One Born-Oppenheimer EFT to rule them all: hybrids, tetraquarks, pentaquarks, quarkonium and doubly heavy baryons

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TECHNISCHE UNIVERSITÄT MÜNCHEN

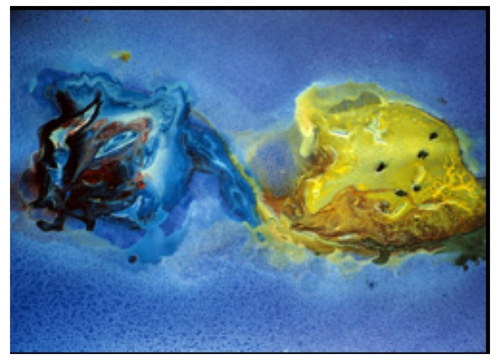
Advanced Grant EFT-XYZ

T30f

TUMQCD LATTICE Collaboration

STRONG 2020

TUM-IAS Focus Group Physics with Effective Field Theories



Quark Confinement and the Hadron Spectrum since 1994

MCQST

MDSI

Munich Data Science Institute

ORIGINS Excellence Cluster

The XYZ states represent a revolution in particle physics:
besides quarkonium we have for the first time tetraquarks, pentaquarks, hybrids

They have the potential to give us information on the fundamental strong force if studied in QCD

They represent a big challenge for theory

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Focus of the talk

We introduce a QCD derived nonrelativistic effective field theory the Born Oppenheimer EFT (BOEFT) that can address in the same framework quarkonium, tetraquarks, pentaquarks, hybrids and doubly heavy baryons

The BOEFT is based on symmetries and factorization
It allows for QCD perturbative calculations at short distance

It factorizes long distance in few flavour independent correlators to be calculated on the lattice

Factorization allows for model independent predictions

Examples of application includes:
Quarkonium, tetraquarks
the X(3872)and the Tcc with insight in their nature
hybrids

Novel tools to bridge perturbative methods with lattice QCD are key to this program, as well as the combination between different EFTs

Based on:

N.B. , A Mohapatra, T. Scirpa, A. Vairo, ‘The nature of X(3827) and Tcc’ in preparation

M. Berwein, N.B. , A. Mohapatra, A. Vairo, 2408.04719, in press on PRD, Editor’s suggestion -> obtains equations for all cases

M. Berwein, N.B. , J. Tarrus, A. Vairo, 1510.04299 -> establishes BOEFT

N.B. , G. Krein, J. Tarrus, A. Vairo, 1707.09647 -> generalizes to all cases

N.B. , W.K. Lai, J. Segovia, J. Tarrus, A. Vairo, 1805.07713

N.B. , W.K. Lai, J. Segovia, J. Tarrus 1908.11699 -> spin corrections (hybrids)

N.B. , W.K. Lai, A. Mohapatra, A. Vairo 2212.09187 -> semi-inclusive decays (hybrids)

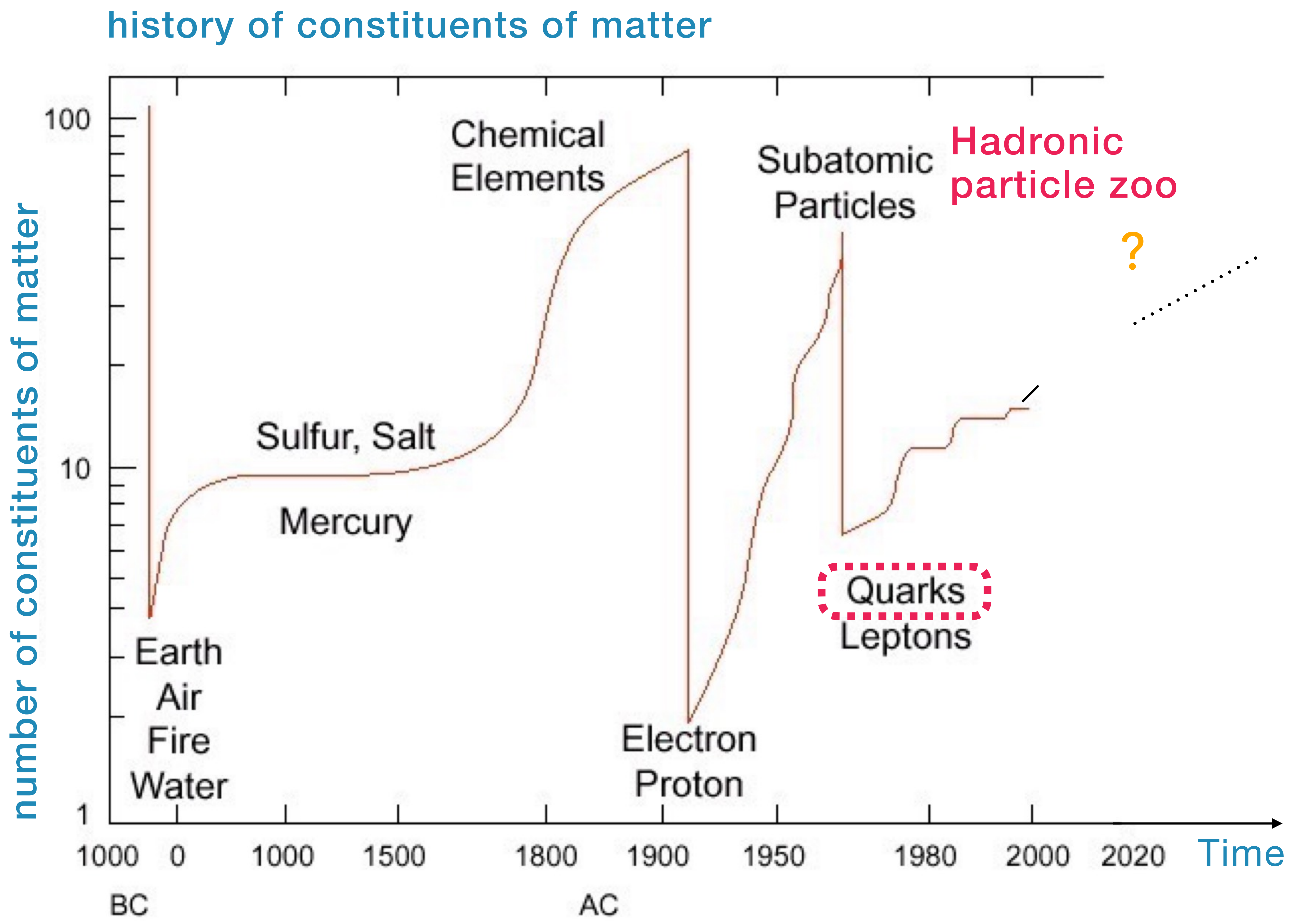
N.B. , J. Soto, A. Pineda, A. Vairo hep-ph/9907240

N.B. , J. Soto, A. Pineda, A. Vairo hep-ph/0410047 -> quarkonium strongly coupled pNRQCD

N.B. , J. Soto, A. Pineda, A. Vairo hep-ph/0410047

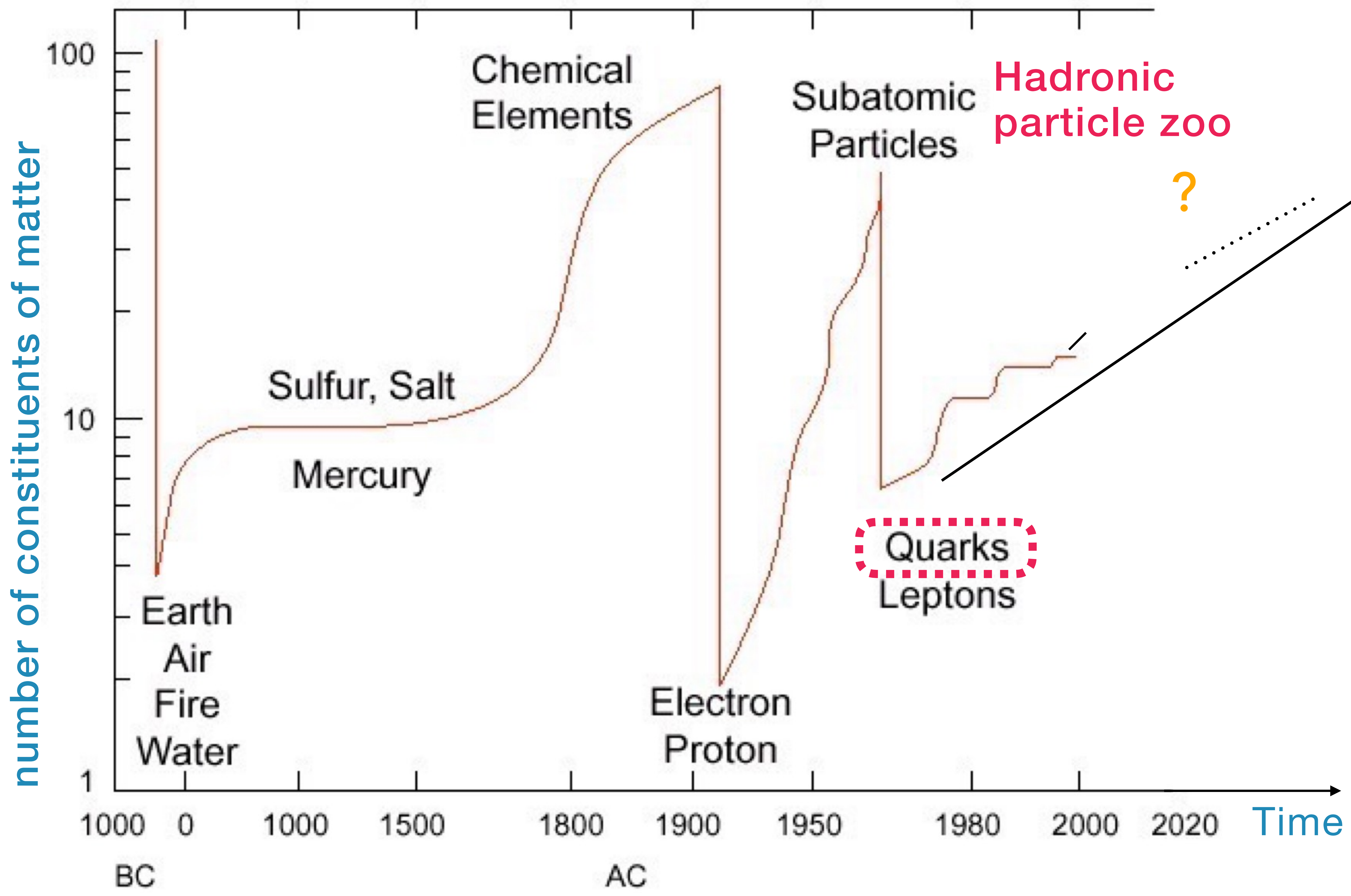
the XYZ are a revolution in particle physics

Constituents of matter and fundamental forces



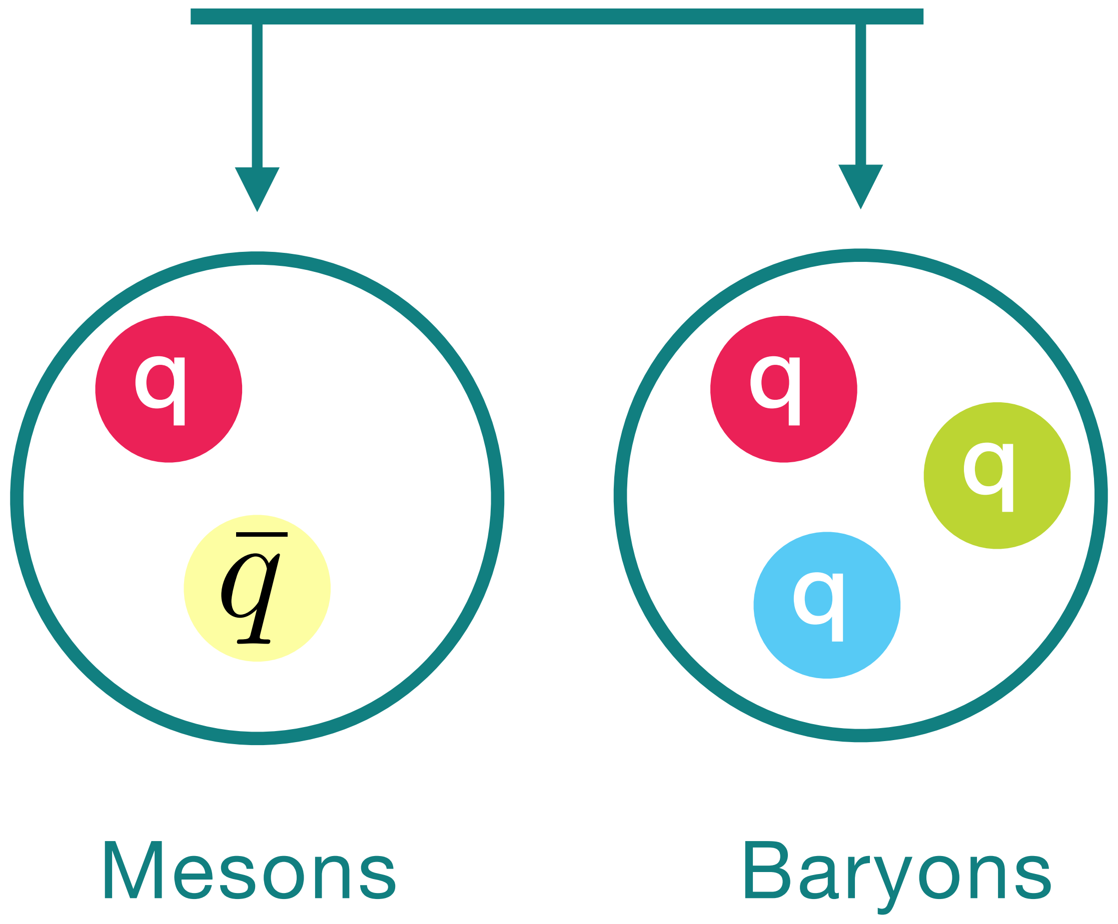
Constituents of matter and fundamental forces

