Computing mass spectra of gauge theories in the Hamiltonian formalism

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collaboration with
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Background: mass spectrum of QCD

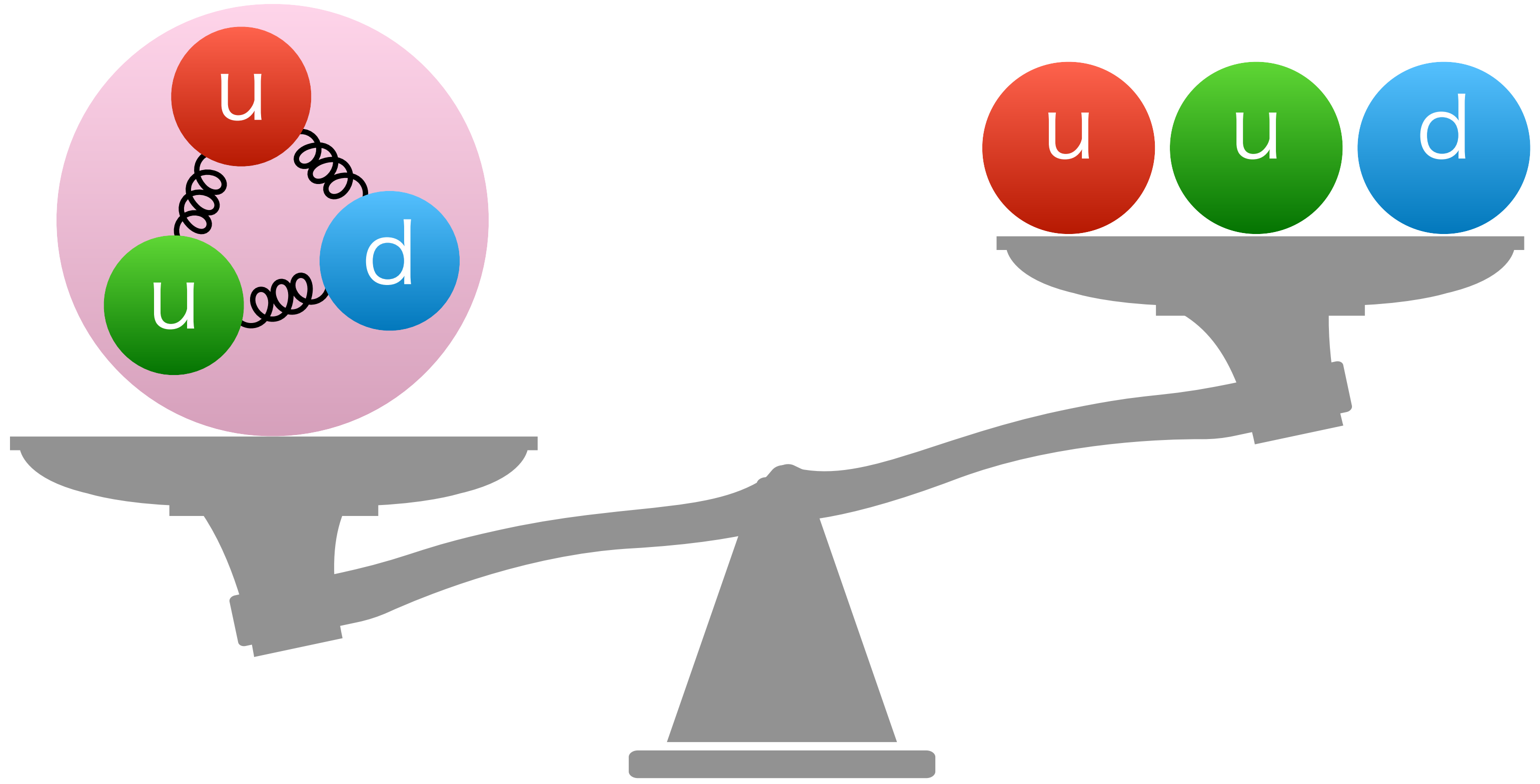
- quark confinement in QCD
  ⋯ low-energy d.o.f. are not quarks but composite particles (hadrons)

- **<u>hadrons are much heavier than quarks</u>**

  u/d quark: $m_u \sim 2$ MeV, $m_d \sim 5$ MeV

  π^+ meson (u, d): $140 \text{ MeV} \gg m_u + m_d$

  proton (u, u, d): $938 \text{ MeV} \gg 2m_u + m_d$



- nonperturbative calc. is essential to understand the properties of hadrons

motivation:
Numerically investigate low-energy spectra of gauge theories such as QCD

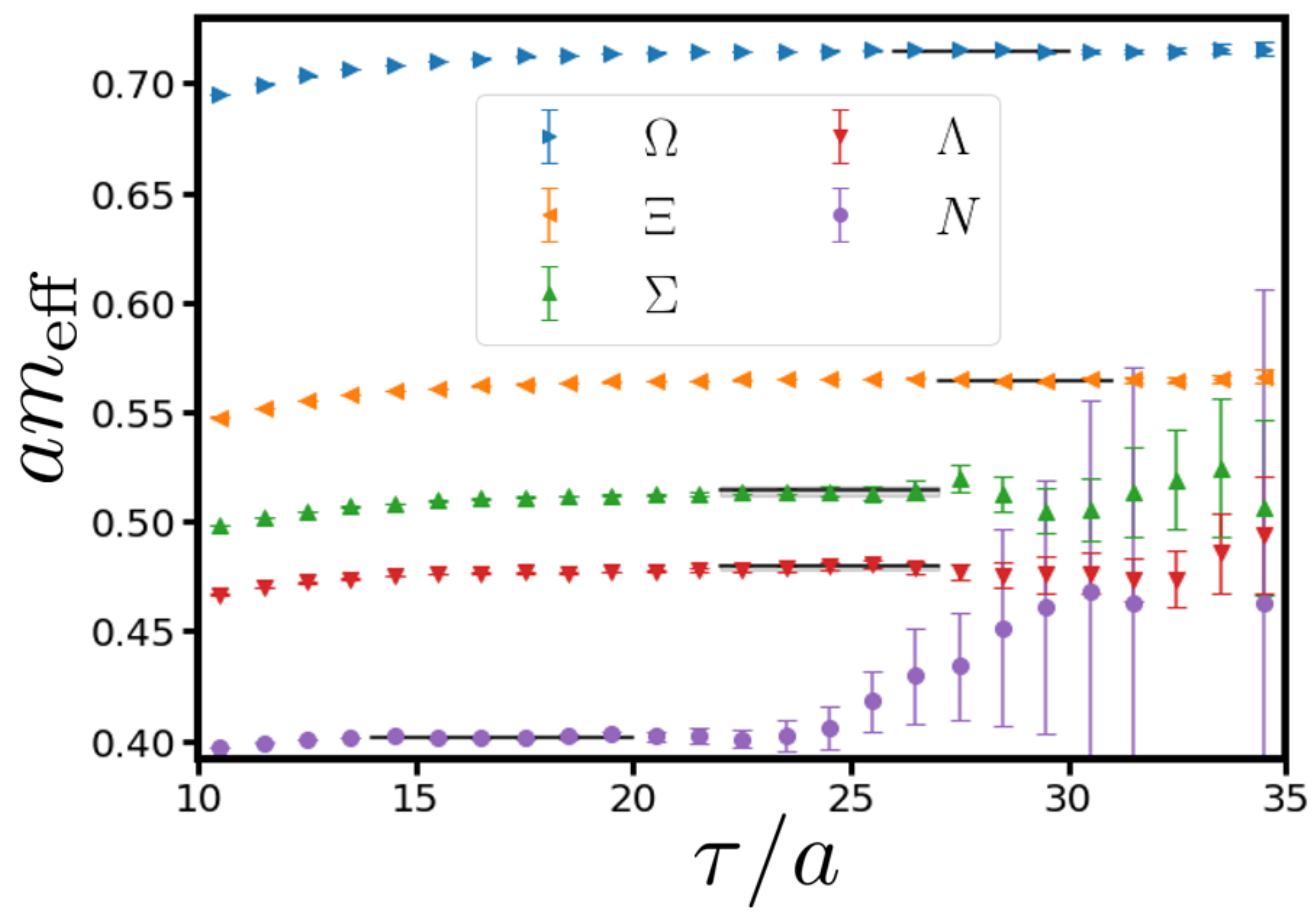
Mass spectrum by lattice QCD

- well-established method:
  Monte Carlo simulation of the lattice gauge theory (Lagrangian formalism)

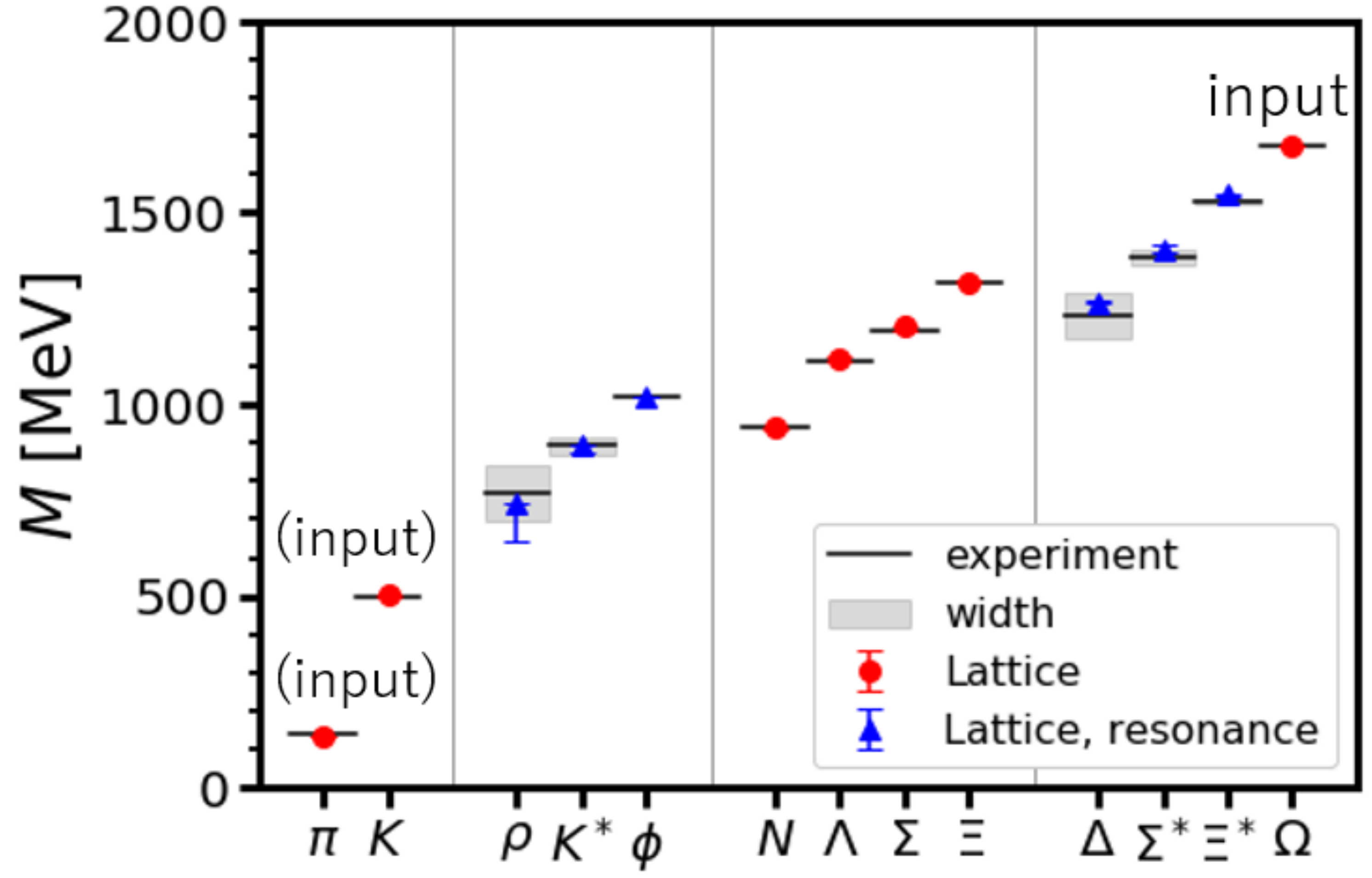
- obtain hadron masses from imaginary-time correlation functions

$$C(\tau) = \sum_x \langle \mathcal{O}(0,0)\, \mathcal{O}(x,\tau) \rangle \sim e^{-M\tau} \quad \text{—>} \quad \text{effective mass: } m_{\rm eff}(\tau) = -\log \frac{C(\tau+1)}{C(\tau)} \simeq M$$

effective mass



hadron spectrum



[HAL QCD collab. (2024)]

Hamiltonian formalism

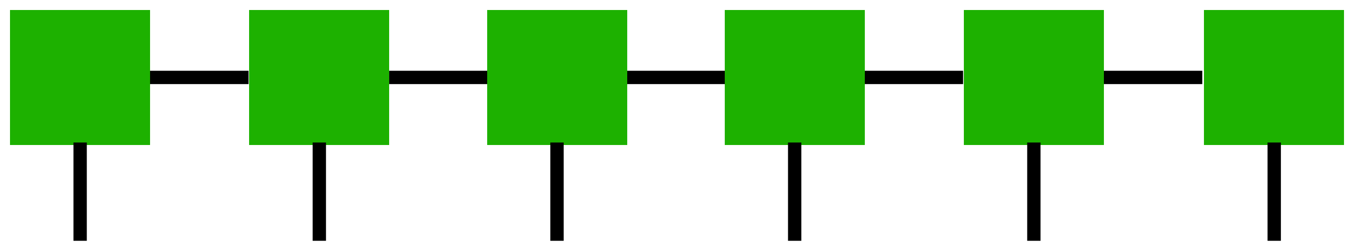
😣 Monte Carlo method cannot be applied to models with complex actions

—> sign problem (finite density QCD, topological term, real-time evolution, ···)

💡 Tensor network and quantum computing in Hamiltonian formalism can be complementary approaches!

👍 free from the sign problem

👍 analyze excited states directly



aim of this work:
computing the hadron mass spectrum in Hamiltonian formalism that is applicable even when the sign problem arises

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Short summary

- JHEP11 (2023) 231:
  demonstrate three distinct methods to compute the mass spectrum at $\theta = 0$

  (1) correlation-function scheme

  (2) one-point-function scheme

  (3) dispersion-relation scheme

- JHEP09 (2024) 155: improve and extend them to the case of $\theta \neq 0$

  (1)+(2) improved one-point-function scheme

  (3) dispersion-relation scheme

- θ-dependent spectra by these schemes are
  consistent with each other and with calculation in the bosonized model

Outline

1. Three mesons in 2-flavor Schwinger model

2. Calculation strategy: MPS and DMRG

3. Correlation-function scheme at $\theta = 0$

4. Improved one-point-function scheme for $\theta \neq 0$

5. Dispersion-relation scheme

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Schwinger model with two fermions

<u>Schwinger model = QED in 1+1d</u>

- simplest nontrivial gauge theory sharing some features with QCD

$$\mathscr{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} + \sum_{f=1}^{N_f}\left[i\bar{\psi}_f\gamma^\mu\left(\partial_\mu + iA_\mu\right)\psi_f - m\bar{\psi}_f\psi_f\right]$$

sign problem if $\theta \neq 0$

- quantum numbers:
  isospin J, parity P, G-parity $G = Ce^{i\pi J_y}$

- P and G are broken at $\theta \neq 0$
  —> η becomes unstable

$N_f = 2$ —> three "mesons"

$\pi_a = -i\bar{\psi}\gamma^5\tau_a\psi$: $J^{PG} = 1^{-+}$

$\sigma = \bar{\psi}\psi$: $J^{PG} = 0^{++}$

$\eta = -i\bar{\psi}\gamma^5\psi$: $J^{PG} = 0^{--}$

Analytic study by bosonization

- $m = 0 \cdots$ 2-flavor Schwinger model = SU(2)$_1$ Wess-Zumino-Witten model

- $m > 0 \cdots$ not exactly solvable, but bosonization analysis is valid for $m \ll g$

$$\mathscr{L} = \frac{1}{2}\left[(\partial\eta)^2 - \mu^2\eta^2\right] + \frac{1}{4\pi}(\partial\varphi)^2 + \frac{e^\gamma}{\pi} m\rho N_\rho \left[\cos\left(\sqrt{2\pi}\eta - \frac{\theta}{2}\right)\cos\varphi\right]$$

[Coleman (1976)]

η : non-compact scalar $\qquad$ φ : 2π-periodic scalar

$N_\rho[\,\cdot\,]$: normal ordering at the scale ρ $\qquad$ $\mu^2 := 2g^2/\pi$

- integrating out η of a mass $O(\mu)$

—> low-energy spectrum is described by sine-Gordon theory

Mass spectrum by bosonization

- optimized perturbation and WKB-type approximation result in

[Coleman (1976)] [Dashen et al. (1975)]

π meson: $M_\pi(\theta) \propto \left| m\sqrt{\mu}\cos\frac{\theta}{2} \right|^{2/3}$

σ meson: $M_\sigma(\theta) = \sqrt{3}M_\pi(\theta)$

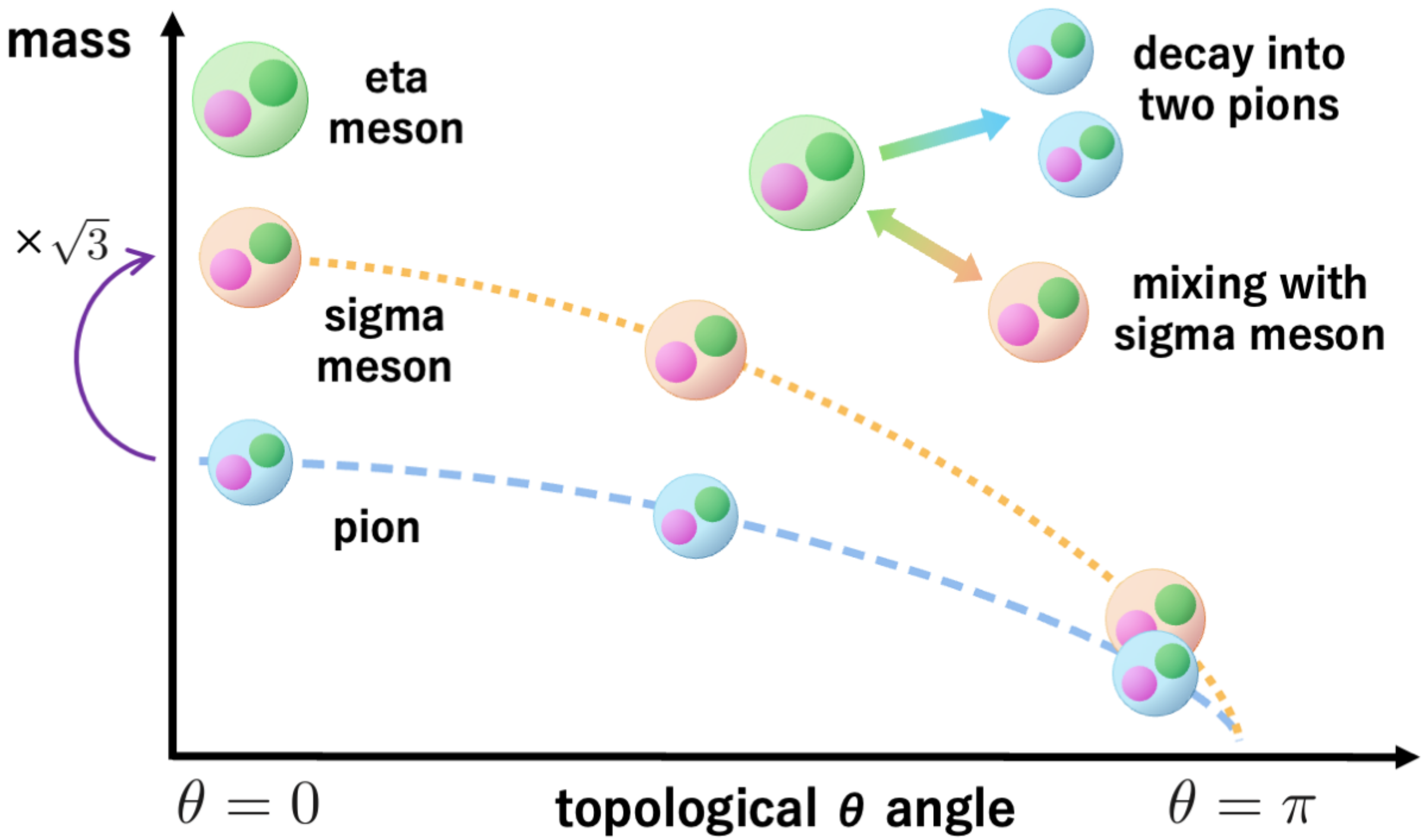
validity of the approximation is not clear
—> check it by first-principles calculation

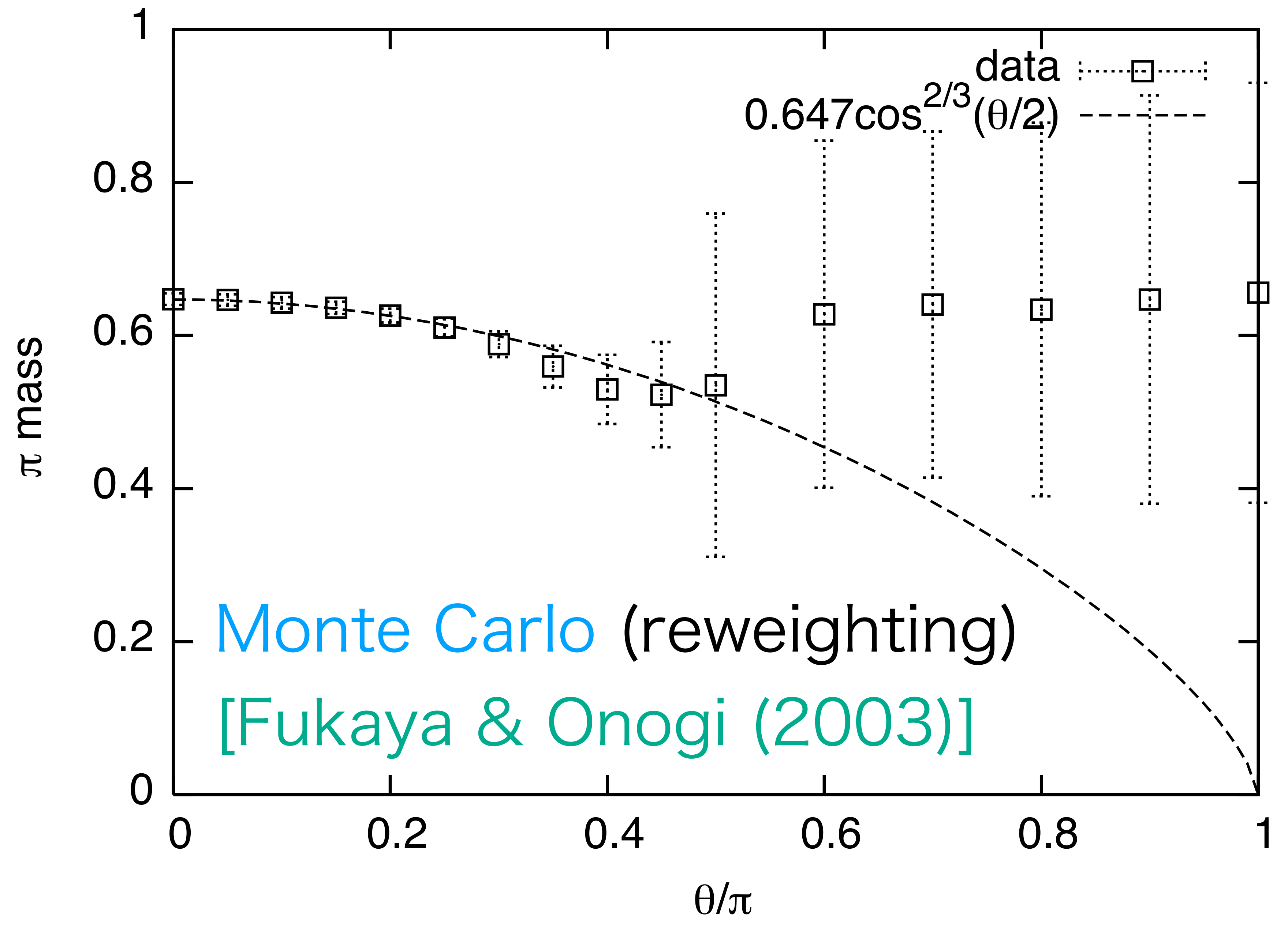
- η meson has a mass of $O(\mu)$ at $\theta = 0$ but becomes unstable at $\theta \neq 0$

⚠️ θ -dependent interaction $\left(\sqrt{2\pi}\sin\frac{\theta}{2}\right)\eta\cos\varphi$ creates

a σ state or $\pi\,\pi$ scattering state from η —> η-σ mixing / $\eta \to \pi\,\pi$ decay

θ-dependent mass spectrum





Lattice 2-flavor Schwinger model

- Hamiltonian on the 1d lattice with the open boundaries [Kogut & Susskind (1975)]

$$H = \frac{g^2 a}{2} \sum_{n=0}^{N-2} \left(L_n + \frac{\theta}{2\pi} \right)^2 + \sum_{f=1}^{N_f} \left[\frac{-i}{2a} \sum_{n=0}^{N-2} \left(\chi_{f,n}^\dagger U_n \chi_{f,n+1} - \chi_{f,n+1}^\dagger U_n^\dagger \chi_{f,n} \right) + m_{\mathrm{lat}} \sum_{n=0}^{N-1} (-1)^n \chi_{f,n}^\dagger \chi_{f,n} \right]$$

$\chi_{f,n}$: staggered fermion, U_n : link variable, L_n : conjugate momentum

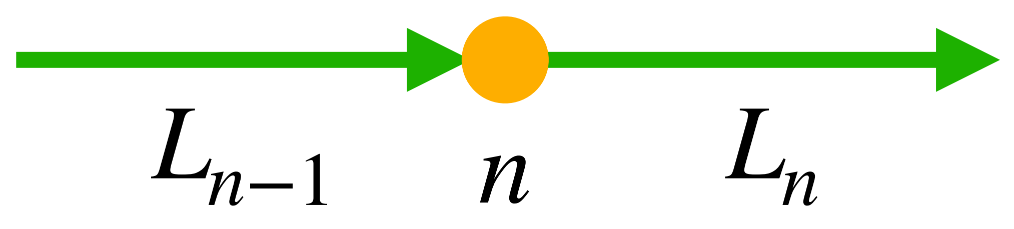
- $m_{\mathrm{lat}} := m - \frac{N_f g^2 a}{8}$ taking $O(a)$ correction into account [Dempsey et al. (2022)]

—> The lattice theory at $m = 0$ maintains the discrete chiral symmetry.

