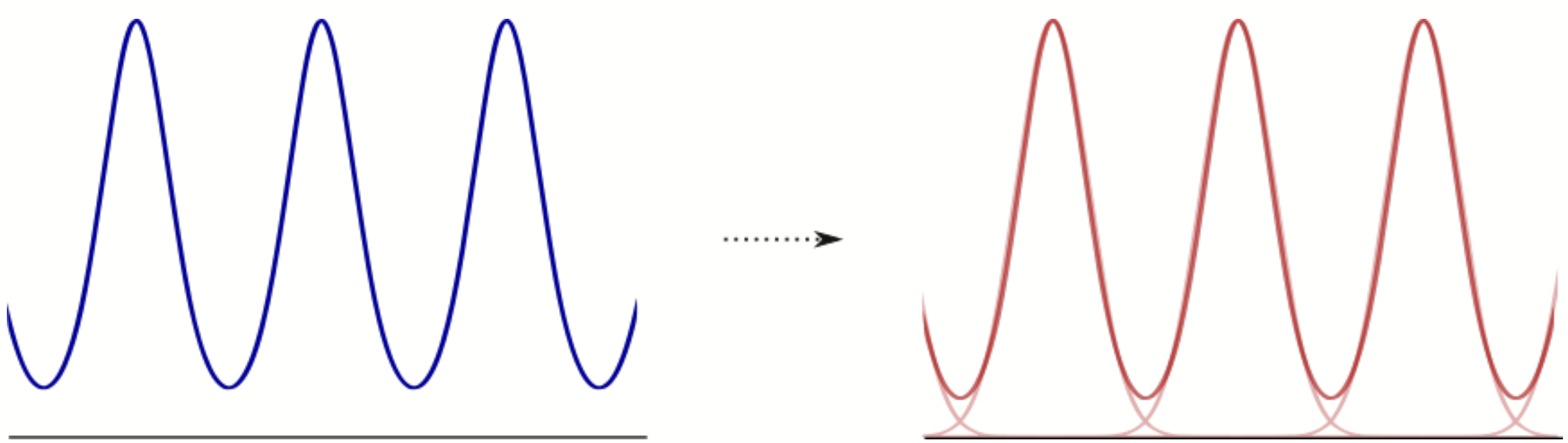
$S^1$ nlsm (XY): Villainization

$$W_{XY}(e^{id\theta_l} + c.c) \approx \sum_{m_l \in \mathbb{Z}} W_1(\gamma_l)$$



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$$v_p := \frac{d\gamma_p}{2\pi} = dm_p \in \mathbb{Z}$$

vortex fugacity

or even a delta function (manifests dual symm)

$S^1$ nlsm (XY): Villainization ***invented by Berezinsky!***

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$S^1$ field on vertex — now U(1) gauge field $A$ on link

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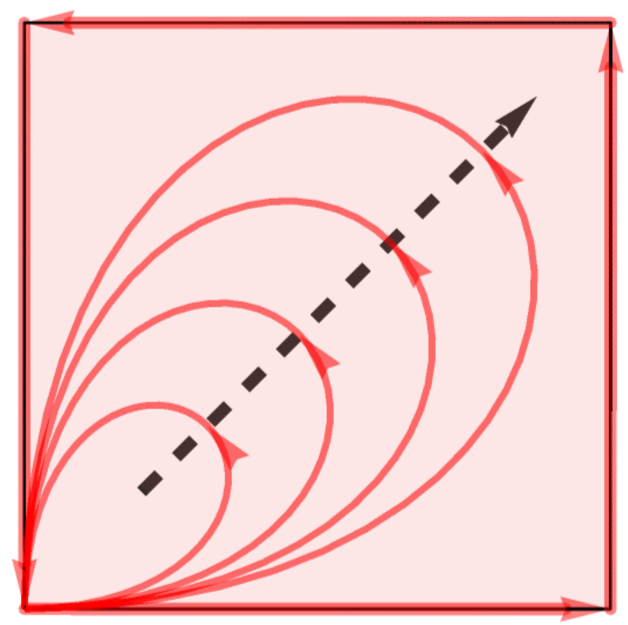

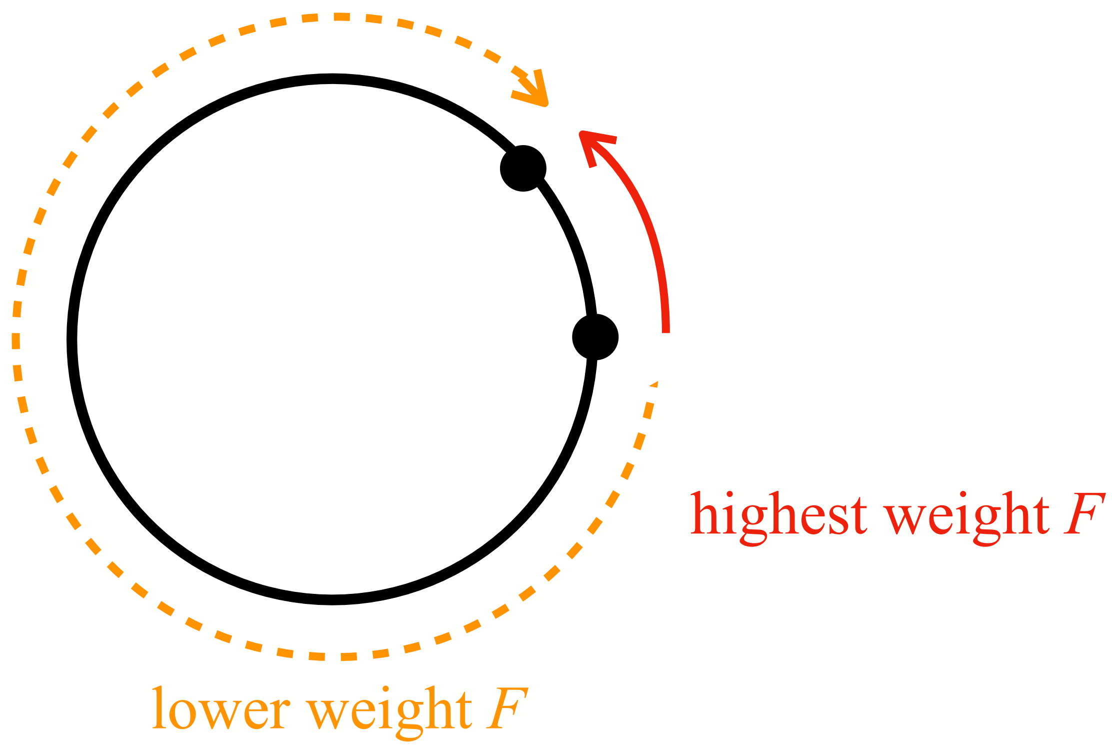

## U(1) lattice gauge theory: Villainization

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winding in 1d — now Dirac quantized flux in 2d

vortex in 2d — now monopole in 3d

*"delooping of cat"*

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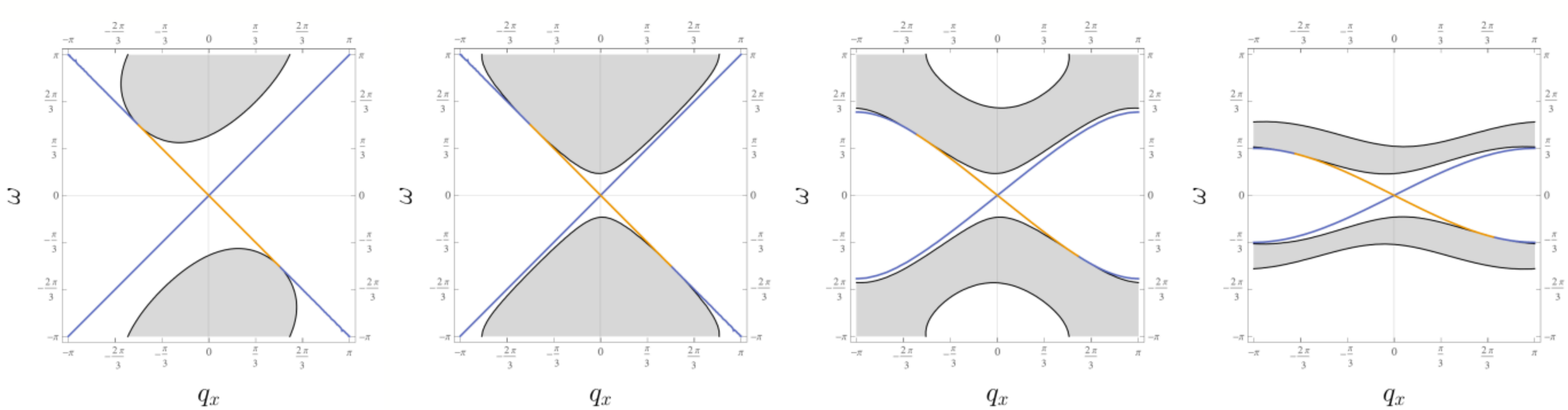
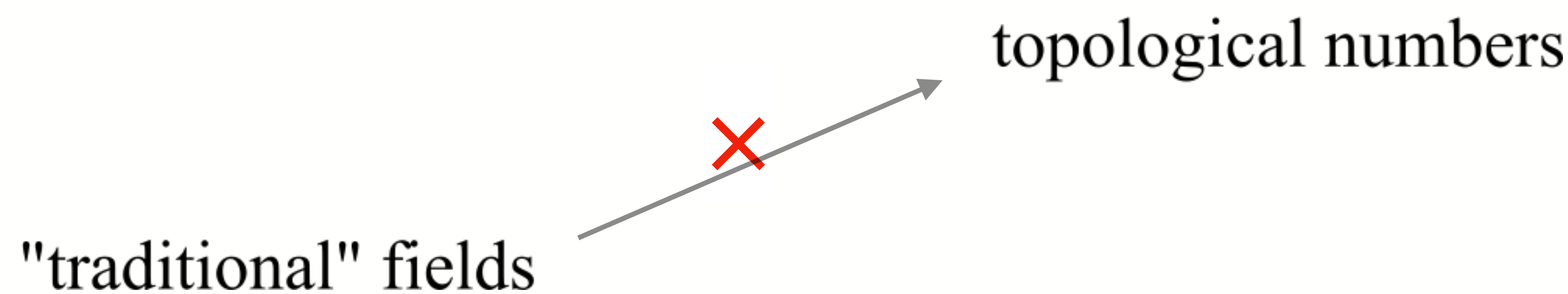
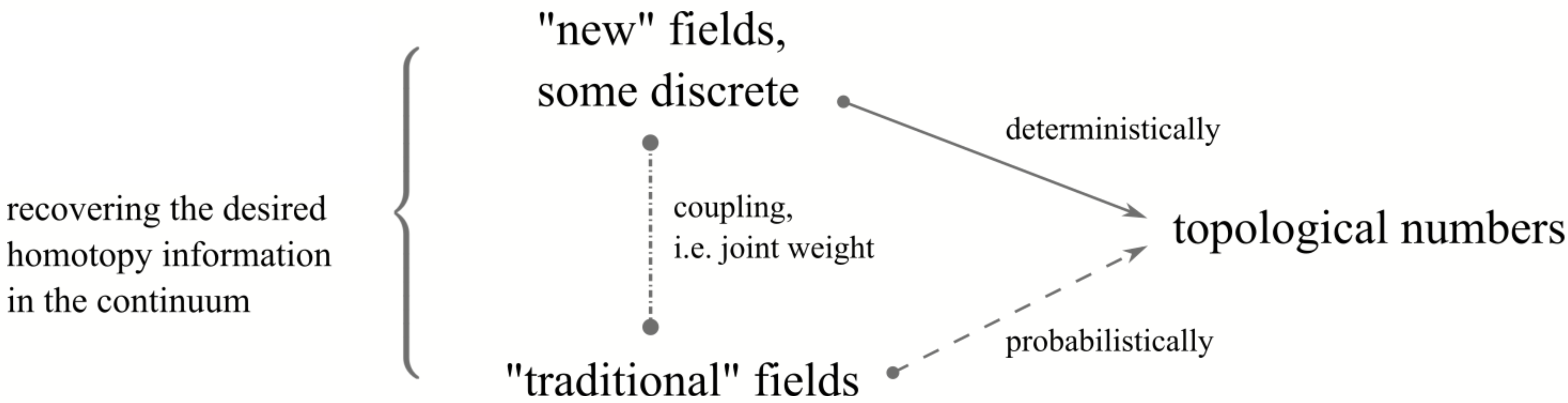
lattice abelian instanton   Tin Sulejmanpasic, Christof Gattringer [1901.02637]

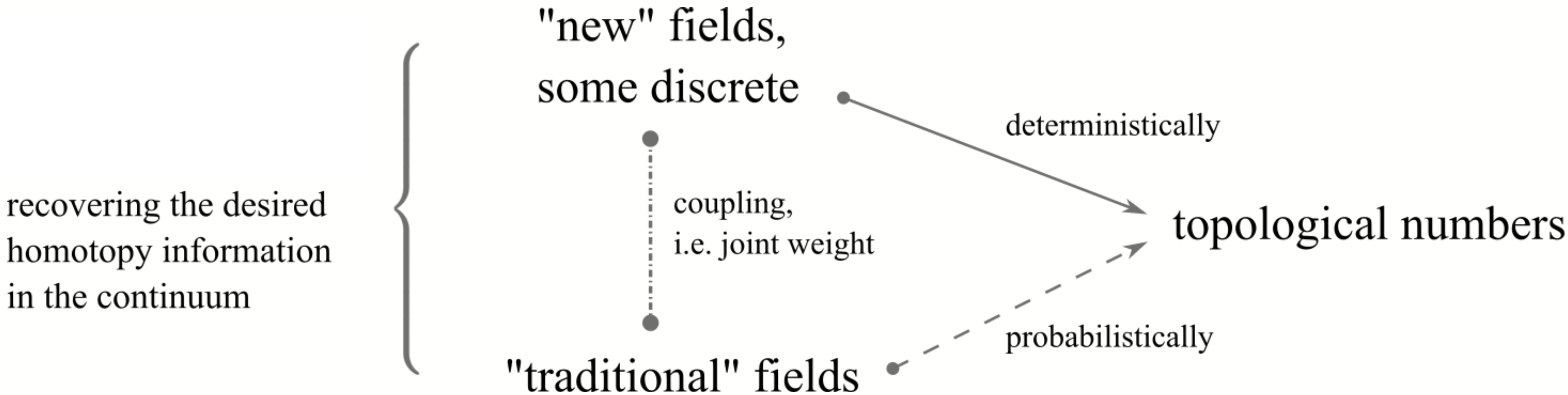
lattice fractional Hall conductivity   JYChen [1902.06756]

lattice U(1) chiral Chern-Simons-Maxwell   Ze-An Xu, JYChen [2410.11034]



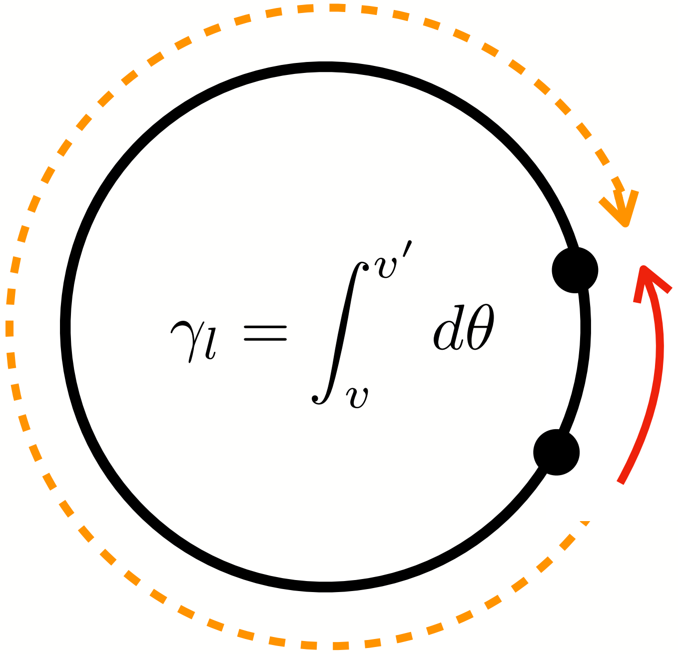

topological numbers
"traditional" fields

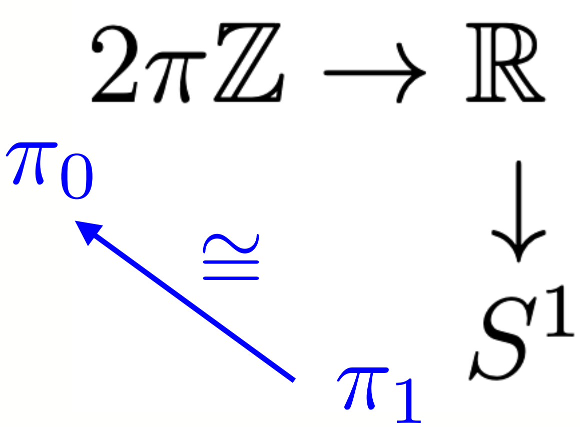

"new" fields,
some discrete
deterministically
coupling,
i.e. joint weight
recovering the desired
homotopy information
in the continuum
topological numbers
probabilistically
"traditional" fields