history of constituents of matter



Quark Model 1964
Gell-Mann Zweig

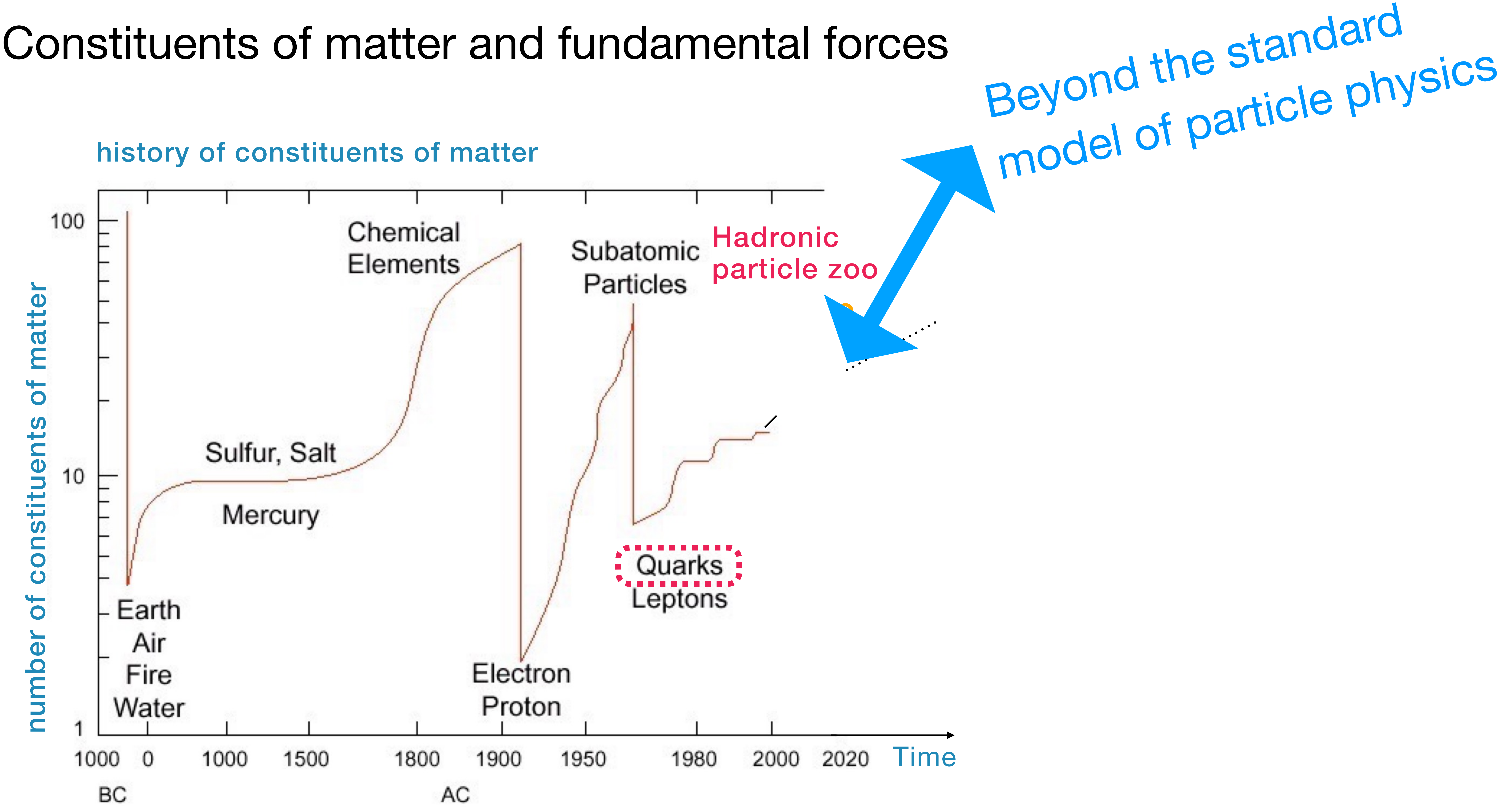
observed only



$qq\bar{q}\bar{q}$ $qqqq\bar{q}\ldots$

possible but not observed

Constituents of matter and fundamental forces



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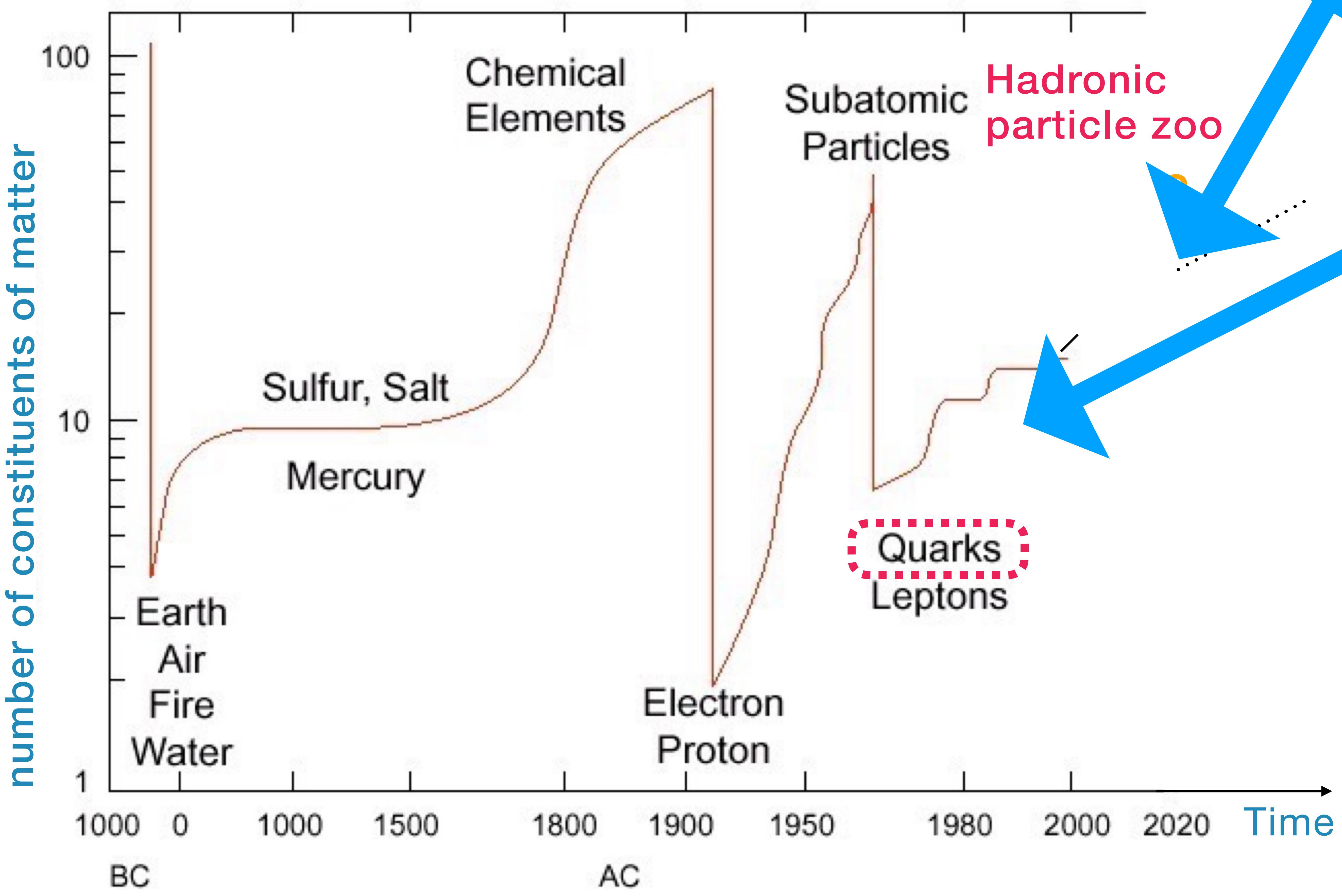


Beyond the standard model of particle physics

Beyond the standard quark model

With the XYZ exotic states discovery, states observed in the sector with two heavy quarks

THE ECONOMIC TIMES

Scientists at CERN observe three "exotic" particles for first time

HARD SCIENCE — JULY 16, 2022

Tetraquarks and pentaquarks: "Unnatural" forms of exotic matter have been found

Scientists have found three new examples of a very exotic form of matter made of quarks. They can yield insights into the early Universe.

JARY 26, 2016 | 3 MIN READ

ysicists May Have Discovered a New "Tetraquark" Particle

Data from the DZero experiment shows evidence of a particle containing four different types of qu

WIRED

'Impossible' Particle Adds a Piece to the Strong Force Puzzle

The unexpected discovery of the double-charm tetraquark gives physicists fresh insight into the strongest of nature's fundamental forces.

INDIA TODAY

Mysterious 'X' particles that formed moments after the big bang found in Large Hadron Collider

CORRIERE DELLA SERA

Nuova straordinaria particella scoperta al Cern: il pentaquark

Consentirà di saperne di più sulla «forza forte» che tiene unite le particelle nel nucleo ... i e sui componenti della materia

BBC

Pentaquarks: scientists find new "exotic" configurations of quarks

Scientists have found new ways in which quarks, the tiniest particles known to humankind, group together.

of unknown nature discovered ... sector with tw

Le Monde

Les surprises du tétraquark, « collage » de particules élémentaires

La découverte d'une nouvelle particule à la structure particulièrement stable pourrait permettre aux chercheurs de vérifier leurs théories sur l'interaction forte.

ZEIT ONLINE

Cern-Forscher entdecken neues Teilchen

Die Physiker am Kernforschungszentrum in Genf haben die Existenz des Pentaquark-Teilchens nachgewiesen. Bislang war es nur in theoretischen ... beschrieben worden.

LHCb discovers longest-lived exotic matter yet

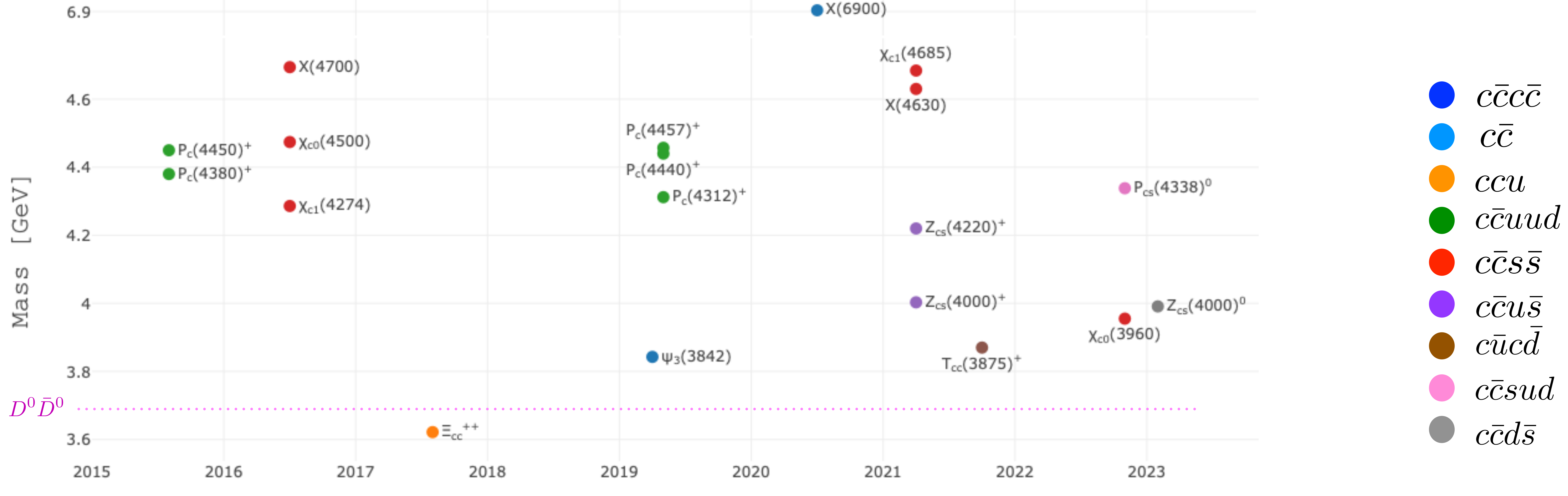
08/04/21 | By Sarah Charley

The newly discovered tetraquark provides a unique window into the interactions of the particles that make up atoms.

symmetry

The present revolutions

XYZ Revolution: Discoveries of Exotic States in the Sector With Two Heavy Quarks: Beyond the Standard Quark Model!



Date of arXiv submission

https://qwg.ph.nat.tum.de/exoticshub/

INTERPLAY AMONG MANY EXPERIMENTS:

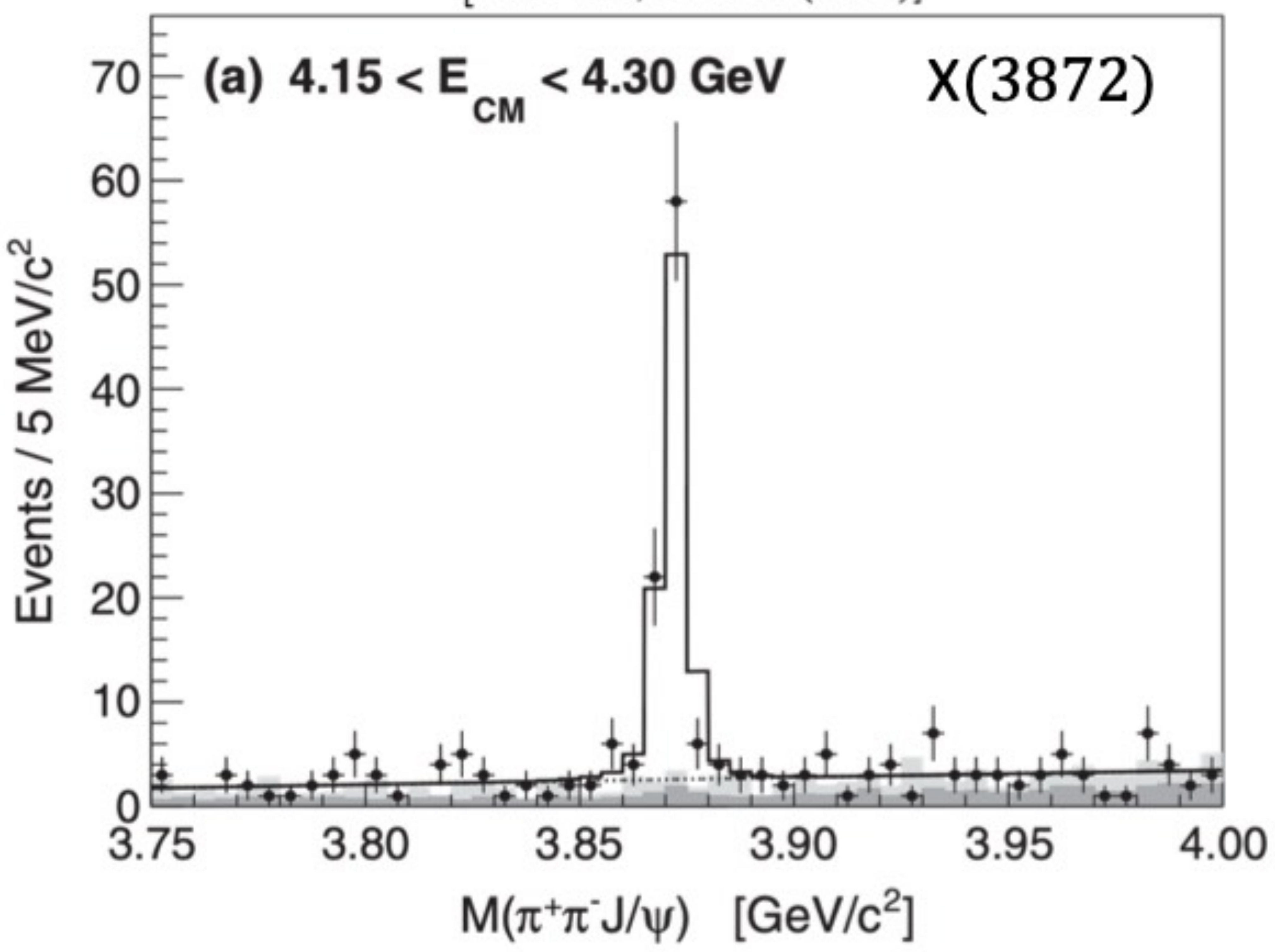
UPCOMING EXPERIMENTS: STCF, Electron ion

X(3872) aka $\chi_{c1}(3872)$

Some surprisingly narrow states even if above/at strong decay thresholds

$e^+e^- \to \gamma X(3872);\quad X(3872) \to \pi^+\pi^- J/\psi$

[PRL 122, 232002 (2019)]



$$J^{PC} = 1^{++} \qquad I = 0$$

Observed in e+e-, B decays, hadroproduction (large cross section 30nb)

Compositeness, radiative decays, production suggest the presence of a compact component

$$M_{X(3872)} - M_{D^0 D^{*0}} = 0.01 \pm 0.14 MeV$$

<-within 100 KeV of the threshold (molecule?)

width of 1 MeV! very small binding energy

XYZ REVOLUTION: A New Spectroscopy Is Born!

X(3872) aka

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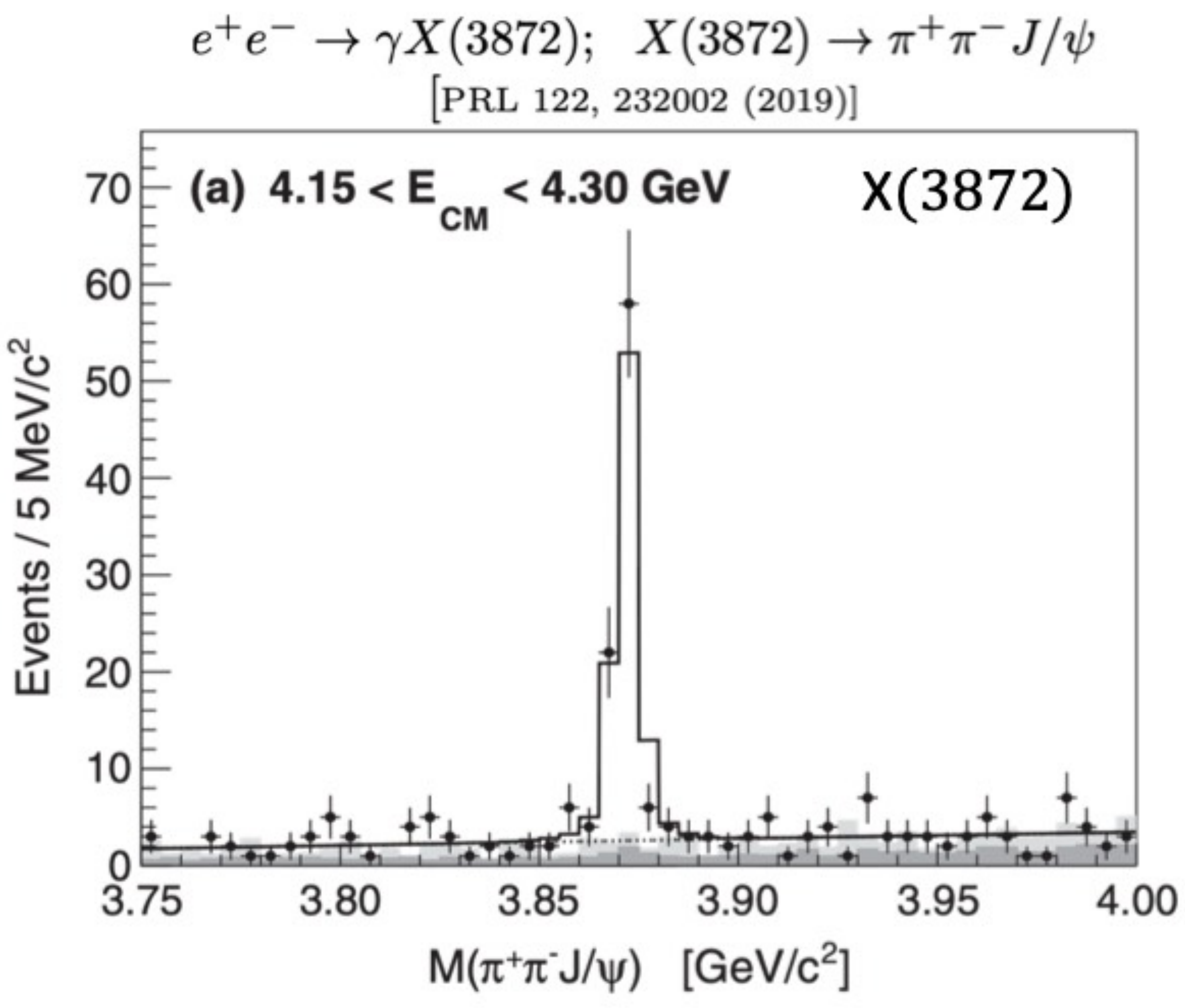