Map to the spin system

- gauge fixing $U_n = 1$ and solving Gauss law $L_n - L_{n-1} = \sum_f \chi^\dagger_{f,n}\chi_{f,n} + (-1)^n - 1$

—> gauge field is eliminated (a specific feature of 1+1d theory with o.b.c)



- Jordan-Wigner transformation for $N_f = 2$

$$\chi_{1,n} = \sigma^-_{1,n}\prod_{j=0}^{n-1}(-\sigma^z_{2,j}\sigma^z_{1,j}), \quad \chi_{2,n} = \sigma^-_{2,n}(-i\sigma^z_{1,n})\prod_{j=0}^{n-1}(-\sigma^z_{2,j}\sigma^z_{1,j})$$

- $\{\chi^\dagger_{f,n}, \chi_{f',n'}\} = \delta_{f,f'}\delta_{n,n'}$ and $\{\chi'_{f,n}, \chi_{f',n'}\} = \{\chi^\dagger_{f,n}, \chi^\dagger_{f',n'}\} = 0$ are reproduced

—> spin Hamiltonian with a finite-dim. Hilbert space (spin-1/2, $N_f \times N$ sites)

Explicit form of the spin Hamiltonian

$$H = H_{\text{gauge}} + H_{\text{kin}} + H_{\text{mass}}$$

$$H_{\text{gauge}} = \frac{g^2 a}{8} \sum_{n=0}^{N-2} \left[\sum_{f=1}^{N_f} \sum_{k=0}^{n} \sigma_{f,k}^z + N_f \frac{(-1)^n + 1}{2} + \frac{\theta}{\pi} \right]^2$$

$$H_{\text{kin}} = \frac{-i}{2a} \sum_{n=0}^{N-2} \left(\sigma_{1,n}^+ \sigma_{2,n}^z \sigma_{1,n+1}^- - \sigma_{1,n}^- \sigma_{2,n}^z \sigma_{1,n+1}^+ + \sigma_{2,n}^+ \sigma_{1,n+1}^z \sigma_{2,n+1}^- - \sigma_{2,n}^- \sigma_{1,n+1}^z \sigma_{2,n+1}^+ \right)$$

$$H_{\text{mass}} = \frac{m_{\text{lat}}}{2} \sum_{f=1}^{N_f} \sum_{n=0}^{N-1} (-1)^n \sigma_{f,n}^z + \frac{m_{\text{lat}}}{2} N_f \frac{1 - (-1)^N}{2}$$

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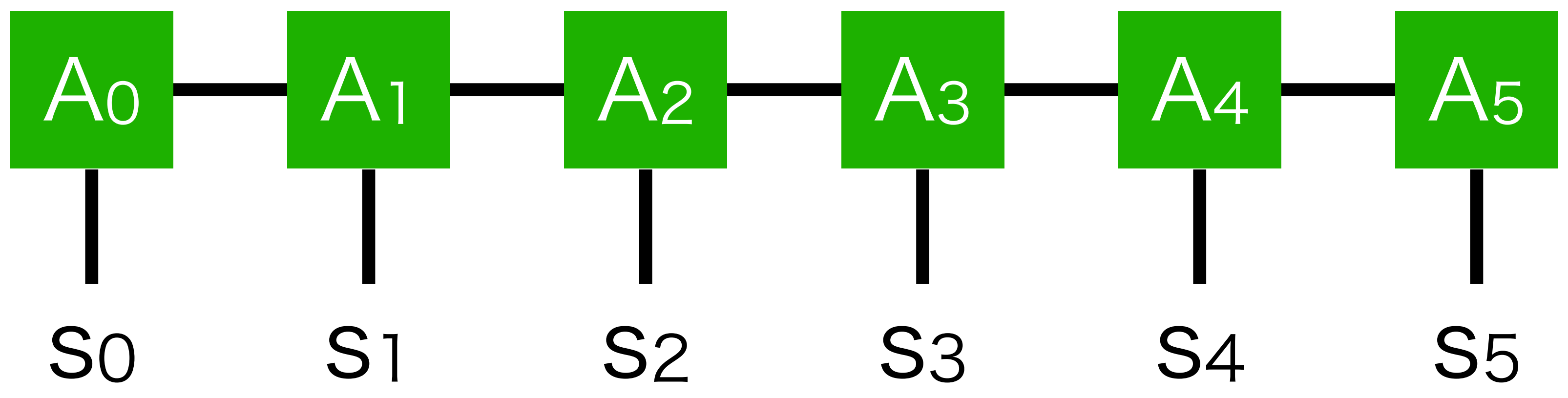
Approximation of states by MPS

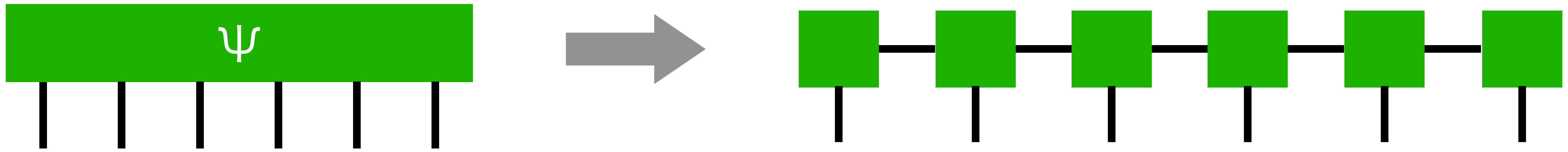


Matrix Product State (MPS)

$$|\Psi\rangle = \sum_{\{s_i\}} \mathrm{Tr}\left[A_0(s_0)\,A_1(s_1)\,\cdots\right] |\,s_0\,s_1\,\cdots\rangle$$

- $A_i(s_i)$: $D_{i-1} \times D_i$ matrix with a spin index $s_i \in \{\uparrow, \downarrow\}$ (D_i : bond dimension)
- Any state can be written as MPS by repeating SVD, but $D_i = O(2^{N/2})$ in general.

$$|\Psi\rangle = \sum_{\{s_i\}} \Psi(s_0, s_1, \cdots)\,|\,s_0\,s_1\,\cdots\rangle$$



- Even with a cutoff $D_i \leq \mathrm{const}$, MPS efficiently approximates low-energy states of 1+1d gapped systems of any size N. → numerical cost = $O(ND^3)$

Density-matrix renormalization group (DMRG)

[White (1992)] [Schollwock (2005)]

<u>variational method to find the ground state using MPS as an ansatz</u>

- cost function: energy $E = \langle \Psi | H | \Psi \rangle$

$$|\Psi\rangle = \sum_{\{s_i\}} \mathrm{Tr}\left[A_0(s_0)\, A_1(s_1)\ \cdots\right] | s_0\, s_1\ \cdots \rangle$$

- update $A_i(s_i)$ iteratively to decrease E
  by local optimization and low-rank approx. with SVD

$A_i(s_i)$: $D_{i-1} \times D_i$ matrix

D_i : bond dimension

- control the accuracy by a cutoff parameter ε
  —> D_i is determined so that (truncation error of SVD) < ε

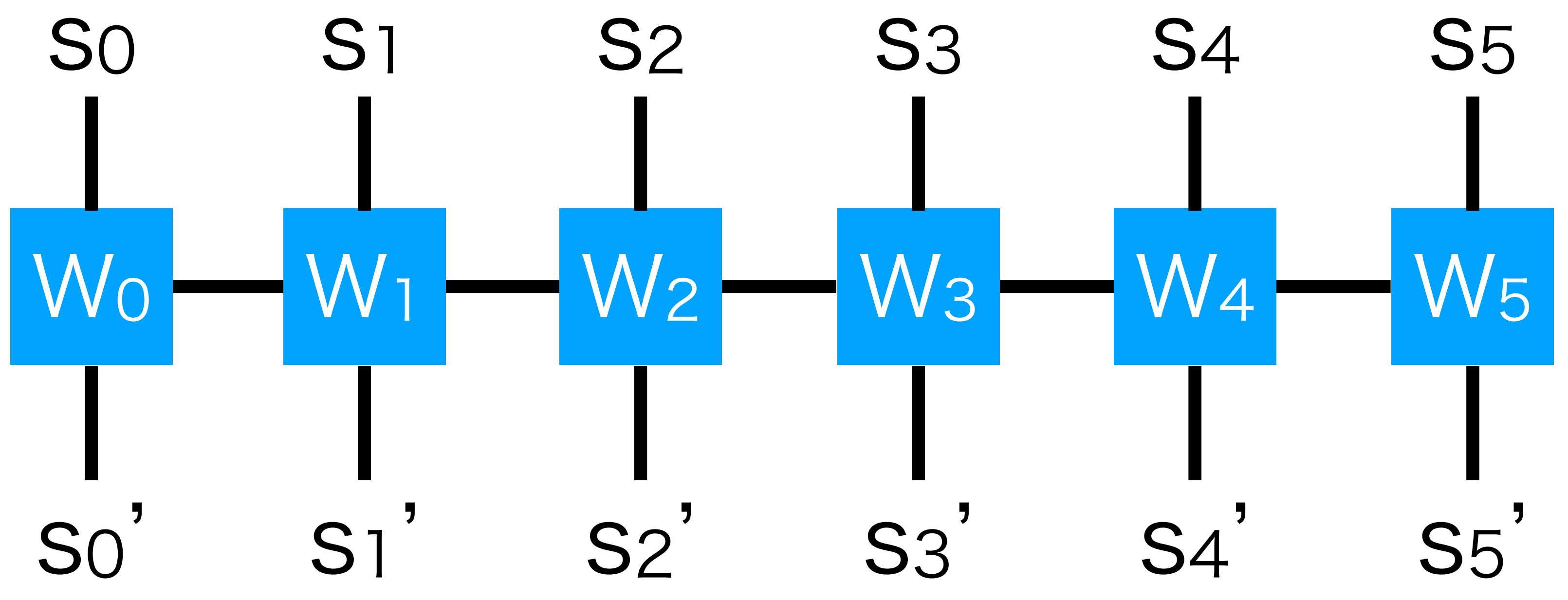
  high accuracy ↔ small ε ↔ large D_i ↔ high cost

The C++ library of ITensor is used in this work. [Fishman et al. (2022)]

Calculation of expectation values

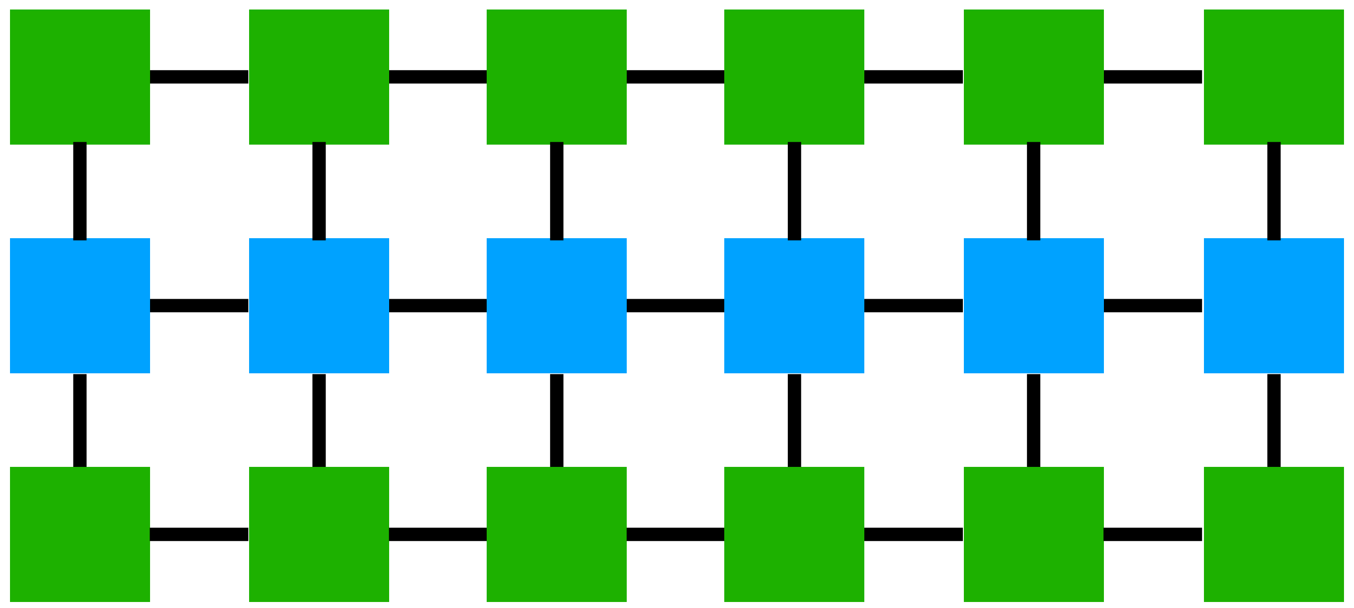
Matrix Product Operator (MPO)

$$\mathcal{O} = \sum_{\{s_i'\}} \sum_{\{s_i\}} \mathrm{Tr}\left[W_0(s_0', s_0) W_1(s_1', s_1) \cdots\right] | s_0' s_1' \cdots \rangle \langle s_0 s_1 \cdots |$$

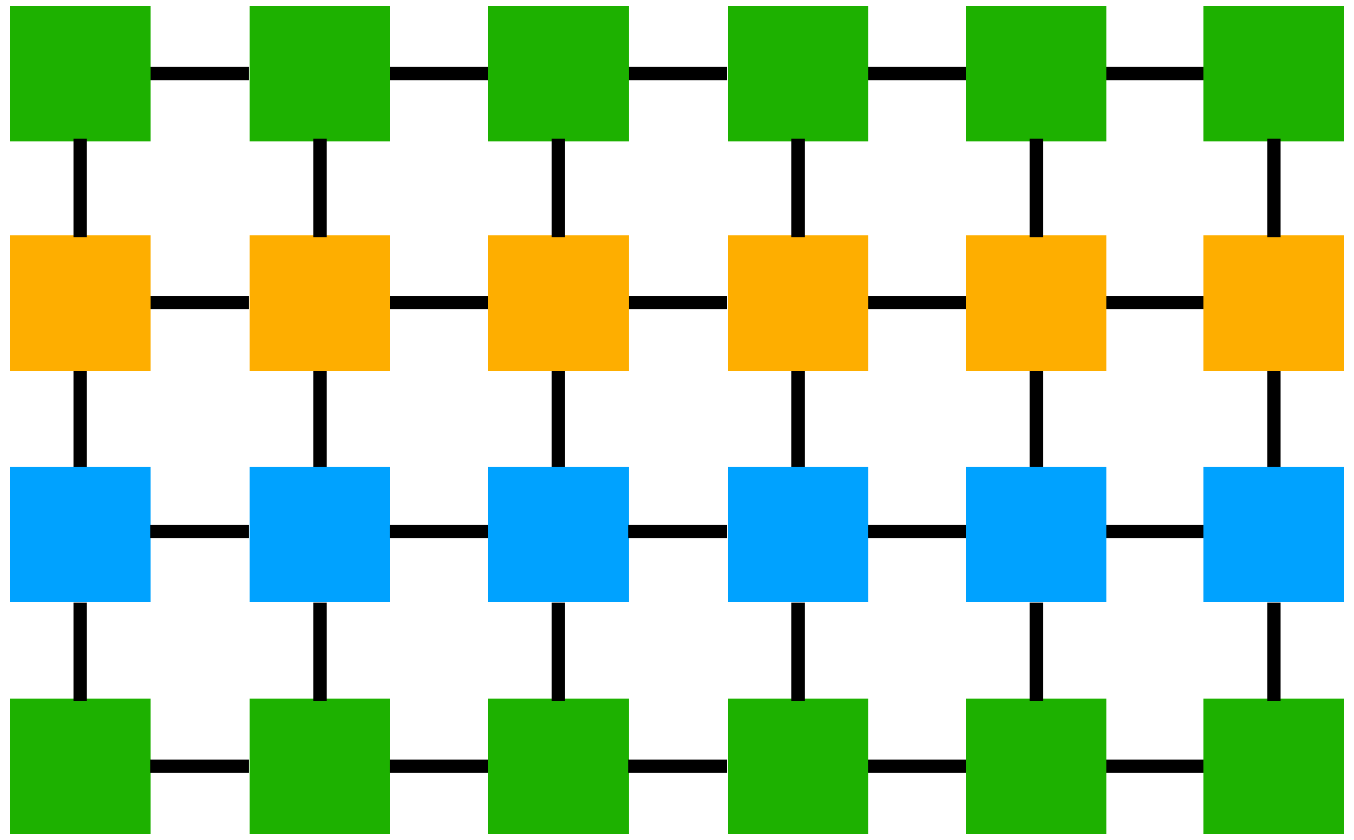


- Expectation values are computed by contracting MPS and MPO

$\langle \Psi | \mathcal{O} | \Psi \rangle$



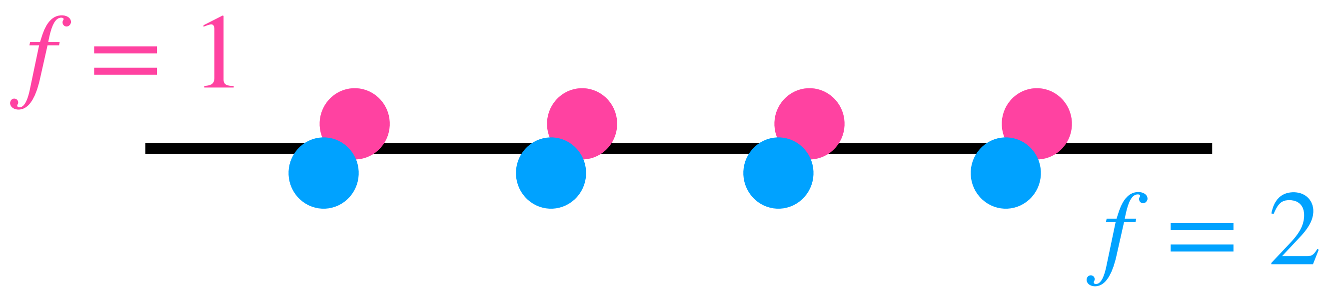
$\langle \Psi | \mathcal{O}_1 \mathcal{O}_2 | \Psi \rangle$



Arrangement of the flavors

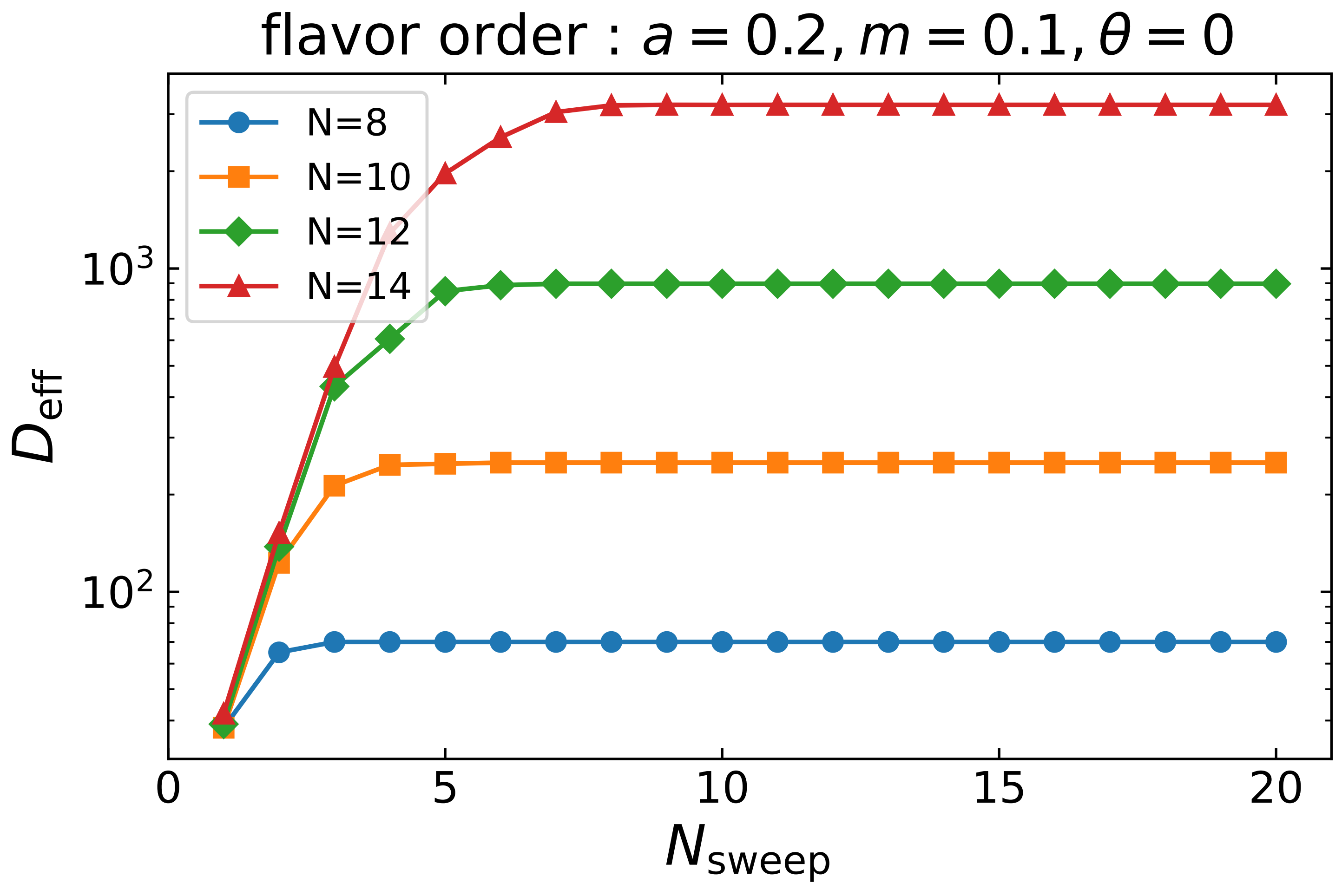
arrange $N_f \times N$ spins on the 1d lattice to use MPS for 2-flavor Schwinger model