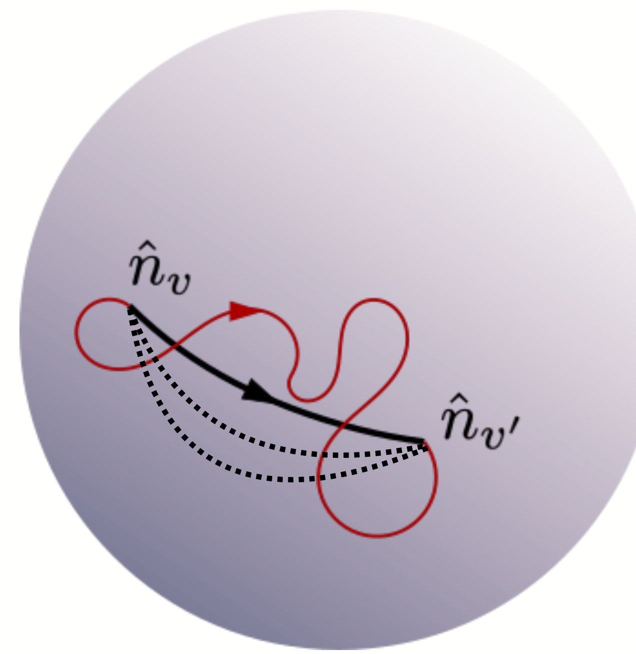

$\pi_1$ and $\pi_2$ physics on lattice (known):
form principal bundles

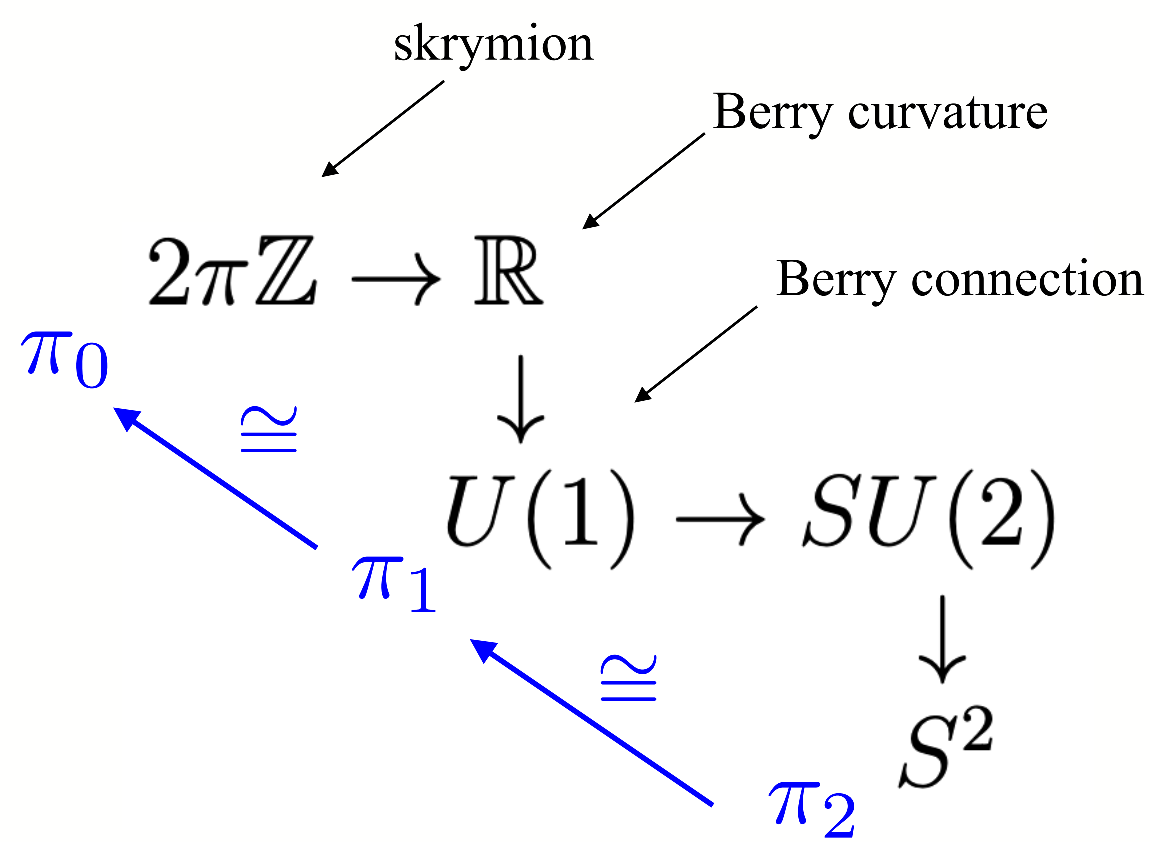
$S^1$ nlsm (XY): Villainization



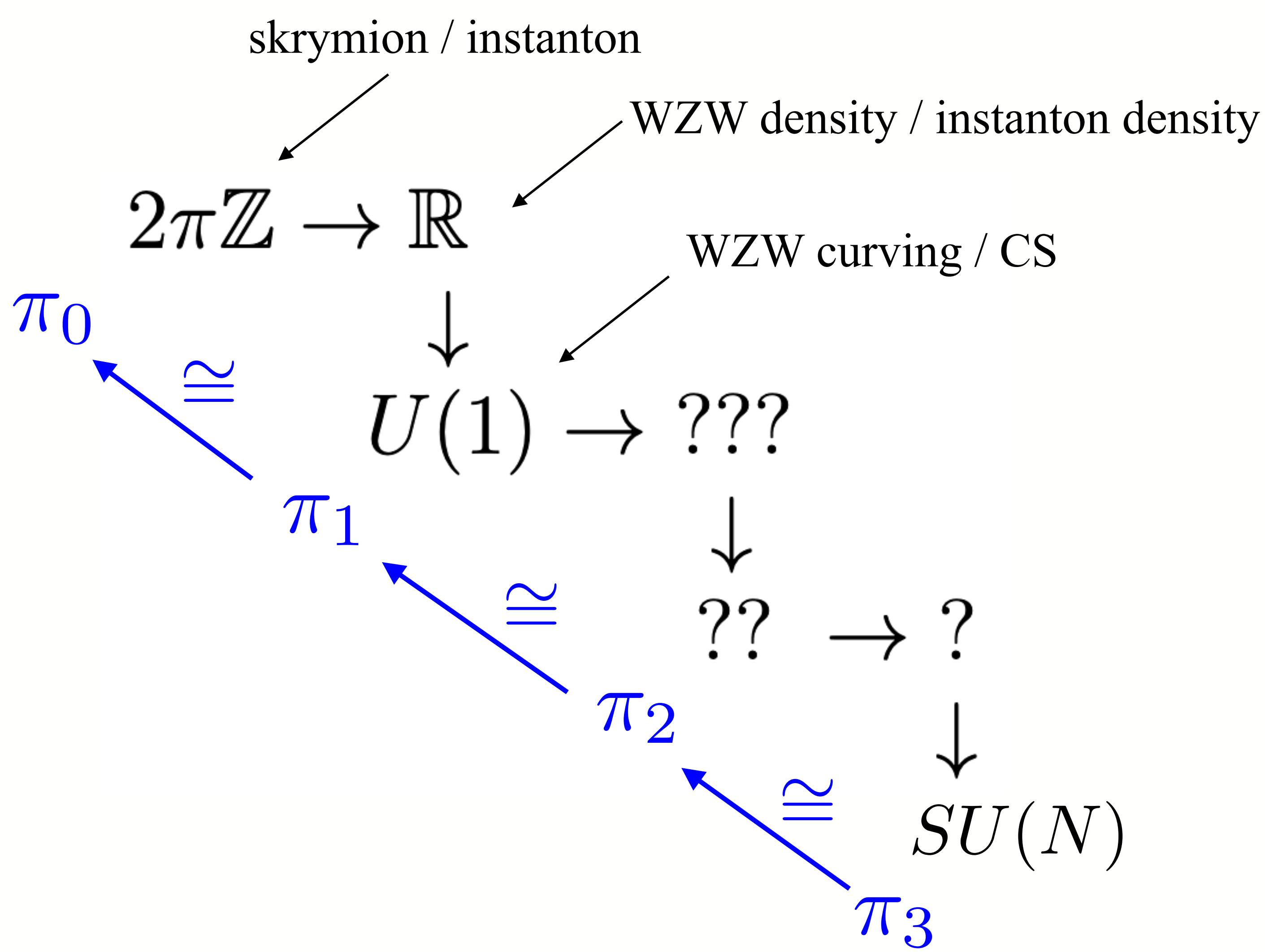




$S^2$ nlsm: spinon-decomposition



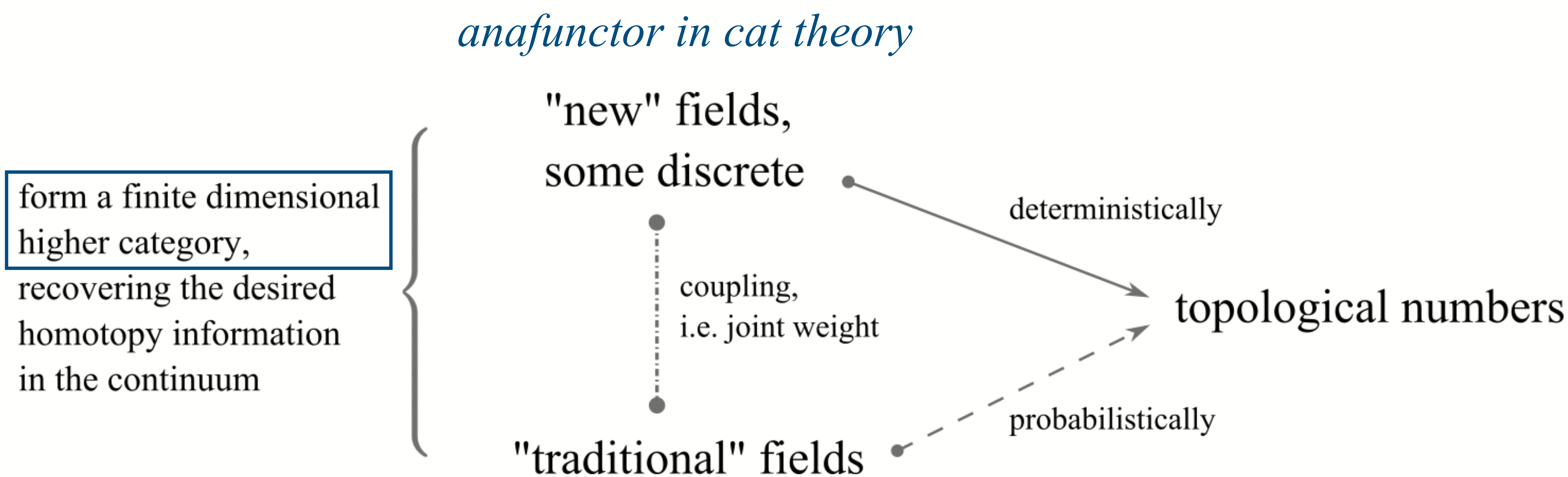



seems now we want




The "??" is on link (nlsm) or plaquette (YM), should compose.

But finite dimensional Lie group always has trivial $\pi_2$ !

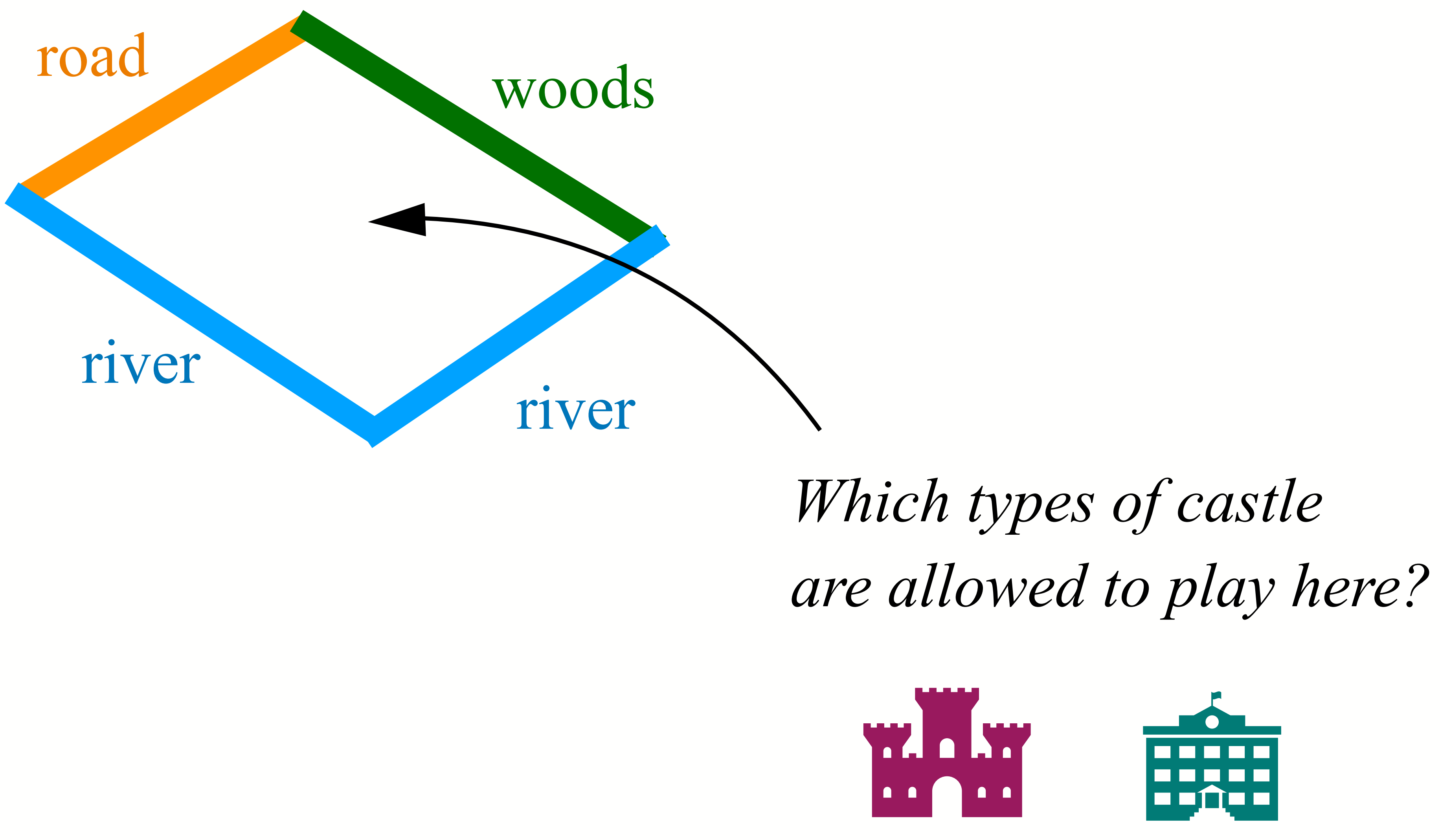



$\pi_1$ and $\pi_2$ physics on lattice (known):
form principal bundles

More general cases:
Mathematically impossible for
group theory / fibre bundles to fulfill goal