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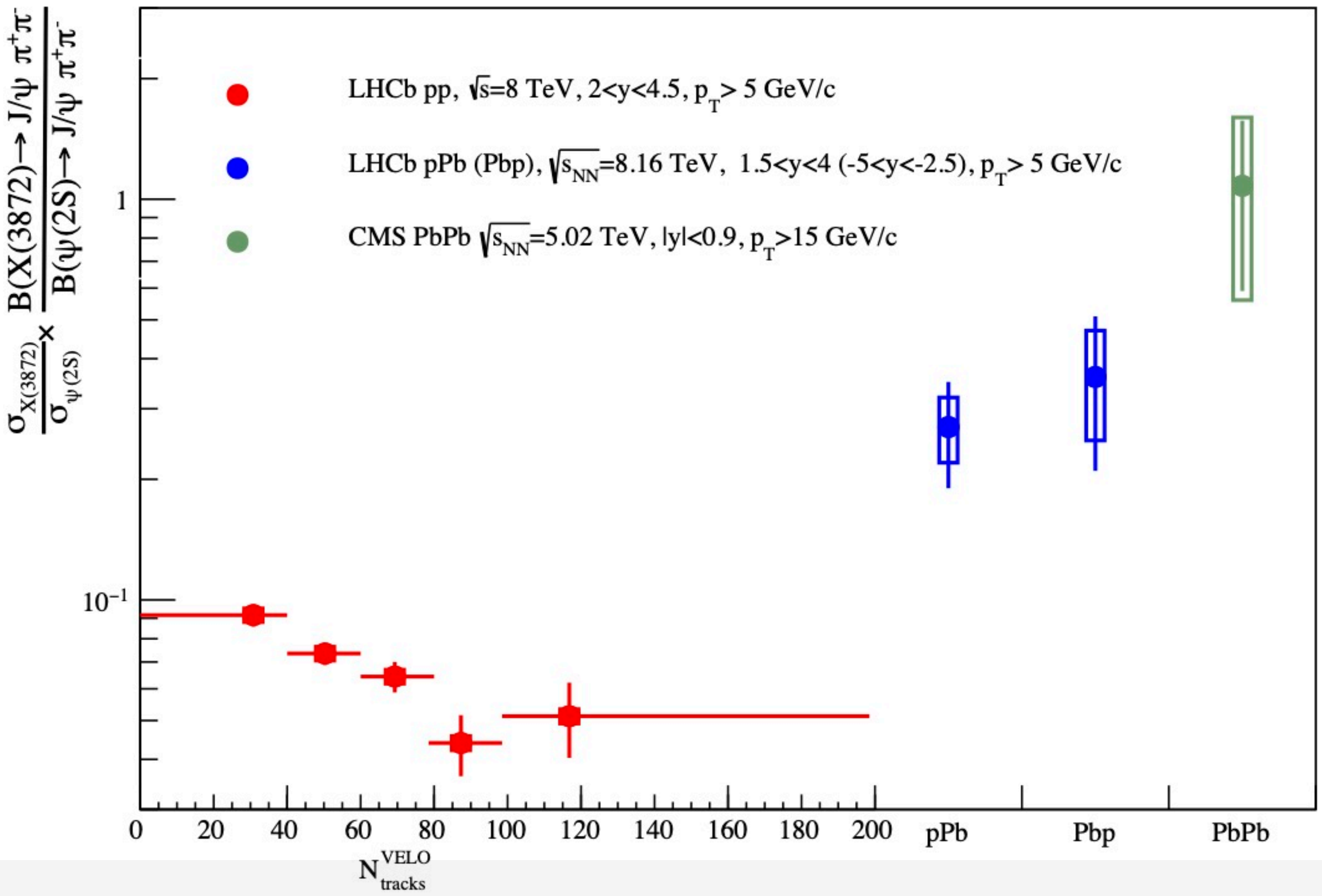
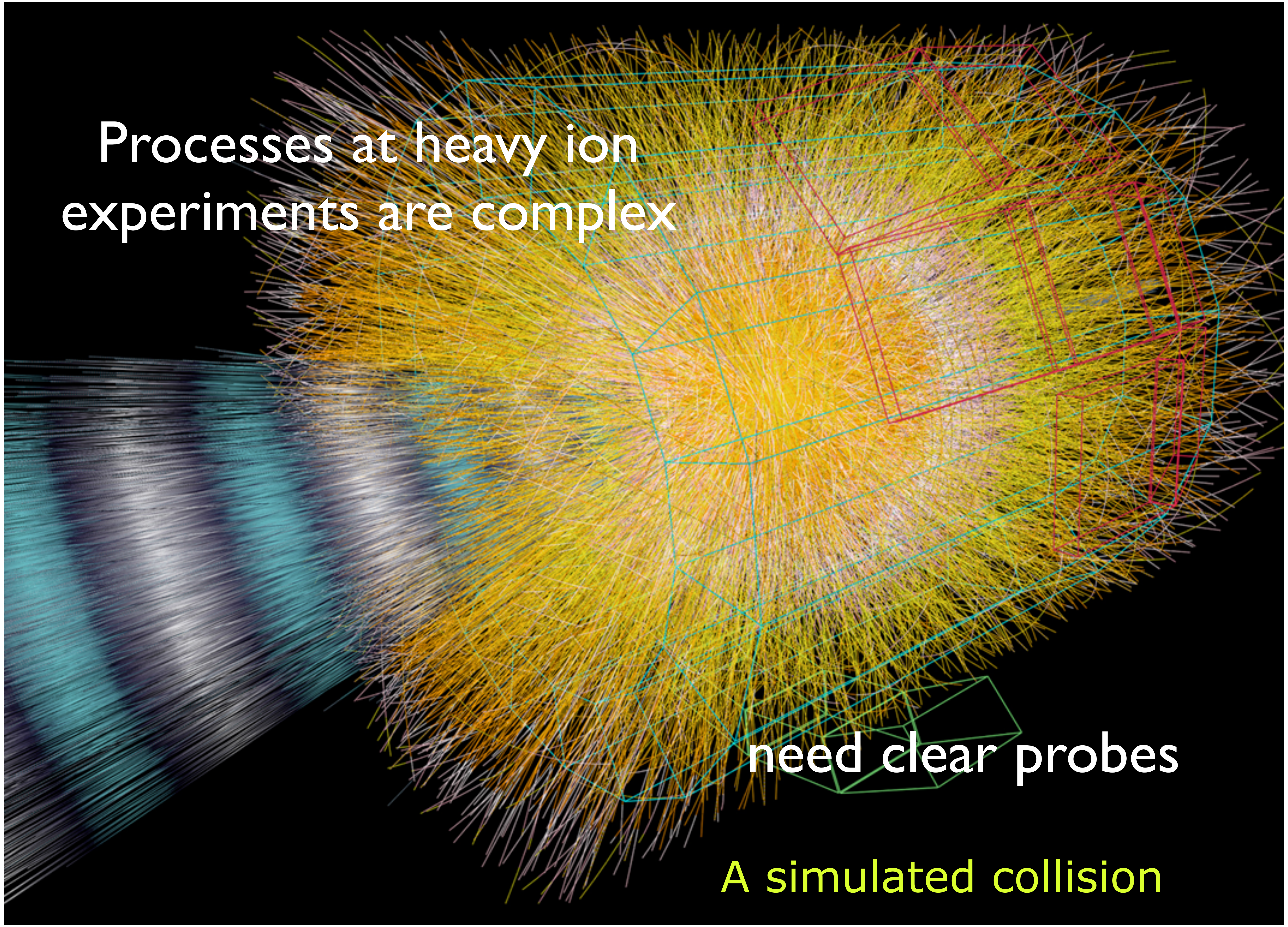
Produced in heavy ions where the deconfined strongly coupled QCD medium (Quark Gluon Plasma-QGP) is formed



New perspectives for XYZ studies!

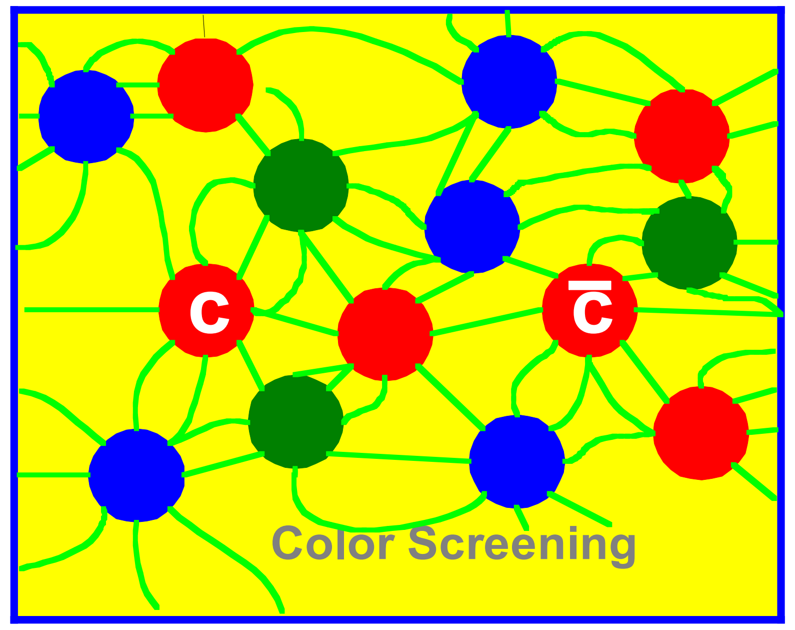
investigated in heavy ions collision at the LHC at CERN and RHIC USA (5.36 TeV per nucleon pair)



The present revolutions: nuclear matter phase diagram

Quarkonia are probe of QGP formation

Matsui Satz 1986 idea **of color screening in medium**

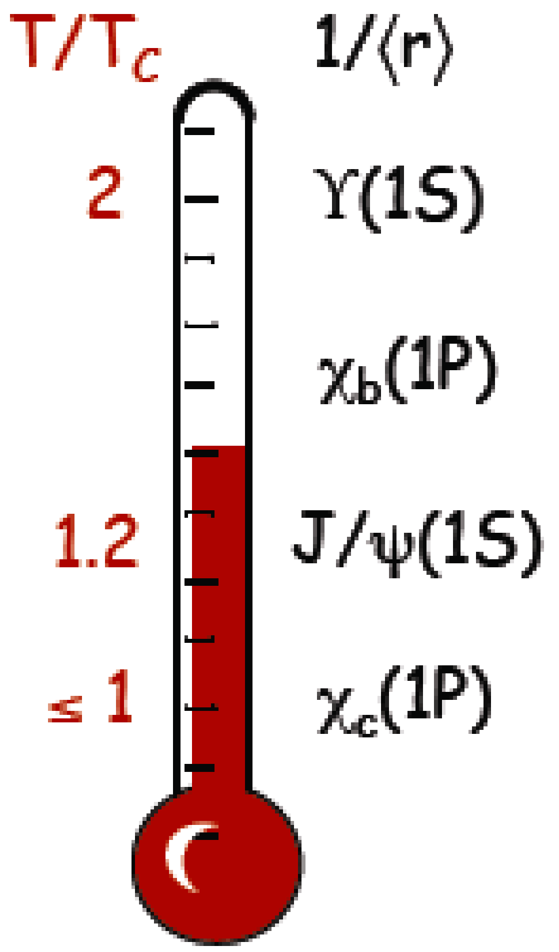


Debye charge screening

$$V(r) \sim -\alpha_s \frac{e^{-m_D r}}{r}$$

$$m_D \sim gT$$

$$r \sim \frac{1}{m_D}$$ → Bound state dissolve



Sequential Melting at different Temperature

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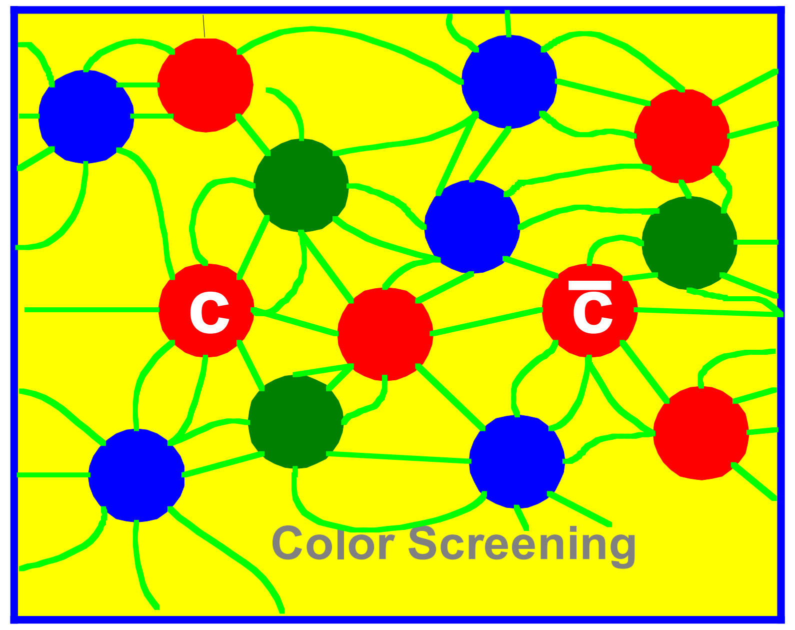

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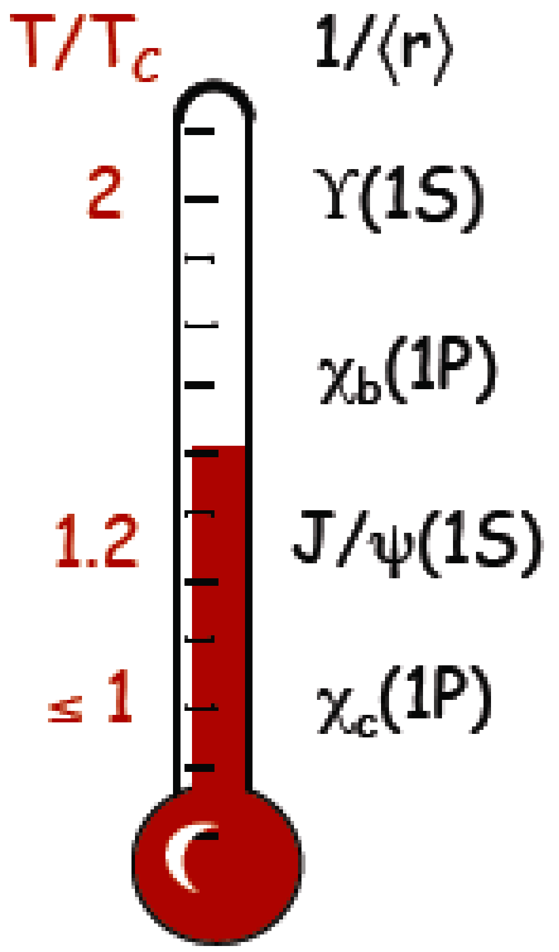

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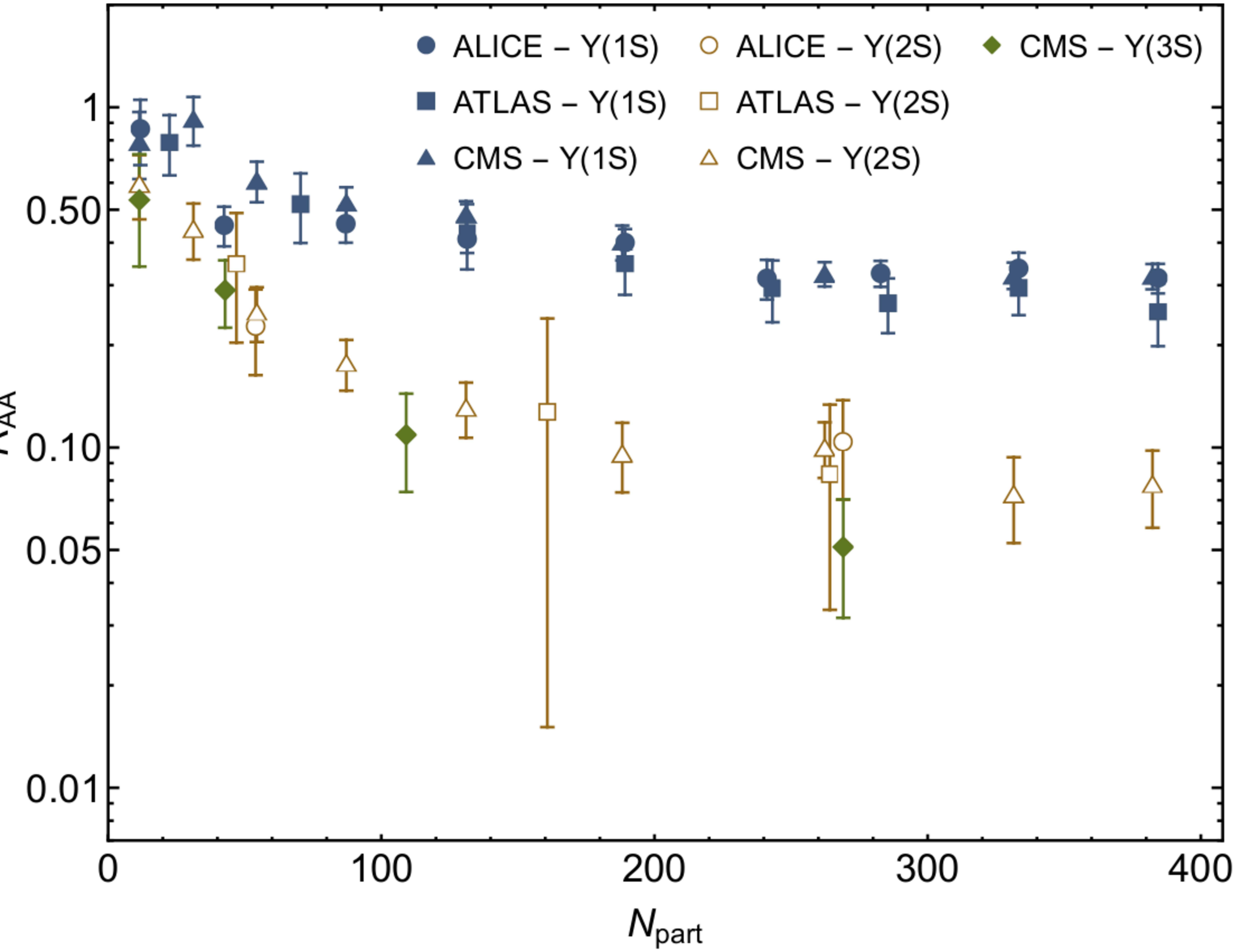
Experimental measurements:

R_{AA} is the nuclear modification factor = yield of quarkonium in PbPb / yield in pp.