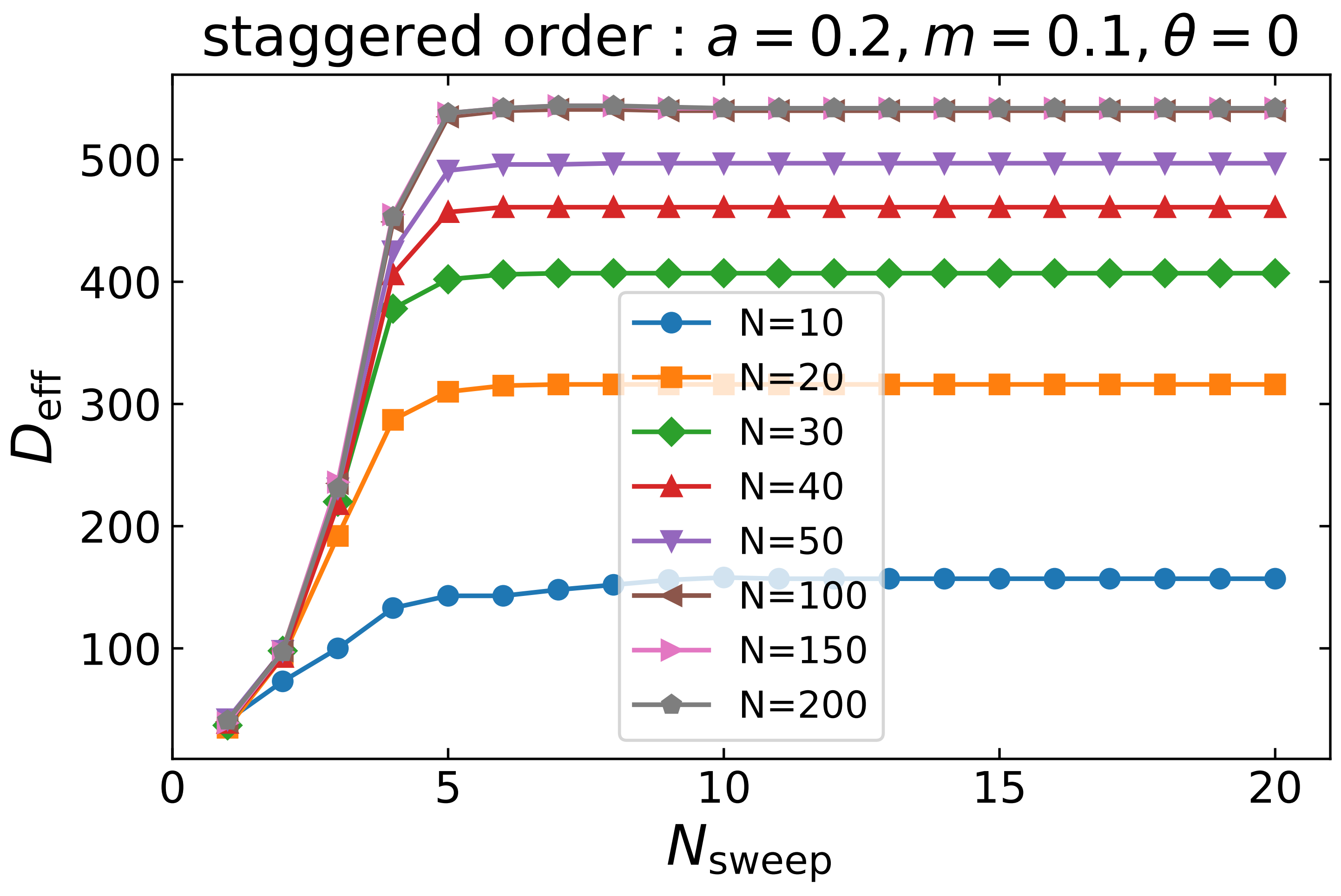
ex.) $N = 4,\ N_f = 2$

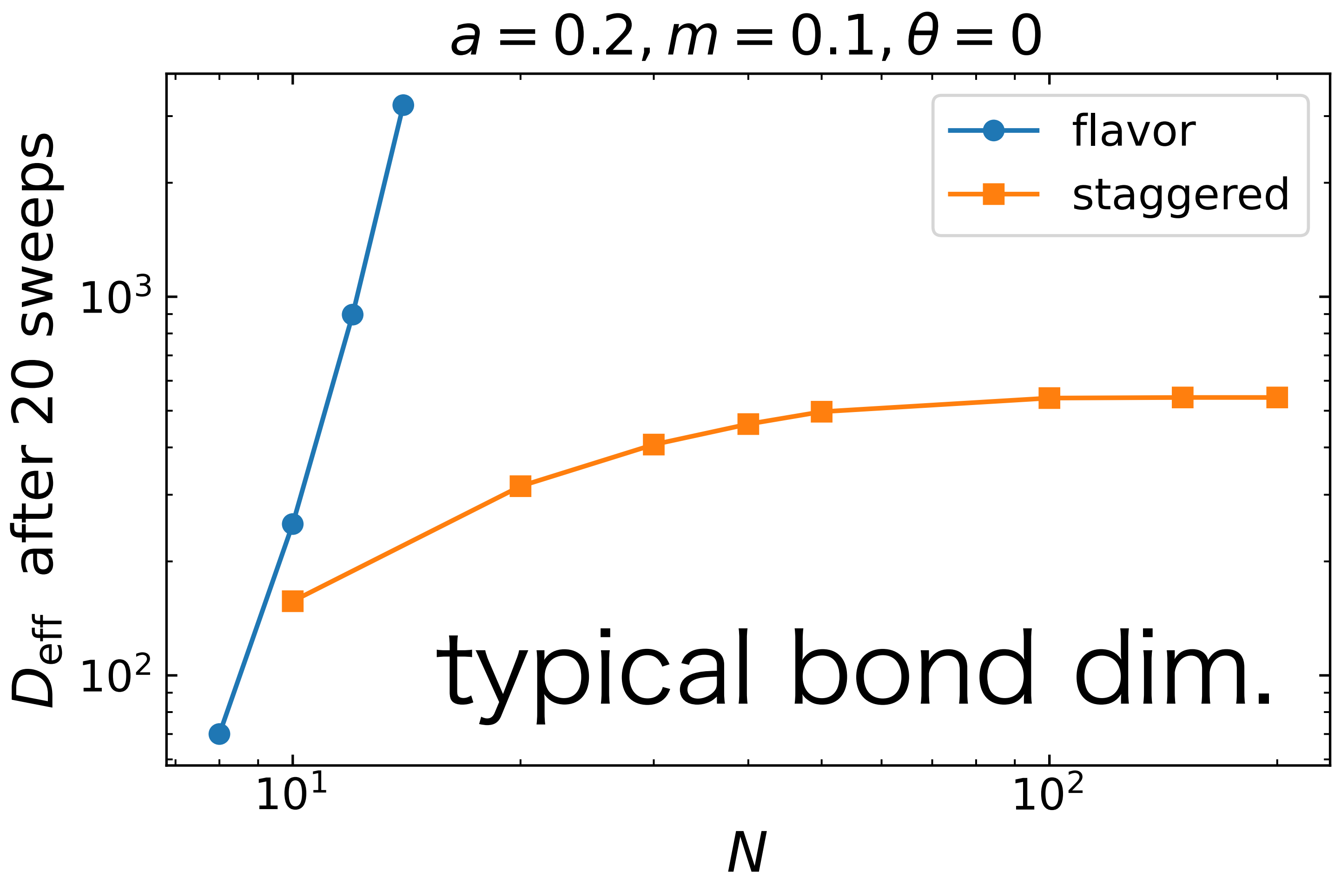


😥 flavor order: $(n, f) \to i = n + N(f-1)$

👍 staggered order: $(n, f) \to i = nN_f + f - 1$







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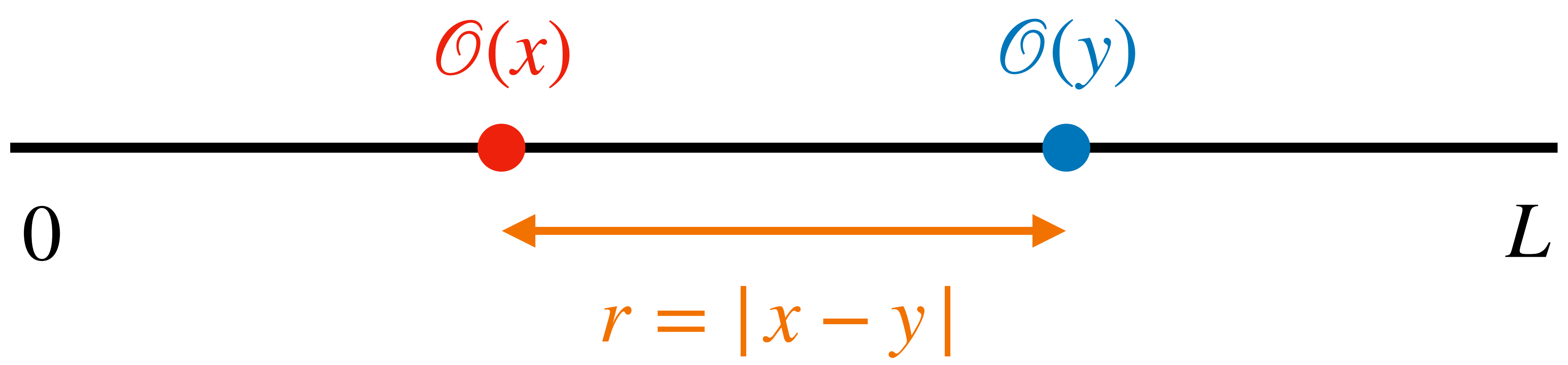
Correlation-function scheme

<u>Extract meson masses from the spatial correlation function as in conventional Lattice QCD</u>

- generate the ground state by DMRG
  and measure the spatial correlation function (connected part)

$$C_{\mathcal{O}}(r) = \langle \mathcal{O}(x)\mathcal{O}(y)\rangle - \langle \mathcal{O}(x)\rangle\langle \mathcal{O}(y)\rangle$$



Pion correlation function at $\theta = 0$

- pion at $\theta = 0$: $C_\pi(r) = \langle \pi(x)\pi(y) \rangle \quad r = |x - y|$
- effective mass: $M_{\pi,\mathrm{eff}}(r) = -\frac{d}{dr} \log C_\pi(r)$

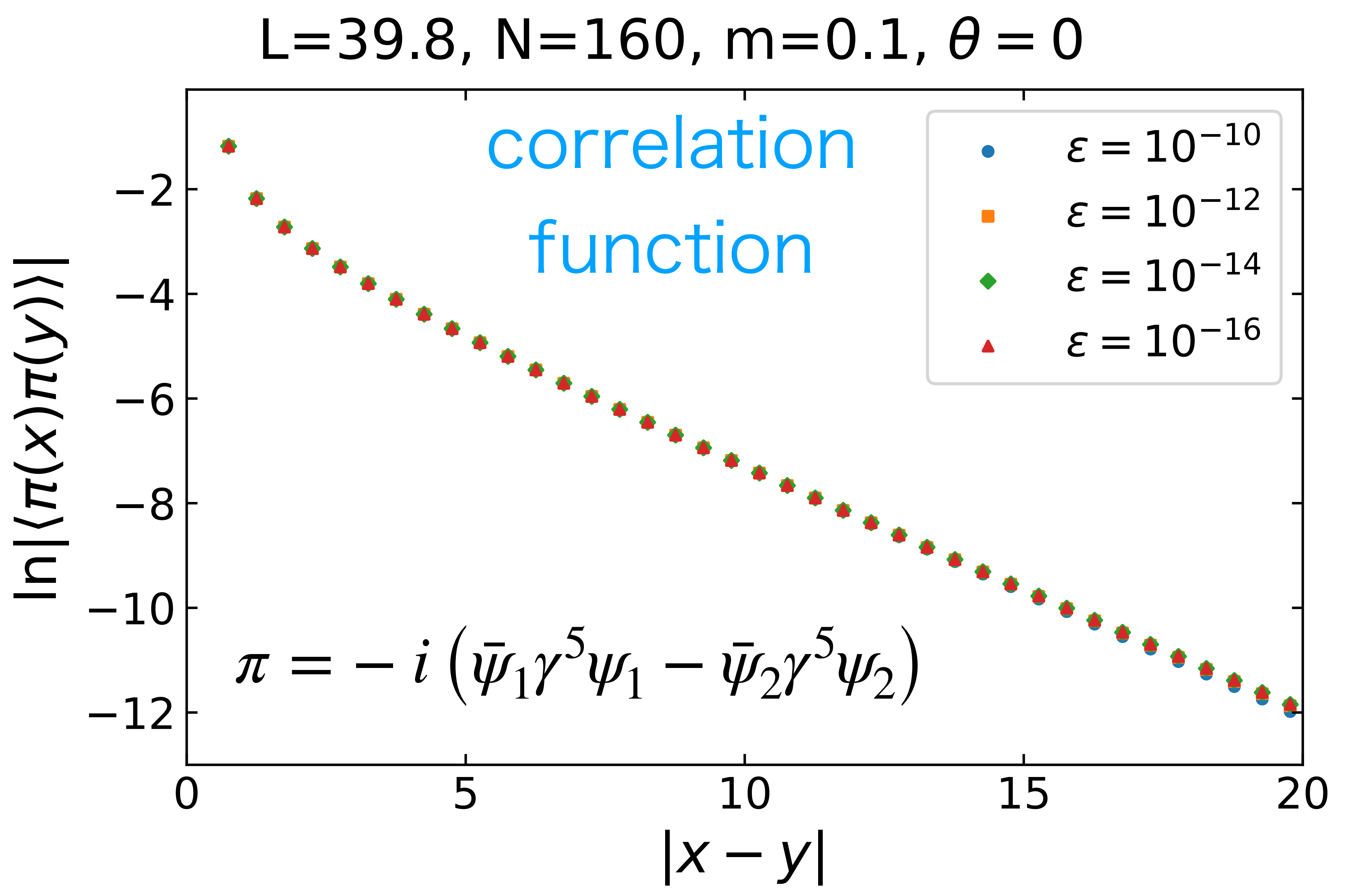
plateau value = pion mass?

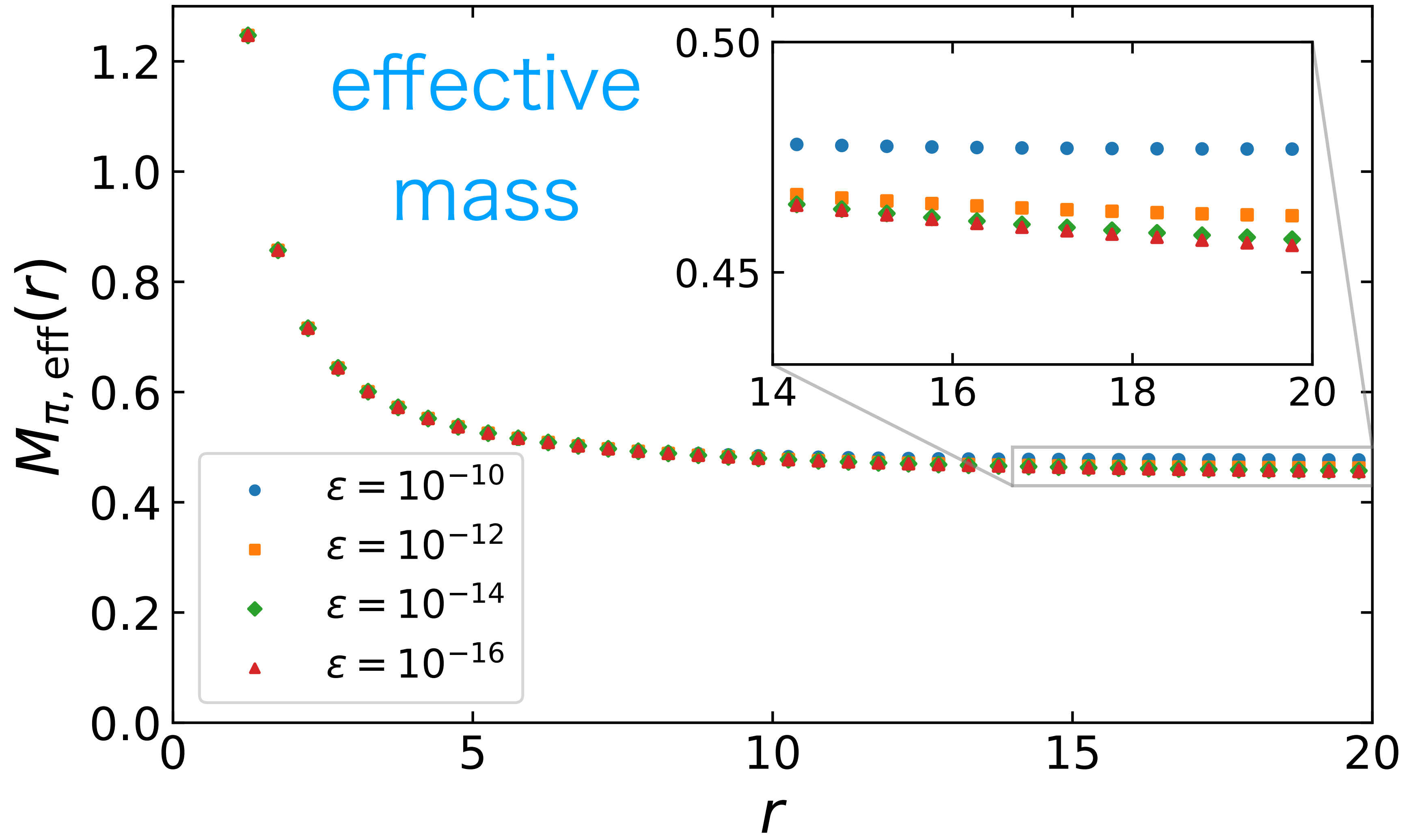
⚠️ plateau behavior gets modified in accurate calc.

$\varepsilon = 10^{-10}$ $(D_i \sim 400)$: $M_{\pi,\mathrm{eff}}(r)$ is almost flat

$\varepsilon = 10^{-16}$ $(D_i \sim 2800)$: $M_{\pi,\mathrm{eff}}(r)$ depends on r

- What's happened?

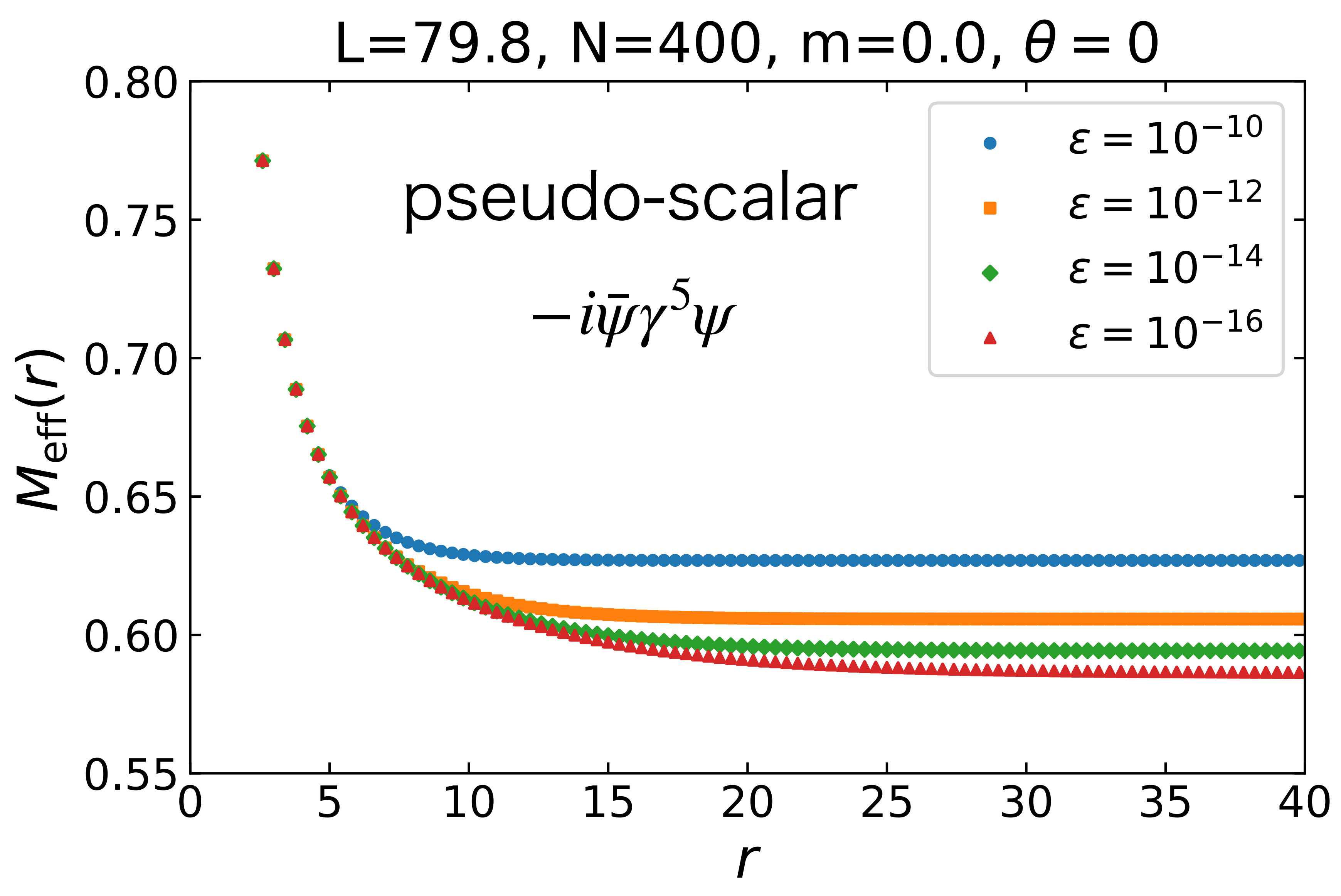




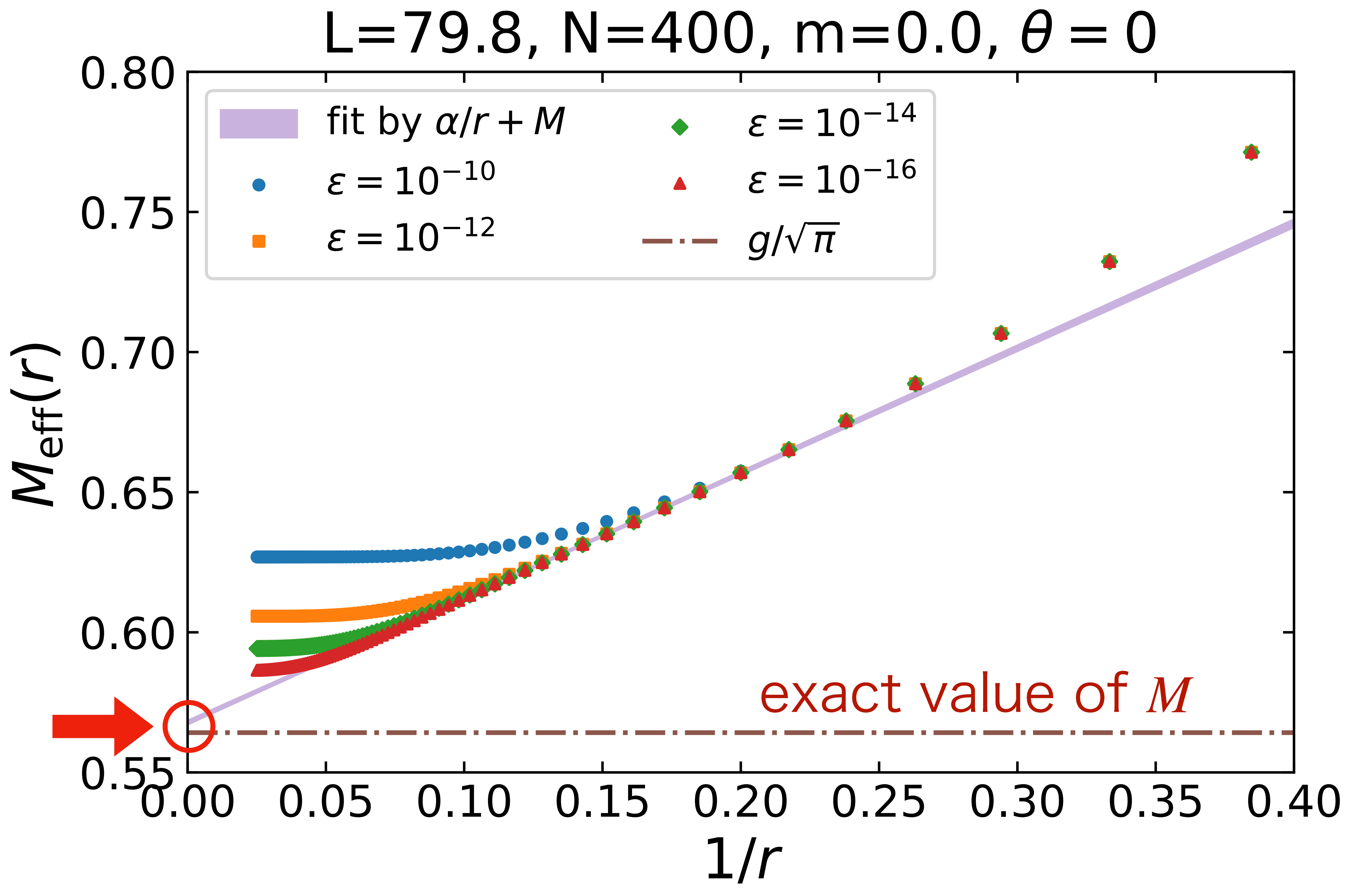
Yukawa-type correlation → 1/r term

cf.) 1+1d free scalar of mass M : $\langle \phi(x,t)\phi(y,t)\rangle \sim \frac{1}{r^\alpha} e^{-Mr} \rightarrow M_{\rm eff}(r) \sim \frac{\alpha}{r} + M \quad \alpha = \frac{1}{2}$