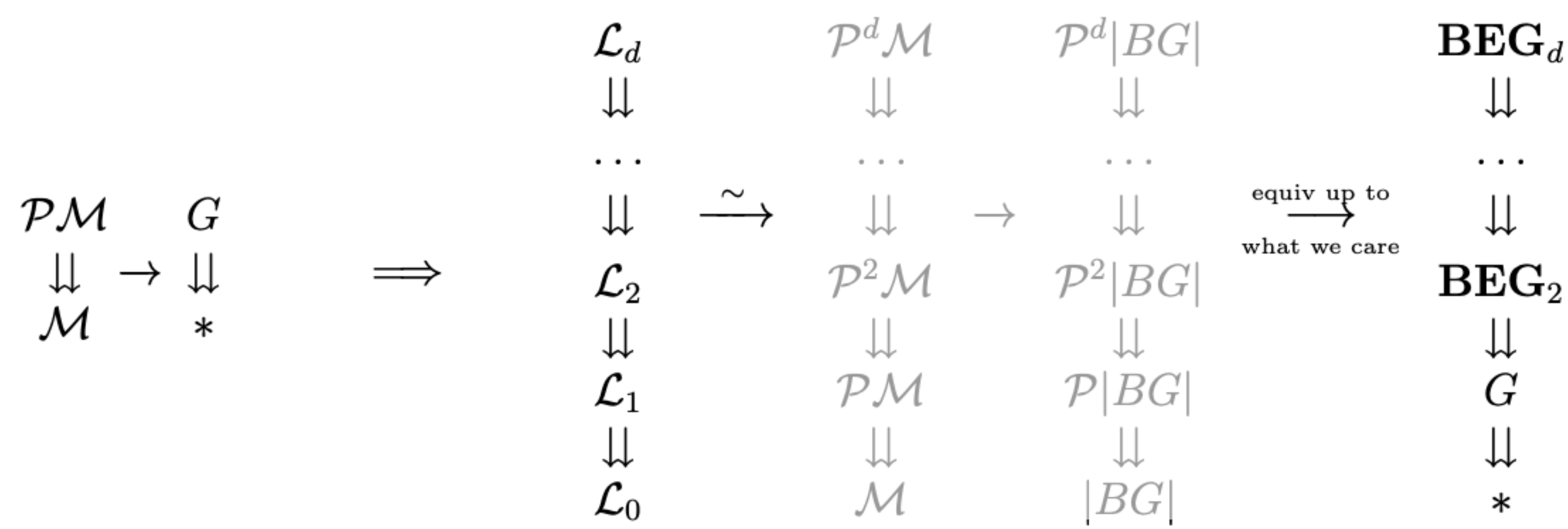
need more flexible “rules of the game”

much like some kind of board game:



continuum gauge field:
*looks simple,*
*but infinite dim. path int.*

lattice gauge field:
*looks complicated,*
*but finite dim. path int.*



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$$
\begin{array}{ccc}
\mathcal{P}\mathcal{M} & & G \\
\downdownarrows & \to & \downdownarrows \\
\mathcal{M} & & *
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccccccc}
\mathcal{L}_d & & \mathcal{P}^d\mathcal{M} & & \mathcal{P}^d|BG| & & \mathbf{BEG}_d \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\cdots & & \cdots & & \cdots & & \cdots \\
\downdownarrows & \xrightarrow{\sim} & \downdownarrows & \to & \downdownarrows & \xrightarrow[\text{what we care}]{\text{equiv up to}} & \downdownarrows \\
\mathcal{L}_2 & & \mathcal{P}^2\mathcal{M} & & \mathcal{P}^2|BG| & & \mathbf{BEG}_2 \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_1 & & \mathcal{P}\mathcal{M} & & \mathcal{P}|BG| & & G \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_0 & & \mathcal{M} & & |BG| & & *
\end{array}
$$

Wilson's theory

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\mathcal{L}_0 & & \mathcal{M} & & |BG| & & *
\end{array}
$$

*A grand picture is unfolding, where*

*Wilson's dream* use lattice QFT to make sense of QFT

*and*

*Grothendieck's dream* use weak higher categories as foundation of homotopy theory

*come into splice.*

*The Big Picture*

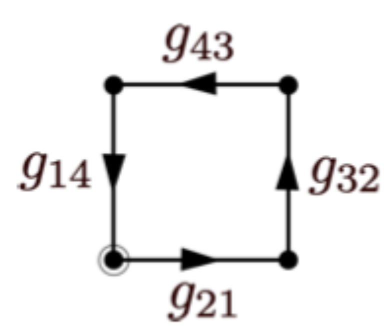
***Refined Lattice Yang-Mills Path Integral***

- *Plaquette d.o.f. and weight*
- *Cube d.o.f. and weight; Chern-Simons saddle*
- *Hypercube d.o.f. and weight; instanton*

*Further Remarks*

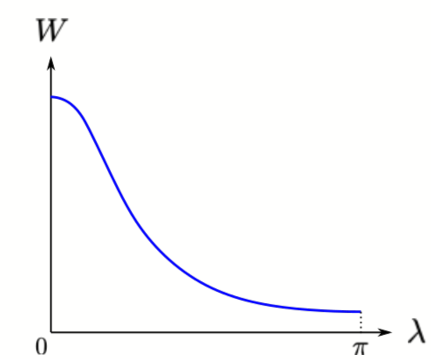
Wilson's traditional lattice gauge theory (we focus on *SU(2)* now)

$$Z \;=\; \left[\prod_{l'}\int_{g_{l'}\in SU(2)}\right]\;\prod_p W(\lambda_p)$$




$$Dg_p = U_p e^{i\lambda_p\sigma^3}U_p^{-1}$$

*W* decreases with $\lambda$



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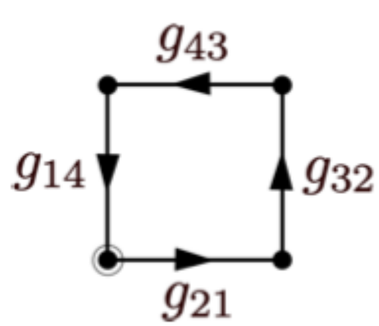


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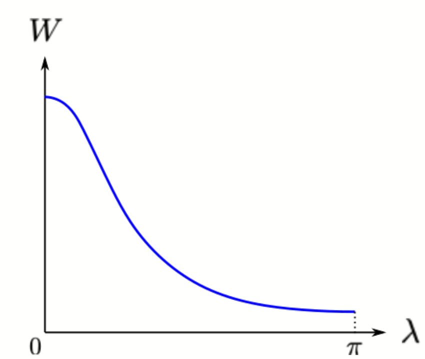


*W* only depends on eigenvalue — gauge invariance

refined version — categorical generalization of Villainization

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

with Peng Zhang
[2411.07195]

refined version — categorical generalization of Villainization

traditional link d.o.f.

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

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refined version — categorical generalization of Villainization

plaquette d.o.f. — NOT group or fibre bundle

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plaquette weight

refined version — categorical generalization of Villainization

cube d.o.f. — dynamical CS phase

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cube weight — CS saddle & CS sensitivity (technical part)

refined version — categorical generalization of Villainization

hypercube d.o.f. — Villainization

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topological theta term

hypercube weight
over instanton density

$$\mathcal{I}_h = \frac{d\mathcal{C}_h}{2\pi} + \iota_h$$

*The Big Picture*

*Refined Lattice Yang-Mills Path Integral*

- ***Plaquette d.o.f. and weight***
- *Cube d.o.f. and weight; Chern-Simons saddle*
- *Hypercube d.o.f. and weight; instanton*

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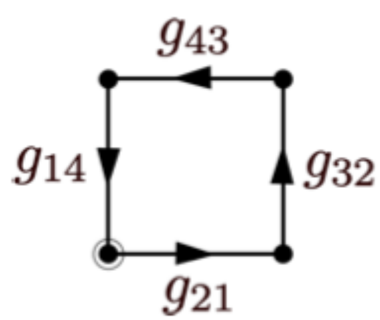
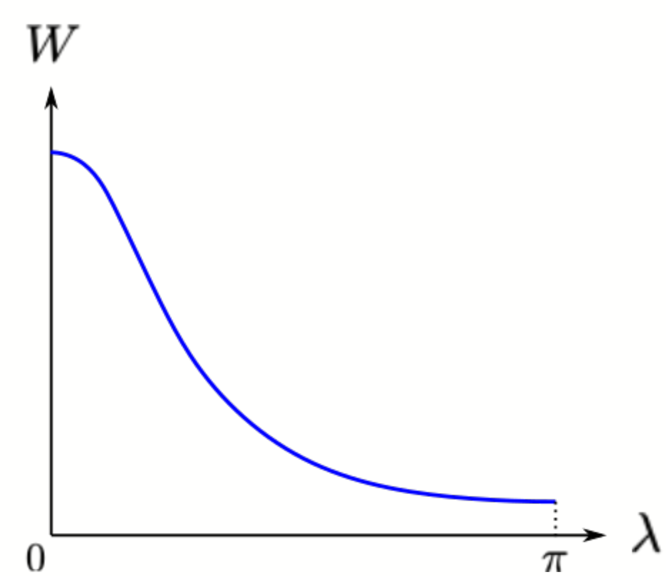
plaquette weight

*traditionally:*



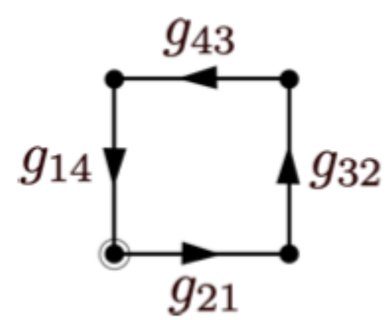

$$Dg_p = U_p e^{i\lambda_p \sigma^3} U_p^{-1} \in SU(2)$$

traditional link weight:




*W* only depends on eigenvalue — gauge invariance

*traditionally:*

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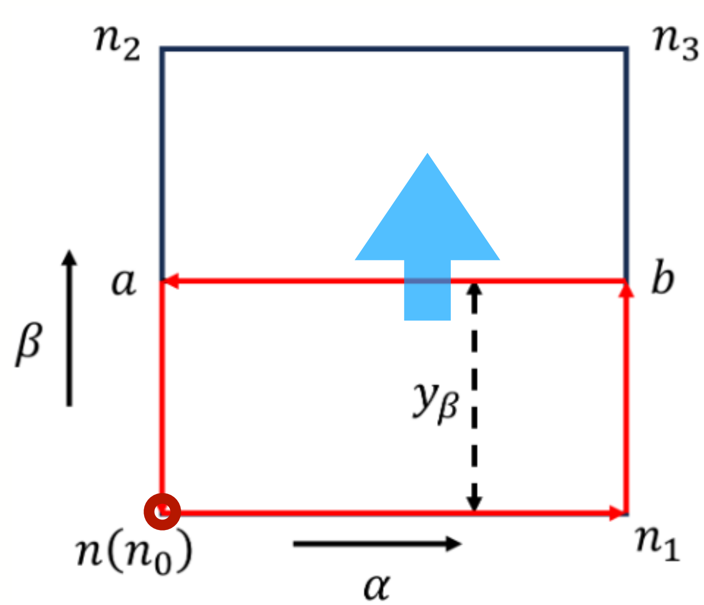
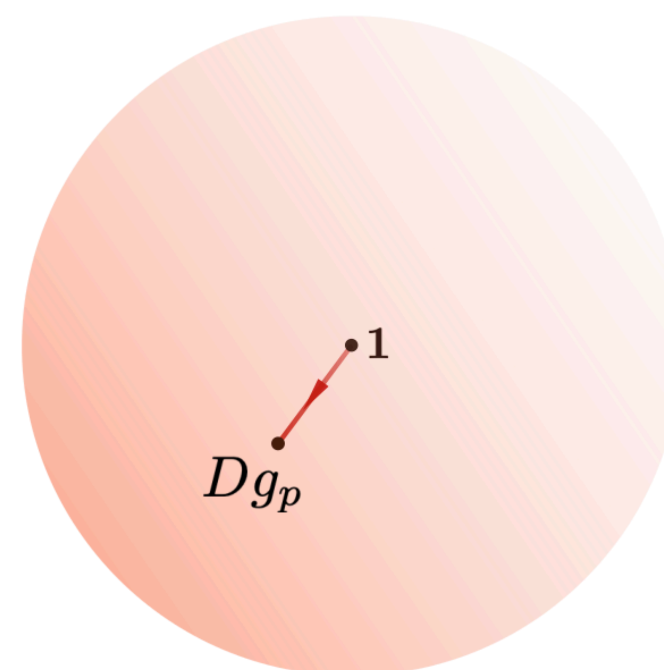
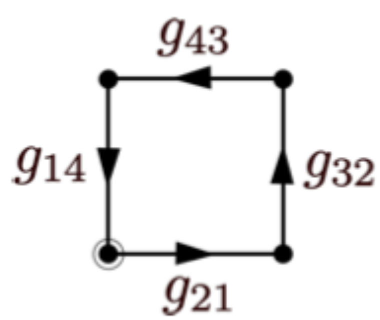



think of the interpolation in *SU(2)*
pictured as a 3-ball with center=+1, surface=-1

*traditionally:*

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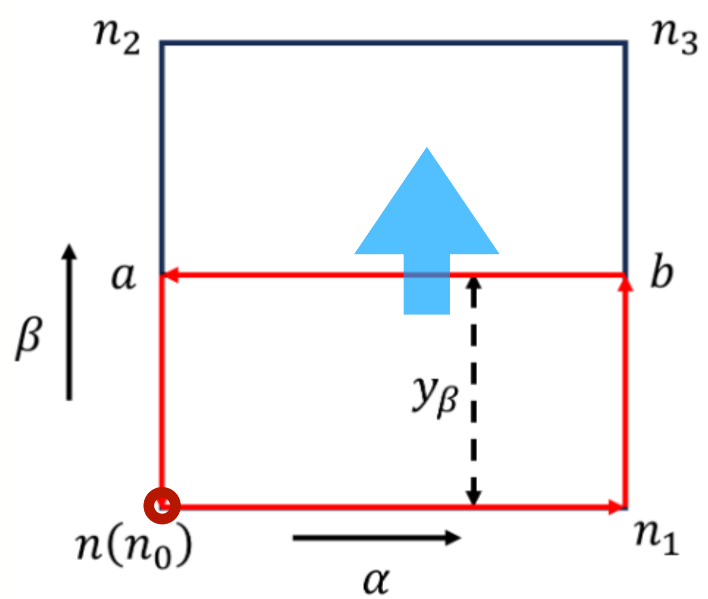
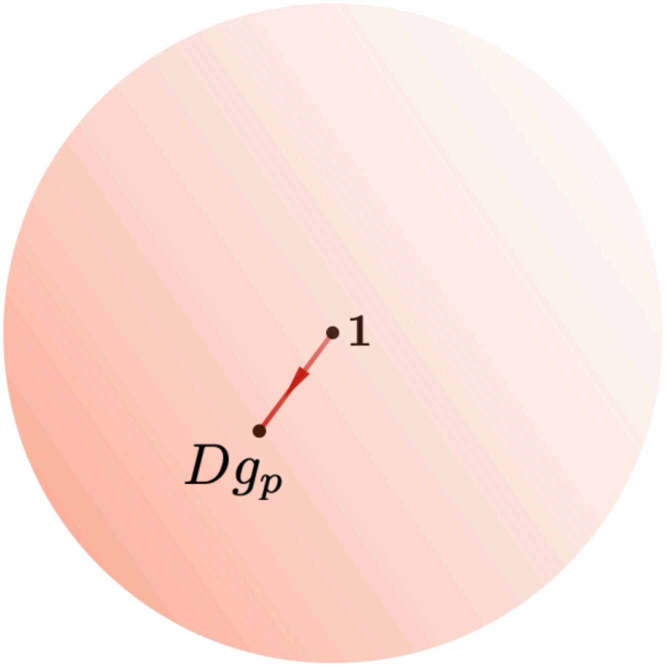
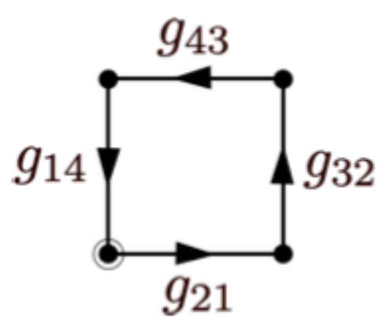



think of the interpolation in *SU(2)*
pictured as a 3-ball with center=+1, surface=-1