CMS PLB 790 (2019) 270

ALICE PLB 822 (2021) 136579

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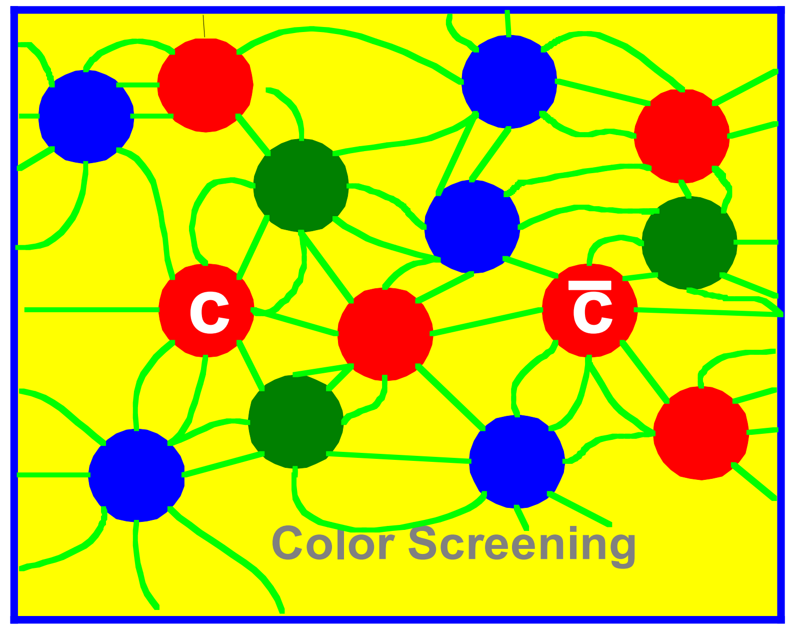

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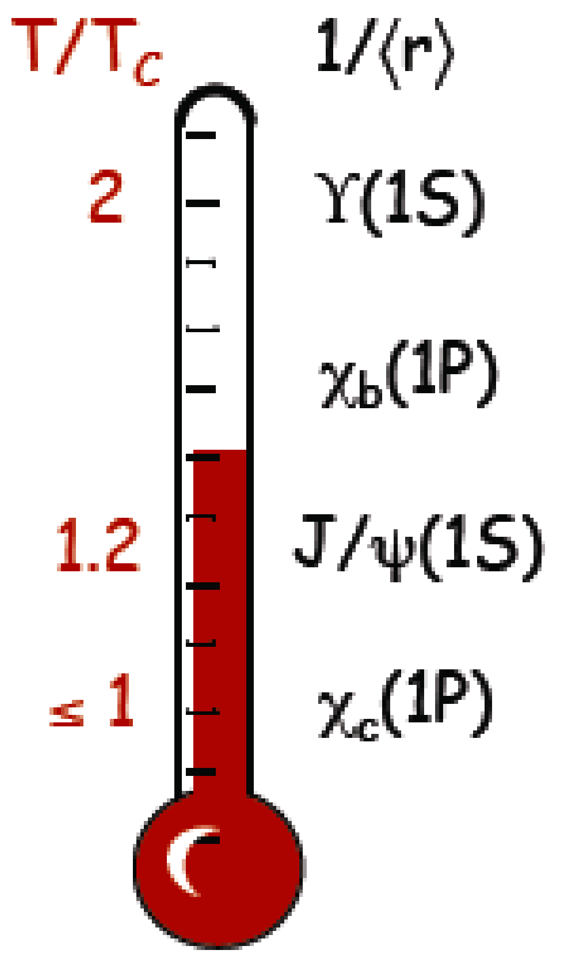

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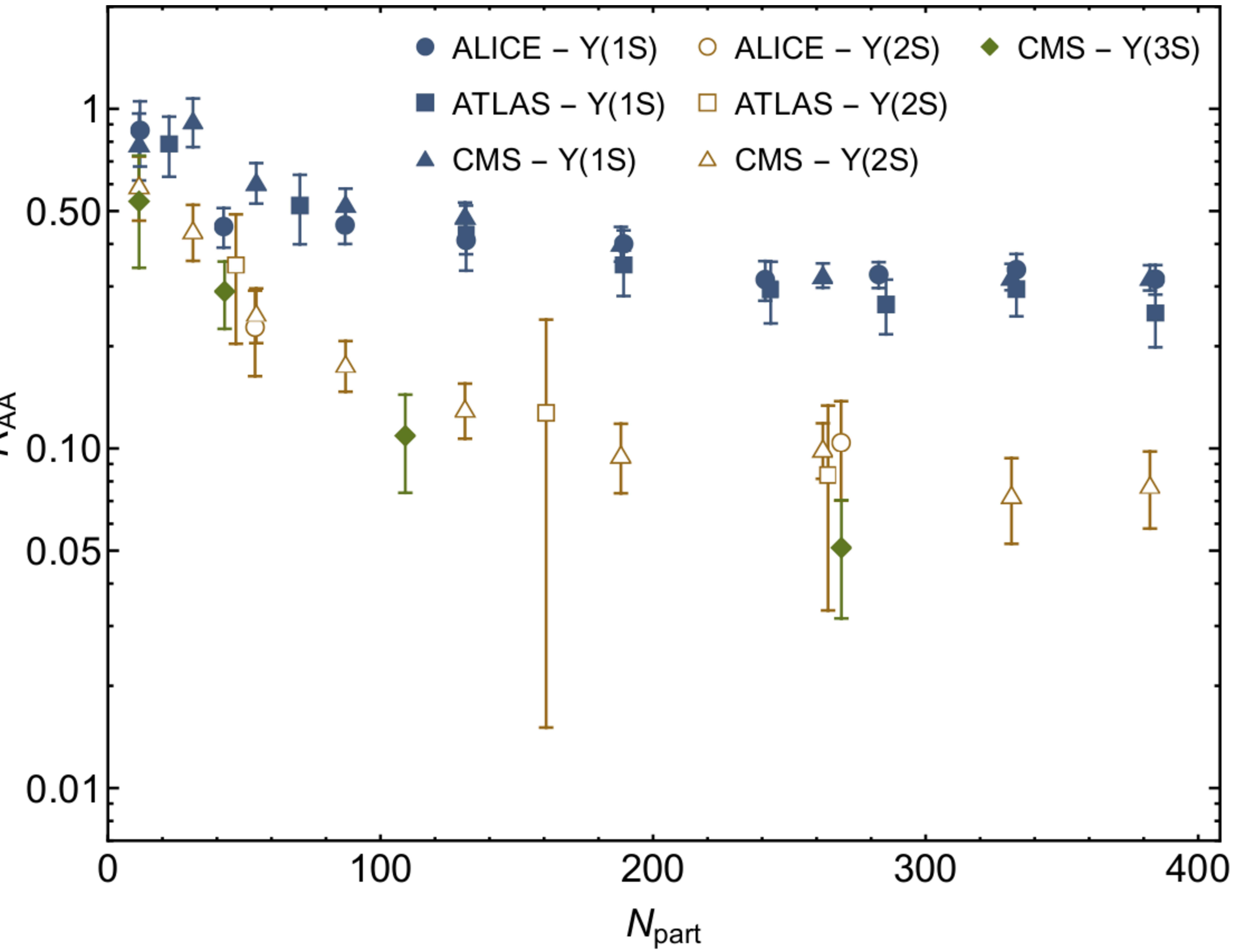

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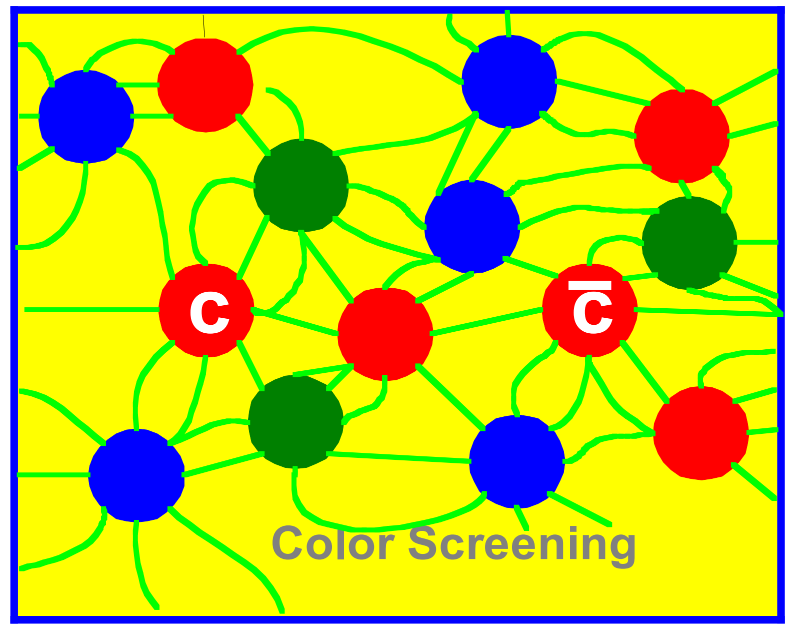

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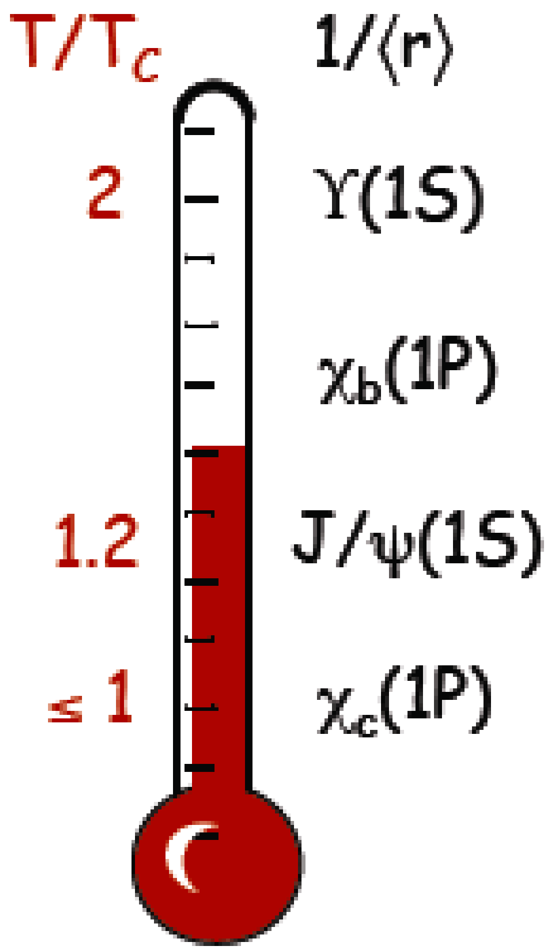

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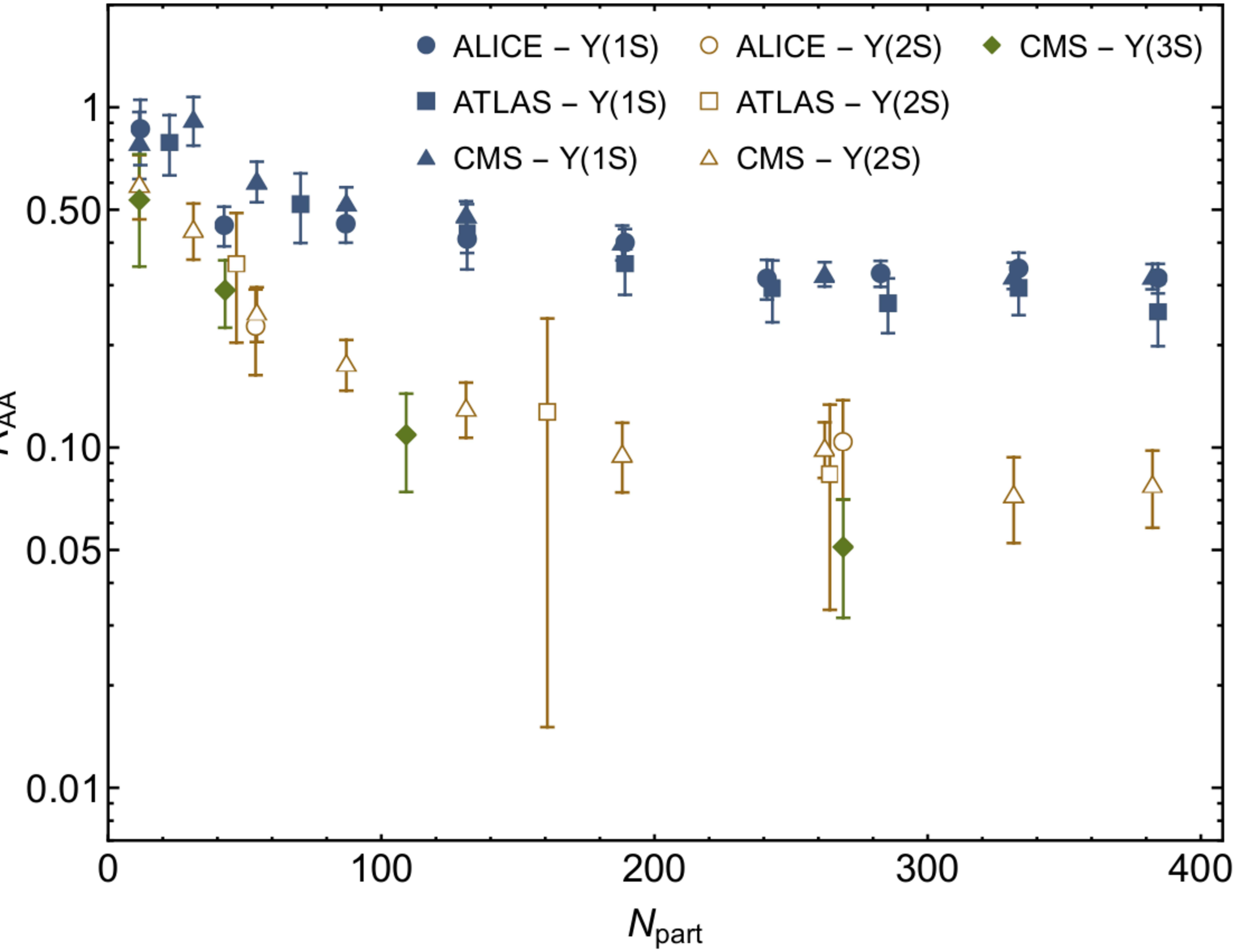

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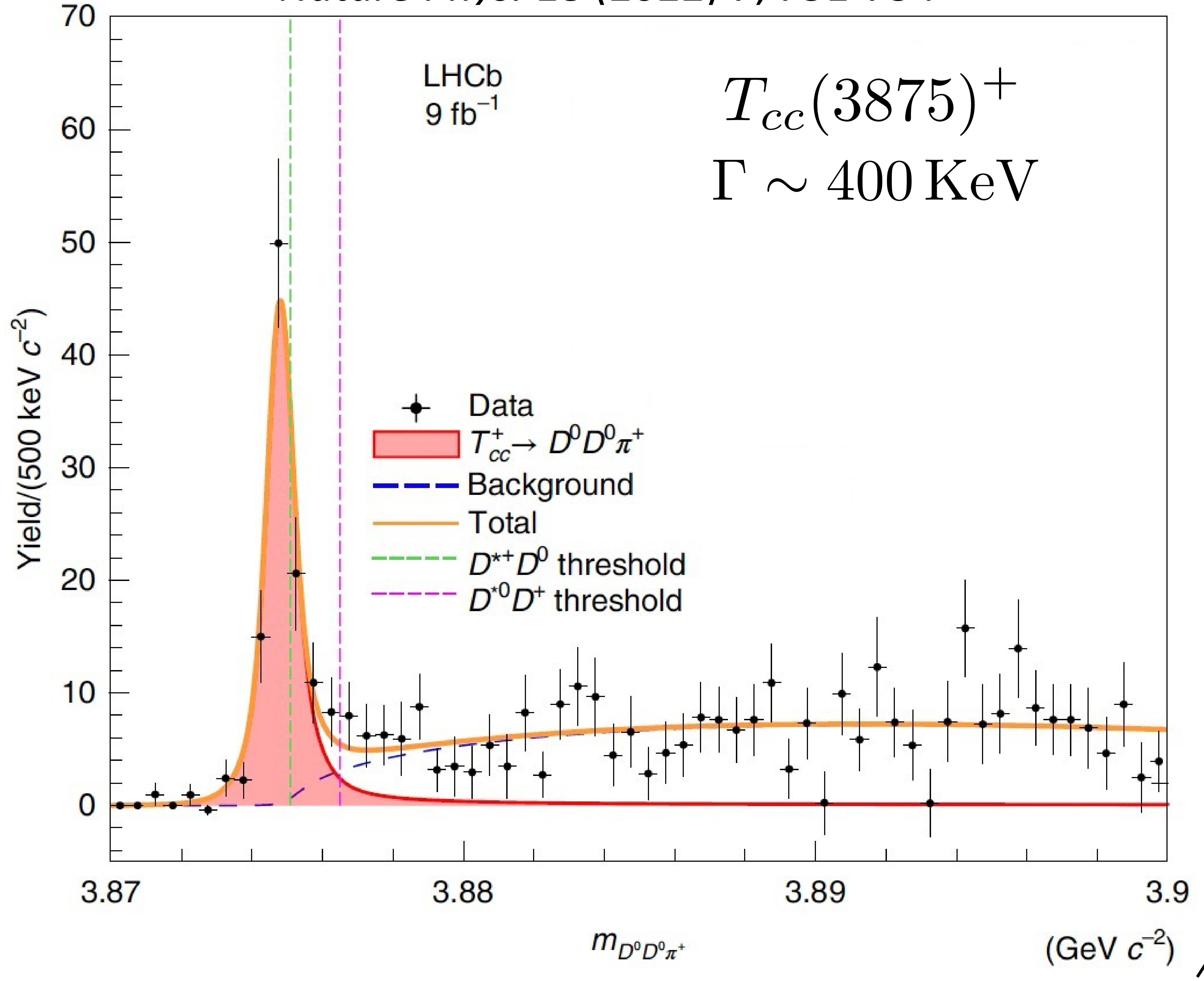
XYZ states are also produced and evolve in heavy ion collisions and should be studied with similar methods

$T_{cc}^{+}(3875)$

The longest lived exotic matter ever found!

Nature Phys. 18 (2022) 7, 751-754



$$J^P = 1^+ \qquad I = 0$$

<-within 300 KeV of the threshold (molecule?)