cf.) massless $N_f = 1$ Schwinger model (exactly solvable):



plot against $\frac{1}{r}$

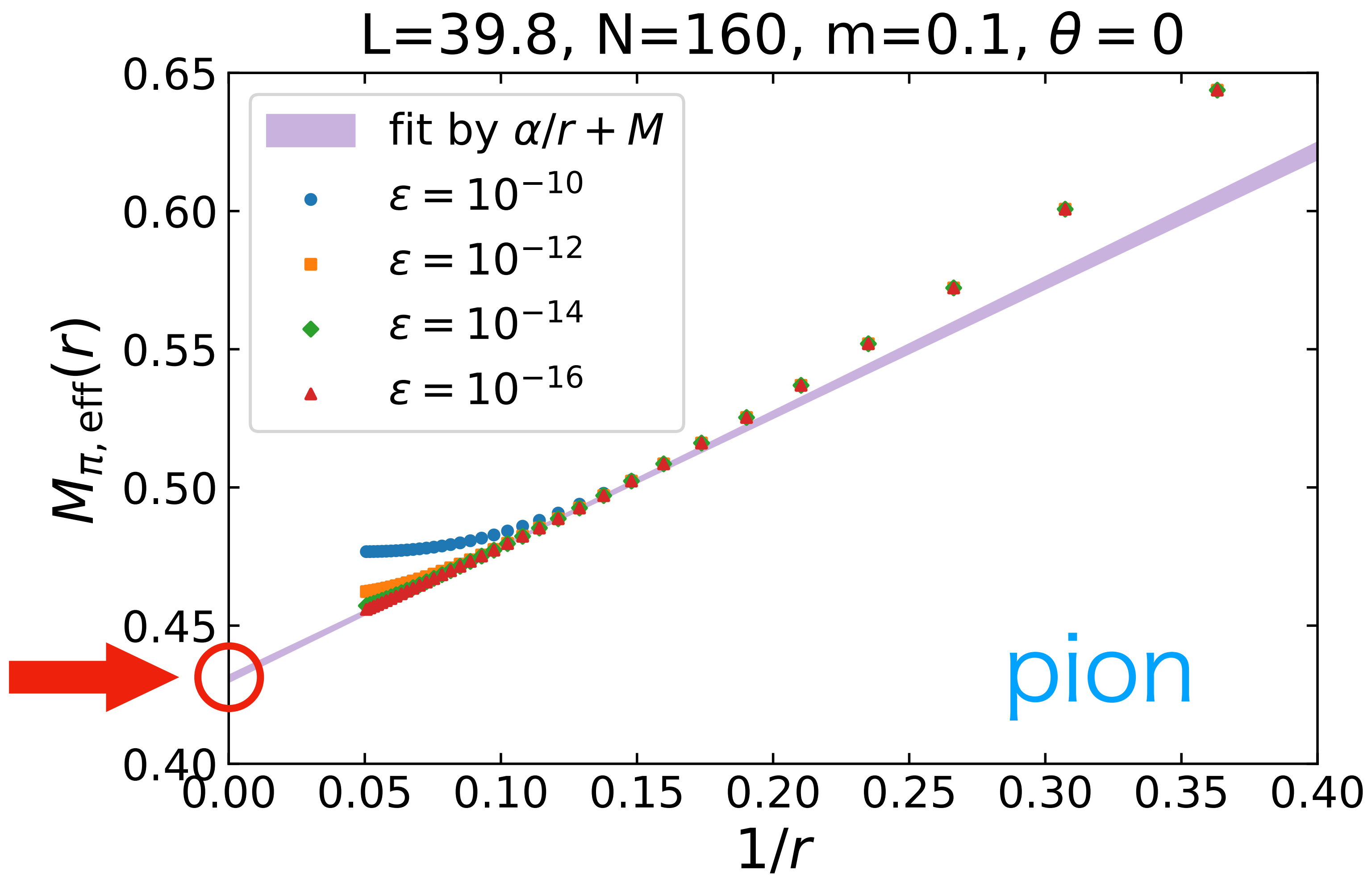


- ε must be small enough to see $1/r$ behavior
- $r \rightarrow \infty$ extrapolation is required

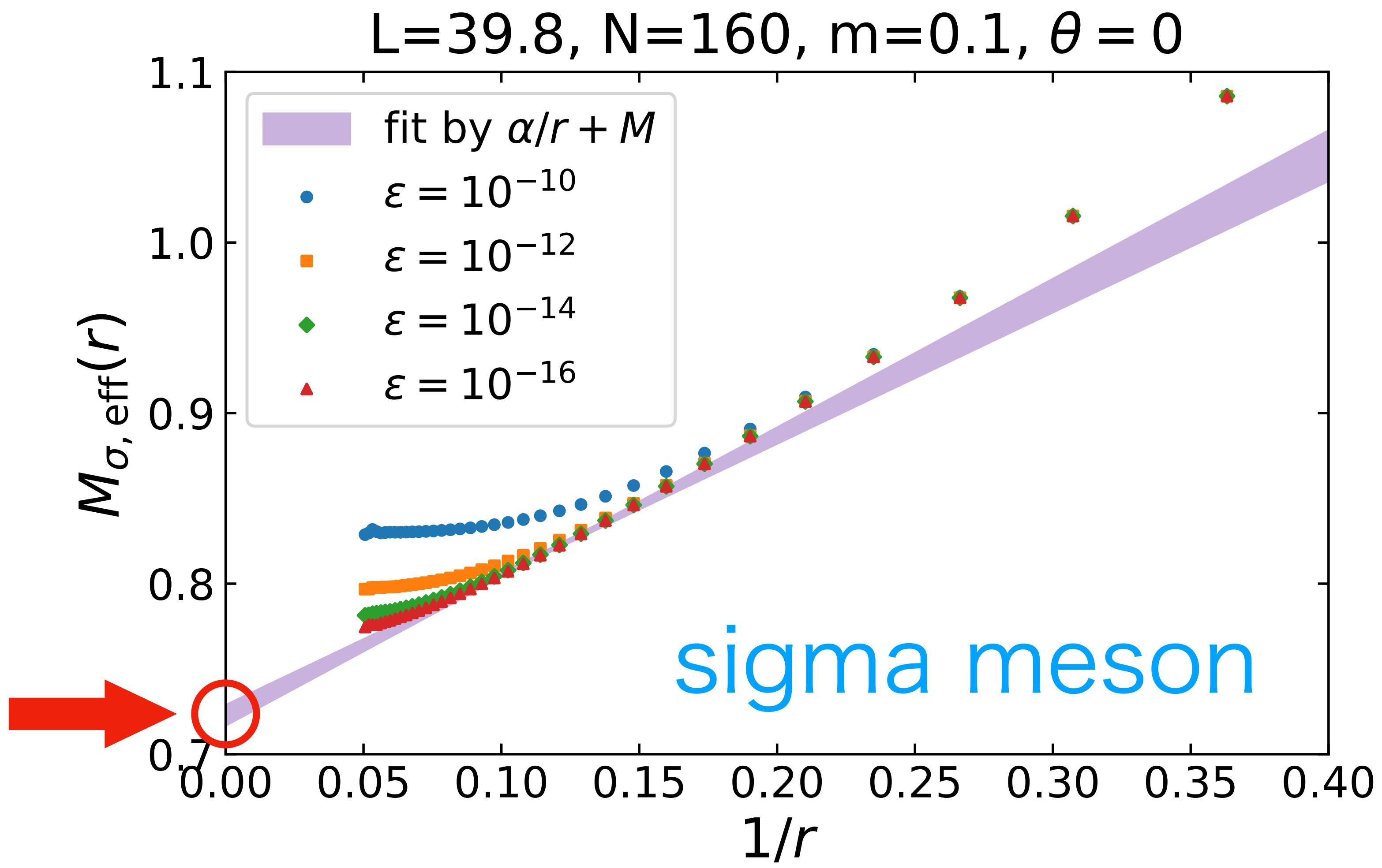
Result of the three mesons at $\theta = 0$

extrapolate the effective mass to $r \to \infty$ using the result for $\varepsilon = 10^{-16}$

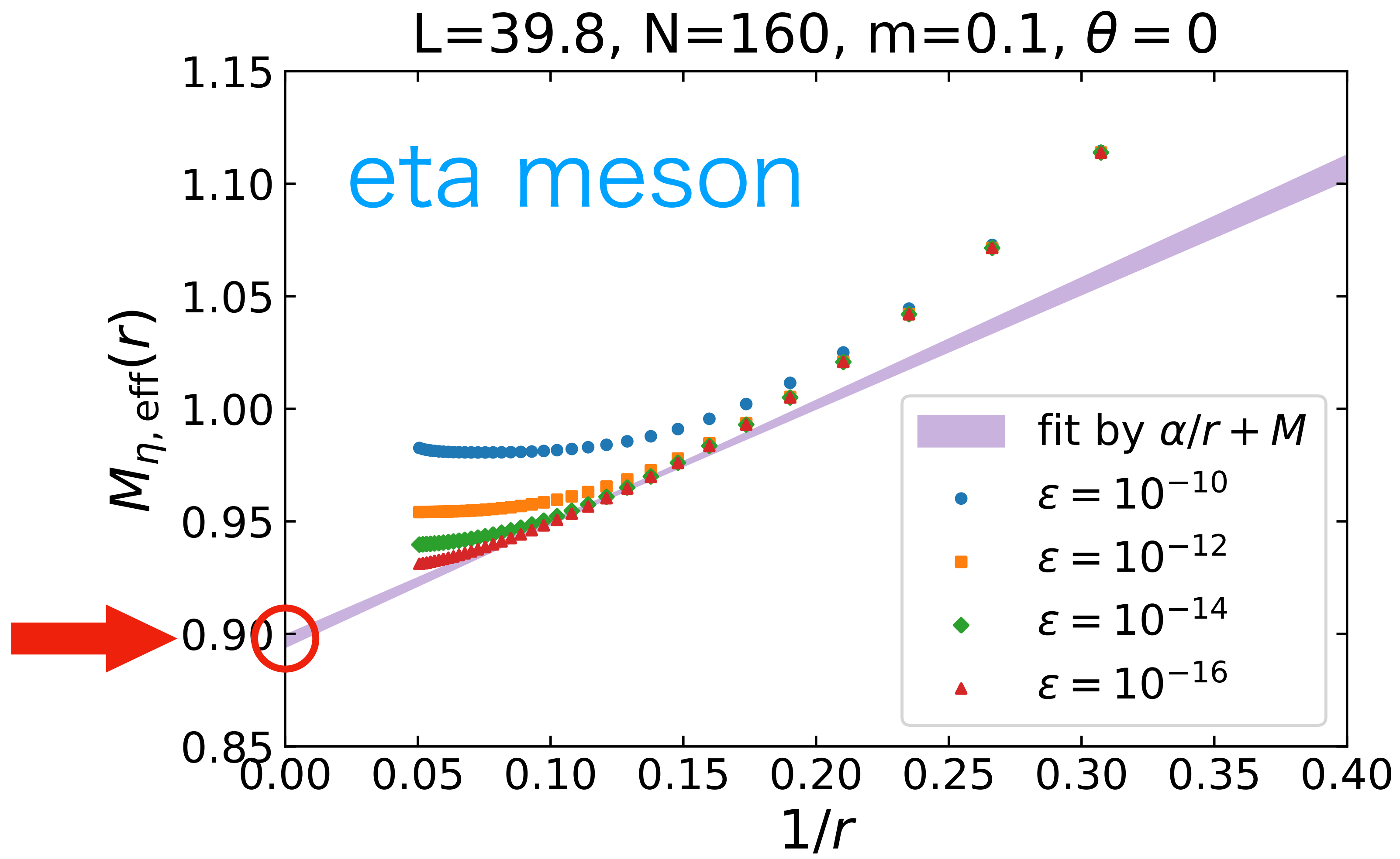
$$\pi = -i\left(\bar{\psi}_1\gamma^5\psi_1 - \bar{\psi}_2\gamma^5\psi_2\right)$$



$$\sigma = \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2$$



$$\eta = -i\left(\bar{\psi}_1\gamma^5\psi_1 + \bar{\psi}_2\gamma^5\psi_2\right)$$



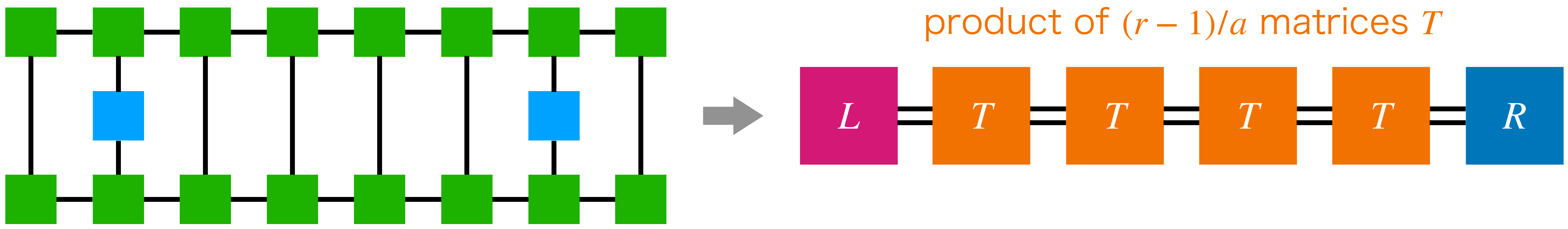
	pion	sigma	eta
M	0.431(1)	0.722(6)	0.899(2)

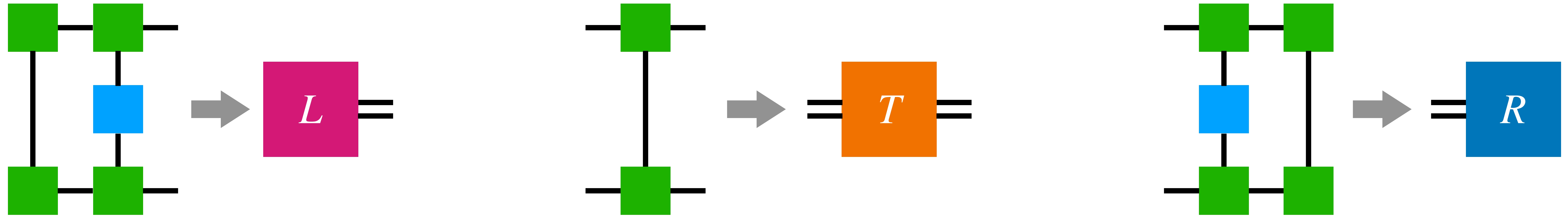
Correlation function from MPS

Assuming the translational invariance of MPS;
correlation functions $C(r)$ ∼ linear sum of exponential functions of $r = |x - y|$

$$C(r) = \langle \mathcal{O}(x)\mathcal{O}(y) \rangle \sim L\, T^{(r-1)/a} R \sim \sum_{k=1}^{D^2} C_k\, \xi_k^{\,r/a}$$

ξ_i : eigenvalue of "transfer matrix" T





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Meson operators at $\theta \neq 0$

define the meson operators at $\theta \neq 0$ taking the axial rotation into account

$$\pi_a = -i\bar{\psi} \exp\left[i\frac{\theta}{2}\gamma^5\right]\gamma^5\tau_a\psi$$

$$\sigma = \bar{\psi} \exp\left[i\left(\frac{\theta}{2} + \omega(\theta)\right)\gamma^5\right]\psi$$

$$\eta = -i\bar{\psi} \exp\left[i\left(\frac{\theta}{2} + \omega(\theta)\right)\gamma^5\right]\gamma^5\psi$$

$\omega(\theta)$: nontrivial correction due to η-σ mixing

Correlation matrix → meson operators

determine θ-dependent mixing angle from numerical results

- diagonalization of the correlation matrix

e.g.) isosinglet sector

$$\langle AB\rangle_c := \langle AB\rangle - \langle A\rangle\langle B\rangle$$

$$\begin{pmatrix} \langle S(x)\,S(y)\rangle_c & \langle S(x)\,PS(y)\rangle_c \\ \langle PS(x)\,S(y)\rangle_c & \langle PS(x)\,PS(y)\rangle_c \end{pmatrix} = R(\delta)^{\mathrm{T}} \begin{pmatrix} \langle \sigma(x)\,\sigma(y)\rangle_c & 0 \\ 0 & \langle \eta(x)\,\eta(y)\rangle_c \end{pmatrix} R(\delta)$$

$$S(x) \leftrightarrow \bar{\psi}\psi(x)$$

$$PS(x) \leftrightarrow -i\bar{\psi}\gamma^5\psi(x)$$

$$R(\delta) = \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix}$$ $\cdots$ rotation matrix with the mixing angle δ

—> define the meson operators by $\begin{pmatrix} \sigma(x) \\ \eta(x) \end{pmatrix} := R(\delta) \begin{pmatrix} S(x) \\ PS(x) \end{pmatrix}$

Result of mixing angle

- **triplet sector**: $\delta_- \approx \theta/2$

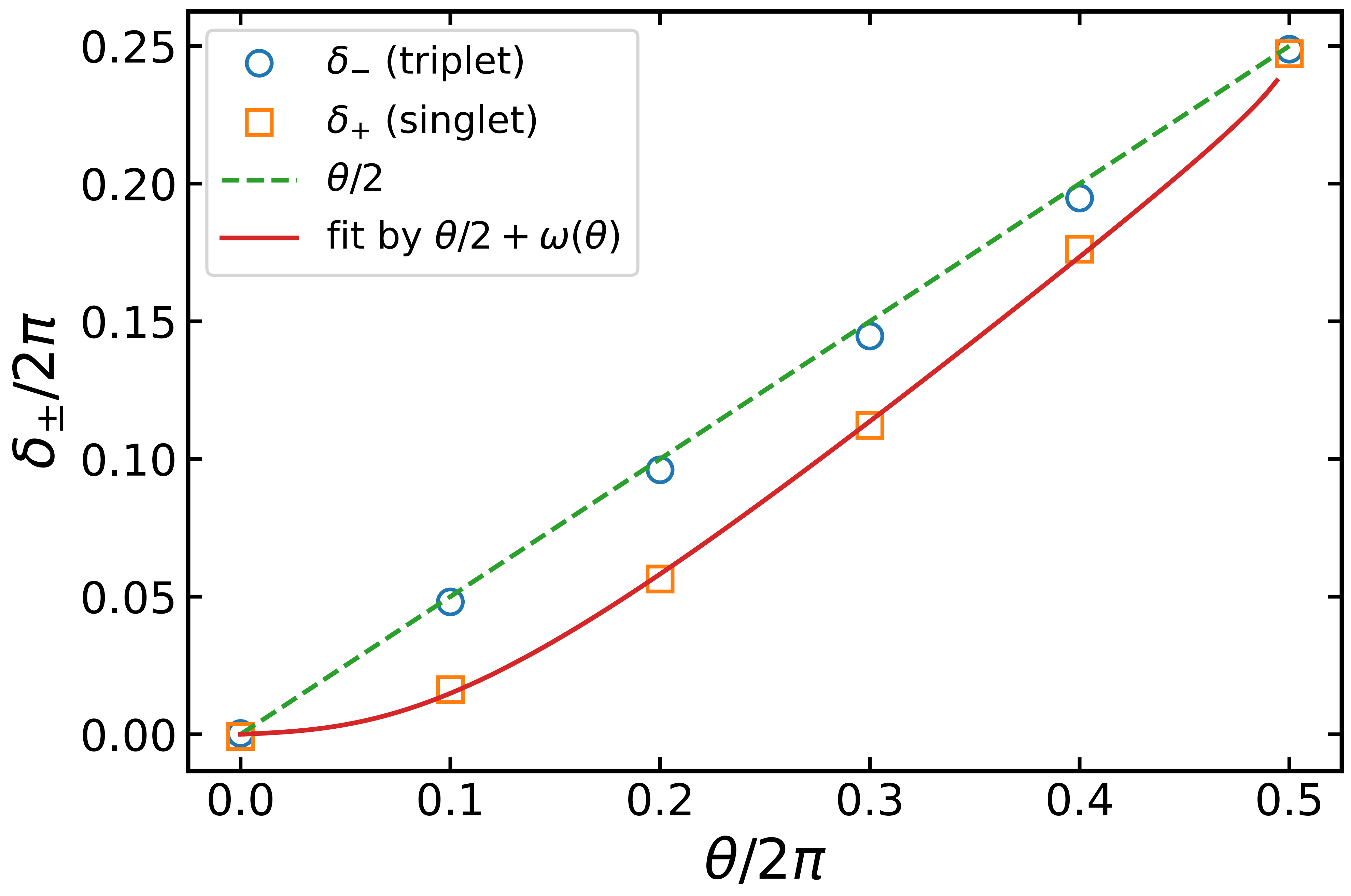
  trivial rotation $\exp\left[i(\theta/2)\gamma^5\right]$

  since there is no mixing partner with π

- **singlet sector**: $\delta_+ \approx \theta/2 + \omega(\theta)$

  correction from η - σ mixing

- The result of δ_+ can be fitted by the function obtained from the bosonized model