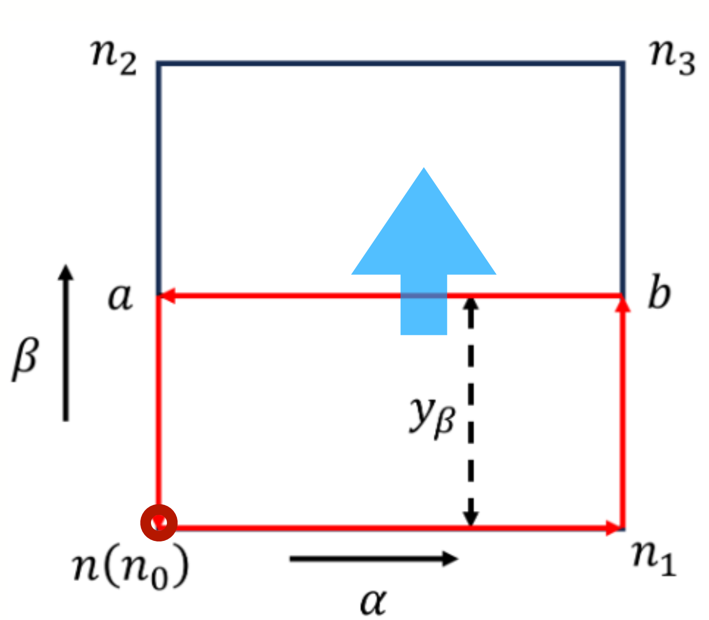
**ambiguous when:** $Dg_p \to -1$

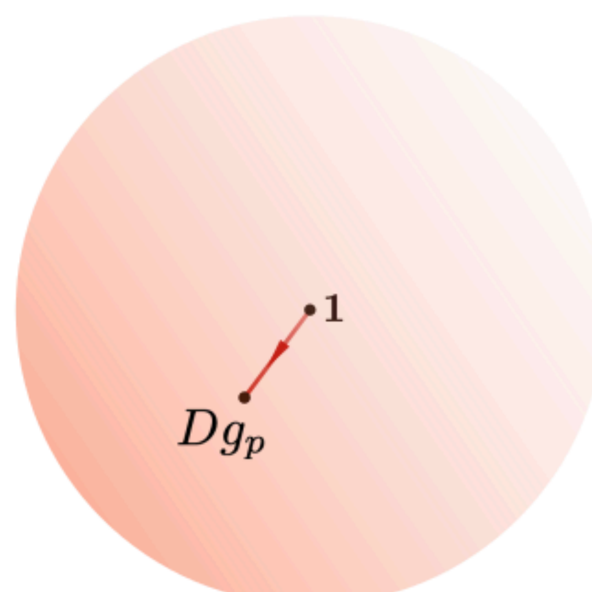
*refined — different ways of interpolation*



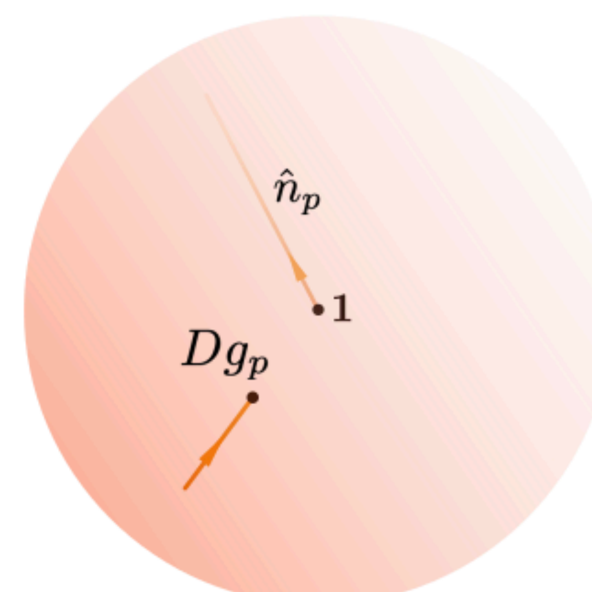

$$Dg_p = U_p e^{i\lambda_p \sigma^3} U_p^{-1} \in SU(2)$$

$$y_p = (Dg_p, m_p, \hat{n}_p) \in Y$$



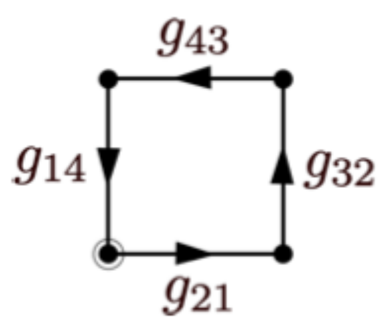




$$m_p = +$$
$$Dg_p \neq -1$$




$$m_p = -$$
$$Dg_p \neq +1$$

*refined — different ways of interpolation*

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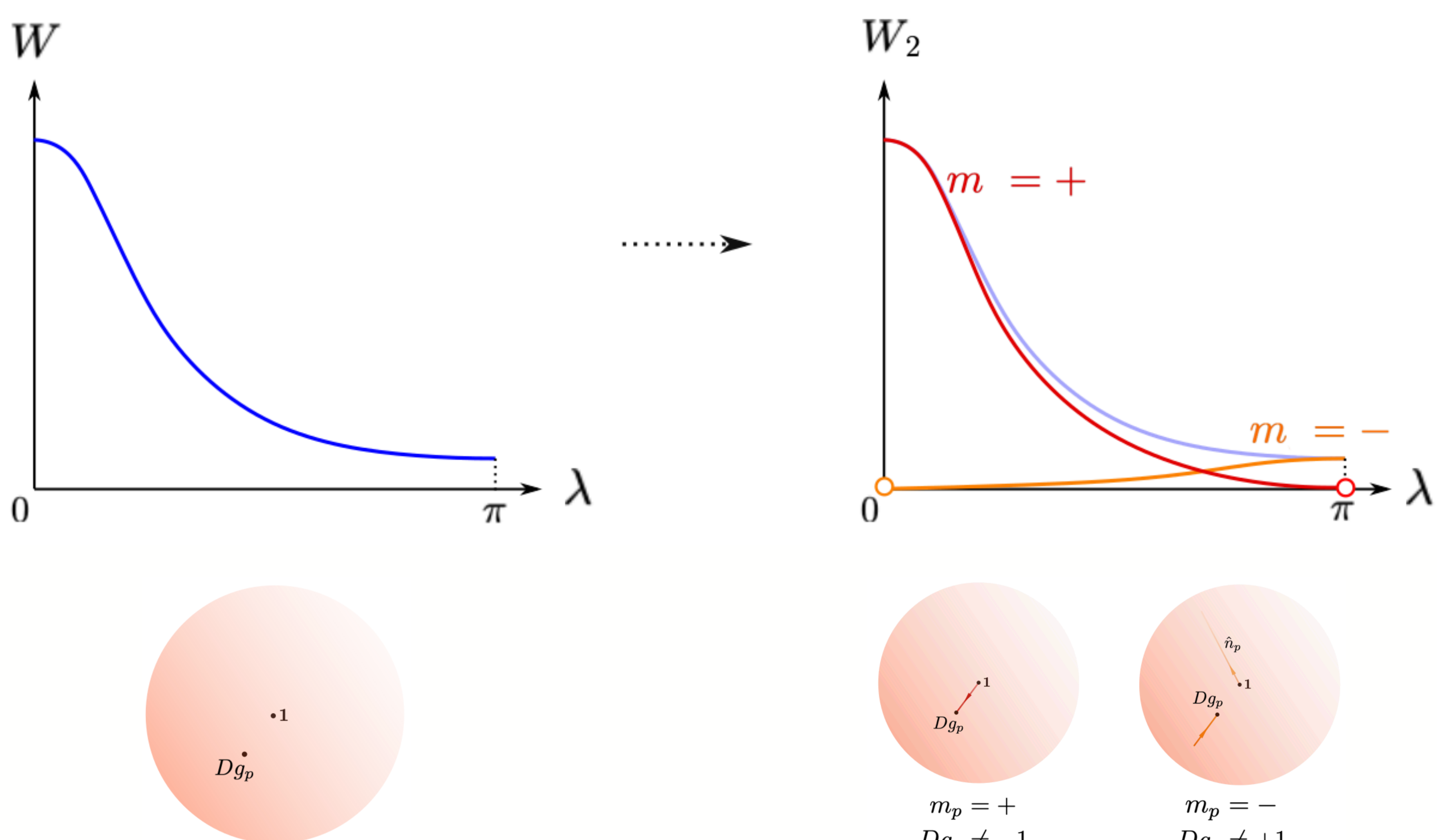
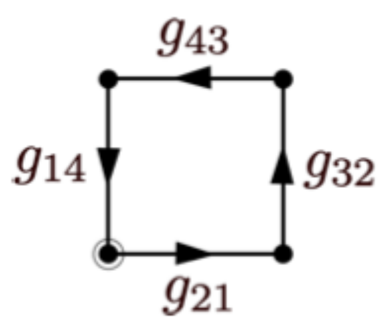


*refined — different ways of interpolation*

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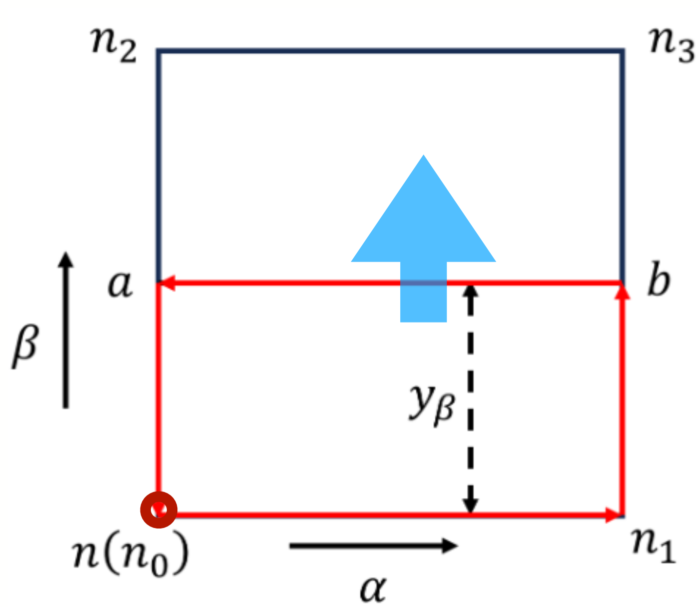


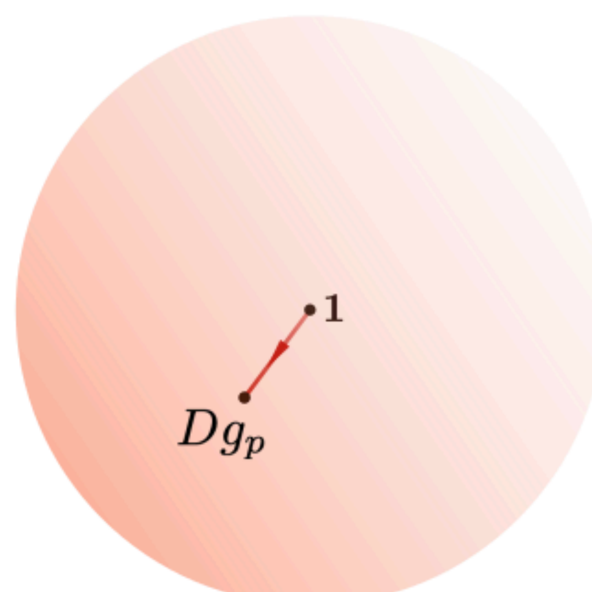


$m_p = +$

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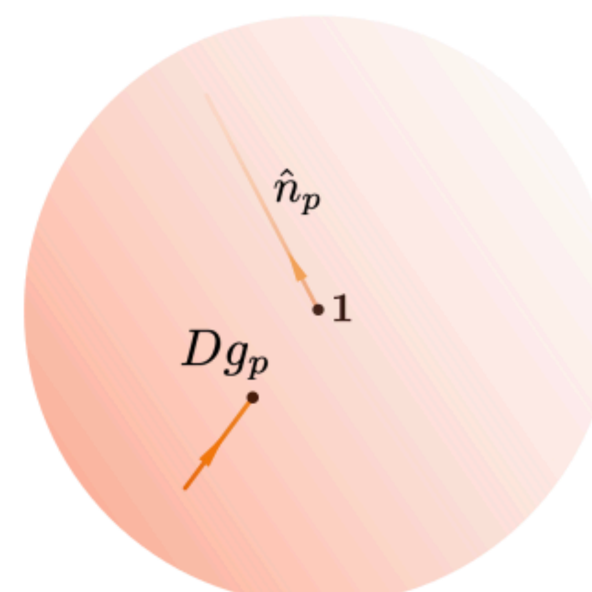
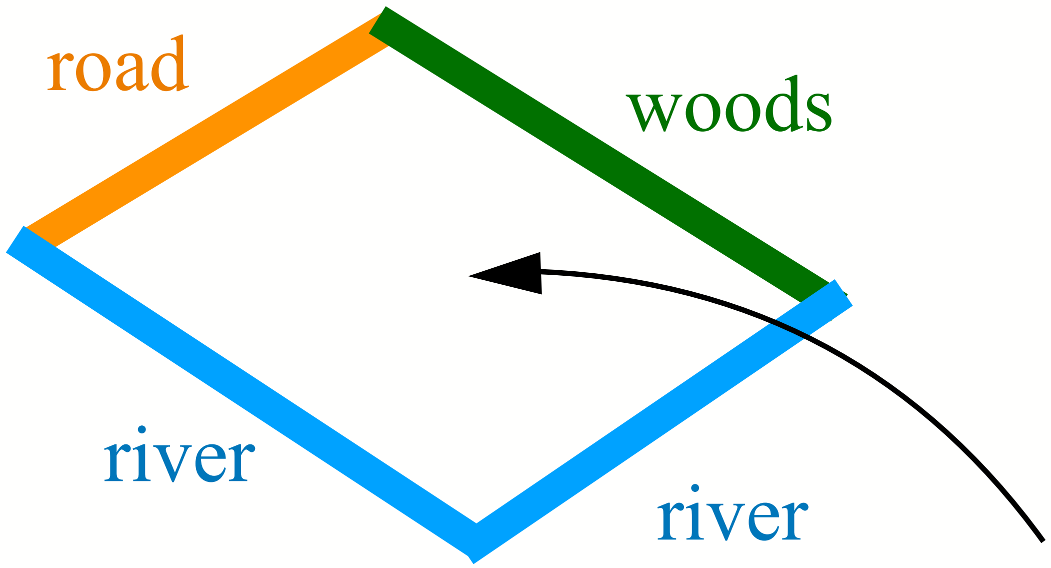



$m_p = -$

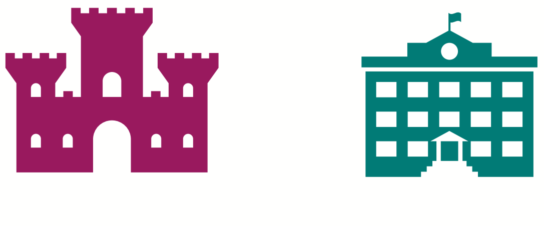
$Dg_p \neq +1$

The *m label does not form a group! And Y not fibre bundle!*

much like some kind of board game:




*Which types of castle are allowed to play here?*



*The Big Picture*

*Refined Lattice Yang-Mills Path Integral*

*Plaquette d.o.f. and weight*

***Cube d.o.f. and weight; Chern-Simons saddle***

*Hypercube d.o.f. and weight; instanton*

*Further Remarks*

refined version — categorical generalization of Villainization

cube d.o.f. — dynamical CS phase

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

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cube weight — CS saddle & CS sensitivity (technical part)

$$W_3(e^{i\mathcal{C}_c}\nu^*_{g_{l\in\partial c},m_{p\in\partial c},\hat{n}_{p\in\partial c}} + c.c.)$$

positive, increasing function

$e^{i\mathcal{C}_c}$ *dynamical CS phase d.o.f. over the cube*

saddle point when $e^{i\mathcal{C}_c} = \nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$

*the dynamical CS phase actually forms a U(1) bundle over the space d.o.f. around the cube*

$$W_3(e^{i\mathcal{C}_c}\nu^*_{g_{l\in\partial c},m_{p\in\partial c},\hat{n}_{p\in\partial c}} + c.c.)$$

positive, increasing function

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saddle point when $e^{i\mathcal{C}_c} = \nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$

CS saddle $\nu/|\nu|$ constructed by standard interpolation of gauge holonomy into cube (~ Lüscher 1982)

so deviation of $e^{i\mathcal{C}_c}$ from CS saddle $e^{i\mathcal{C}_c^{(0)}}$

~ deviation from standard interpolation

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~ deviation from standard interpolation

CS sensitivity $|\nu|$ approaches 0 when standard interpolation into cube becomes ambiguous

To construct CS saddle $\nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$, we need:

— a standard interpolation of gauge holonomy into cube

— for a given interpolation, an evaluation of the cube's CS phase

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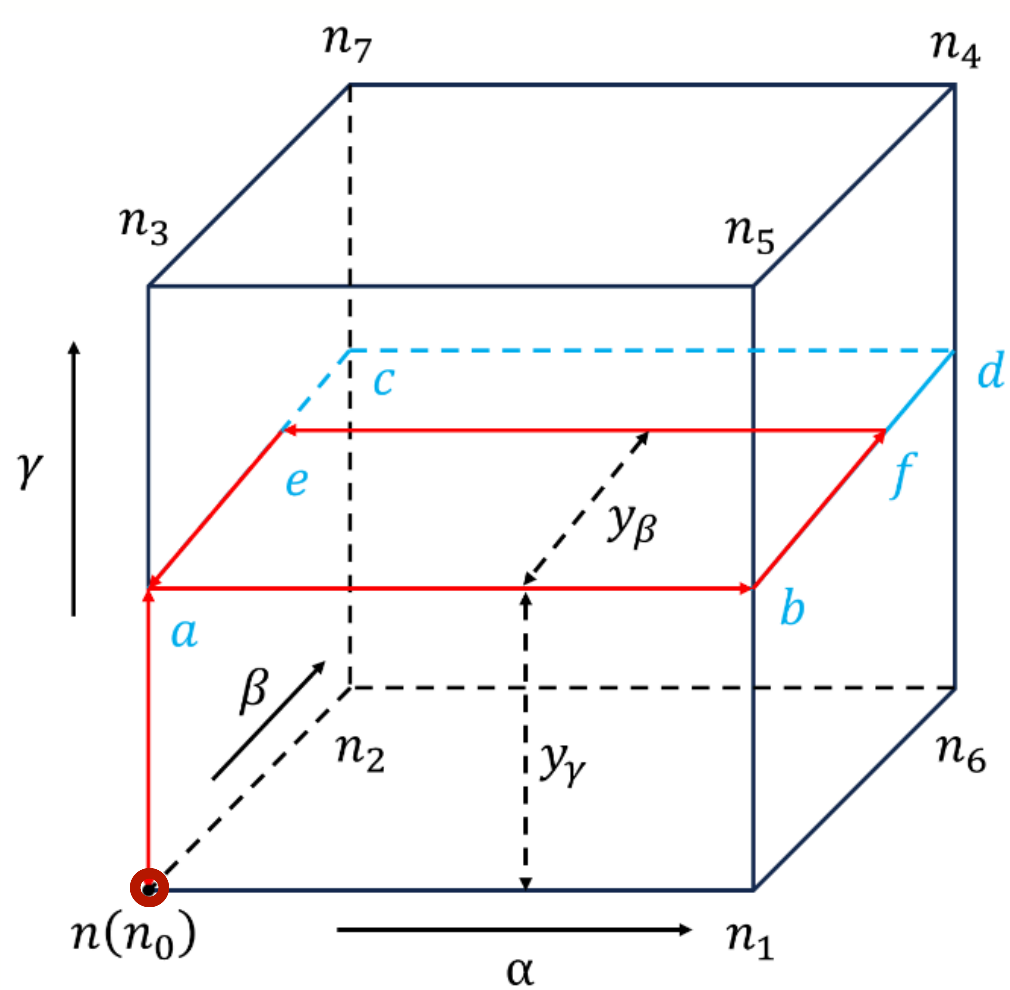
— a standard interpolation of gauge holonomy into cube

— for a given interpolation, an evaluation of the cube's CS phase

*crucial*: we only consider interpolation of Wilson loops starting at vertices
—to avoid worrying about unnecessary gauge choices in the interior of cube




given d.o.f. and interpolations on the plaquettes around

construct a standard interpolation into the interior of cube

CS sensitivity $|\nu|$ approaches 0 when standard interpolation ambiguous

To construct CS saddle $\nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$, we need:

— a standard interpolation of gauge holonomy into cube

— for a given interpolation, an evaluation of the cube's CS phase

in fact we only need the CS phase over all cubes on the boundary of hypercube

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in fact we only need the CS phase over all cubes on the boundary of hypercube

in cube weight $$e^{i\mathcal{C}_c}\nu^* = e^{i(\mathcal{C}-\mathcal{C}^{(0)})_c}|\nu| = e^{i\tilde{\mathcal{C}}_c}|\nu|$$

instanton density over hypercube $$\mathcal{I}_h = \frac{d\mathcal{C}_h}{2\pi} + \iota_h = \frac{d\tilde{\mathcal{C}}_h + d\mathcal{C}_h^{(0)}}{2\pi} + \iota_h$$

only need $d\mathcal{C}_h^{(0)} \mod 2\pi$

To construct CS saddle $\nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$, we need:

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— for a given interpolation, an evaluation of the cube's CS phase

Lüscher has an expression for instanton density over a hypercube given the interpolation in the cubes around

$$\frac{\varepsilon_{\mu\nu\rho\sigma}}{24\pi^2}\Bigg\{\int_{c(n+\hat{\mu},\mu)} \mathrm{d}^3x \mathrm{Tr}[(S_{n+\hat{\mu},\mu})^{-1}\partial_\nu S_{n+\hat{\mu},\mu}(S_{n+\hat{\mu},\mu})^{-1}\partial_\rho S_{n+\hat{\mu},\mu}(S_{n+\hat{\mu},\mu})^{-1}\partial_\sigma S_{n+\hat{\mu},\mu}]$$
$$+\int_{p(n+\hat{\mu}+\hat{\nu},\mu,\nu)} \mathrm{d}^2x \mathrm{Tr}[P_{n+\hat{\mu}+\hat{\nu},\mu\nu}\partial_\rho(P_{n+\hat{\mu}+\hat{\nu},\mu\nu})^{-1}(R_{n+\hat{\mu},\mu;\nu})^{-1}\partial_\sigma R_{n+\hat{\mu},\mu;\nu}]$$
$$-\int_{c(n,\mu)} \mathrm{d}^3x \mathrm{Tr}[(S_{n,\mu})^{-1}\partial_\nu S_{n,\mu}(S_{n,\mu})^{-1}\partial_\rho S_{n,\mu}(S_{n,\mu})^{-1}\partial_\sigma S_{n,\mu}]$$
$$-\int_{p(n+\hat{\nu},\mu,\nu)} \mathrm{d}^2x \mathrm{Tr}[P_{n+\hat{\nu},\mu\nu}\partial_\rho(P_{n+\hat{\nu},\mu\nu})^{-1}(R_{n,\mu;\nu})^{-1}\partial_\sigma R_{n,\mu;\nu}]\Bigg\},$$

but now we are using it differently:

—we allow general gauge holonomies, not just those close to 1

—only the phase, not a real number, since integer part depends on gauge

—CS phase saddle, not "the CS phase"; the latter fluctuates

—re-express the phase saddle in a manifestly gauge invariant form
to avoid unnecessary gauge choices ambiguities in the interior of cube

To construct CS saddle $\nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$, we need:

— a standard interpolation of gauge holonomy into cube

— for a given interpolation, an evaluation of the cube's CS phase

$$
\begin{aligned}
&-\frac{\epsilon^{1\mu\nu\rho}}{12\pi}\int_{c(n+\hat{1},1)} \mathrm{d}^3x\,\mathbf{tr}\left[T(x)^{-1}\partial_\mu T(x)T(x)^{-1}\partial_\nu T(x)T(x)^{-1}\partial_\rho T(x)\right] \\
&-\frac{\epsilon^{1\mu\nu\rho}}{4\pi}\int_{p(n+\hat{1}+\hat{\mu},1,\mu)} \mathrm{d}^2x\,\mathbf{tr}\left[T(x)^{-1}\left(R_{n,1;\mu}(x-\hat{1})\partial_\nu R_{n+\hat{\mu},\mu;1}(x)\partial_\rho R_{n+\hat{1},1;\mu}(x)^{-1}\right.\right. \\
&\qquad\qquad +\partial_\nu R_{n,1;\mu}(x-\hat{1})R_{n+\hat{\mu},\mu;1}(x)\partial_\rho R_{n+\hat{1},1;\mu}(x)^{-1} \\
&\qquad\qquad \left.\left.+\partial_\nu R_{n,1;\mu}(x-\hat{1})\partial_\rho R_{n+\hat{\mu},\mu;1}(x)R_{n+\hat{1},1;\mu}(x)^{-1}\right)\right] \\
&\mod 2\pi\ .
\end{aligned}
$$

CS phase saddle around hypercube *manifestly* gauge inv

beautiful relation between CS in the space of gauge fields
and WZW in the space of Wilson lines

$$W_3(e^{i\mathcal{C}_c}\nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.)$$

positive, increasing function

$e^{i\mathcal{C}_c}$ *dynamical CS phase d.o.f. over the cube*

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topological theta term

hypercube weight
over instanton density

$$\mathcal{I}_h = \frac{d\mathcal{C}_h}{2\pi} + \iota_h$$

Villainize the U(1) dynamical CS field on cube

instanton density over hypercube

$$\mathcal{I}_h = \frac{d\mathcal{C}_h}{2\pi} + \iota_h = \frac{d\tilde{\mathcal{C}}_h + d\mathcal{C}_h^{(0)}}{2\pi} + \iota_h \in \mathbb{R}$$

instanton number

$$I := \oint_{4d} \mathcal{I} = \sum_h \mathcal{I}_h = \sum_h \iota_h \in \mathbb{Z}$$

hypercube weight decreases with instanton density

$$W_4(\mathcal{I}_h)$$

(5d cell Yang monopole $d\mathcal{I}$ )

*“delooped CS bundle 2-gerbe (as a cubicial weak 4-group)” construction*

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

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*"delooped CS bundle 2-gerbe (as a cubicial weak 4-group)" construction*

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*"delooped CS bundle 2-gerbe (as a cubicial weak 4-group)" construction*

cube d.o.f. — dynamical CS phase

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

cube weight — CS saddle & CS sensitivity (technical part)

*"delooped CS bundle 2-gerbe (as a cubicial weak 4-group)" construction*

plaquette d.o.f. — NOT group or fibre bundle

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

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plaquette weight

*"delooped CS bundle 2-gerbe (as a cubicial weak 4-group)" construction*

traditional link d.o.f.

$$Z_\Theta = \boxed{\left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right]} \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

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*The Big Picture*

*Refined Lattice Yang-Mills Path Integral*

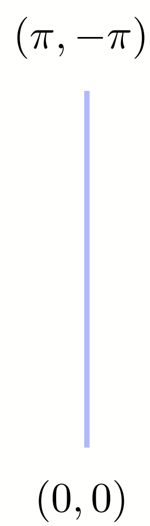
*Plaquette d.o.f. and weight*

*Cube d.o.f. and weight; Chern-Simons saddle*

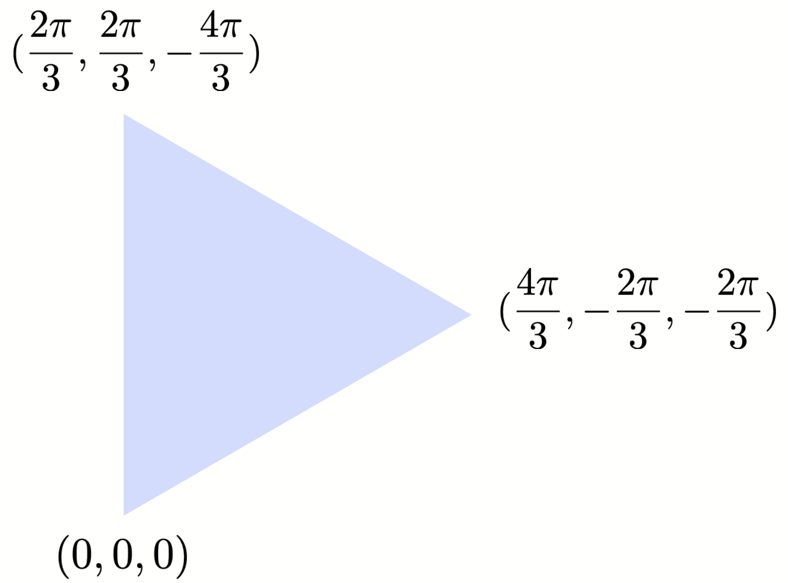
*Hypercube d.o.f. and weight; instanton*

***Further Remarks***

— Generalization to SU(N)




SU(2)




SU(3)

Weyl alcove in Cartan subalgebra (space of eigenvalues)

— Generalization to SU(N)

— Optimization of the weight functions in numerics (not my expertise)

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better control in renormalization
(recall vortex fugacity in BKT)

relation to Symanzik improvement?