<-width of 48 KeV!

$$M_{T_{cc}(3875)^+} - (M_{D^{*+}} + M_{D^0}) = -0.27 \pm 0.06 MeV$$

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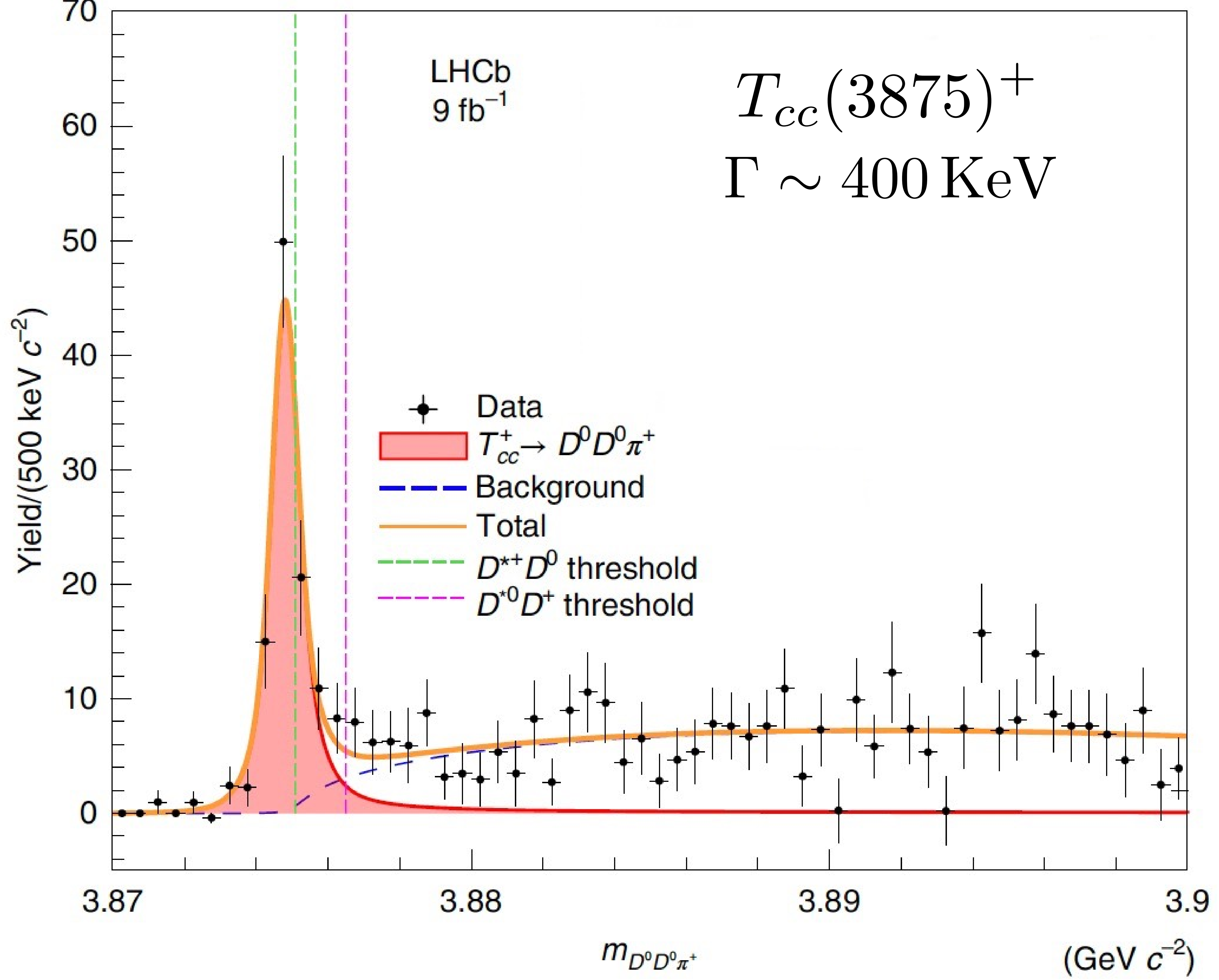

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XYZs not merely composite particles, have unique properties can give us information about the strong force

—>Novel strongly correlated exotics systems

The present revolution: XYZ a great theoretical challenge

Close/above threshold new degrees of freedom like glue and light quarks are nonperturbative part in the binding.

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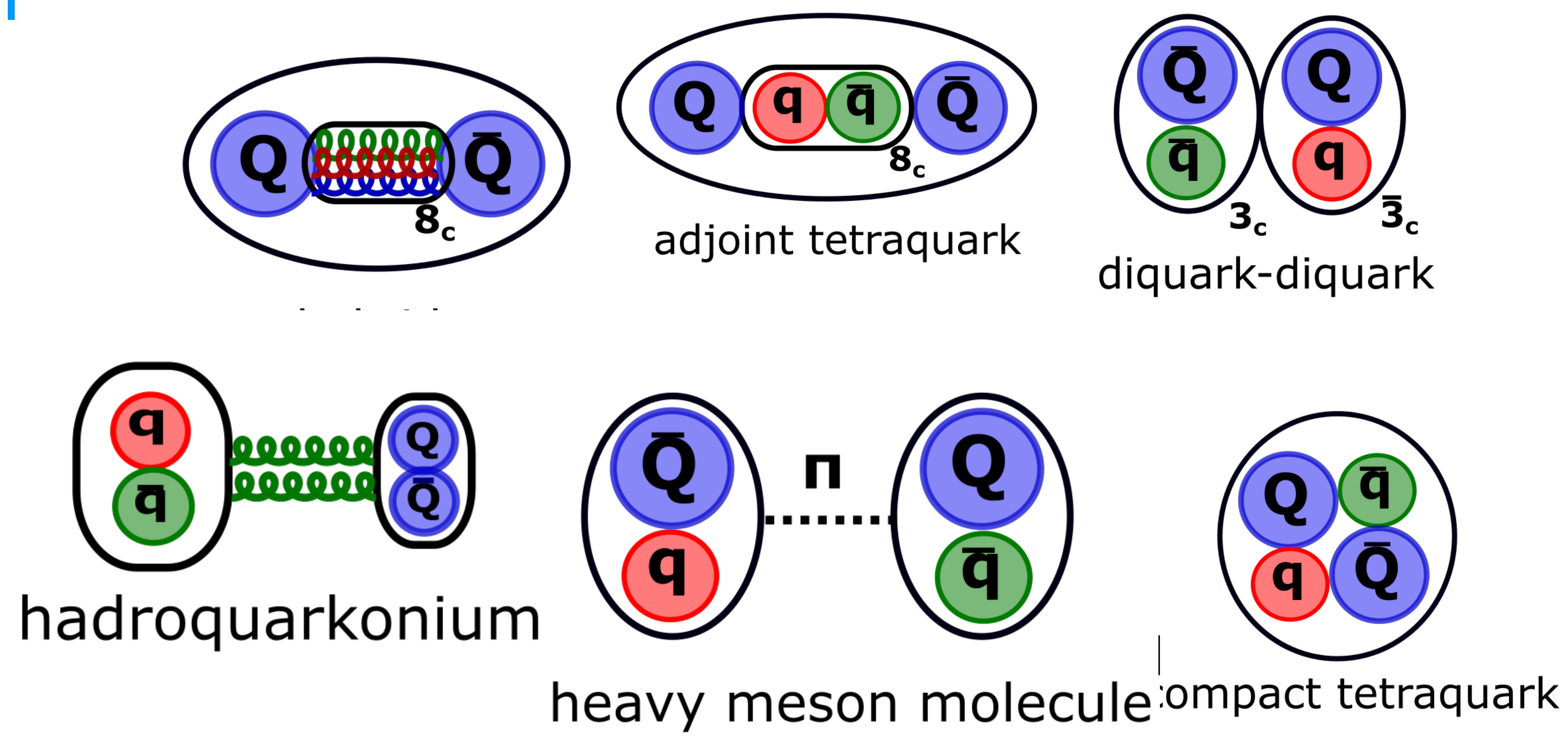
- **Models** assume some special degrees of freedom and a model interaction



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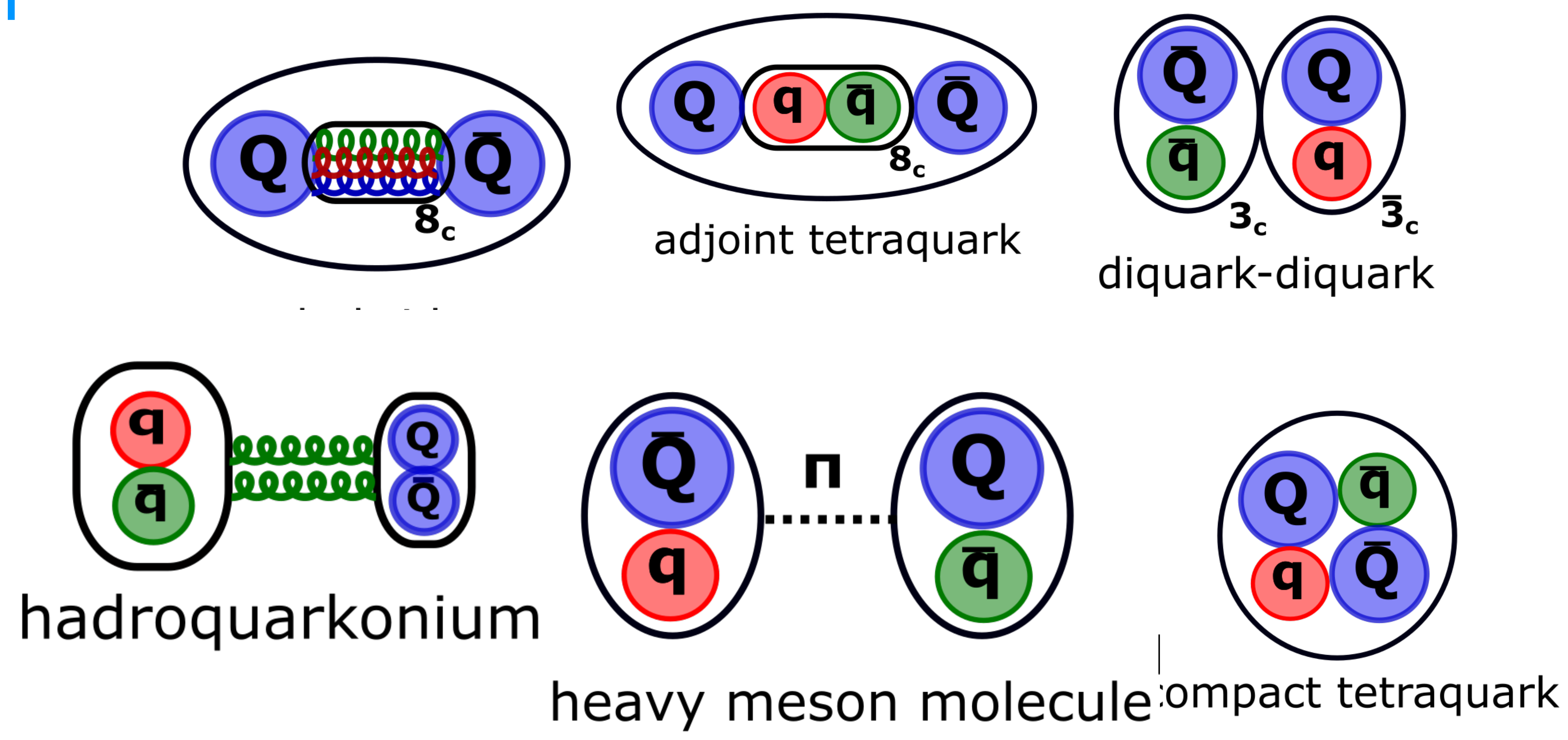

- On the nature of the X(3872) two models in particular compete: molecule versus tetraquark

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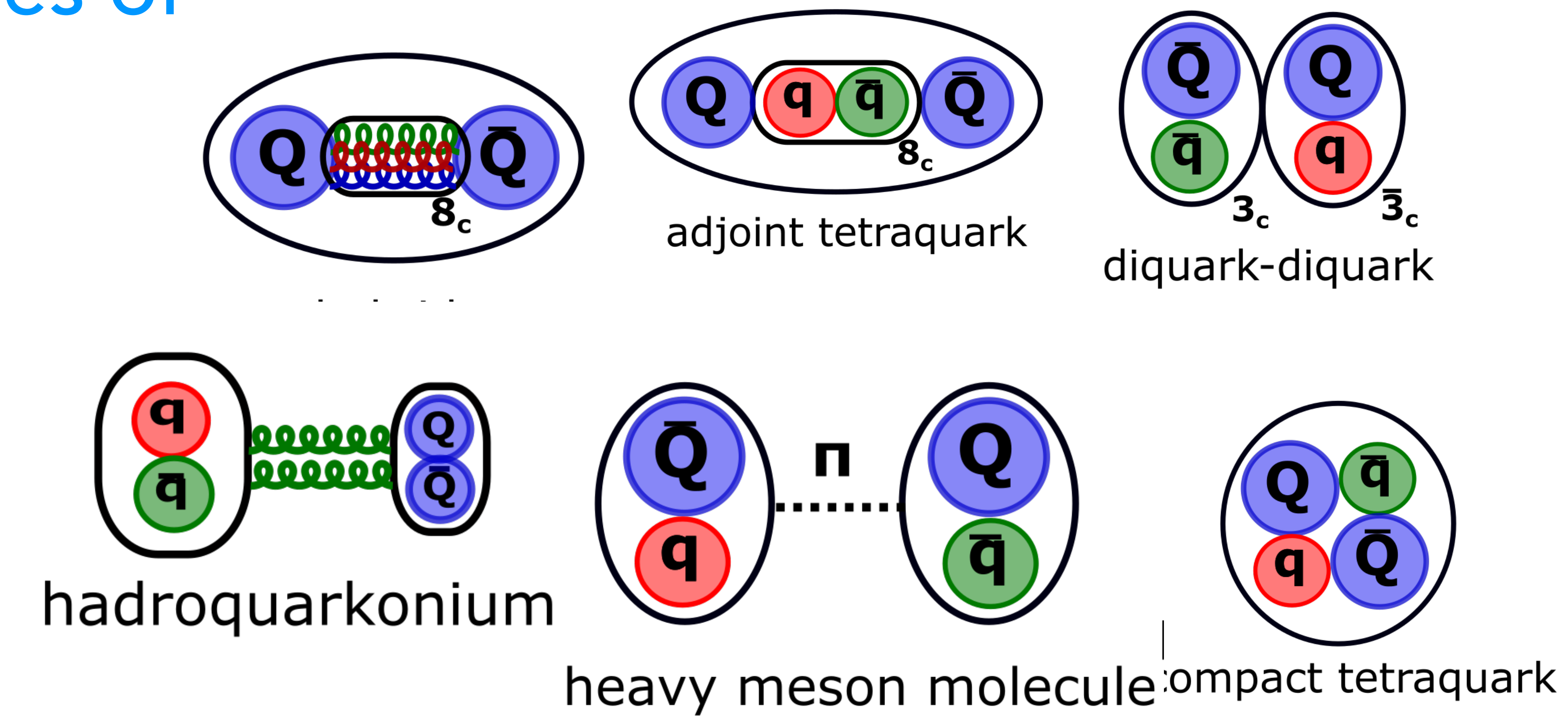
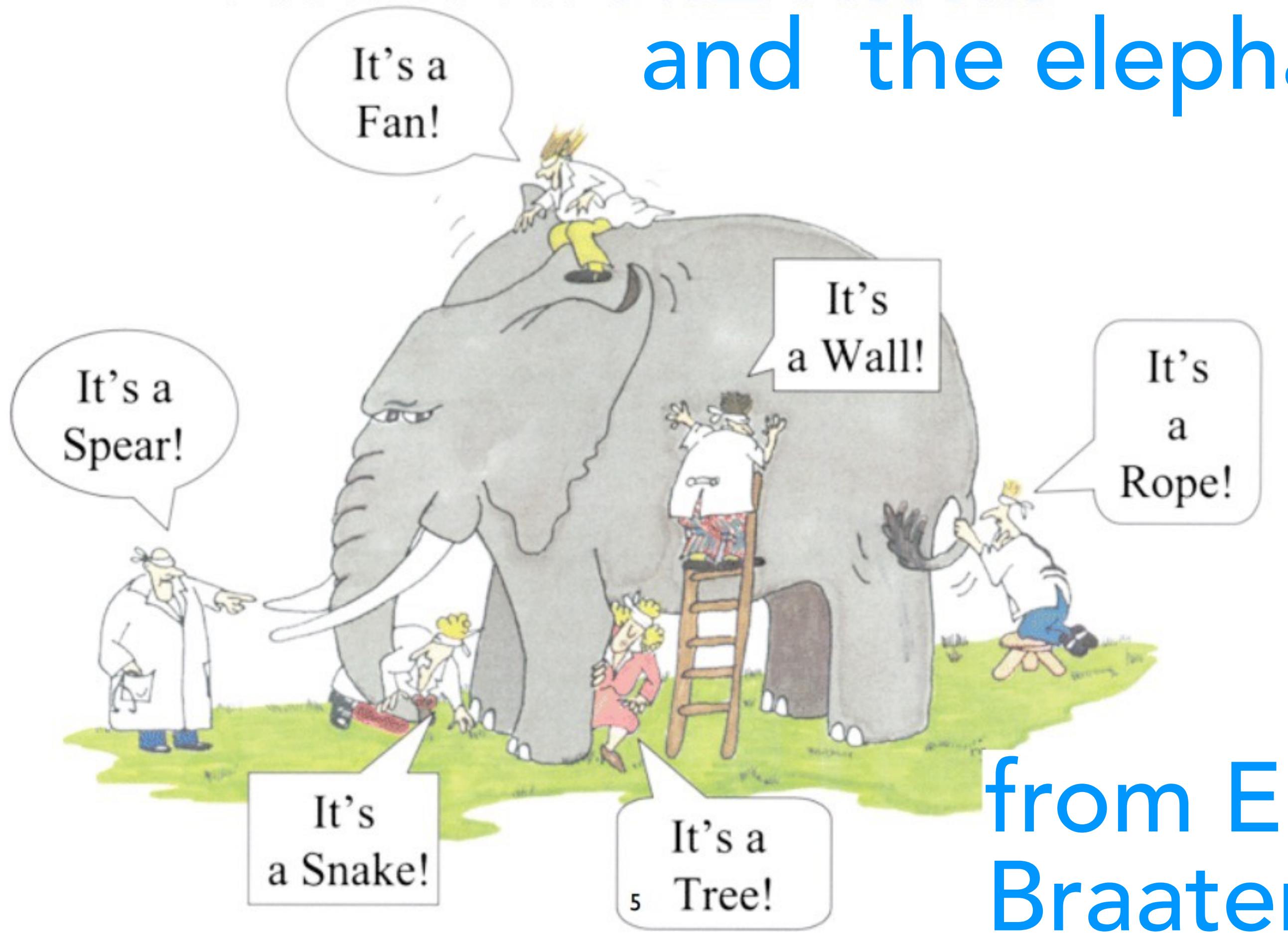