$\omega(\theta)$ is given by diagonalizing the mass matrix

$$\mathcal{M} \propto \begin{pmatrix} 1 & A\sin(\theta/2)\,|\cos(\theta/2)|^{1/3} \\ A\sin(\theta/2)\,|\cos(\theta/2)|^{1/3} & B\,|\cos(\theta/2)|^{4/3} \end{pmatrix} = R(\omega(\theta))^{\mathrm{T}}\Lambda\, R(\omega(\theta))$$

fitting result: $A = -0.23(2), \quad B = 0.76(4)$

One-point-function scheme

<u>**Regarding the boundary (defect) as the source of mesons, obtain the masses from the one-point functions**</u>

- 1pt. function $\langle \mathcal{O}(x) \rangle_{\mathrm{obc}}$ measures the correlation with the boundary state $|\mathrm{bdry}\rangle$

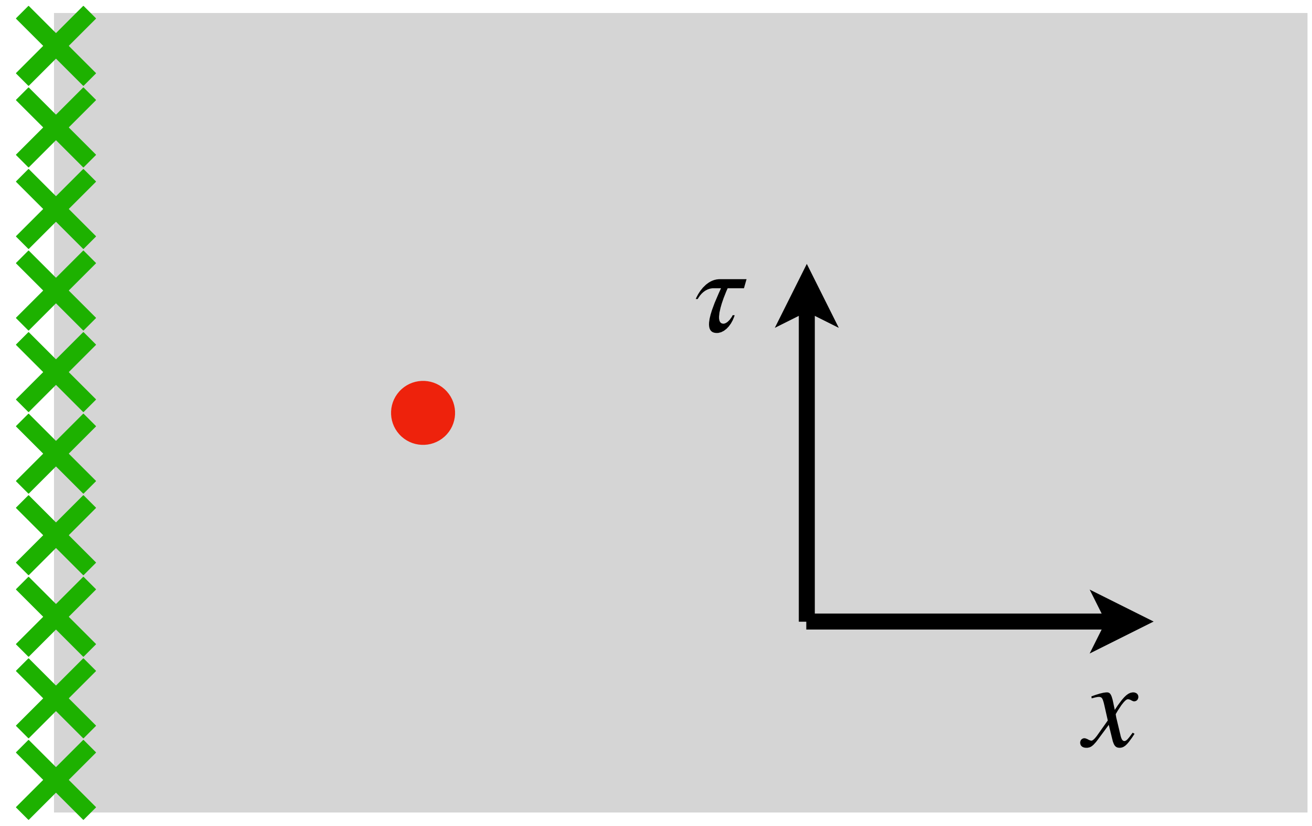
- $|\mathrm{bdry}\rangle$ has translational invariance in time direction

  —> zero-momentum projection —> exponential decay

$p_\tau |\mathrm{bdry}\rangle = 0$

$$\langle \mathcal{O}(x) \rangle_{\mathrm{obc}} \sim \langle \mathrm{bdry} | e^{-Hx} \mathcal{O} | 0 \rangle_{\mathrm{bulk}} \sim e^{-Mx}$$


Euclidean space



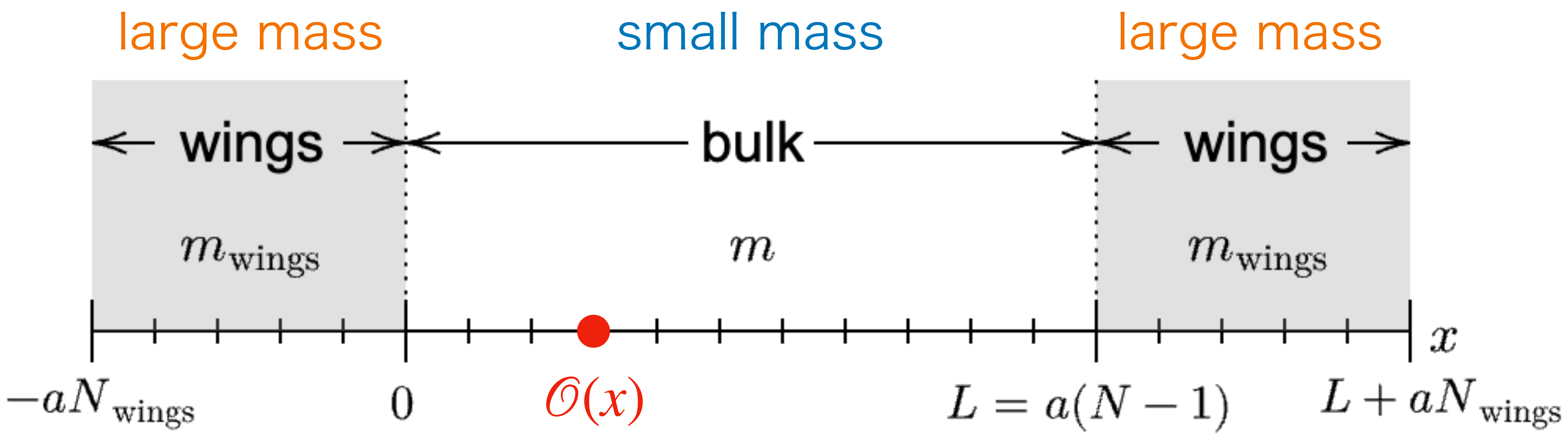
👍 truncation effect is much smaller

cf.) wall source method

Some technical improvement

- We attach “the wings” to the lattice to control the boundary condition flexibly

  e.g.) Dirichlet b.c. $\cdots$ $m_{\text{wings}} \gg m$



- The boundary must have the same quantum number as the target meson

Result of sigma and eta mesons

- For the singlet mesons, we set the Dirichlet b.c. $m_{\text{wings}} = m_0 \gg m$
- Assuming $\langle \mathcal{O}(x) \rangle \sim Ae^{-Mx} + Be^{-(M+\Delta M)x}$,

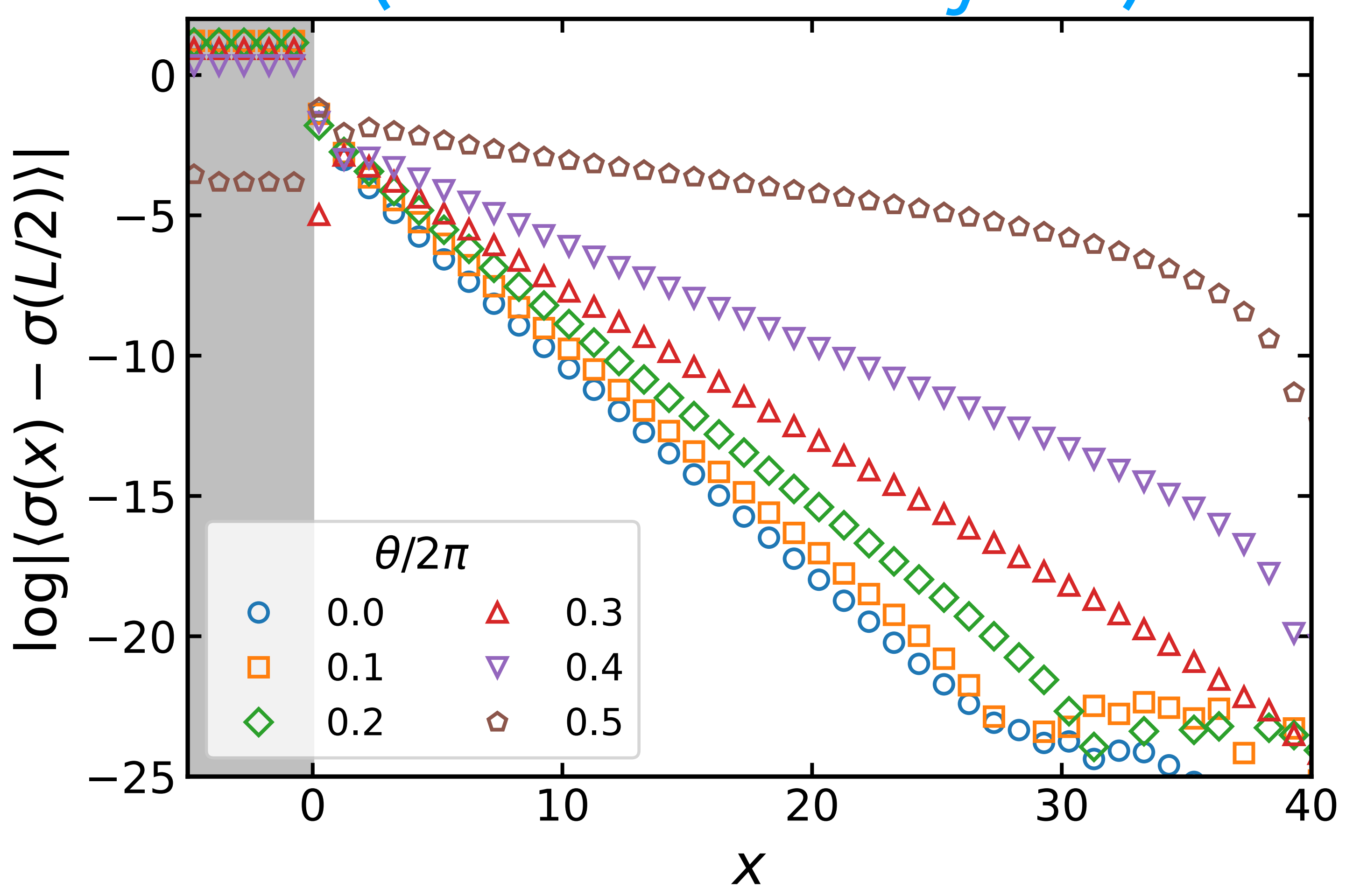
  we fit the effective mass by $M + \dfrac{\Delta M}{1 + Ce^{\Delta Mx}}$ to obtain M

$N = 320,\ a = 0.25$

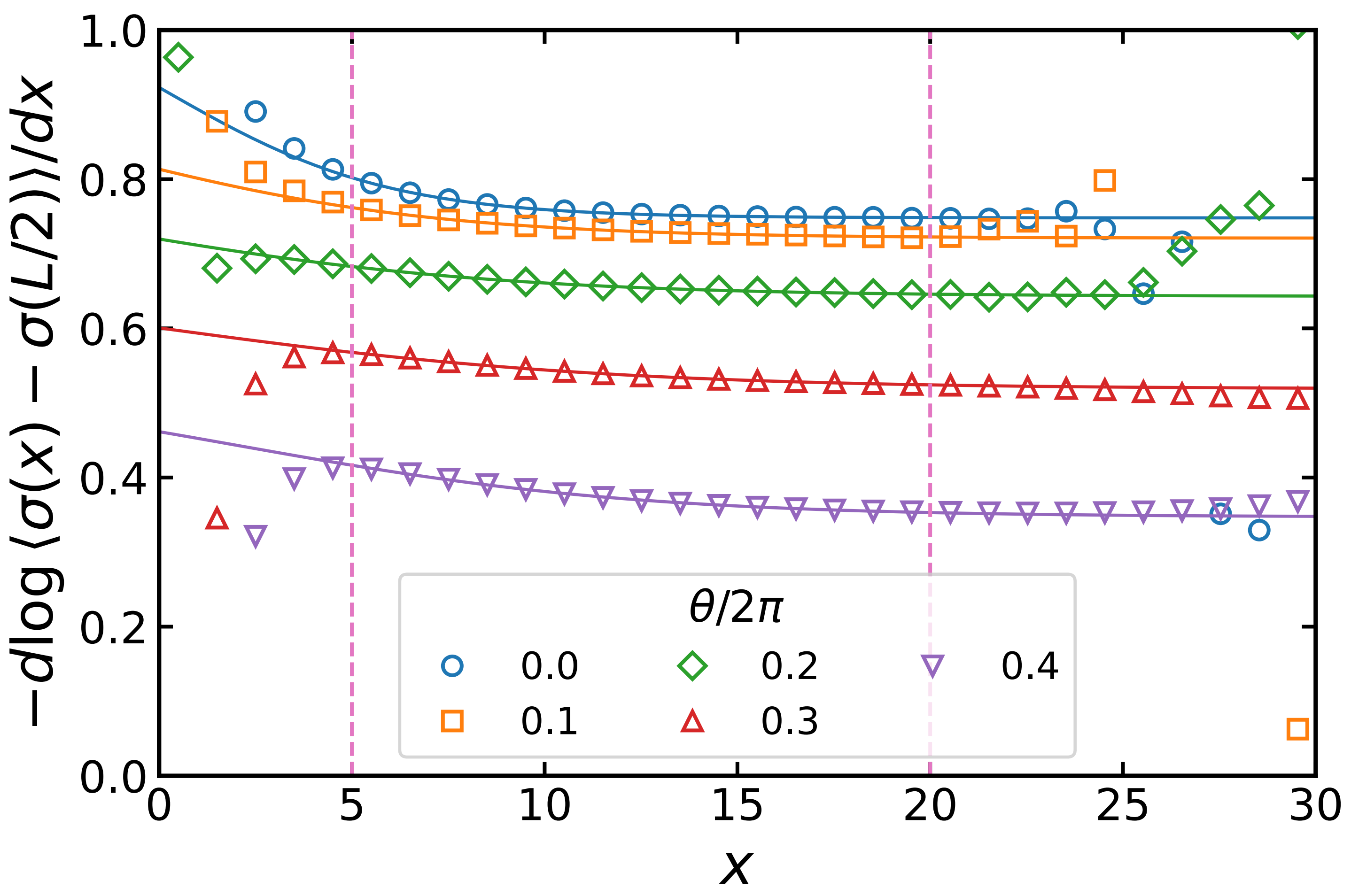
$m = 0.1,\ m_0 = 10$

sigma meson

(stable at any θ)

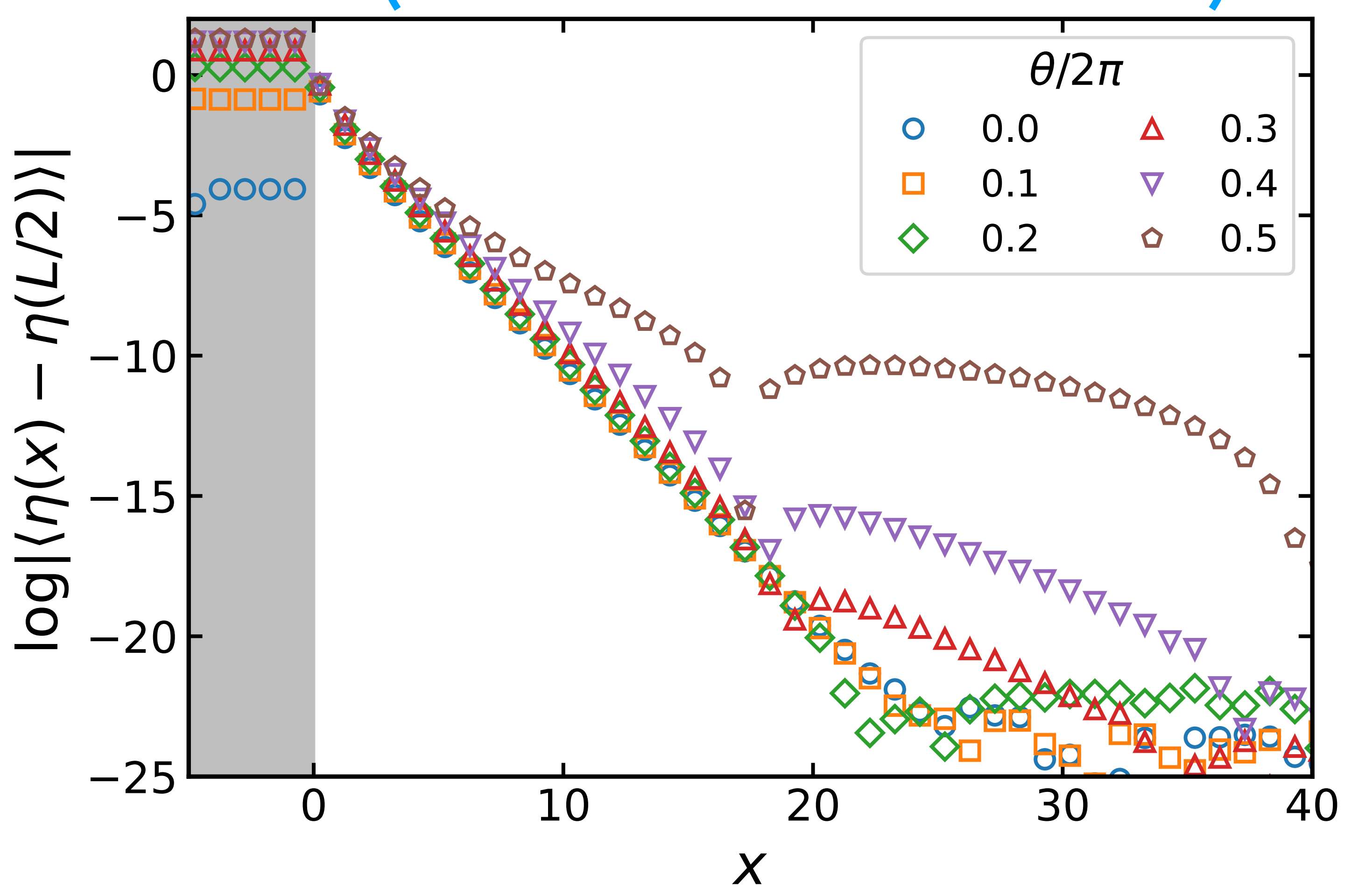


effective mass



eta meson

(unstable at $\theta \neq 0$)



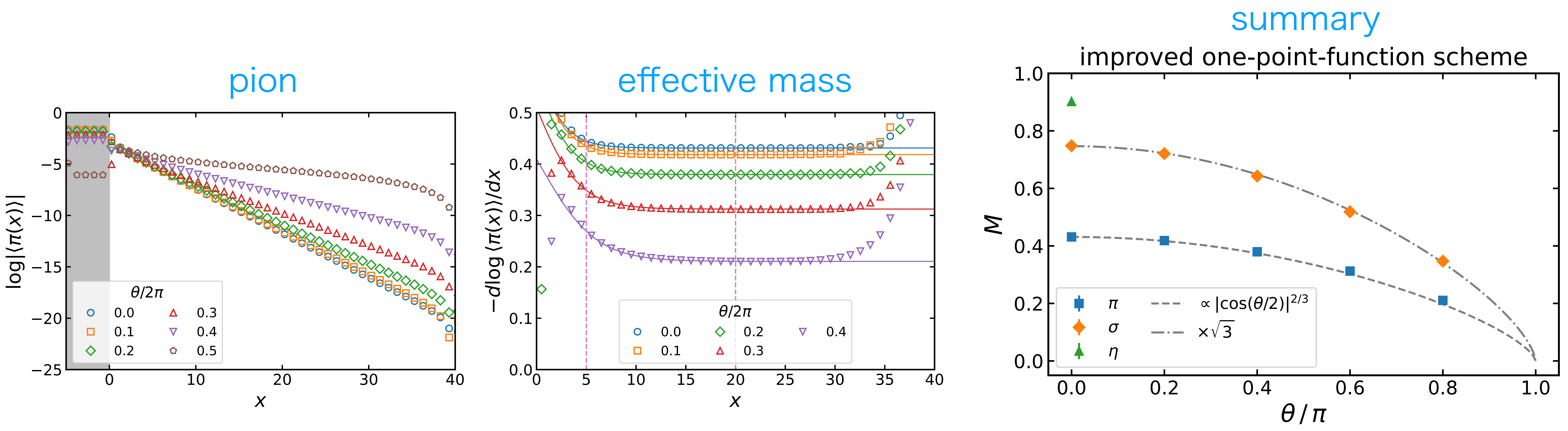
Result of pion

⚠️ $\langle \pi(x) \rangle = 0$ for the Dirichlet b.c. since such a boundary is isospin singlet

- We apply a flavor-asymmetric twist $m_{\text{wings}} = m_0 \exp(\pm i \Delta \gamma^5)$ in the wings to induce the isospin-breaking effect

$N = 320,\ a = 0.25$

$m = 0.1,\ m_0 = 10,\ \Delta = 0.1$



Outline

1. Three mesons in 2-flavor Schwinger model
2. Calculation strategy: MPS and DMRG
3. Correlation-function scheme at $\theta = 0$
4. Improved one-point-function scheme for $\theta \neq 0$
5. **Dispersion-relation scheme**
6. Summary

Dispersion-relation scheme

Obtain the dispersion relation $E = \sqrt{K^2 + M^2}$ directly

from the excited states (momentum excitations of the mesons)

- generate the excited states using DMRG
  and identify the type of meson via the quantum numbers
- measure the energy E and the total momentum $K = \sum_f \int dx\, \psi_f^\dagger (i\partial_x - A_1)\psi_f$
- $[H, K] \neq 0$ due to the open boundary,
  however, K is useful as an approximated operator.