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— Similar construction for $S^3$ pion NLsM, get $\pi_3$ baryons
but still need 4d WZW term, due to $\pi_5$

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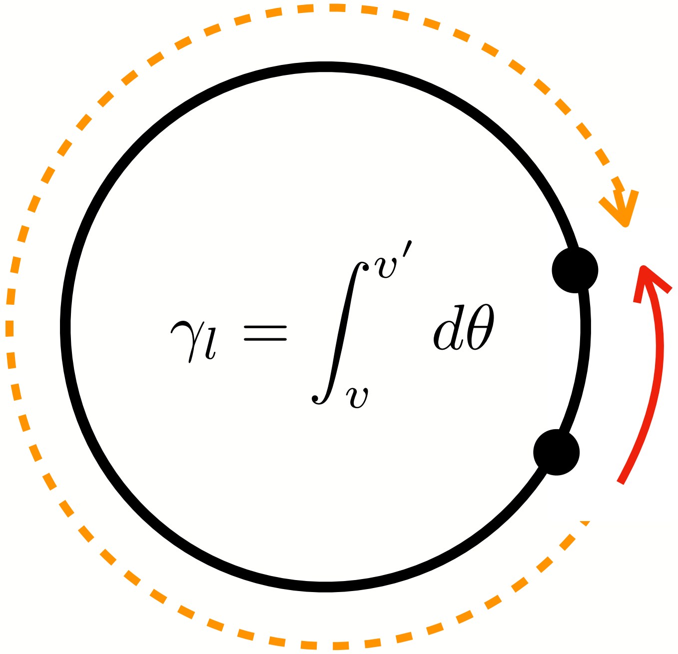
— A lot of beautiful, deep themes in cat theory

thank you

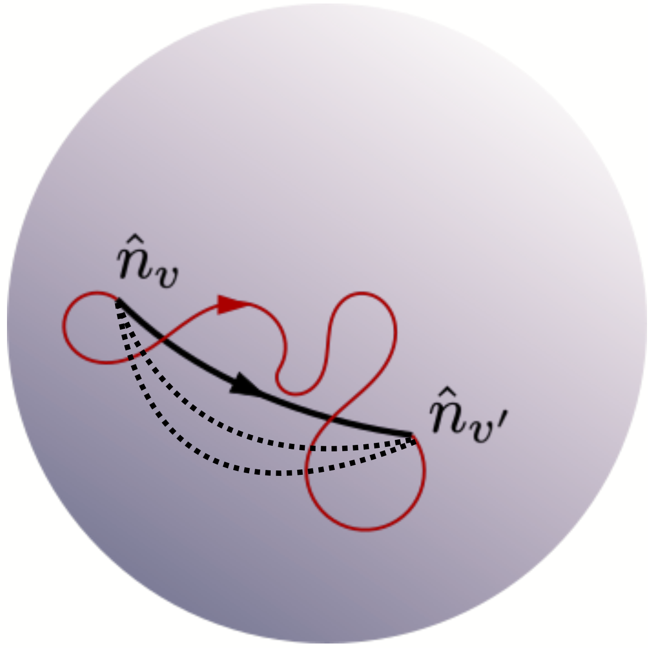
# back up

**why cat(egory) necessary**

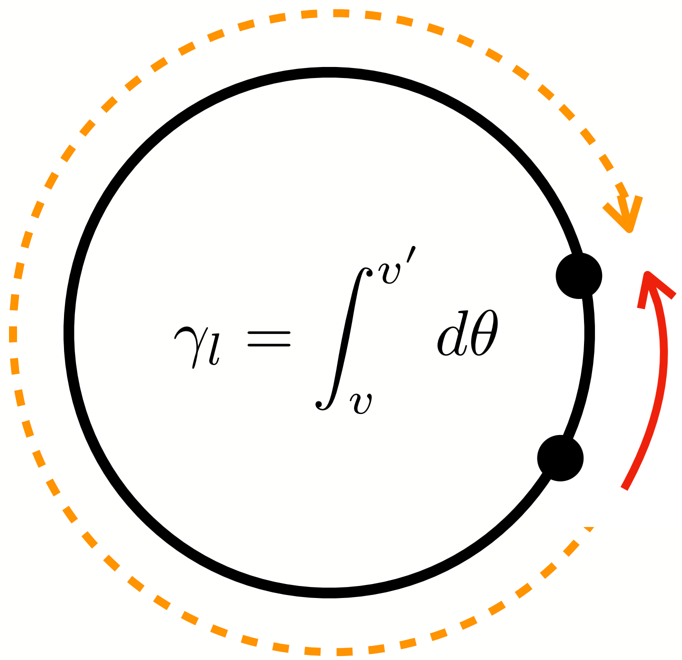
$S^1$ nlsm (XY): Villainization




$S^2$ nlsm: spinon-decomposition

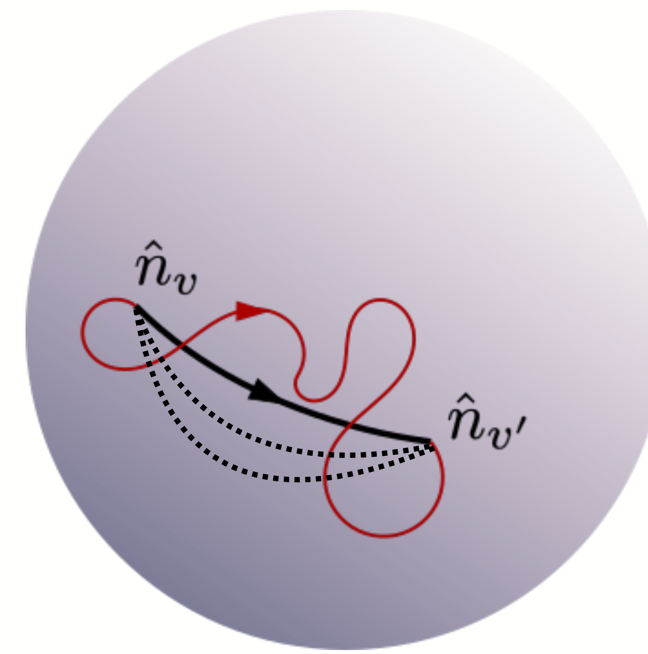


$S^1$ nlsm (XY): Villainization




$$
\begin{array}{ccc}
2\pi\mathbb{Z} & \rightarrow & \mathbb{R} \\
 & & \downarrow \\
 & & S^1
\end{array}
$$

$S^2$ nlsm: spinon-decomposition




skrymion

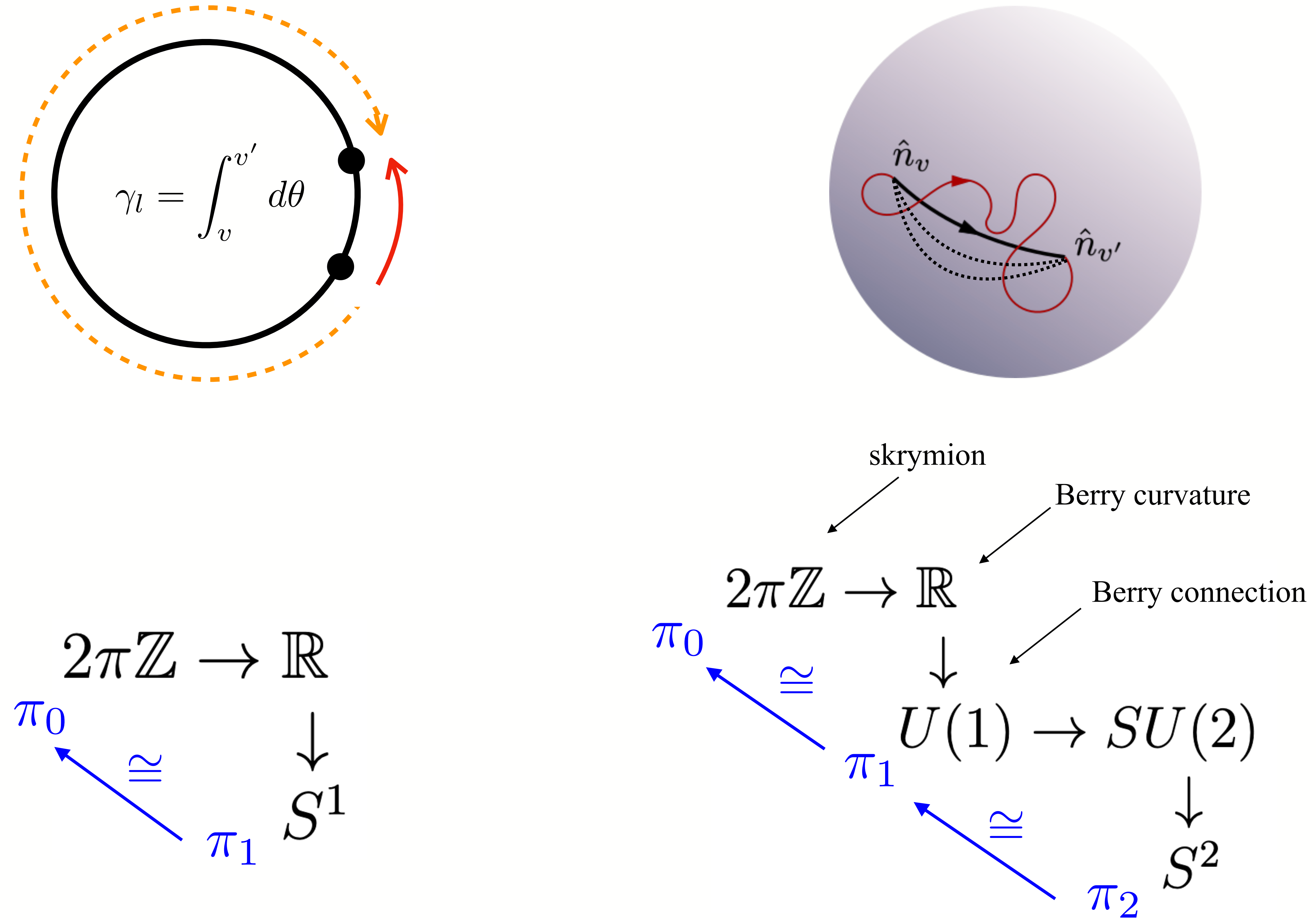
Berry curvature

Berry connection

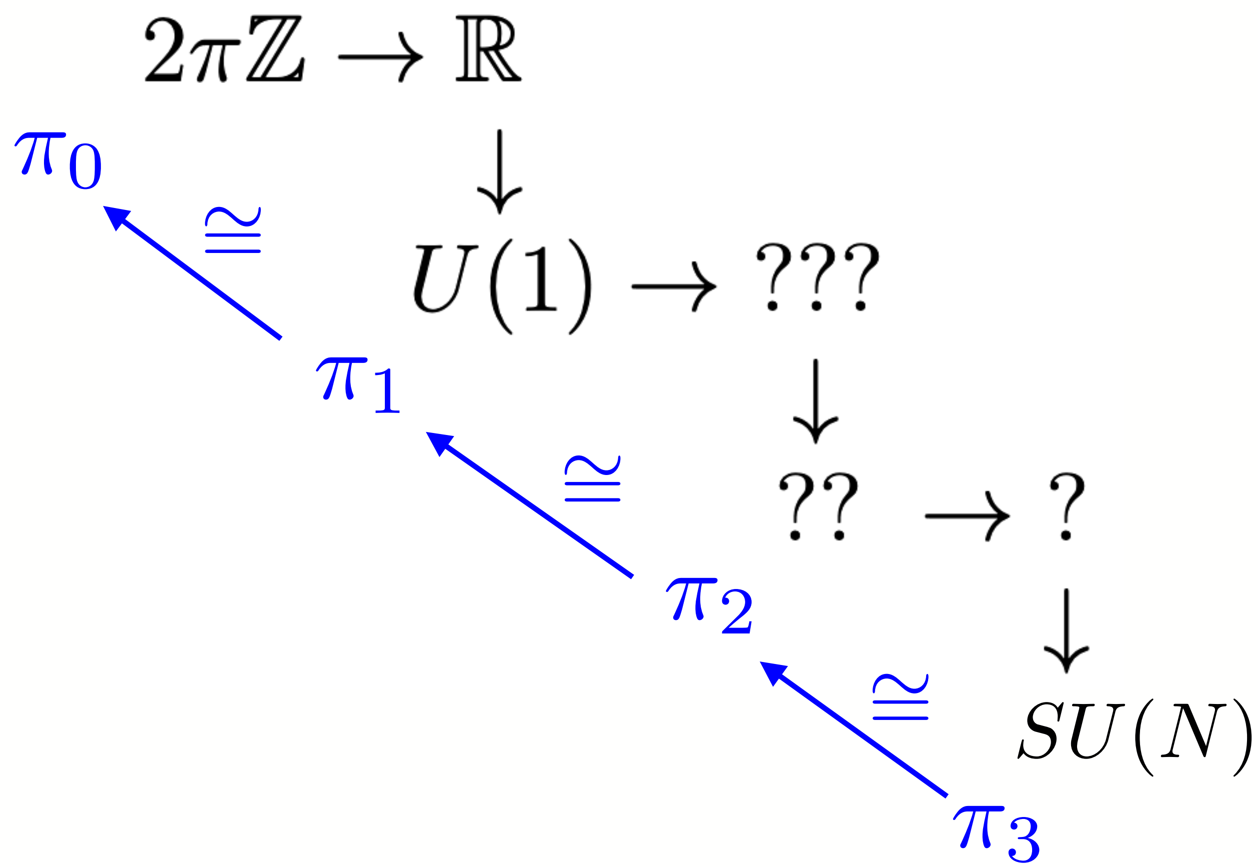
$$
\begin{array}{ccccc}
2\pi\mathbb{Z} & \rightarrow & \mathbb{R} & & \\
 & & \downarrow & & \\
 & & U(1) & \rightarrow & SU(2) \\
 & & & & \downarrow \\
 & & & & S^2
\end{array}
$$