the blind men and the elephant



from E. Braaten

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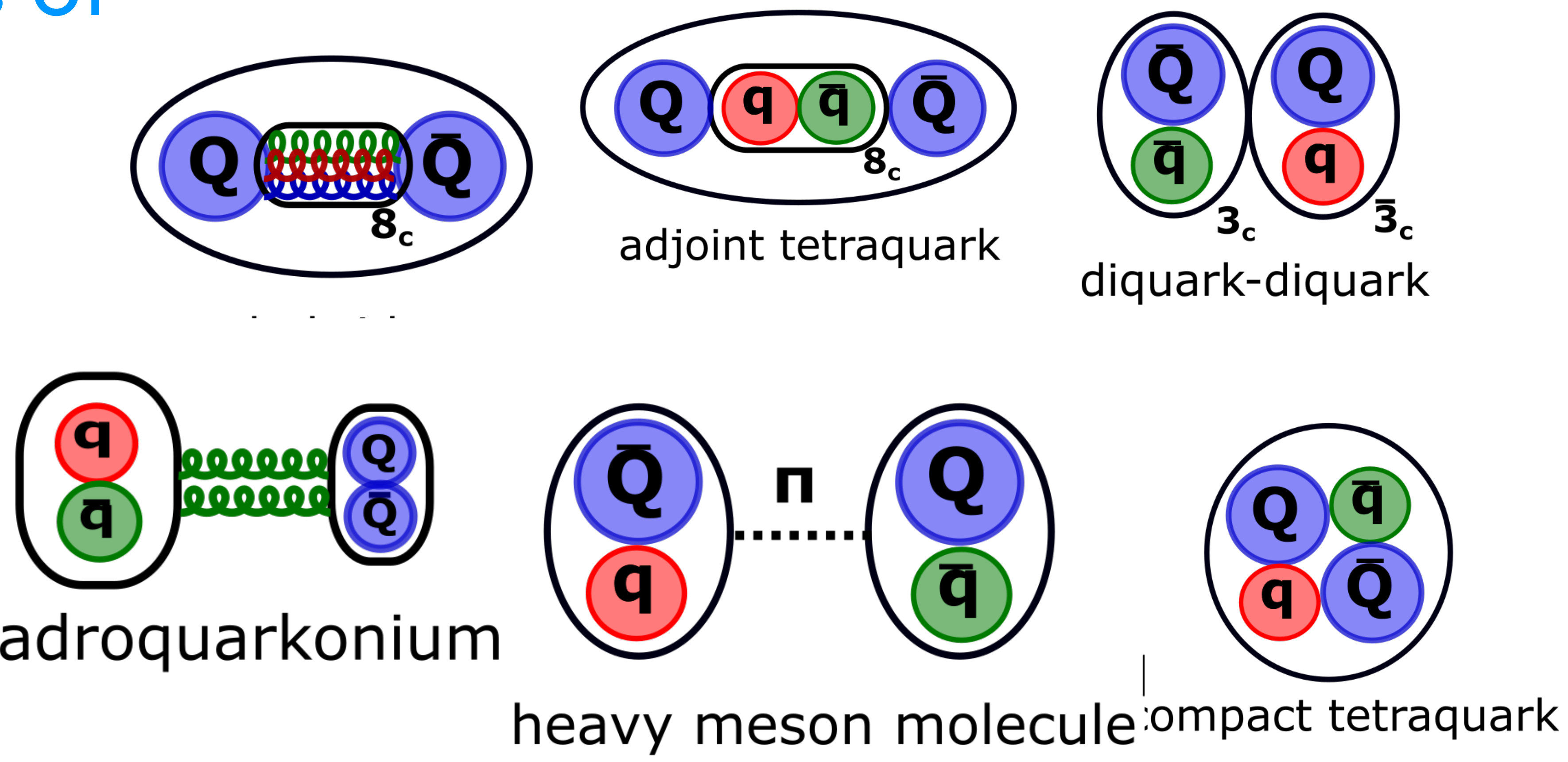

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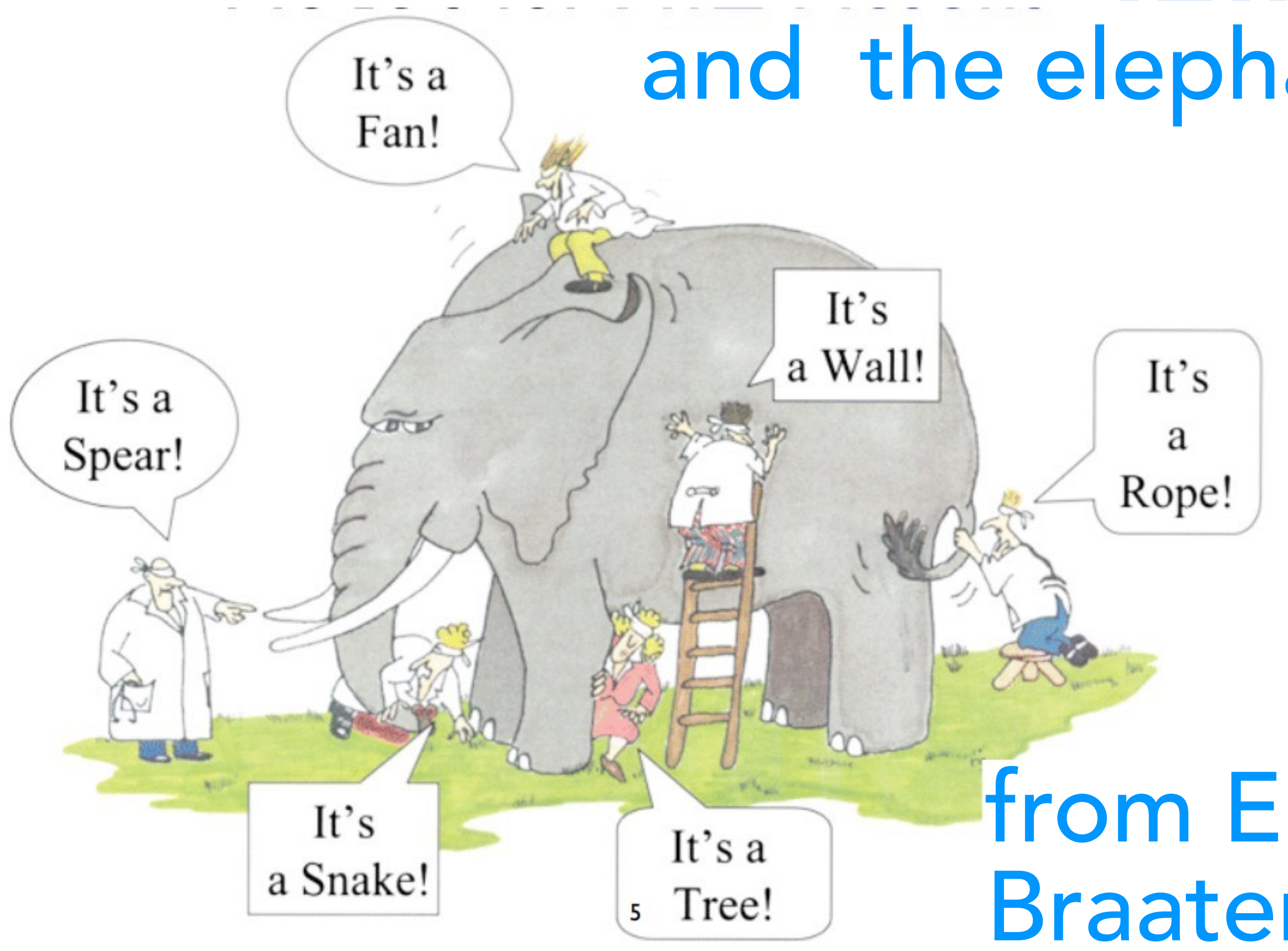

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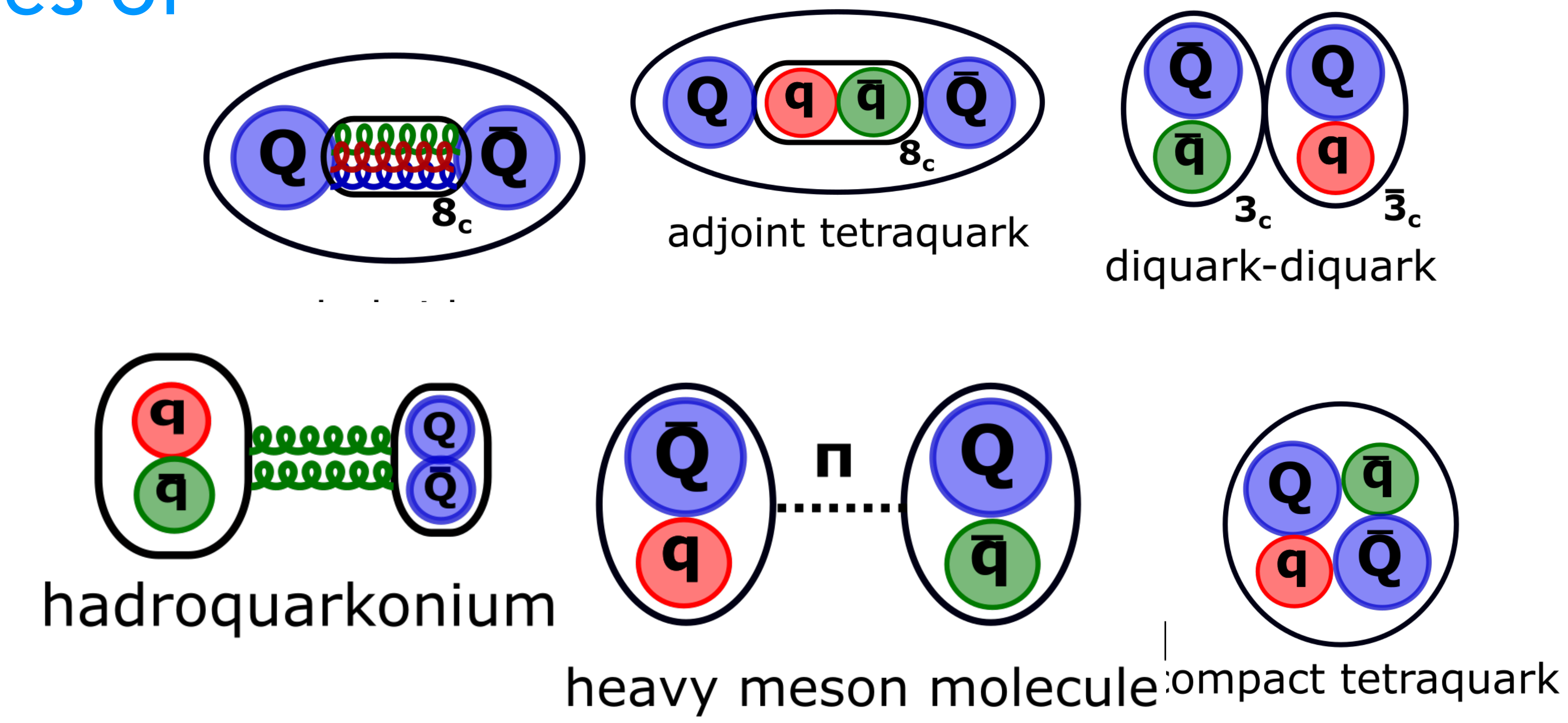

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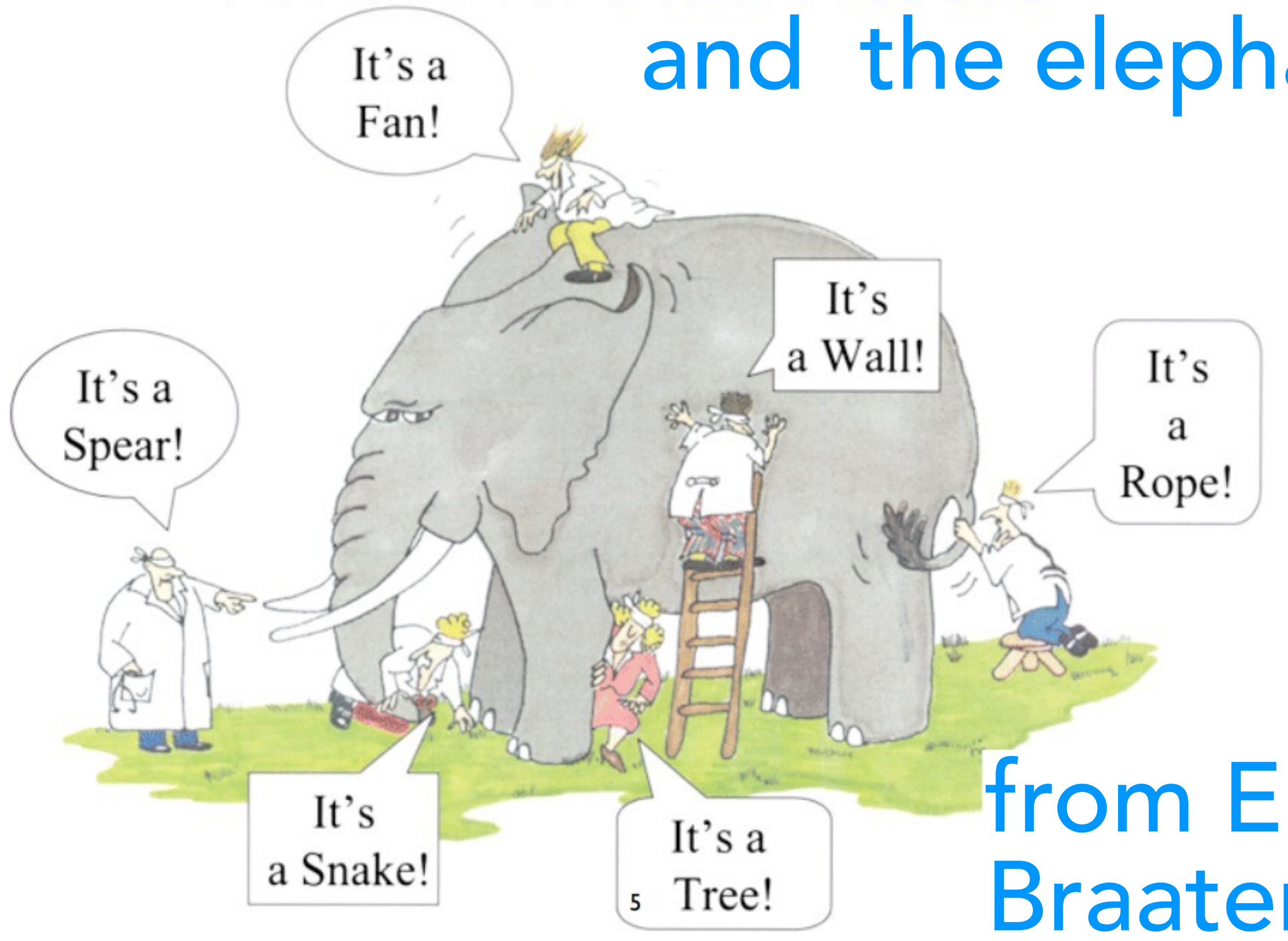

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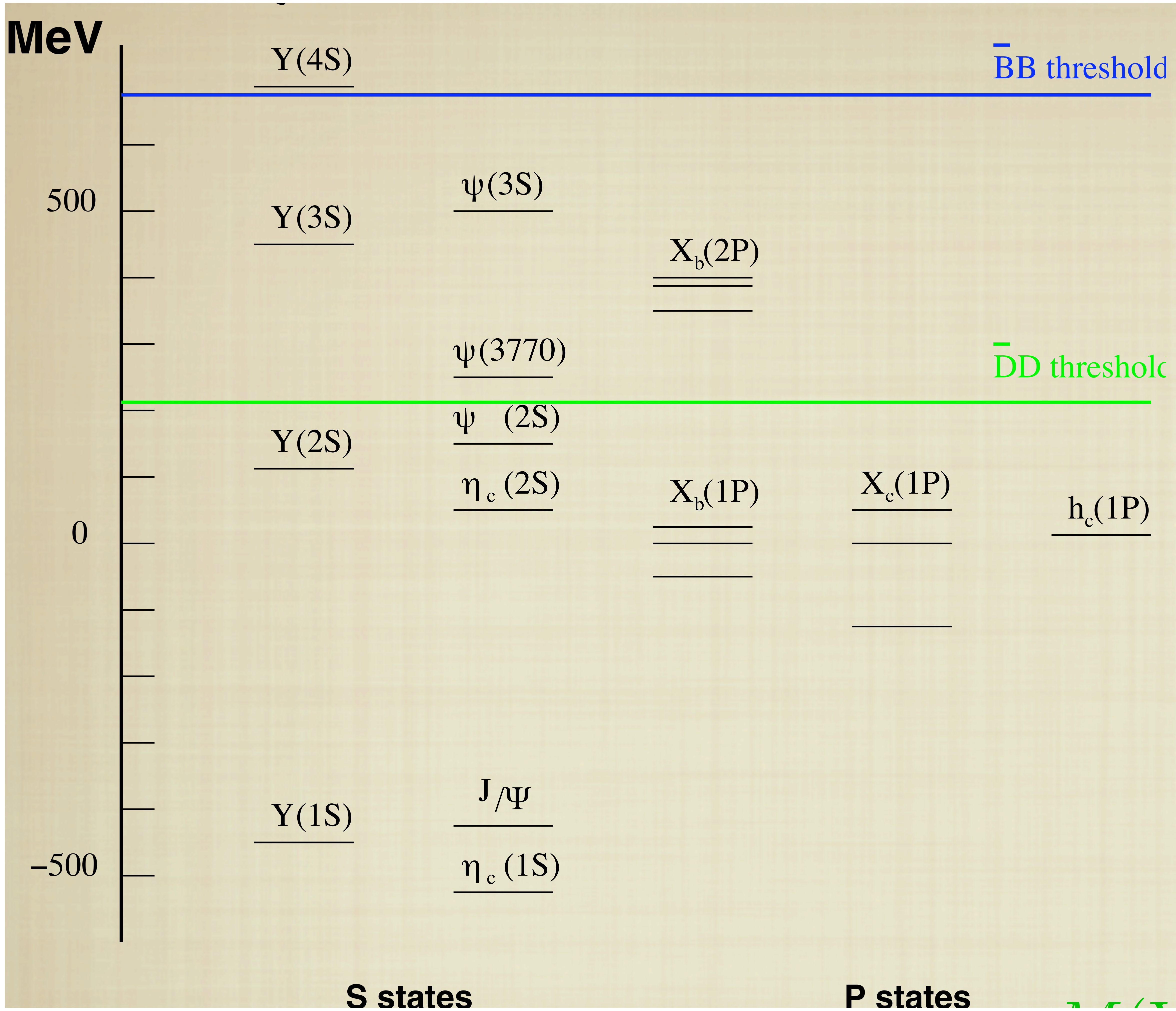
A flexible approach rooted in QCD that can address all properties of XYZ, spectra, transitions, production, propagation in medium is needed allowing also to study the nature of the QCD force