DMRG for excited states

- ℓ-th excited state $|\Psi_\ell\rangle$

  = the lowest energy eigenstate under the orthogonality condition

  $\langle\Psi_{\ell'}|\Psi_\ell\rangle = 0$ for $\ell' = 0, 1, \cdots, \ell - 1$

- obtained by DMRG adding the projection term to the Hamiltonian

$$H_\ell = H + W \sum_{\ell'=0}^{\ell-1} |\Psi_{\ell'}\rangle\langle\Psi_{\ell'}| \qquad W > 0$$

[Banuls et al. (2013)]

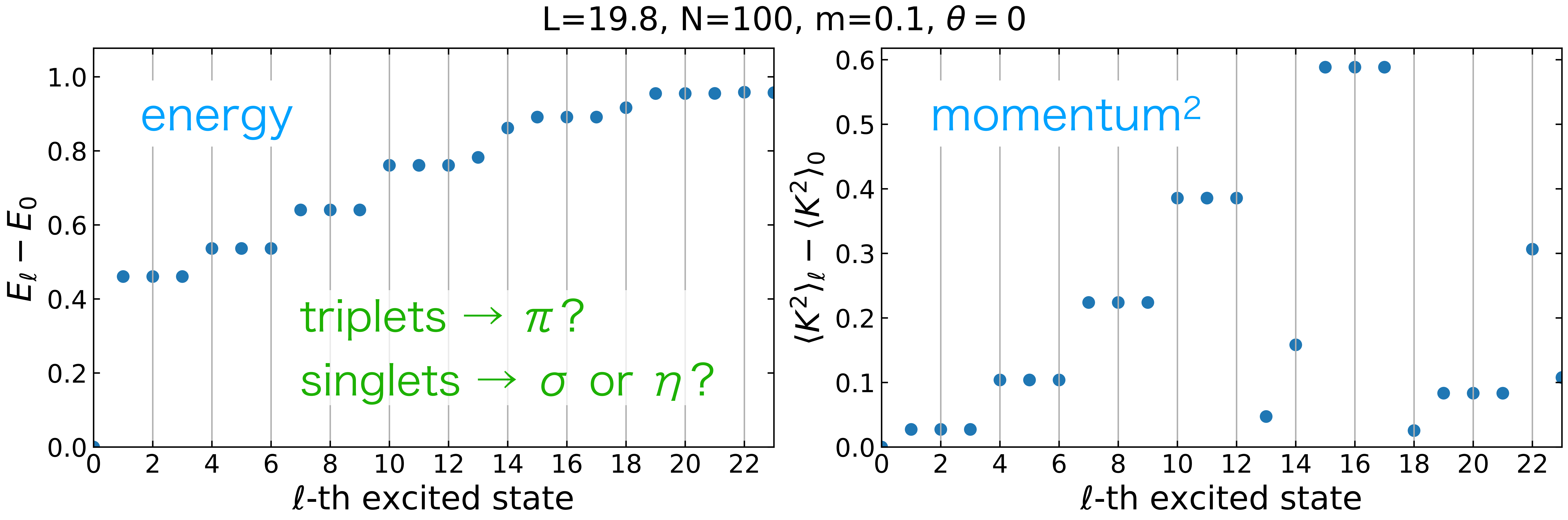
—> cost function: $\langle\Psi_\ell|H|\Psi_\ell\rangle + W \sum_{\ell'=0}^{\ell-1} \left|\langle\Psi_{\ell'}|\Psi_\ell\rangle\right|^2$

- The excited state can be generated step by step from the bottom.

Energy spectrum at $\theta = 0$

- energy gap $\Delta E_\ell = E_\ell - E_0$ and momentum square $\Delta K_\ell^2 = \langle K^2 \rangle_\ell - \langle K^2 \rangle_0$

- the states can be identified by measuring quantum numbers: $\mathbf{J}^2$, J_z, $G = Ce^{i\pi J_y}$



Quantum numbers

- isospin:

$$J_a = \frac{1}{2}\int dx\ \psi^\dagger \tau^a \psi \longrightarrow (\mathbf{J}^2,\ J_z) \qquad [H, \mathbf{J}^2] = [H, J_z] = 0$$

- charge conjugation:

  = exchange even/odd sites and flip each spin

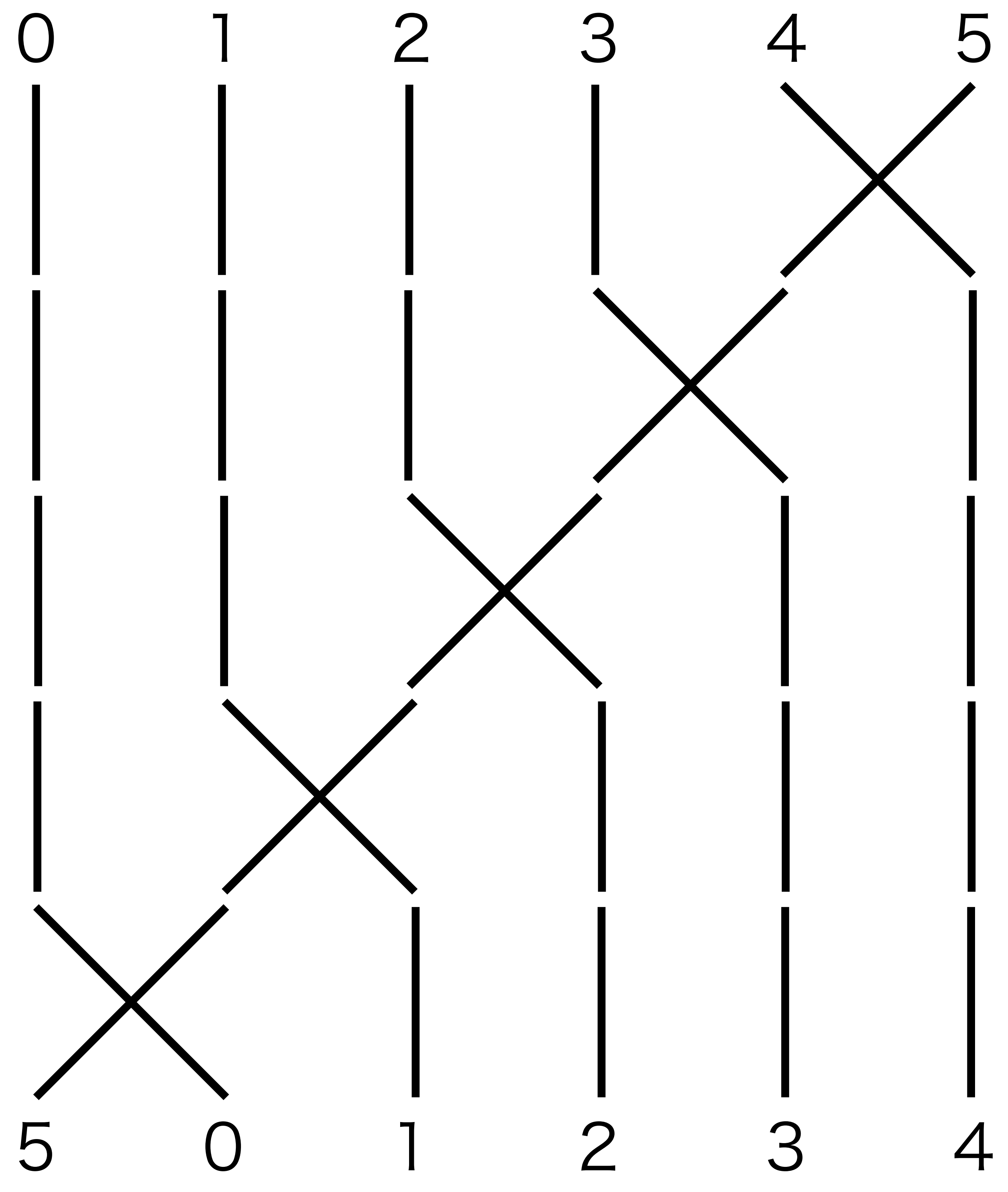
  = 1-site translation and σ^x operator

$$C := \prod_{f=1}^{N_f}\left(\prod_{n=0}^{N-1}\sigma^x_{f,n}\right)\left(\prod_{n=0}^{N-2}(\mathrm{SWAP})_{f;N-2-n,N-1-n}\right)$$

  $[H, C] \neq 0$ due to the boundary

- G-parity: $G = C\exp(i\pi J_y)$

1-site translation



$$(\mathrm{SWAP})_{f;j,k} = \frac{1}{2}\left(\mathbf{1}_{f,j}\mathbf{1}_{f,k} + \sum_a \sigma^a_{f,j}\sigma^a_{f,k}\right)$$

Result of quantum numbers

- triplets: $\mathbf{J}^2 = 2, \;\; J_z = (0, \pm 1), \;\; G > 0$

  —> pion ($J^{PG} = 1^{-+}$)

- singlets: $\mathbf{J}^2 = 0, \;\; J_z = 0,$

  $G > 0$ ($\ell = 13,14,22$) —> sigma meson ($J^{PG} = 0^{++}$)

  $G < 0$ ($\ell = 18,23$) —> eta meson ($J^{PG} = 0^{--}$)

singlets

ℓ	$\boldsymbol{J}^2$	J_z	G
0	0.00000003	-0.00000000	0.27984227
13	0.00000003	0.00000000	0.27865844
14	0.00000003	0.00000000	0.27508176
18	0.00000028	0.00000006	-0.27390909
22	0.00001537	0.00000115	0.26678987
23	0.00003607	-0.00000482	-0.27664779

triplets

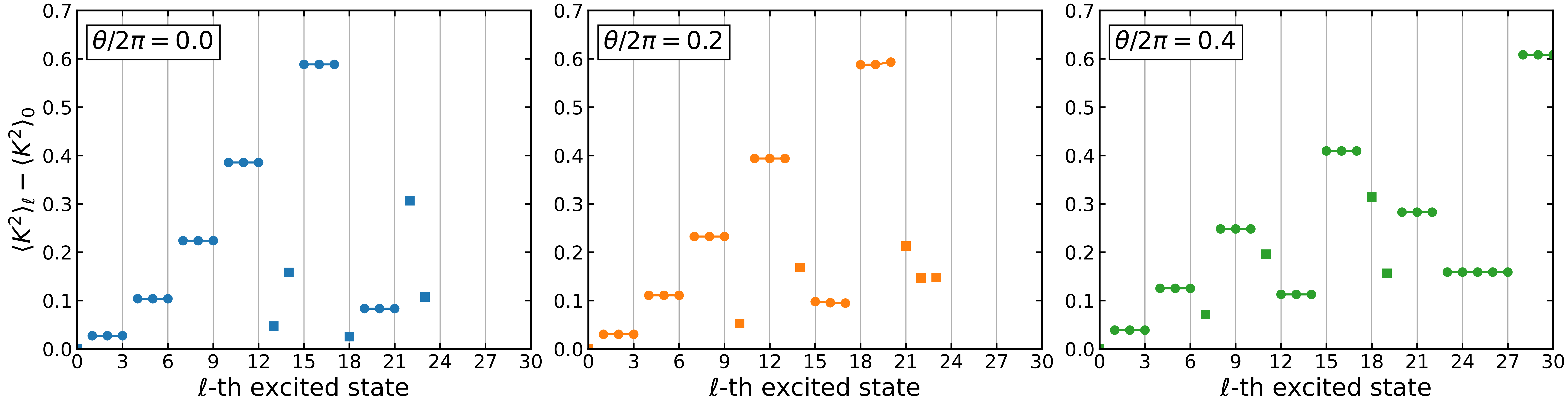
ℓ	$\boldsymbol{J}^2$	J_z	G
1	2.00000004	0.99999997	0.27872443
2	2.00000012	-0.00000000	0.27872416
3	2.00000004	-0.99999996	0.27872443
4	2.00000007	0.99999999	0.27736066
5	2.00000006	0.00000000	0.27736104
6	2.00000009	-0.99999998	0.27736066
7	2.00000010	1.00000000	0.27536687
8	2.00000002	0.00000000	0.27536702
9	2.00000007	-0.99999998	0.27536687
10	2.00000007	0.99999998	0.27356274
11	2.00000005	0.00000001	0.27356277
12	2.00000007	-0.99999999	0.27356274
15	1.99999942	0.99999966	0.27173470
16	2.00000052	0.00000000	0.27173482
17	2.00000015	-1.00000003	0.27173470
19	2.00009067	1.00004377	0.27717104
20	2.00002578	-0.00000004	0.27717020
21	2.00003465	-1.00001622	0.27717104

Extension to $\theta \neq 0$

- G-parity is no longer the quantum number —> η disappears
- singlet projection to obtain σ with reasonable computational cost

$$H_\ell = H + W \sum_{\ell'=0}^{\ell-1} |\Psi_{\ell'}\rangle\langle\Psi_{\ell'}| + W_J \mathbf{J}^2$$

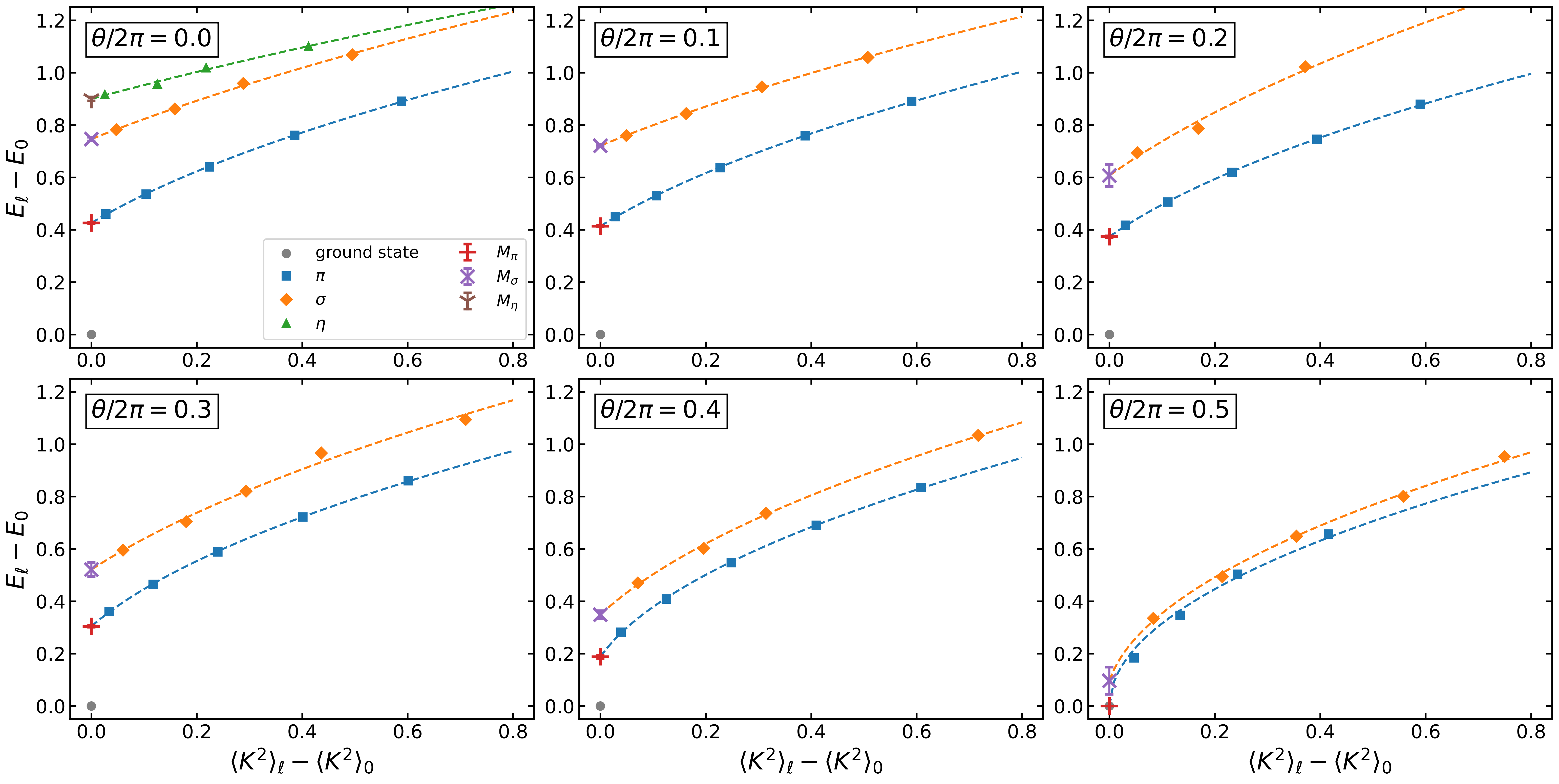
momentum^2



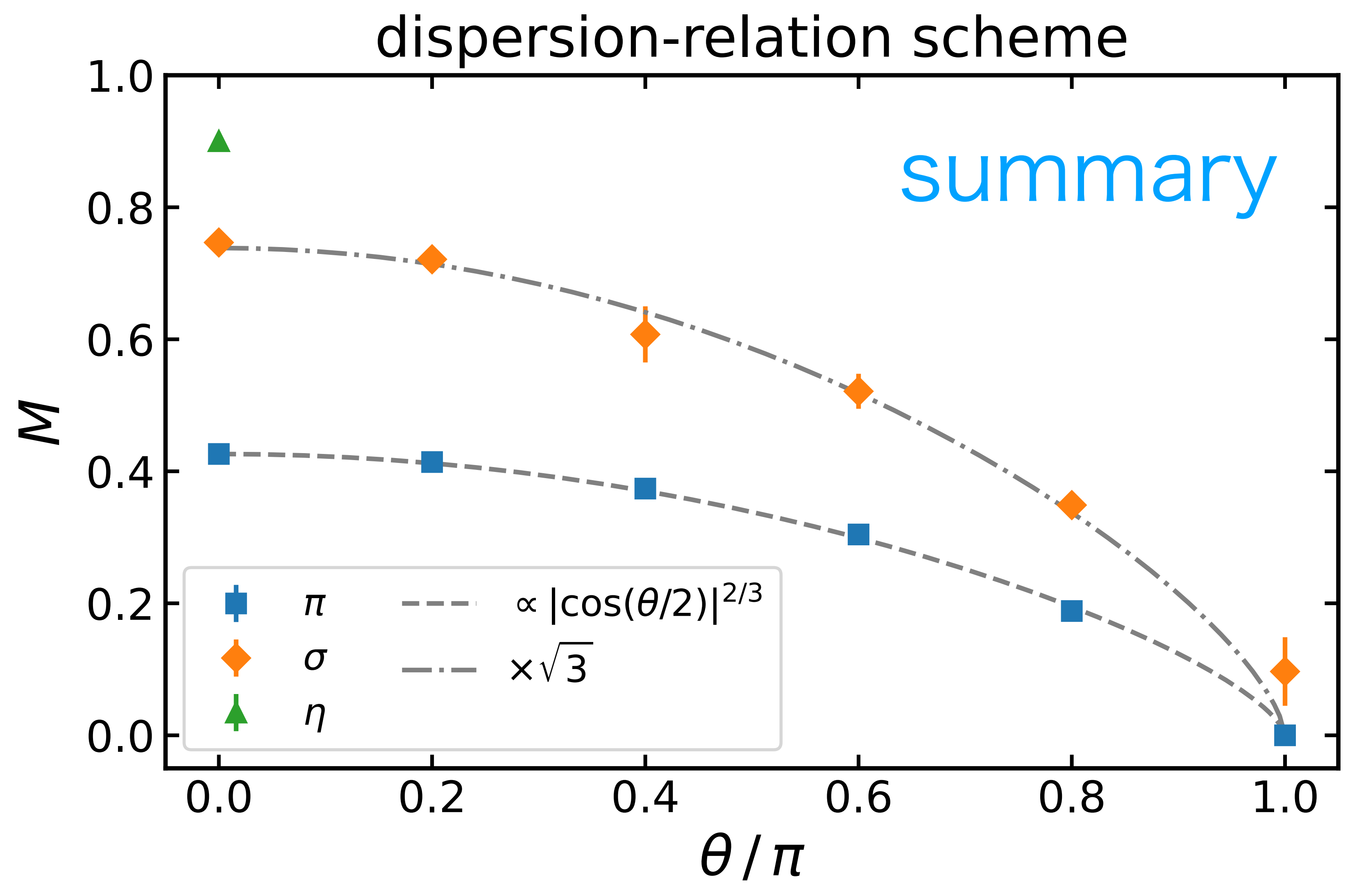
Result of dispersion relation

- plot ΔE_ℓ against ΔK_ℓ^2 and fit the data by $\Delta E = \sqrt{b^2 \Delta K^2 + M^2}$ for each meson

energy vs momentum²



Around $\theta/2\pi = 0.2$, σ is contaminated by a remnant of η due to the mixing



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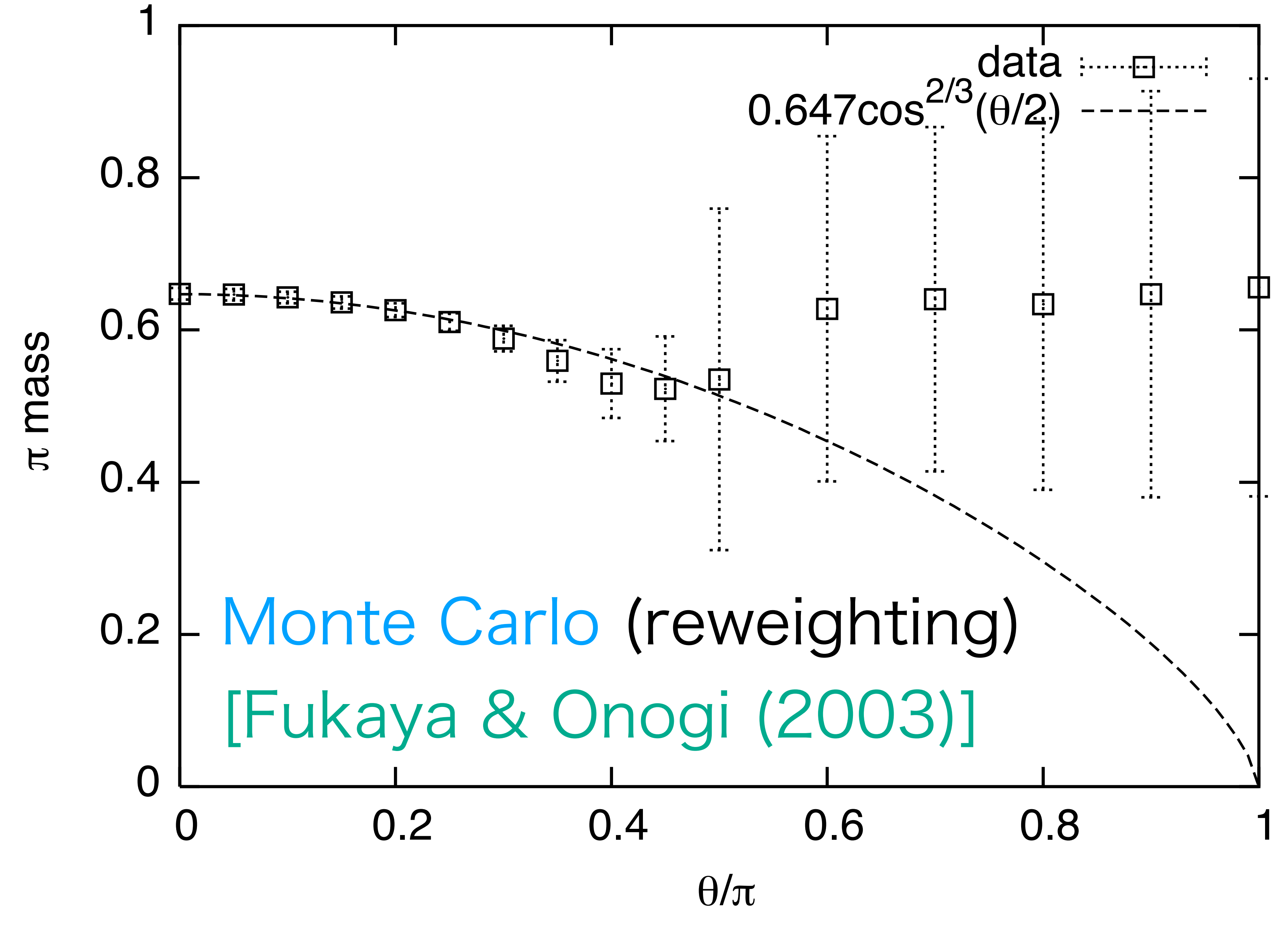
6. **Summary**