$S^1$ nlsm (XY): Villainization

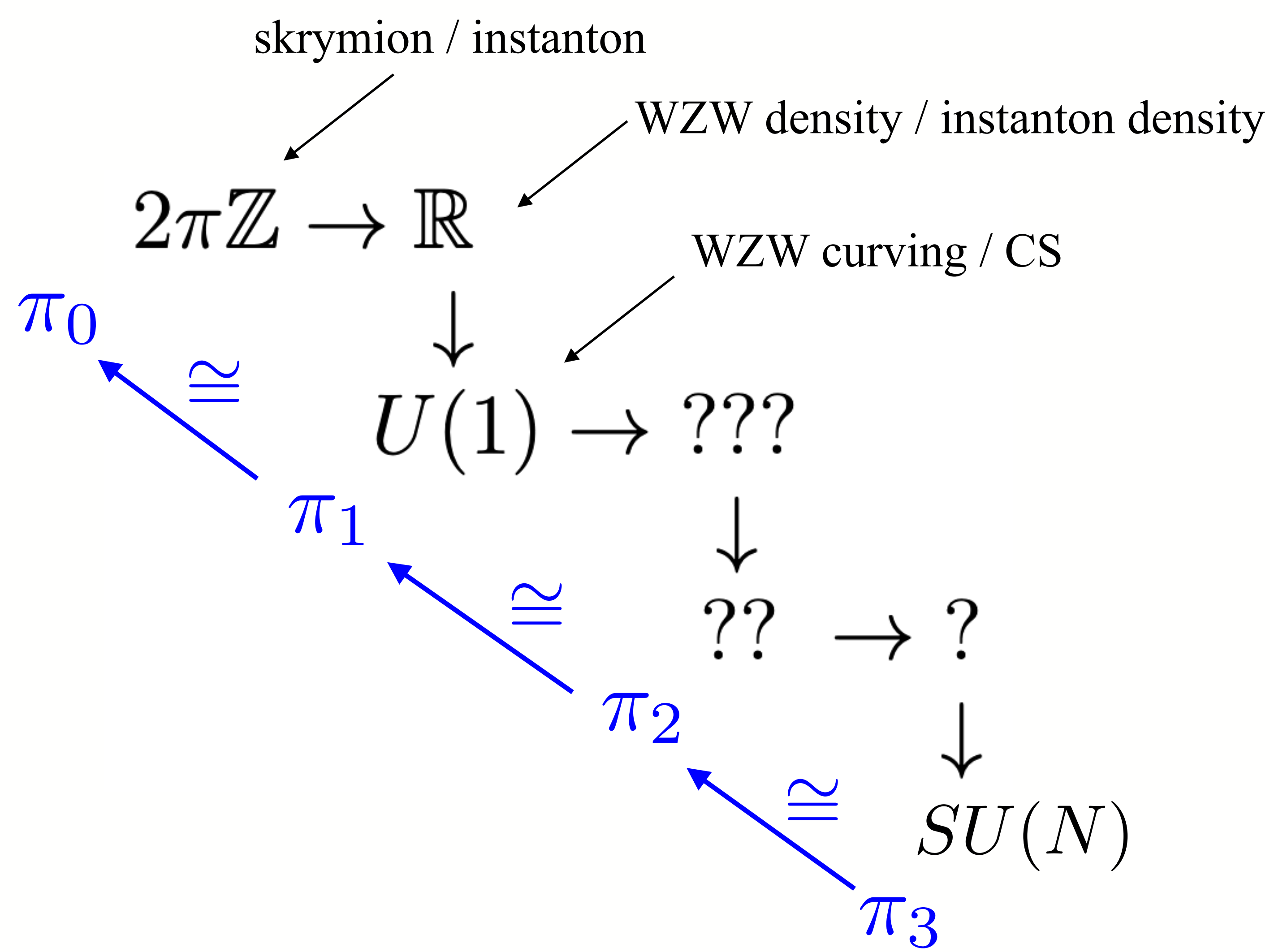
$S^2$ nlsm: spinon-decomposition



seems now we want



seems now we want



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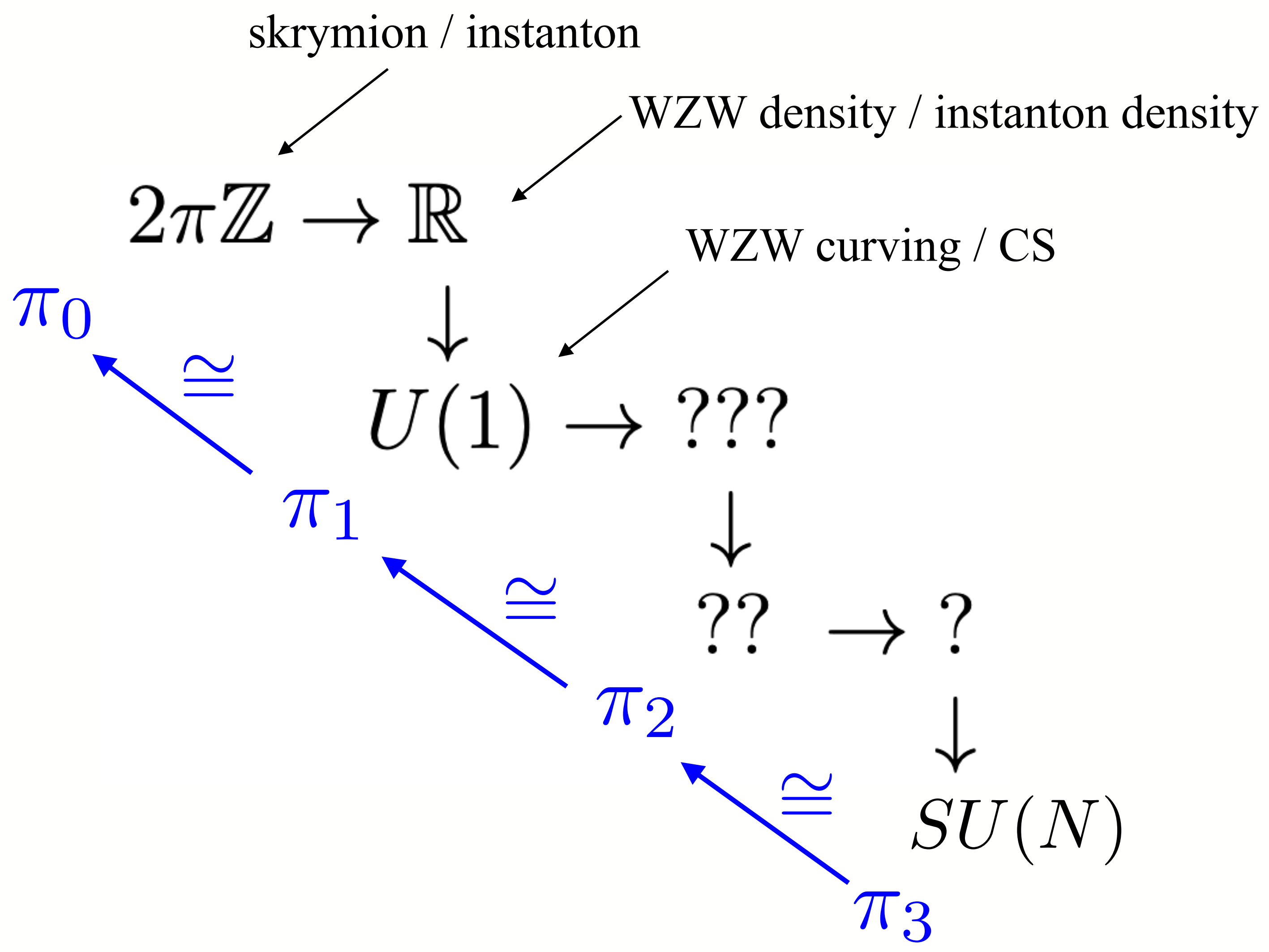


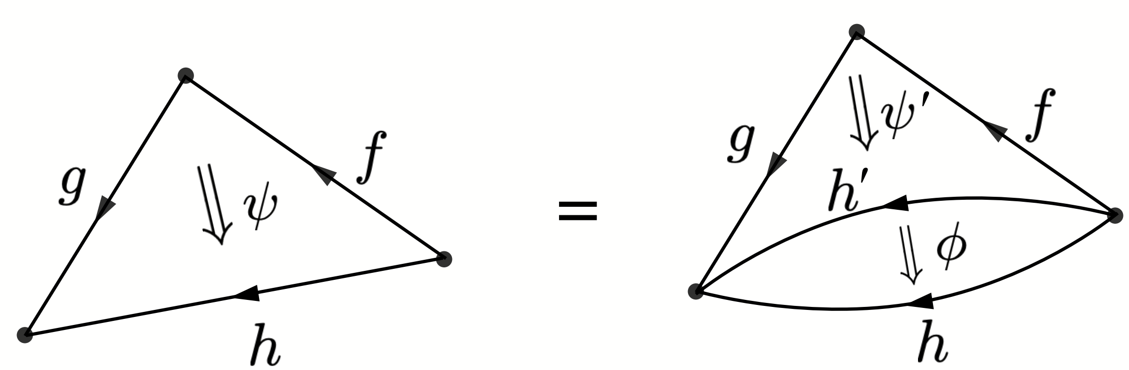

The "??" is on link (for nlsm), should be able to compose.

But finite dimensional Lie group always has trivial $\pi_2$ !

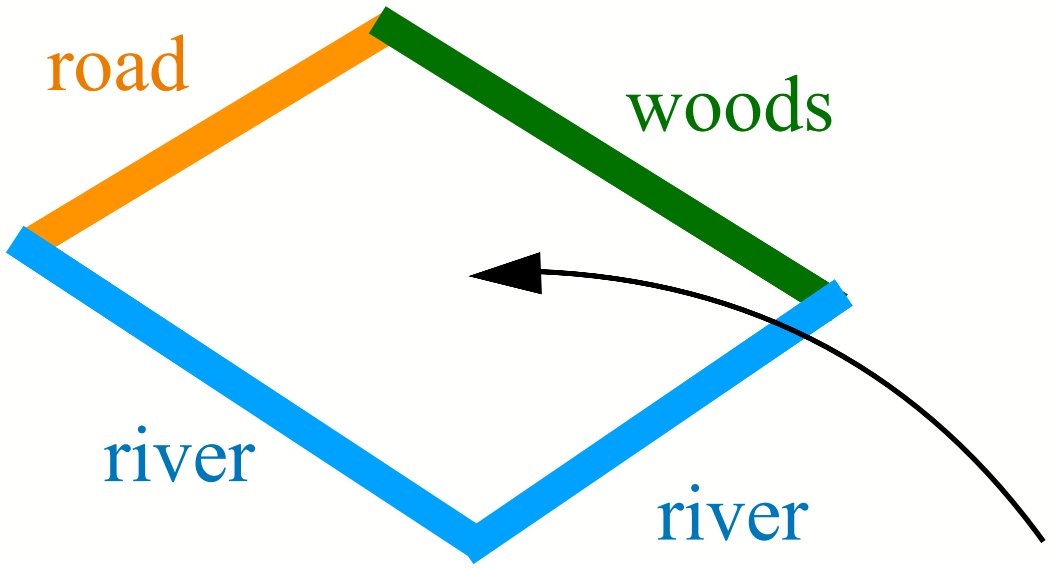
We need:

coverings that are not fibre bundles

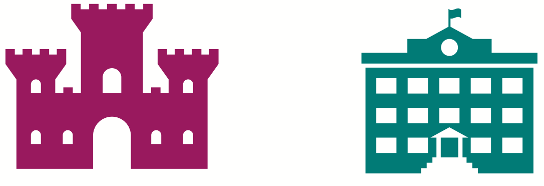
d.o.f. that can be composed, but <u>result is non-unique, and can talk about relations between different results</u>
— more flexible "game rules" than groups



much like some kind of board game:

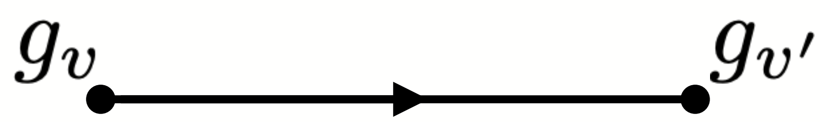



*Which types of castle*
*are allowed to play here?*



# pion non-linear sigma model

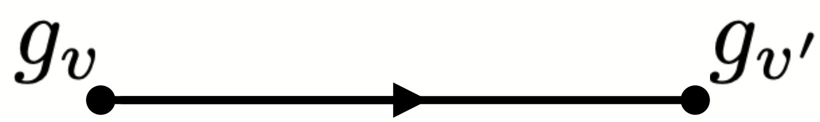
$S^3$ nlsm: 2d WZW, 3d skyrmion, 4d hedgehog




traditional link field: $g_{v'} g_v^{-1} \in SU(2)$

new link field: $y_l = \left(g_{v'} g_v^{-1}, m_l, \hat{n}_l\right) \in Y$ which covers $SU(2)$

## $S^3$ nlsm: 2d WZW, 3d skyrmion, 4d hedgehog

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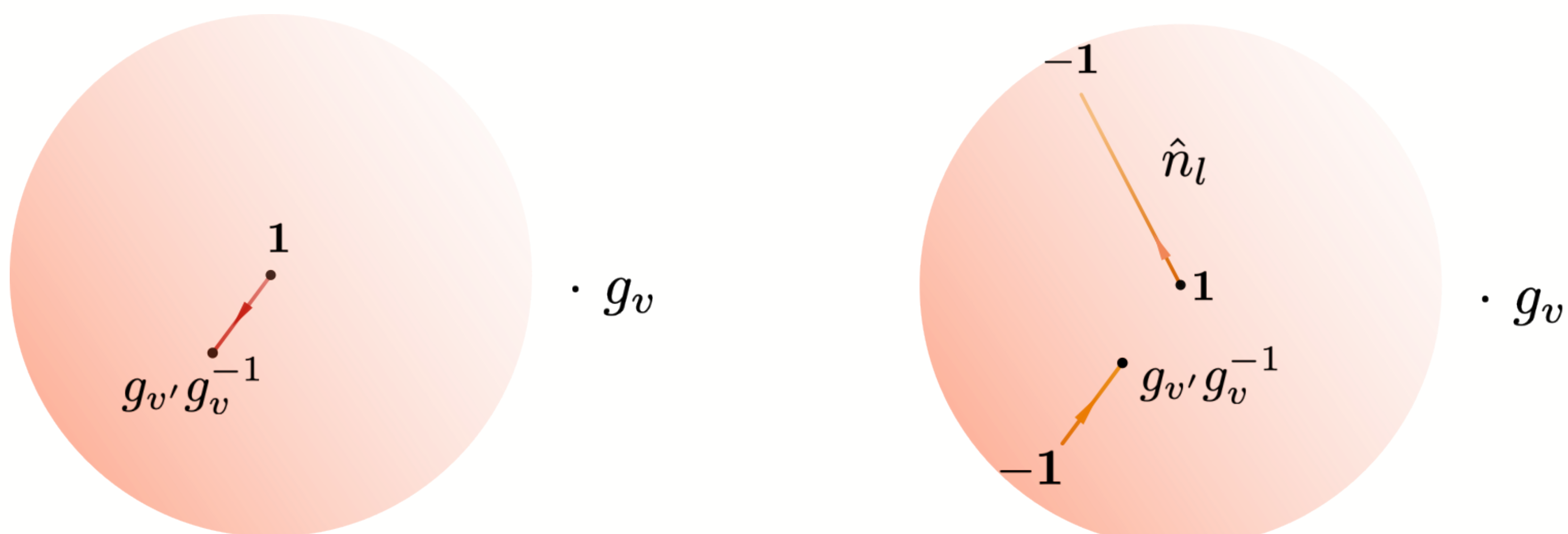
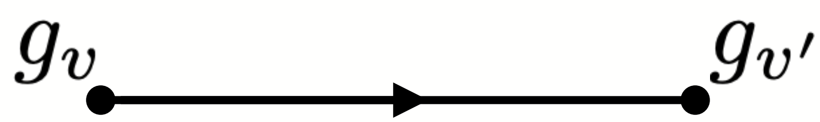



$m_l = +$
$(g_{v'} g_v^{-1} \neq -\mathbf{1})$

$m_l = -$
$(g_{v'} g_v^{-1} \neq +\mathbf{1})$

$$Y = (SU(2) \backslash \{-\mathbf{1}\}) \sqcup \left(SU(2) \backslash \{+\mathbf{1}\} \times S^2\right)$$

$S^3$ nlsm: 2d WZW, 3d skyrmion, 4d hedgehog

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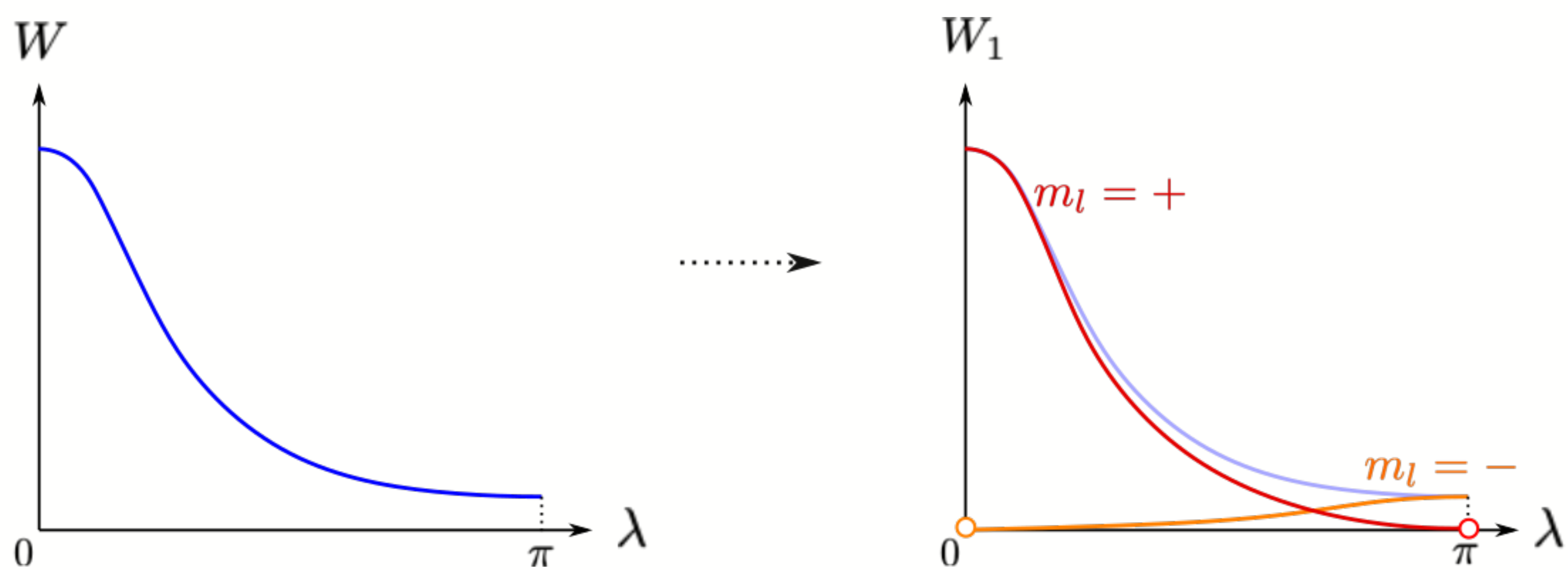
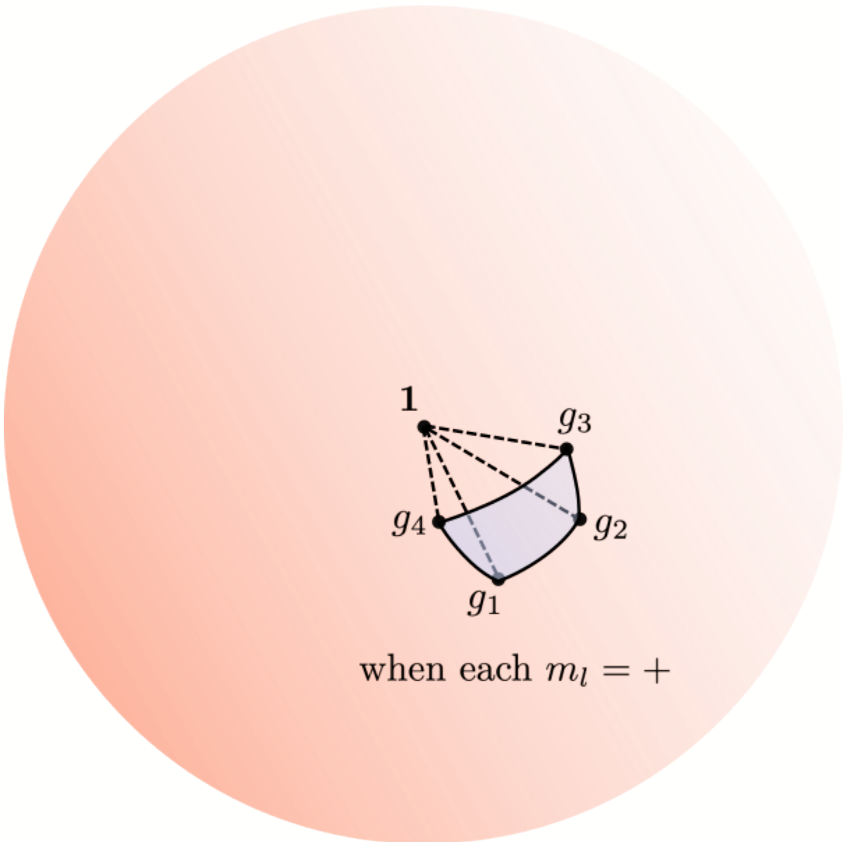