Nonrelativistic EFTs simplify the problem for a multiscale system,

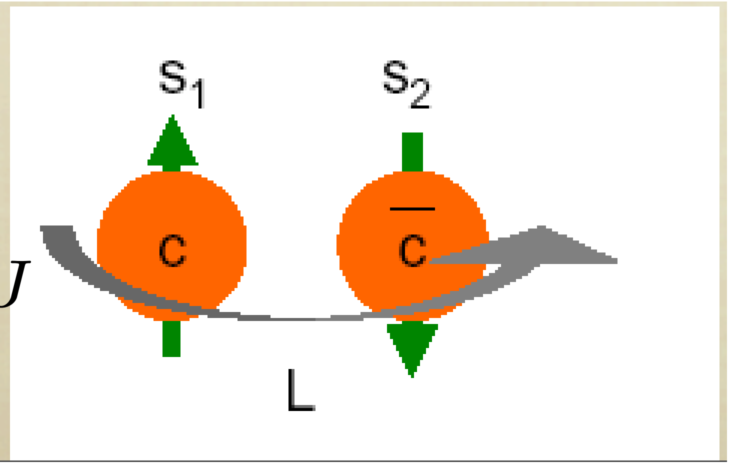
-> make the expansion in the scales explicit at the Lagrangian level

Quarkonium scales



Levels normalized with respect to $\chi_b(1P)$ and $\chi_c(1P)$

$^{2S+1}L_J$



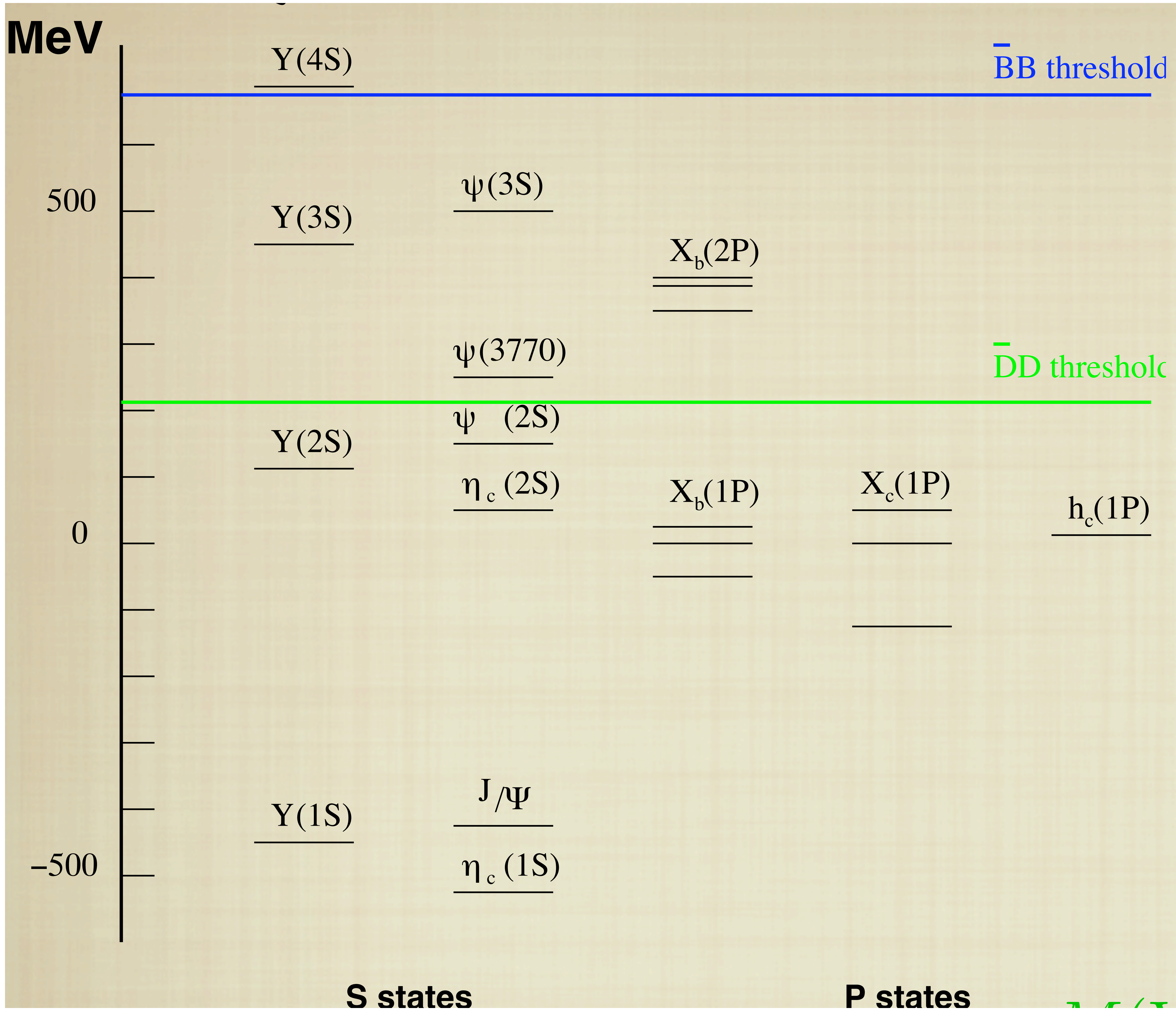
$$M(\Upsilon(1S)) = 9.460\,\mathrm{GeV}$$
$$M(J/\psi) = 3.097\,\mathrm{GeV}$$

The mass scale is perturbative

$$m_Q \gg \Lambda_{\mathrm{QCD}}$$

$$m_b \simeq 5\,\mathrm{GeV};\, m_c \simeq 1.5\,\mathrm{GeV}$$

Quarkonium scales



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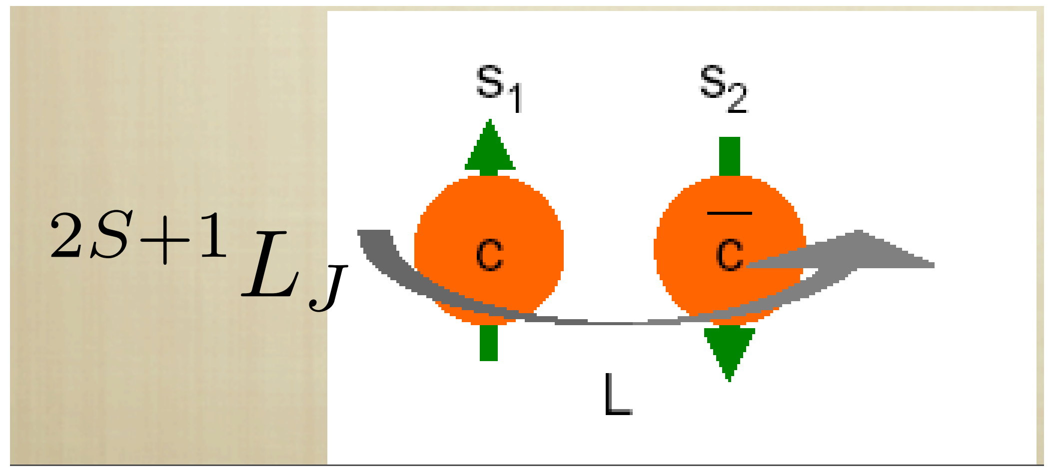

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The system is nonrelativistic (NR)

$$\Delta E \sim mv^2, \Delta_{fs} E \sim mv^4$$

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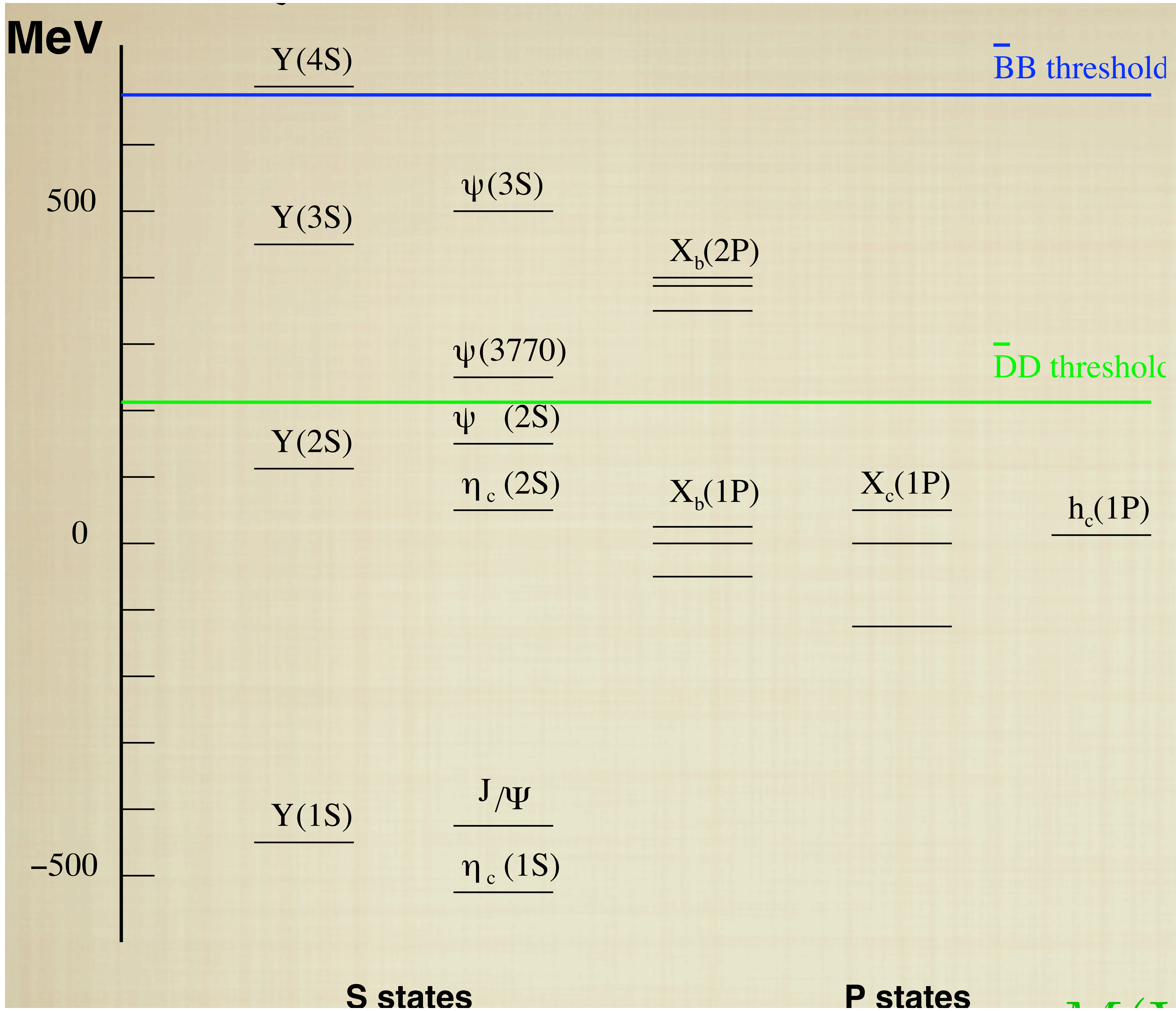

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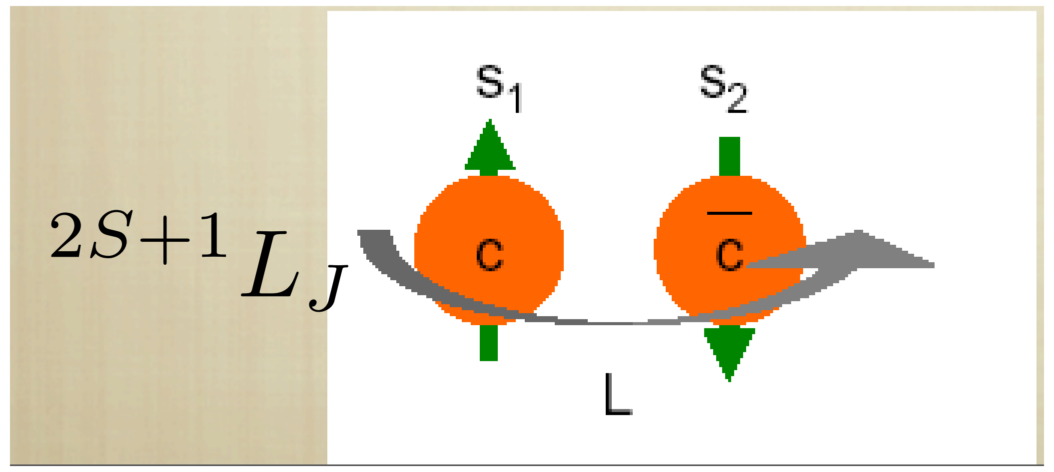

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NR bound states have at least 3 scales

$$m \gg mv \gg mv^2 \qquad v \ll 1$$

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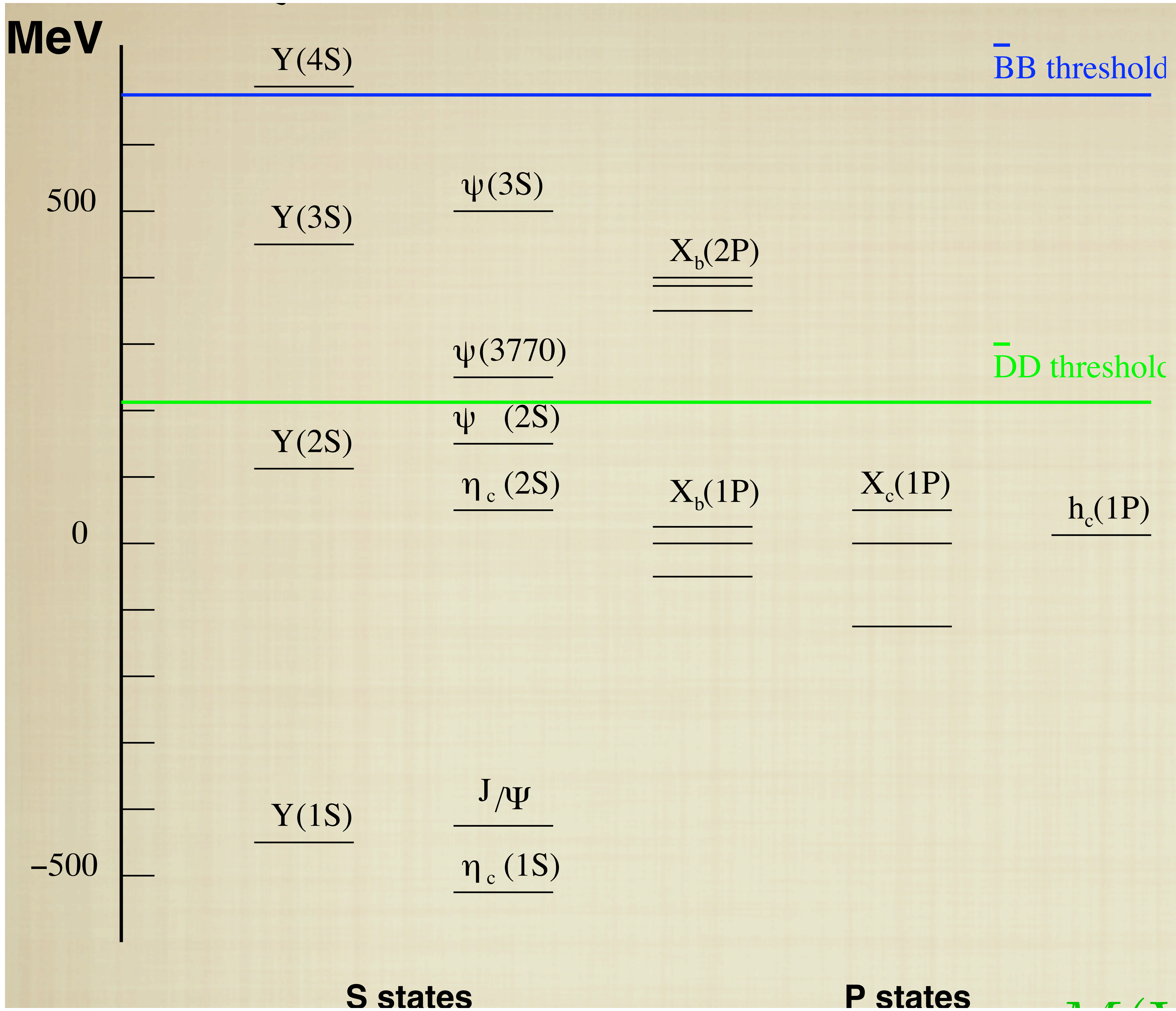

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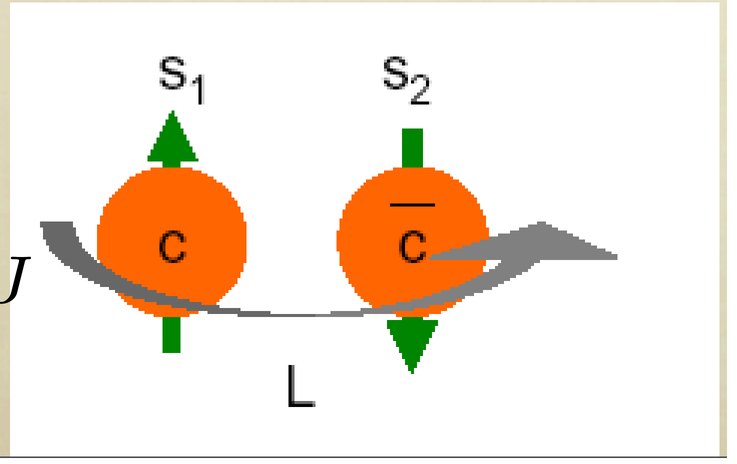