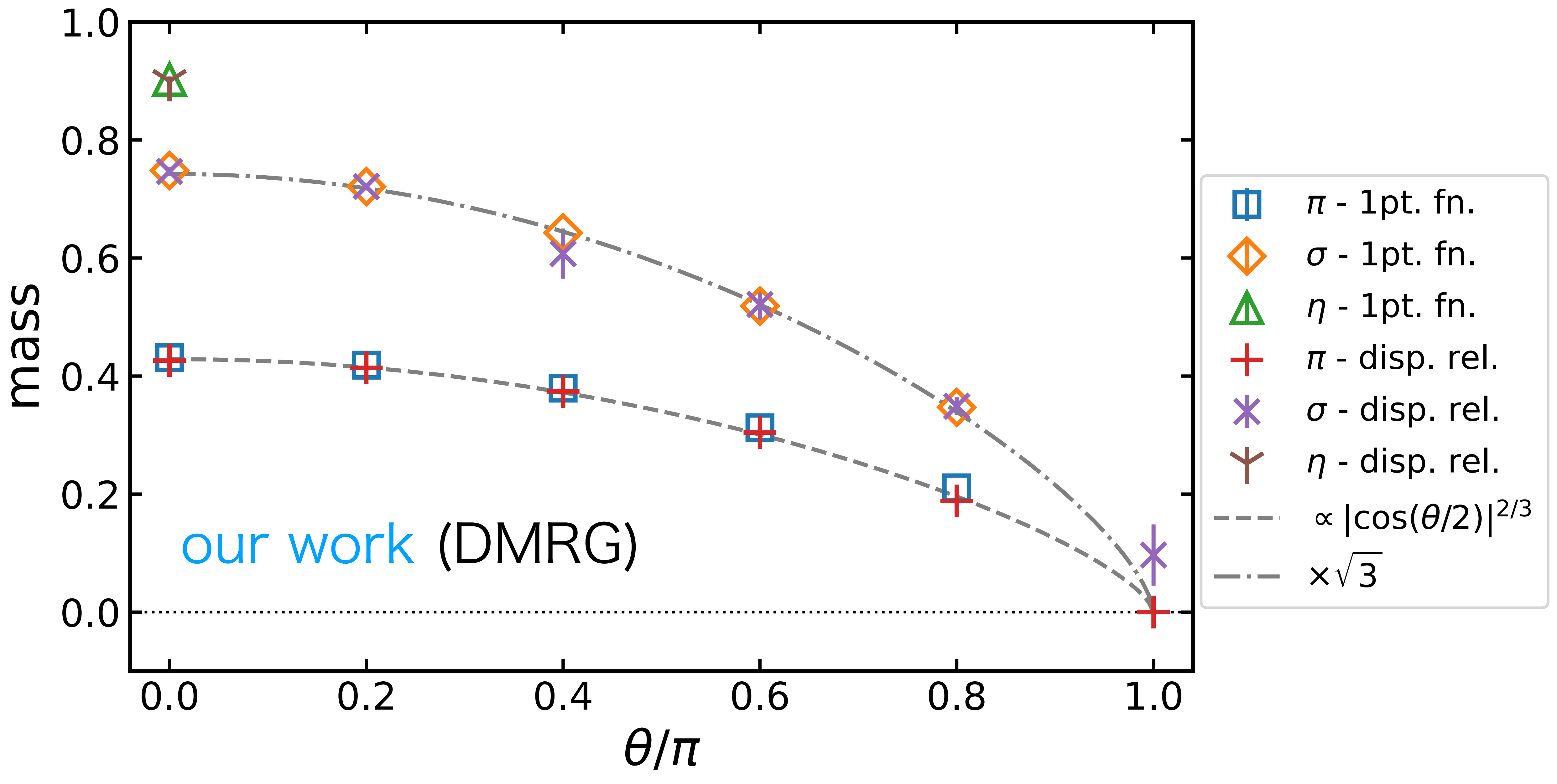
Summary

- The two schemes give consistent results and look promising
- consistent with predictions by the bosonization [Coleman (1976)] [Dashen et al. (1975)]

$$M_\pi(\theta) \propto |\cos(\theta/2)|^{2/3} \qquad M_\sigma(\theta)/M_\pi(\theta) = \sqrt{3}$$



applicable even in large θ region!



Discussion

(1) correlation-function scheme

👍 generic method applicable to any case / off-diagonal elements

😥 sensitive to the bond dimension of MPS —> 😊 quantum computer?

(2) (improved) one-point-function scheme

👍 NOT sensitive to the bond dimension / easy to compute

😥 only the lowest state of the same quantum number as the boundary

(3) dispersion-relation scheme

👍 obtain various states heuristically / directly see wave functions

😥 how to generate excited states efficiently?

Future prospect

- Extension to 2+1 dimensions, where the gauge field is dynamical
- Application to the model with chemical potential:
  How the spectrum changes in the high-density region?
- Analyses using the wave functions of the excited states:
  scattering problem, entanglement property, etc.

Thank you for listening.

CFT-like behavior at $\theta = \pi$

<u>bond dim. of MPS grows up with N at $\theta = \pi \to$ gapless?</u>

cf.) bond dim. D bounds the entanglement entropy of MPS: $S_{\mathrm{EE}} \lesssim \log D$

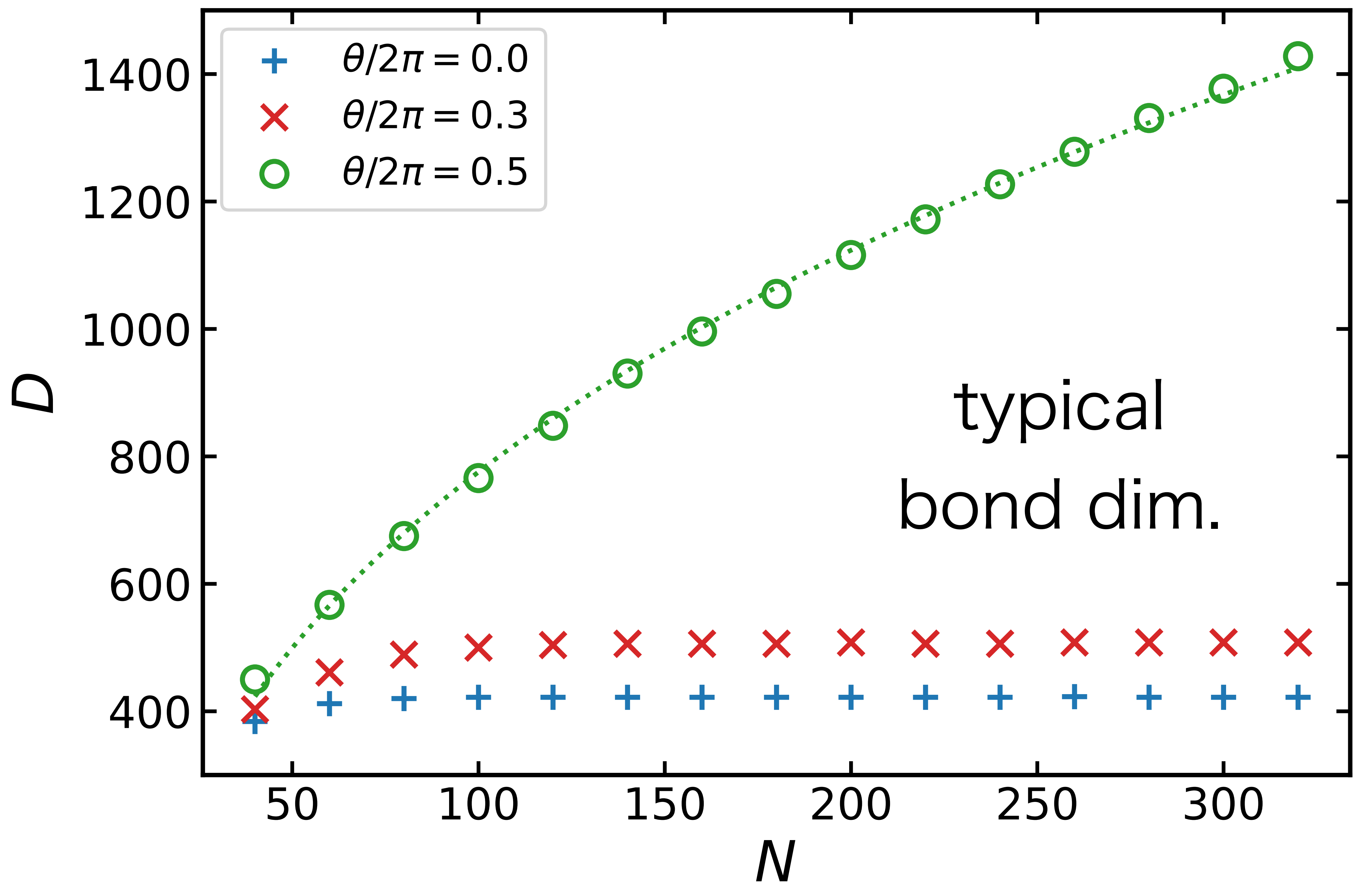
1+1d **gapped** : $S_{\mathrm{EE}} \sim \mathrm{const}.$

—> D is independent of the size N

1+1d **gapless** : $S_{\mathrm{EE}} \sim \frac{c}{3} \log N$

—> increases by power $D \sim N^{c/3}$

- central charge $c = 1$ in this case
  (deviation due to the finite a exists)



CFT-like behavior at $\theta = \pi$

- At $\theta = \pi$, the mass gap is exponentially small $\sim e^{-\# g^2/m^2}$ [Coleman (1976)] [Dempsey et al. (2024)]

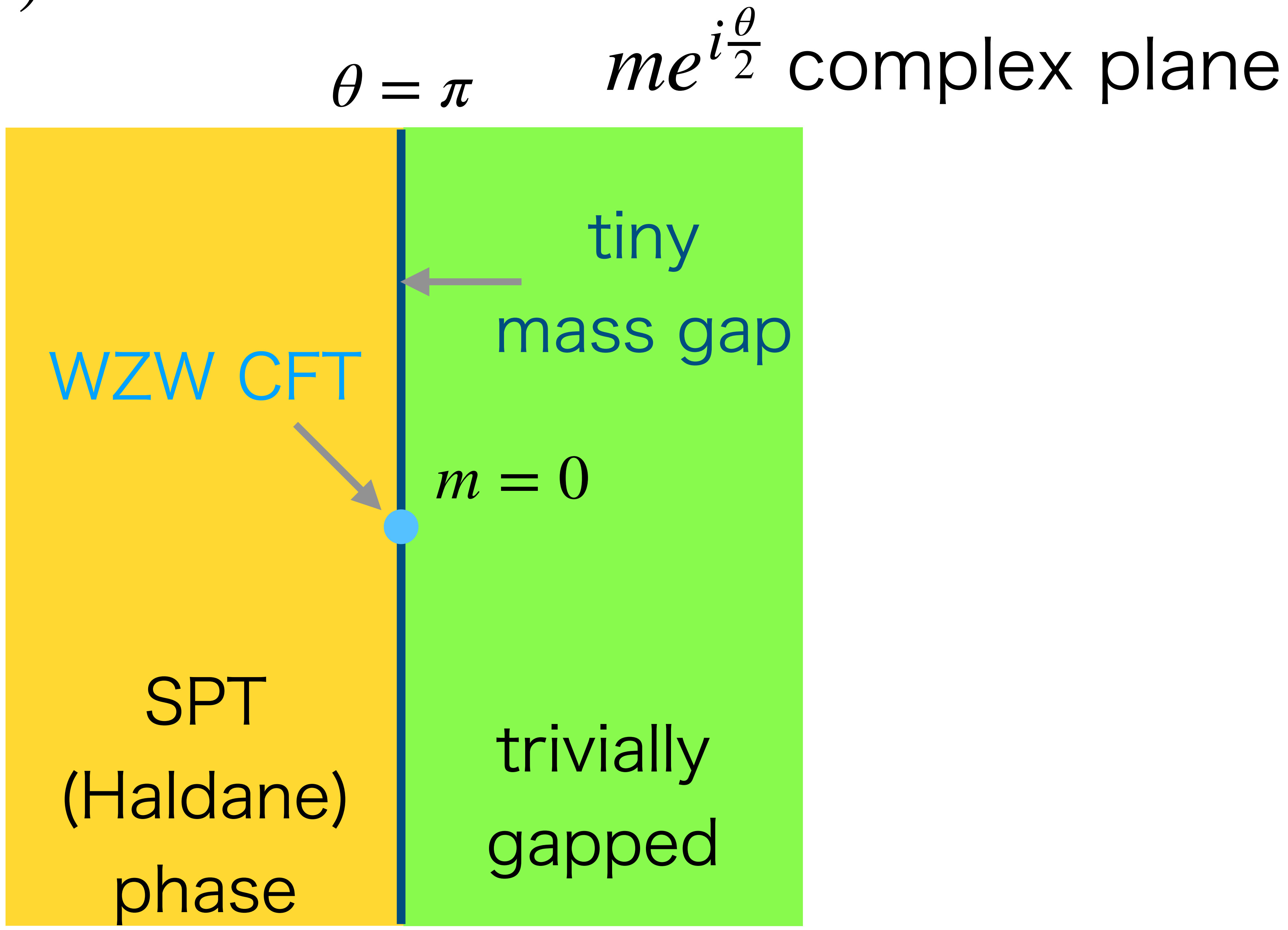
  cf.) SU(2)$_1$ WZW model with marginally relevant $J_L J_R$ deformation

- For the finite size L, the energy scale below $O(1/L)$ is invisible

  —> system is CFT-like if $L < e^{\# g^2/m^2}/g$

  our setup : $L = 80,\ m = 0.1,\ g = 1$

- compare the numerical result of 1pt. functions with the analytic calc. of WZW model

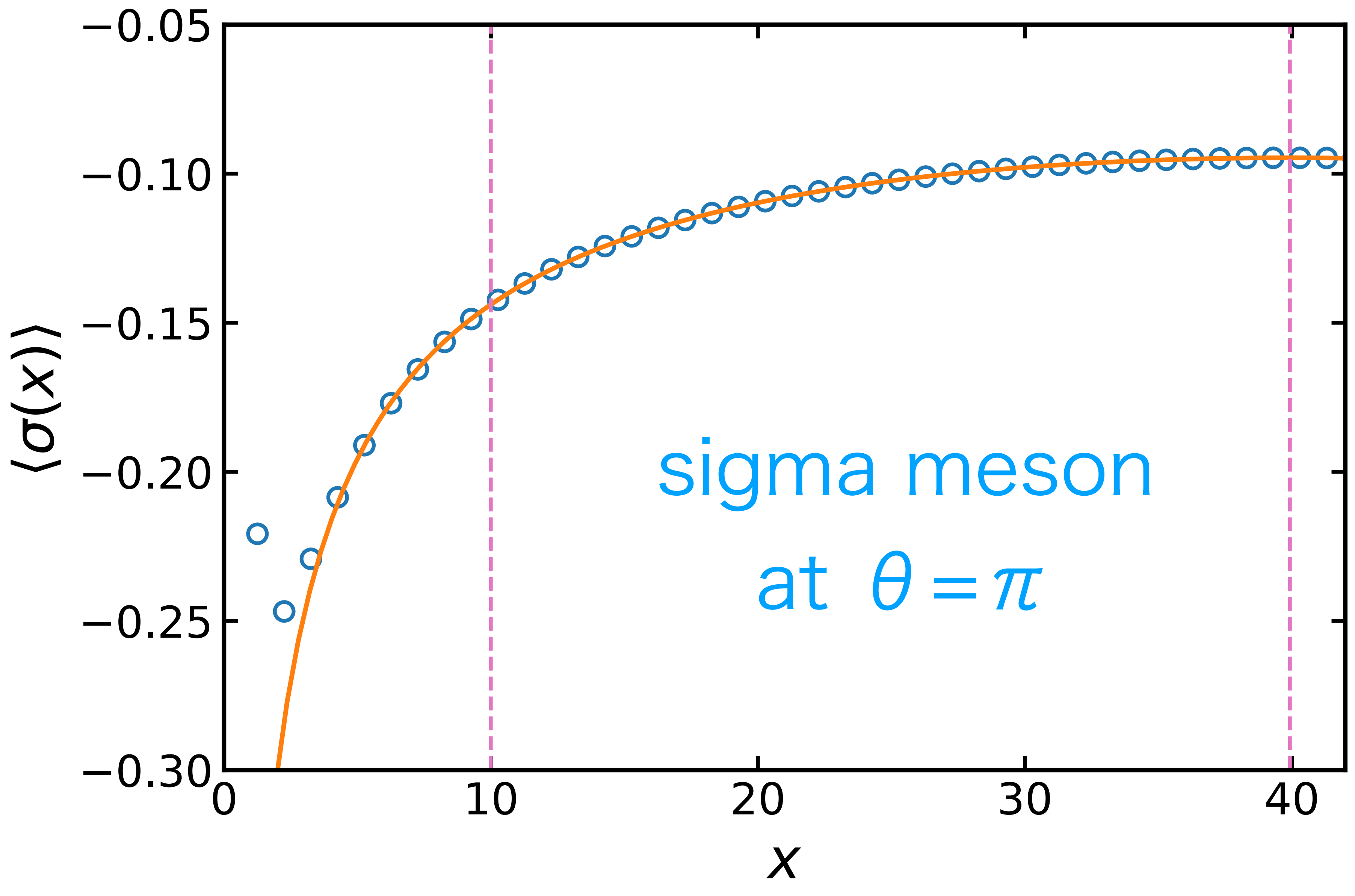


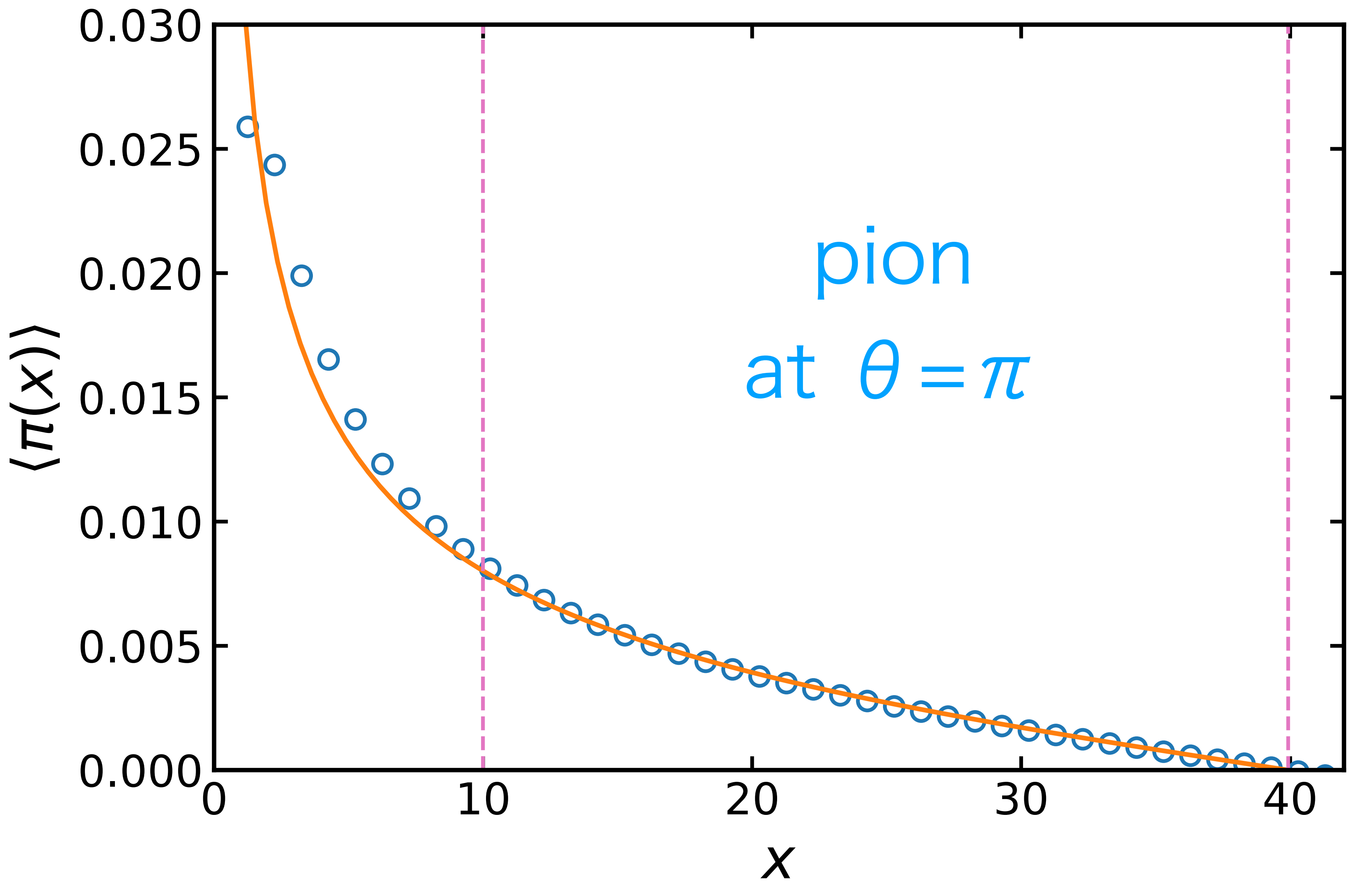
1pt. function of π and σ at $\theta=\pi$