$S^3$ nlsm: 2d WZW, 3d skyrmion, 4d hedgehog

plaquette field: given the vertex fields and link fields,
want to sample the surface, but not too much details:
deviation from min surface captured by U(1) — WZW integral

$$W_2(e^{i\mathcal{W}_p}\mu^*_{g_{v\in\partial p},m_{l\in\partial p},\hat{n}_{l\in\partial p}} + c.c.)$$



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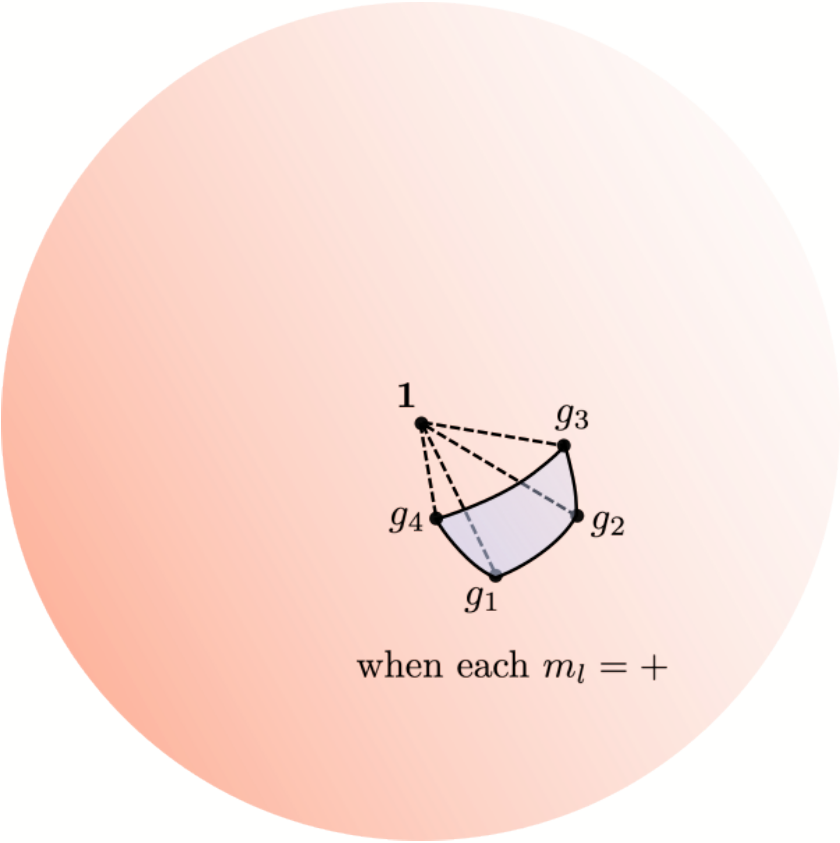
phase of $\mu$: volume of pyramid

$|\mu|$ decreases as loop grows,
$|\mu|$=0 when min surf ambiguous



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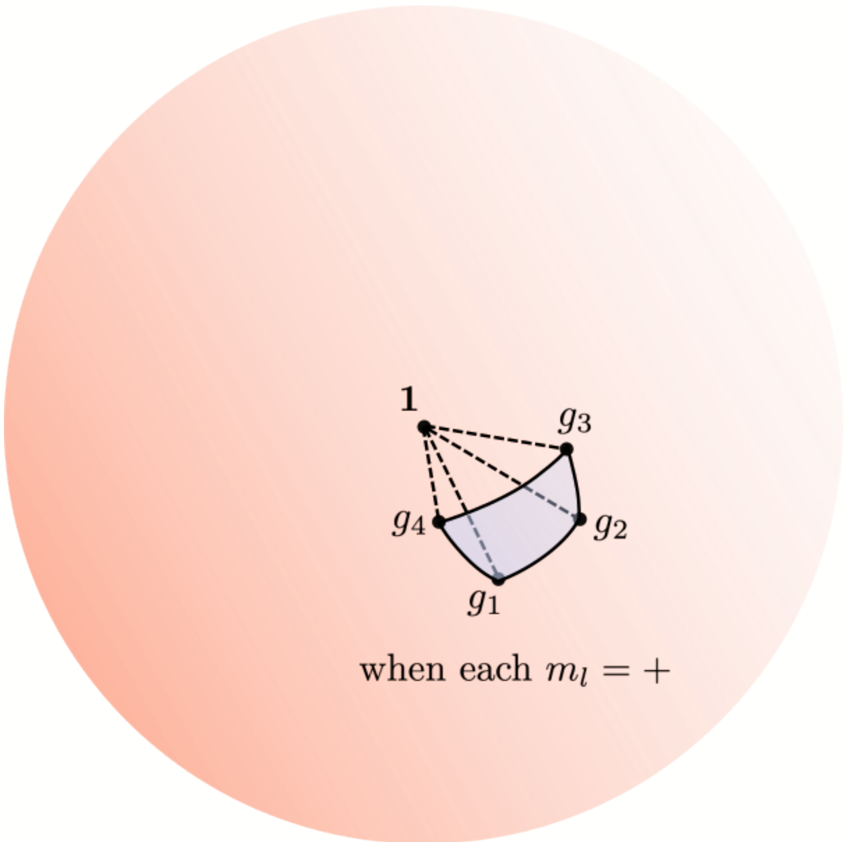


2d lattice WZW: $e^{ik\sum_p \mathcal{W}_p}$

$S^3$ nlsm: 2d WZW, 3d skyrmion, 4d hedgehog

cube field: Villainize the U(1) WZW field on plaquette

skrymion density $\mathcal{S}_c := d\mathcal{W}_c/2\pi + s_c \in \mathbb{R}$

cube weight can be e.g. $e^{-V\mathcal{S}_c^2/2}$

3d theta term: $e^{i\Theta \sum_c \mathcal{S}_c}$

$S^3$ nlsm: 2d WZW, 3d skyrmion, 4d hedgehog

cube field: Villainize the U(1) WZW field on plaquette

skrymion density $\mathcal{S}_c := d\mathcal{W}_c/2\pi + s_c \in \mathbb{R}$

cube weight can be e.g. $e^{-V\mathcal{S}_c^2/2}$ (being Gaussian is not crucial)

3d theta term: $e^{i\Theta \sum_c \mathcal{S}_c}$

hypercube defect: $d\mathcal{S}_h = ds_h \in \mathbb{Z}$ baryon non-conservation

if forbid this by Lagrange multiplier $e^{i\phi_h d\mathcal{S}_h}$

then the U(1) baryon conservation is manifest

**a general relation between**

**continuum QFT and lattice QFT**

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \qquad \Longrightarrow$$

# systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \qquad \Longrightarrow$$

$$\mathcal{L}_0 \hookrightarrow \mathcal{M} \to \mathcal{T}$$

traditional lattice theory

lost info

# systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \quad \Longrightarrow \quad \begin{array}{ccc} \mathcal{P}^d\mathcal{M} & \to & \mathcal{P}^d\mathcal{T} \\ \downdownarrows & & \downdownarrows \\ \vdots & \cdots & \vdots \\ \downdownarrows & & \downdownarrows \\ \mathcal{P}^2\mathcal{M} & \to & \mathcal{P}^2\mathcal{T} \\ \downdownarrows & & \downdownarrows \\ \mathcal{P}\mathcal{M} & \to & \mathcal{P}\mathcal{T} \\ \downdownarrows & & \downdownarrows \\ \mathcal{M} & \to & \mathcal{T} \end{array}$$

still continuum theory
no new info

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \qquad \Longrightarrow \qquad \begin{array}{ccccc} & & \mathcal{P}^d\mathcal{M} & \to & \mathcal{P}^d\mathcal{T} \\ & & \downdownarrows & & \downdownarrows \\ & & \vdots & \cdots & \vdots \\ & & \downdownarrows & & \downdownarrows \\ & & \mathcal{P}^2\mathcal{M} & \to & \mathcal{P}^2\mathcal{T} \\ & & \downdownarrows & & \downdownarrows \\ & & \mathcal{P}\mathcal{M} & \to & \mathcal{P}\mathcal{T} \\ & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_0 & \hookrightarrow & \mathcal{M} & \to & \mathcal{T} \end{array}$$

traditional lattice theory

lost info

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more info

but still lost info

# systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \qquad \Longrightarrow \qquad \begin{array}{ccccc} & & \mathcal{P}^d\mathcal{M} & \to & \mathcal{P}^d\mathcal{T} \\ & & \downdownarrows & & \downdownarrows \\ & & \vdots & \cdots & \vdots \\ & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_2 & \hookrightarrow & \mathcal{P}^2\mathcal{M} & \to & \mathcal{P}^2\mathcal{T} \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_1 & \hookrightarrow & \mathcal{P}\mathcal{M} & \to & \mathcal{P}\mathcal{T} \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_0 & \hookrightarrow & \mathcal{M} & \to & \mathcal{T} \end{array}$$

more info

but still lost info

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field in continuum nlsm

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$$\mathcal{M} \to \mathcal{T} \qquad \Longrightarrow \qquad \begin{array}{ccccc} \mathcal{L}_d & \hookrightarrow & \mathcal{P}^d\mathcal{M} & \to & \mathcal{P}^d\mathcal{T} \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_2 & \hookrightarrow & \mathcal{P}^2\mathcal{M} & \to & \mathcal{P}^2\mathcal{T} \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_1 & \hookrightarrow & \mathcal{P}\mathcal{M} & \to & \mathcal{P}\mathcal{T} \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_0 & \hookrightarrow & \mathcal{M} & \to & \mathcal{T} \end{array}$$

still continuum theory, no new info
but lattice perspective, no lost info

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \qquad \Longrightarrow$$

inf dim

$$\begin{array}{ccccc}
\mathcal{L}_d & \hookrightarrow & \mathcal{P}^d\mathcal{M} & \to & \mathcal{P}^d\mathcal{T} \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\vdots & \cdots & \vdots & \cdots & \vdots \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_2 & \hookrightarrow & \mathcal{P}^2\mathcal{M} & \to & \mathcal{P}^2\mathcal{T} \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_1 & \hookrightarrow & \mathcal{P}\mathcal{M} & \to & \mathcal{P}\mathcal{T} \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_0 & \hookrightarrow & \mathcal{M} & \to & \mathcal{T}
\end{array}$$

still continuum theory
no new info
but lattice perspective

# systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

find finite dim equiv

$$
\mathcal{M} \to \mathcal{T} \quad \Longrightarrow \quad
\begin{array}{c}
\mathcal{L}_d \\ \downdownarrows \\ \cdots \\ \downdownarrows \\ \mathcal{L}_2 \\ \downdownarrows \\ \mathcal{L}_1 \\ \downdownarrows \\ \mathcal{L}_0
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\mathcal{P}^d\mathcal{M} \\ \downdownarrows \\ \cdots \\ \downdownarrows \\ \mathcal{P}^2\mathcal{M} \\ \downdownarrows \\ \mathcal{P}\mathcal{M} \\ \downdownarrows \\ \mathcal{M}
\end{array}
\longrightarrow
\boxed{
\begin{array}{c}
\mathcal{P}^d\mathcal{T} \\ \downdownarrows \\ \cdots \\ \downdownarrows \\ \mathcal{P}^2\mathcal{T} \\ \downdownarrows \\ \mathcal{P}\mathcal{T} \\ \downdownarrows \\ \mathcal{T}
\end{array}
\xrightarrow[\text{what we care}]{\text{equiv up to}}
\begin{array}{c}
\mathbf{ET}_d \\ \downdownarrows \\ \cdots \\ \downdownarrows \\ \mathbf{ET}_2 \\ \downdownarrows \\ \mathbf{ET}_1 \\ \downdownarrows \\ \mathcal{T}
\end{array}
}
$$

vague phys problem becomes
well-posed math problem !

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \quad \Longrightarrow \quad \begin{array}{c} \mathcal{L}_d \\ \downdownarrows \\ \cdots \\ \downdownarrows \\ \mathcal{L}_2 \\ \downdownarrows \\ \mathcal{L}_1 \\ \downdownarrows \\ \mathcal{L}_0 \end{array} \quad \longrightarrow \quad \begin{array}{c} \mathbf{ET}_d \\ \downdownarrows \\ \cdots \\ \downdownarrows \\ \mathbf{ET}_2 \\ \downdownarrows \\ \mathbf{ET}_1 \\ \downdownarrows \\ \mathcal{T} \end{array}$$

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$
\mathcal{M} \to \mathcal{T} \quad \Longrightarrow \quad
\begin{array}{ccccccc}
\mathcal{L}_d & & \mathcal{P}^d\mathcal{M} & & \mathcal{P}^d\mathcal{T} & & \mathbf{ET}_d \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\cdots & & \cdots & & \cdots & & \cdots \\
\downdownarrows & \xrightarrow{\sim} & \downdownarrows & \longrightarrow & \downdownarrows & \xrightarrow[\text{what we care}]{\text{equiv up to}} & \downdownarrows \\
\mathcal{L}_2 & & \mathcal{P}^2\mathcal{M} & & \mathcal{P}^2\mathcal{T} & & \mathbf{ET}_2 \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_1 & & \mathcal{P}\mathcal{M} & & \mathcal{P}\mathcal{T} & & \mathbf{ET}_1 \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_0 & & \mathcal{M} & & \mathcal{T} & & \mathcal{T}
\end{array}
$$

Resonates with development of homotopy theory in math!

*Grothendieck's dream* in his *Pursuing Stacks*

# systematically rethink about lattice QFT

field in continuum Yang-Mills

what we need on lattice

$$
\begin{array}{ccc}
\mathcal{PM} & & G \\
\downdownarrows & \rightarrow & \downdownarrows \\
\mathcal{M} & & *
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccccccc}
\mathcal{L}_d & & \mathcal{P}^d\mathcal{M} & & \mathcal{P}^d|BG| & & \mathbf{BEG}_d \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\cdots & & \cdots & & \cdots & & \cdots \\
\downdownarrows & \xrightarrow{\sim} & \downdownarrows & \rightarrow & \downdownarrows & \xrightarrow[\text{what we care}]{\text{equiv up to}} & \downdownarrows \\
\mathcal{L}_2 & & \mathcal{P}^2\mathcal{M} & & \mathcal{P}^2|BG| & & \mathbf{BEG}_2 \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_1 & & \mathcal{PM} & & \mathcal{P}|BG| & & G \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_0 & & \mathcal{M} & & |BG| & & *
\end{array}
$$

Take $\mathcal{T} = G$ and then "deloop" (involves Yang-Baxter equation) to get Yang-Mills

Yang-Baxter *automatically* resolved by the physically intuitive construction