- Dirichlet b.c. $\rightarrow \langle \sigma(x) \rangle \propto \dfrac{1}{\sqrt{\sin(\pi x/L)}}$
- isospin-breaking b.c. $\rightarrow \langle \pi(x) \rangle \propto \dfrac{\sin[\Delta(1-2x/L)]}{\sqrt{\sin(\pi x/L)}}$

mirror-image method

cf.) appendix A of JHEP09 (2024) 155

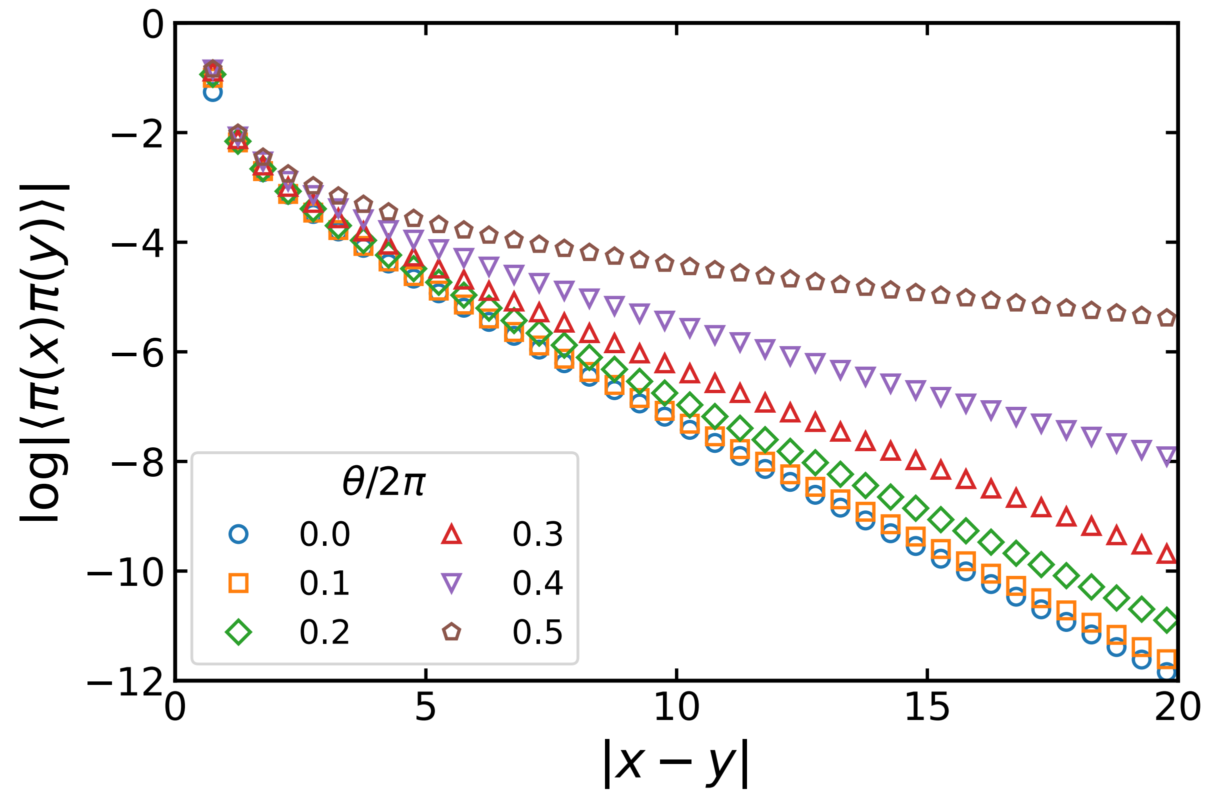


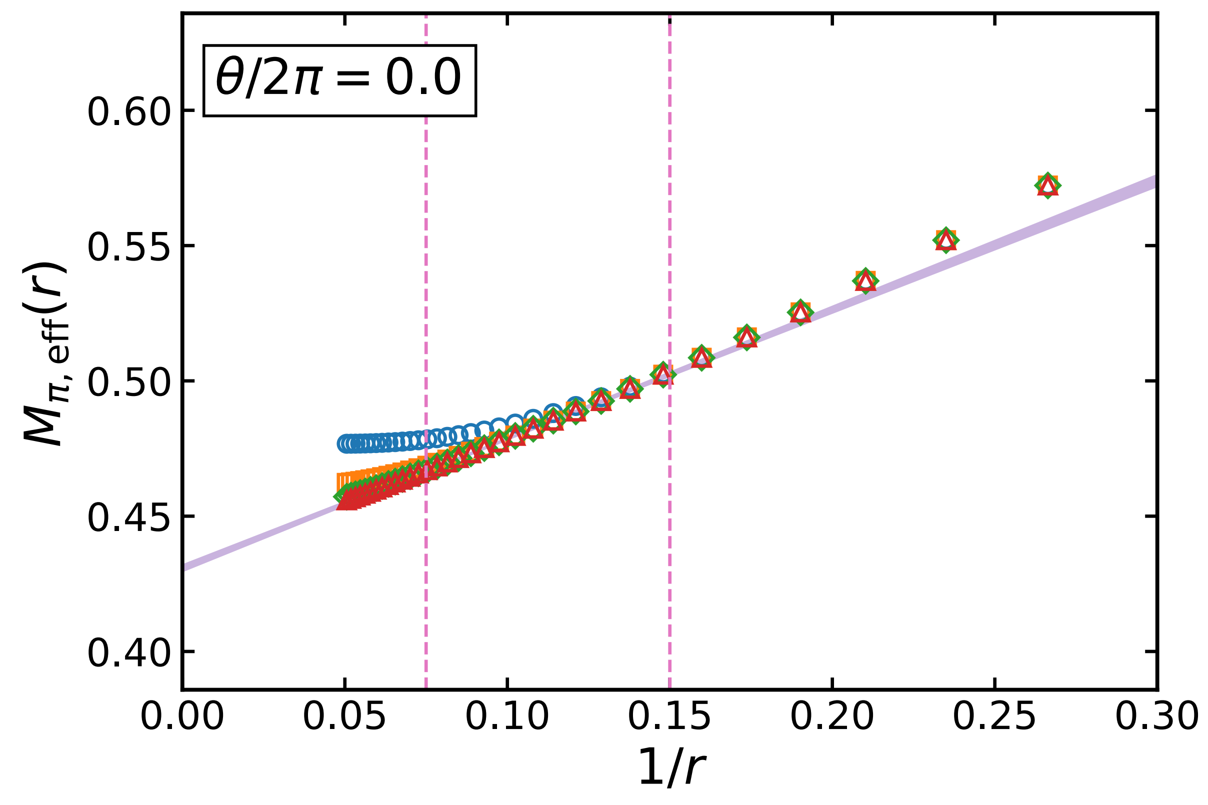


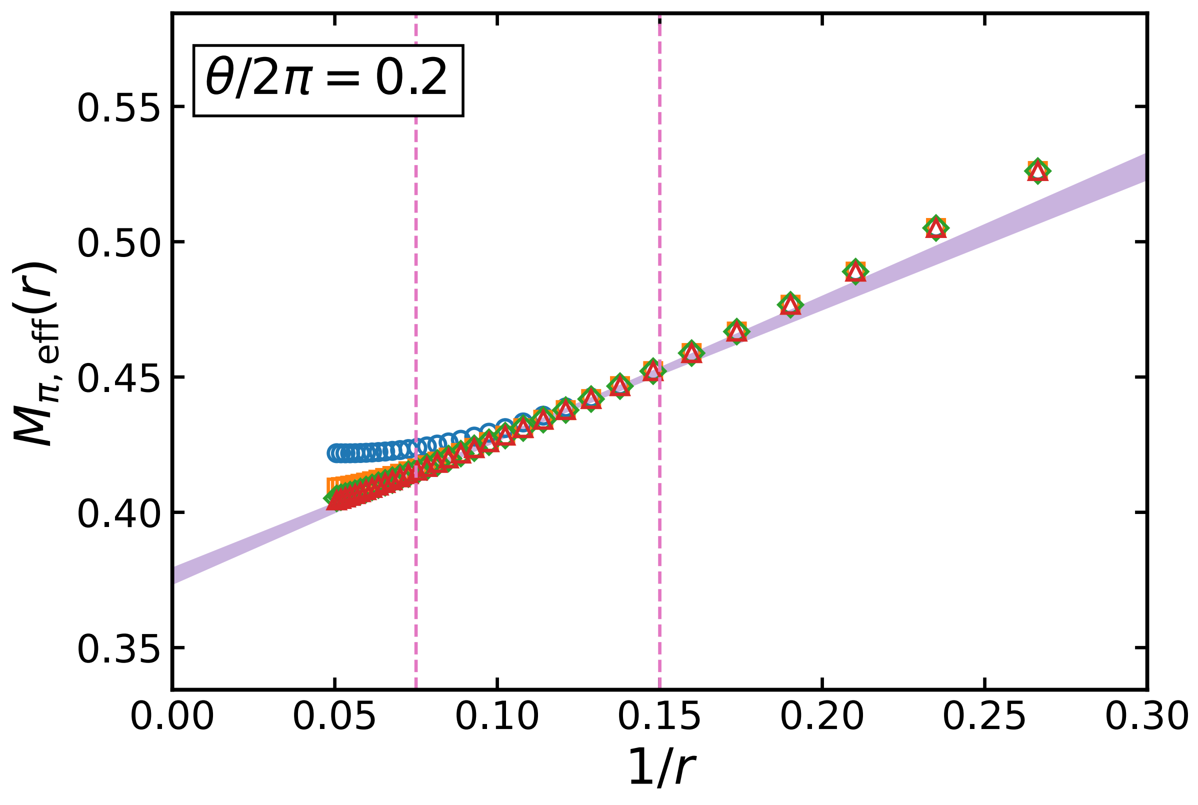
consistent with WZW model in the bulk

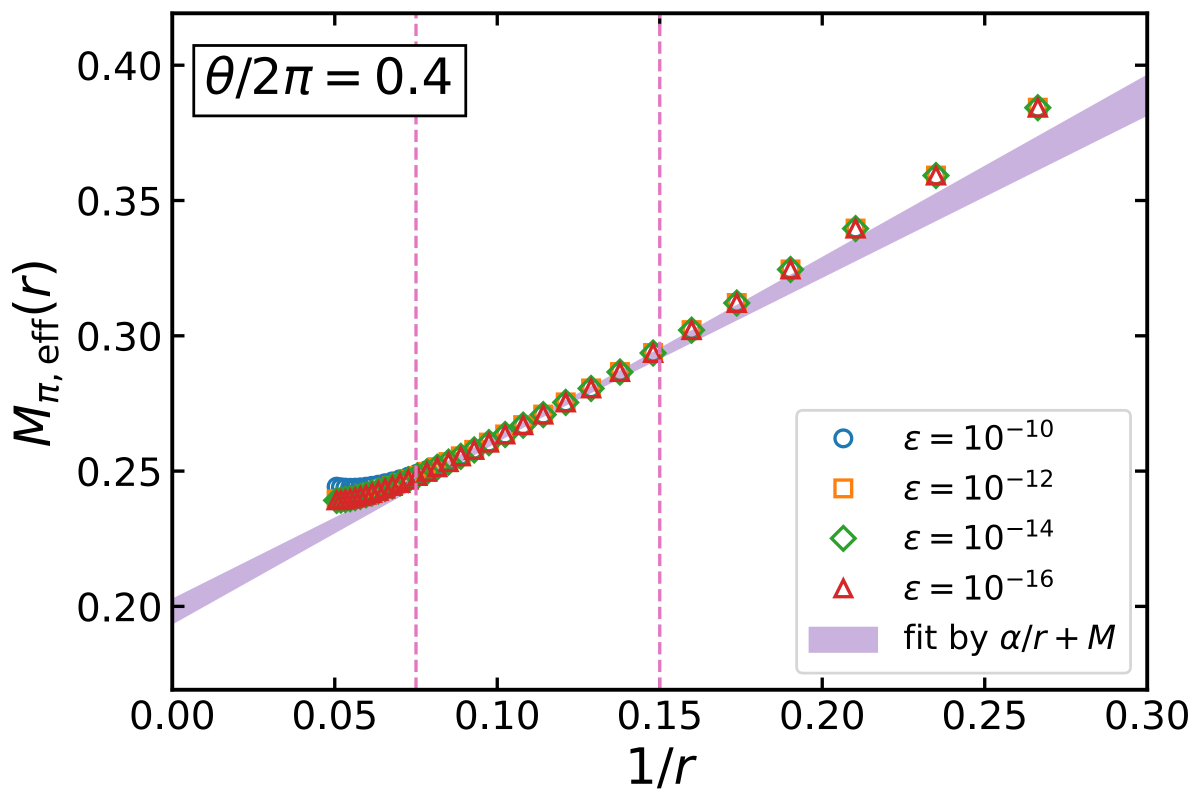
Correlation-function sheme

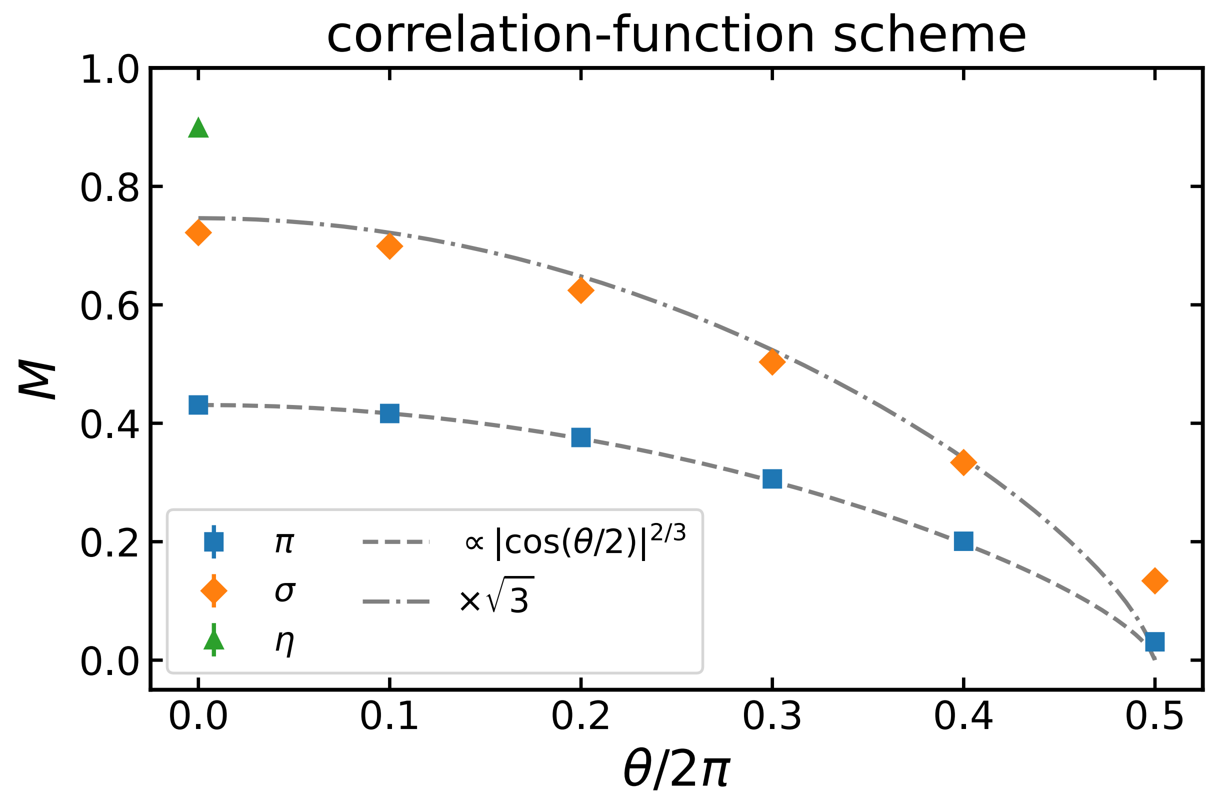
- spatial 2-point correlation function: $C_\pi(r) = \langle \pi(x)\pi(y) \rangle \sim \frac{1}{r^\alpha} e^{-Mr}$ $\quad r = |x - y|$
- effective mass: $M_{\pi,\mathrm{eff}}(r) = -\frac{d}{dr} \log C_\pi(r) \sim \frac{\alpha}{r} + M$
- 1/r behavior is observed only when the bond dim. is large
- mass is given by $r \to \infty$ extrapolation







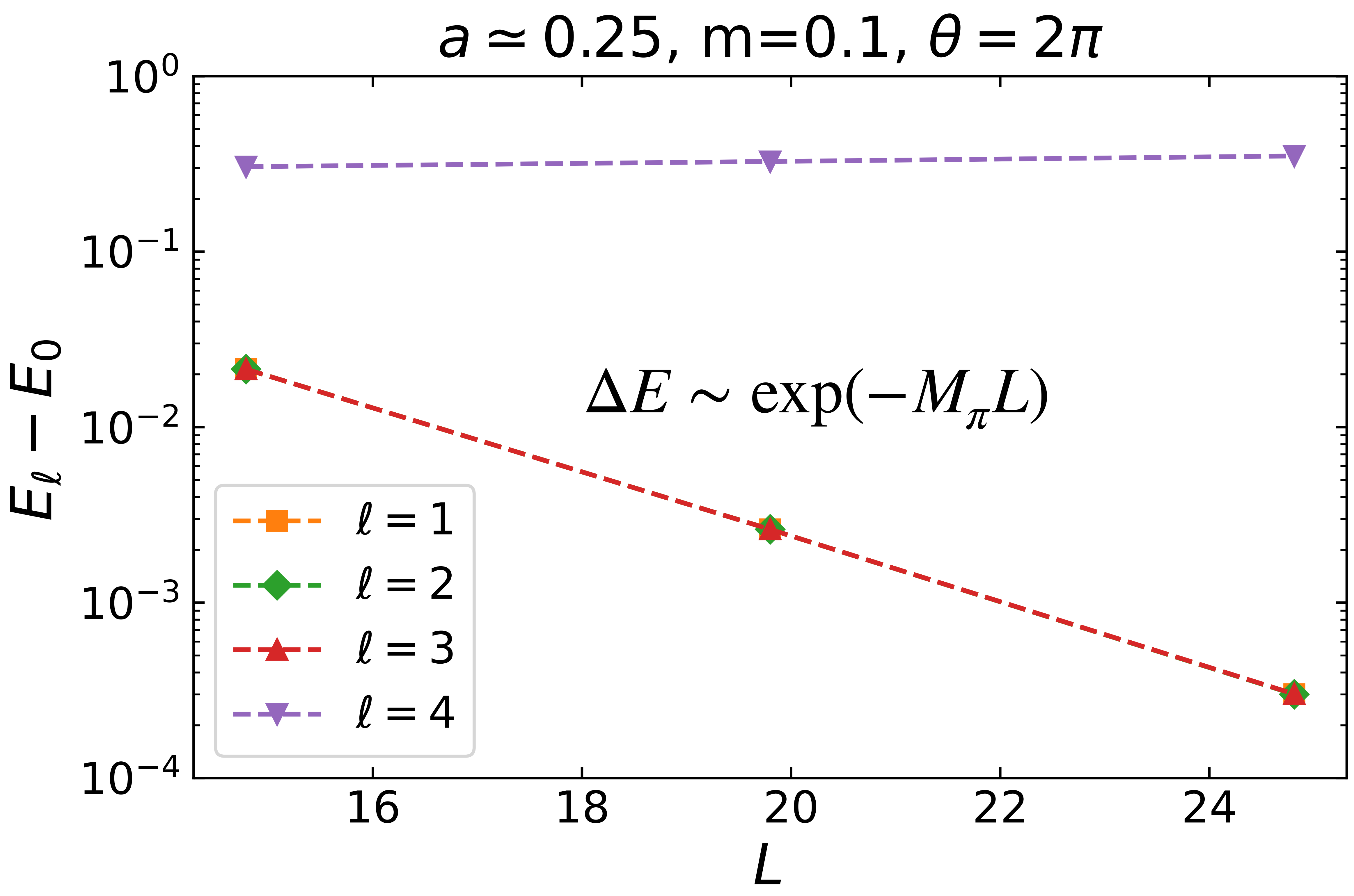




Degeneracy of the ground states

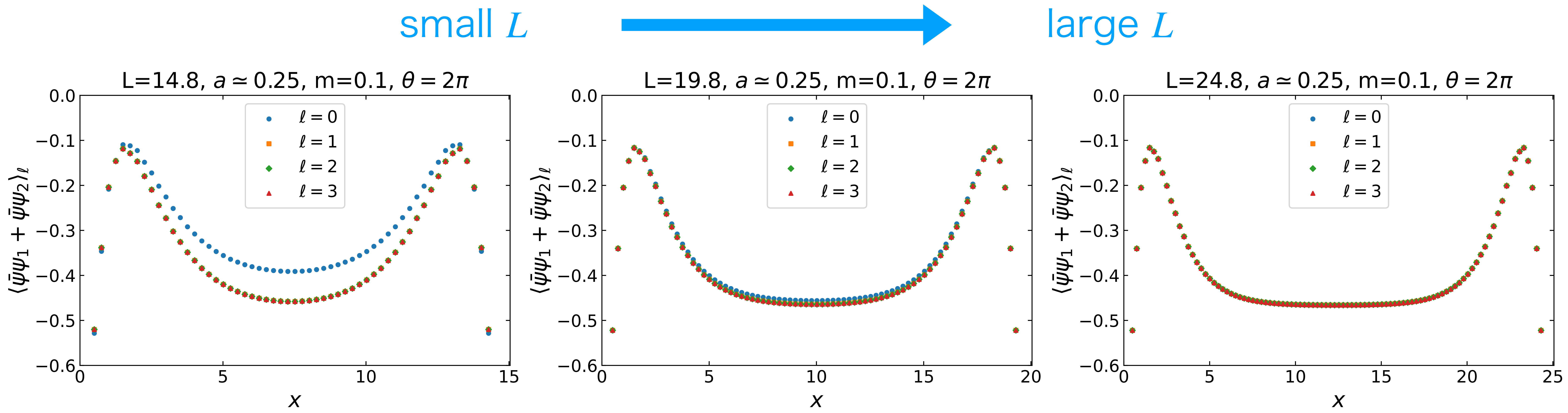
- one ground state + three 1st excited states are observed by DMRG at $\theta = 2\pi$.
- energy gap $\sim \exp(-M_\pi L) \to 0$
- solve $\Delta E_\ell = C_0 + \exp(-ML + C_1)$ for $\ell = 1$;
  $M = 0.41767,\ C_0 = -0.00002,\ C_1 = 2.33326$
- cf.) $M_\pi = 0.4175(9)$ by 1pt-fn. scheme
- DMRG is hard when L is small or $\theta \to \pi+$

energy gap of the ℓ-th excited state



Local observables

- local scalar condensate $\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2$ (isospin singlet) at $\theta = 2\pi$
- degeneracy in $L \to \infty$

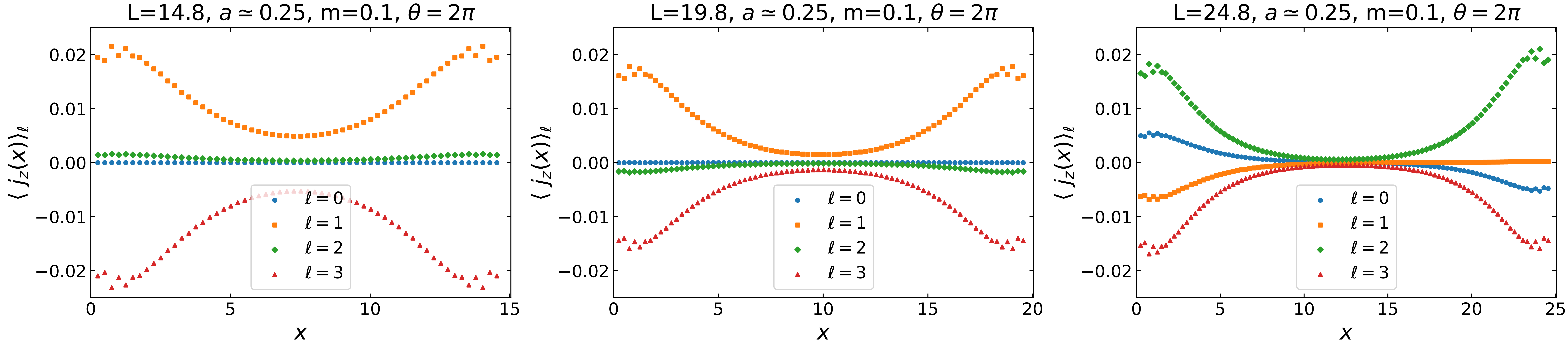


Local isospin

- local isospin $j_z(x) = \frac{1}{2}(\psi_1^\dagger\psi_1 - \psi_2^\dagger\psi_2)$ at $\theta = 2\pi$
- finite L : singlet + triplet —> $L \to \infty$: doublet × doublet

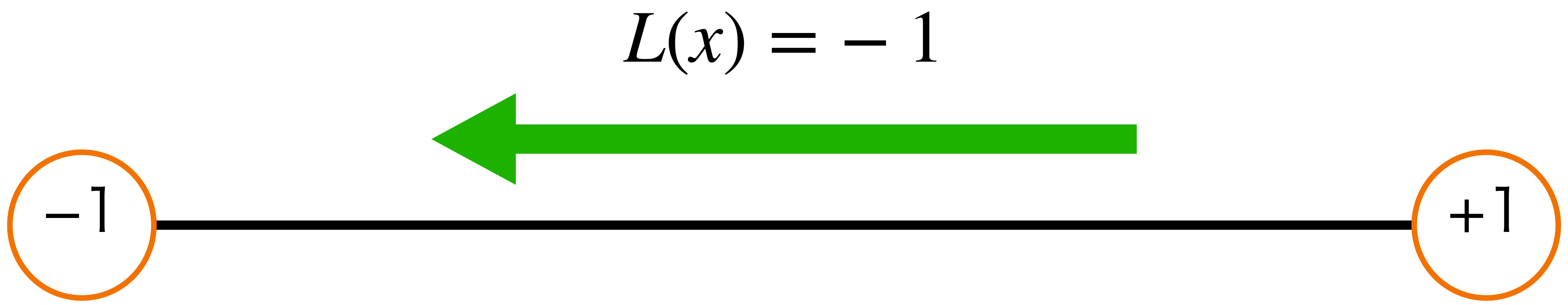
  interaction is suppressed exponentially and the edge modes are decoupled

small L → large L



Electric charge and electric field

- charge density: $\rho(x) = \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2$
- induced electric field: $L(x) = \int_0^x dy\, \rho(y)$



cancel the background electric field

$E = \theta/2\pi = +1$ from θ term