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$$mv \sim r^{-1}$$

The system is nonrelativistic (NR)

$$\Delta E \sim mv^2, \Delta_{fs} E \sim mv^4$$

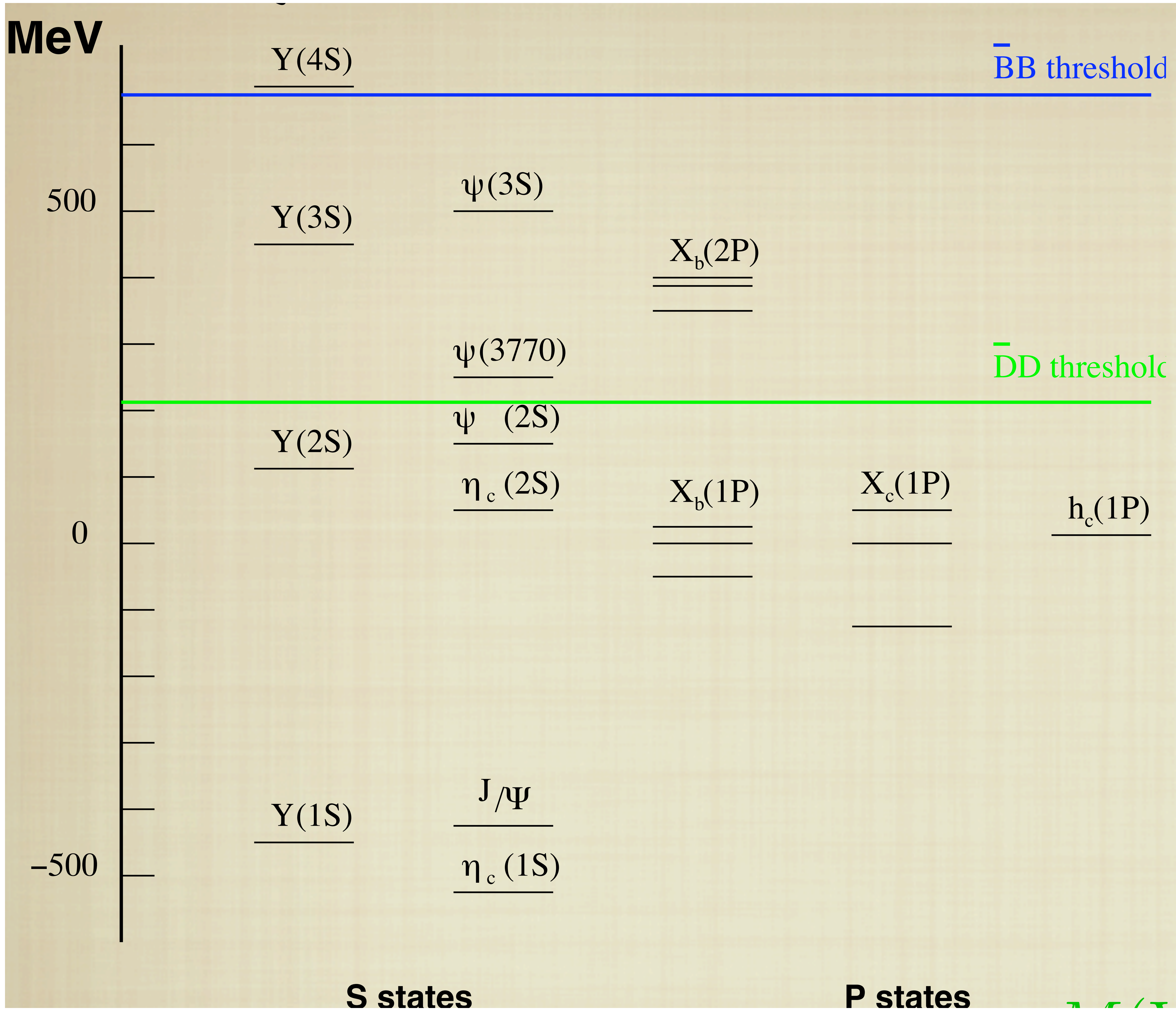
$$v_b^2 \sim 0.1, v_c^2 \sim 0.3$$

The mass scale is perturbative

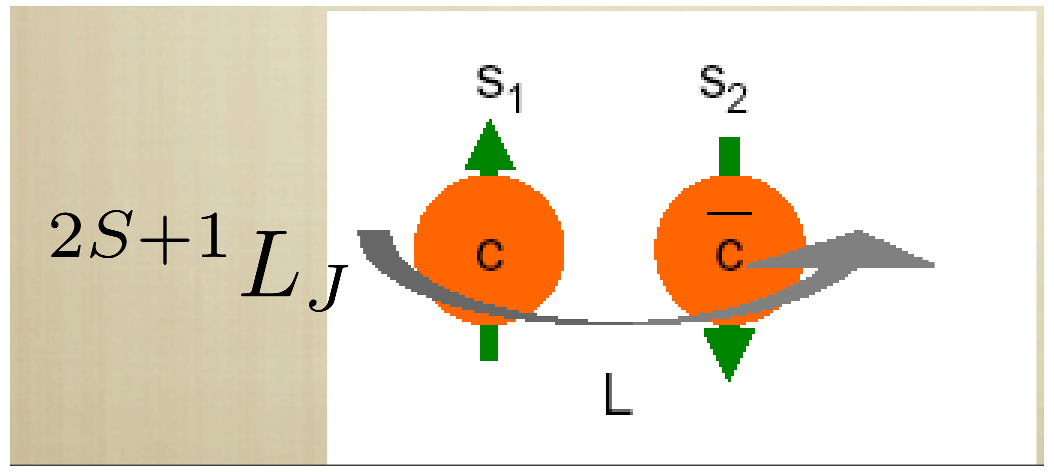
$$m_Q \gg \Lambda_{\mathrm{QCD}}$$

$$m_b \simeq 5\,\mathrm{GeV}; m_c \simeq 1.5\,\mathrm{GeV}$$

Quarkonium scales



Levels normalized with respect to $\chi_b(1P)$ and $\chi_c(1P)$



$M(\Upsilon(1S)) = 9.460\,\text{GeV}$

$M(J/\psi) = 3.097\,\text{GeV}$

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and Λ_{QCD}

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Strongly coupled pNRQCD and Born Oppenheimer EFT

A nonperturbative problem: construct a pNREFT description on the basis of scale separations and symmetries

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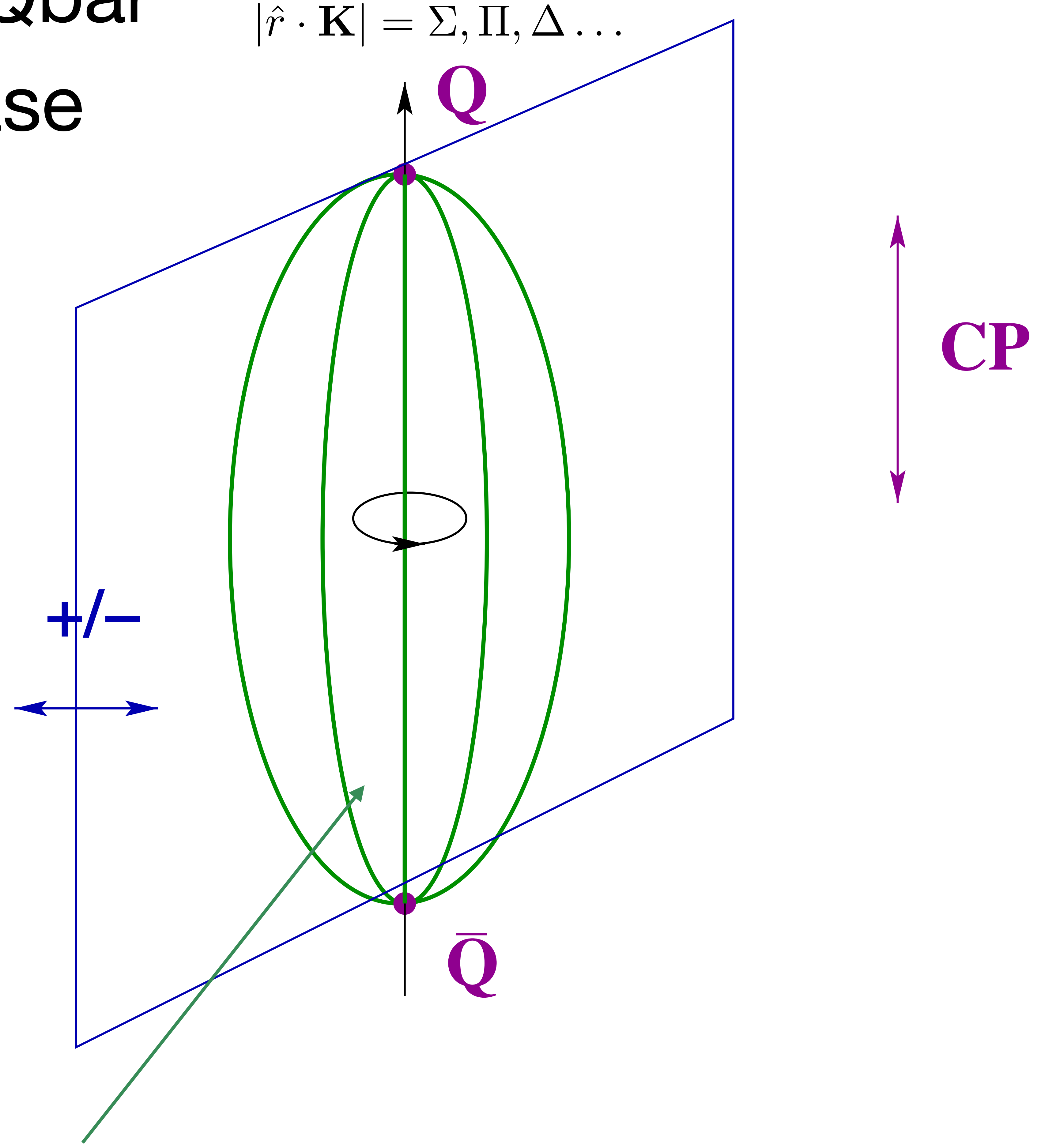
Two heavy quarks with large mass $m \gg \Lambda_{\rm QCD}$ and residual scale separation $\Lambda_{\rm QCD} \gg E$

QQbar case



Nonperturbative light degrees of freedom
glue and light quarks

produce a hierarchy of NRQCD static energies identified by the quantum number of $D_{\infty h}$

Irreducible representations of $D_{\infty h}$

- $\boldsymbol{K}$: angular momentum of light d.o.f.
  $\lambda = \hat{\boldsymbol{r}} \cdot \boldsymbol{K} = 0, \pm 1, \pm 2, \pm 3, \ldots$
  $\Lambda = |\lambda| = 0, 1, 2, 3, \ldots \; (\Sigma,\ \Pi,\ \Delta,\ \Phi,\ \ldots)$ Λ_η^σ
- Eigenvalue of CP: $\eta = +1\,(g),\ -1\,(u)$
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$\mathbf{r} = Q\bar{Q}$ distance $\quad \mathbf{R} =$ center of mass

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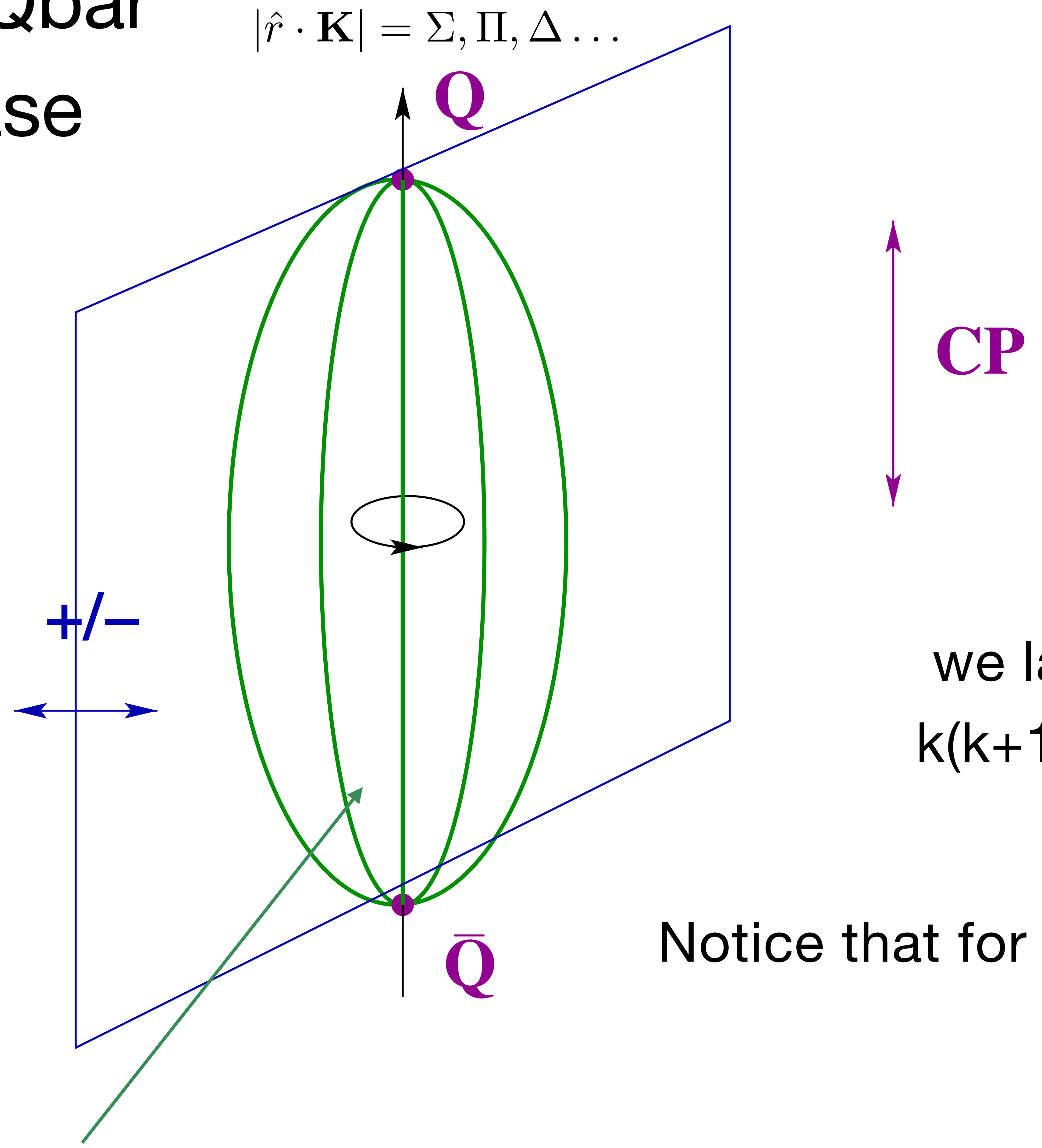

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we label the Light Degrees of Freedom (LDF) by $\kappa = \{k^{PC}, f\}$ where k(k+1) is the eigenvalue of the K^2 and we add the flavour quantum number

Notice that for r->0 the cylindrical symmetric becomes spherical (O(3) X C)

Nonperturbative light degrees of freedom glue and light quarks

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Strongly coupled pNRQCD and Born Oppenheimer EFT

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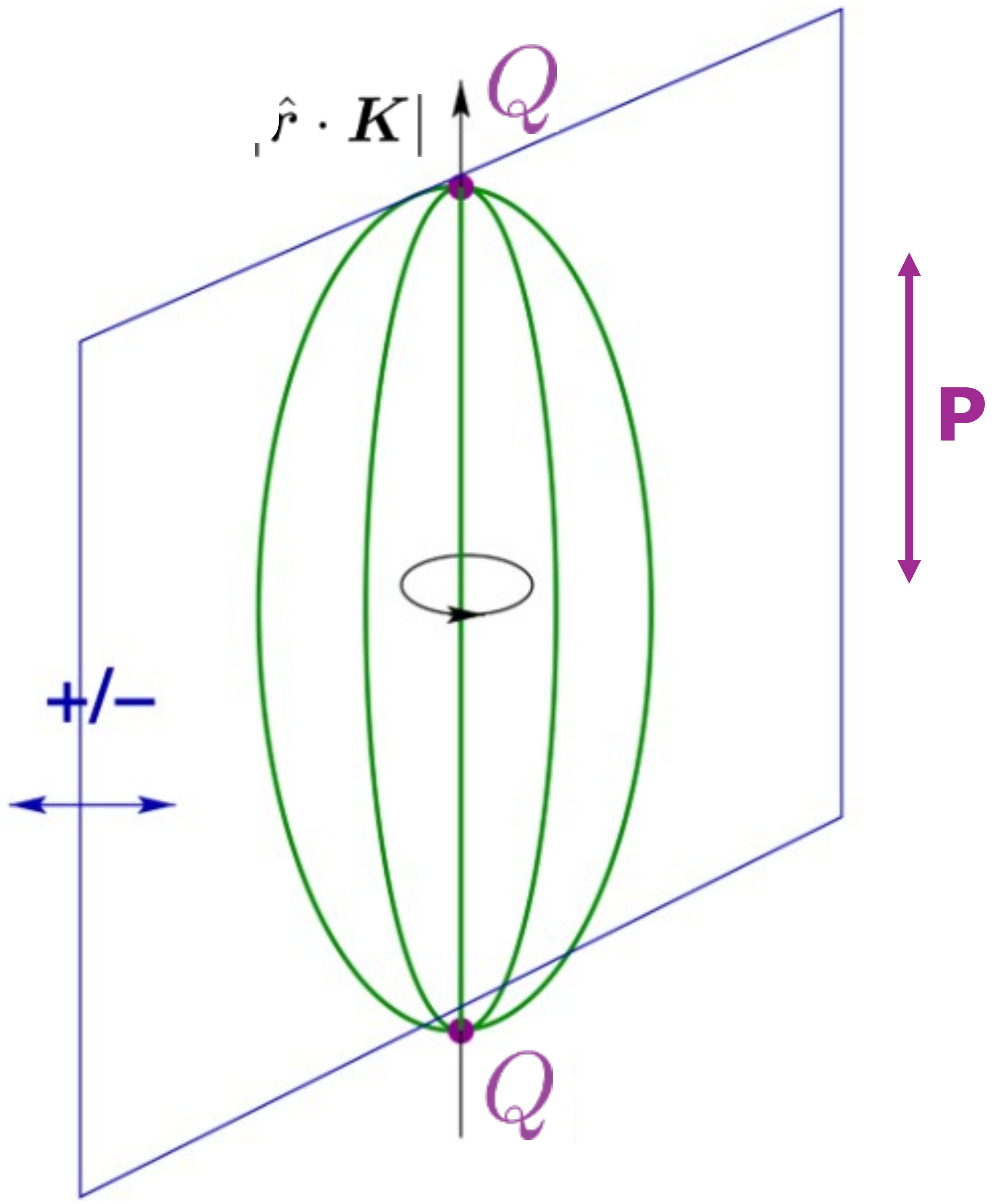

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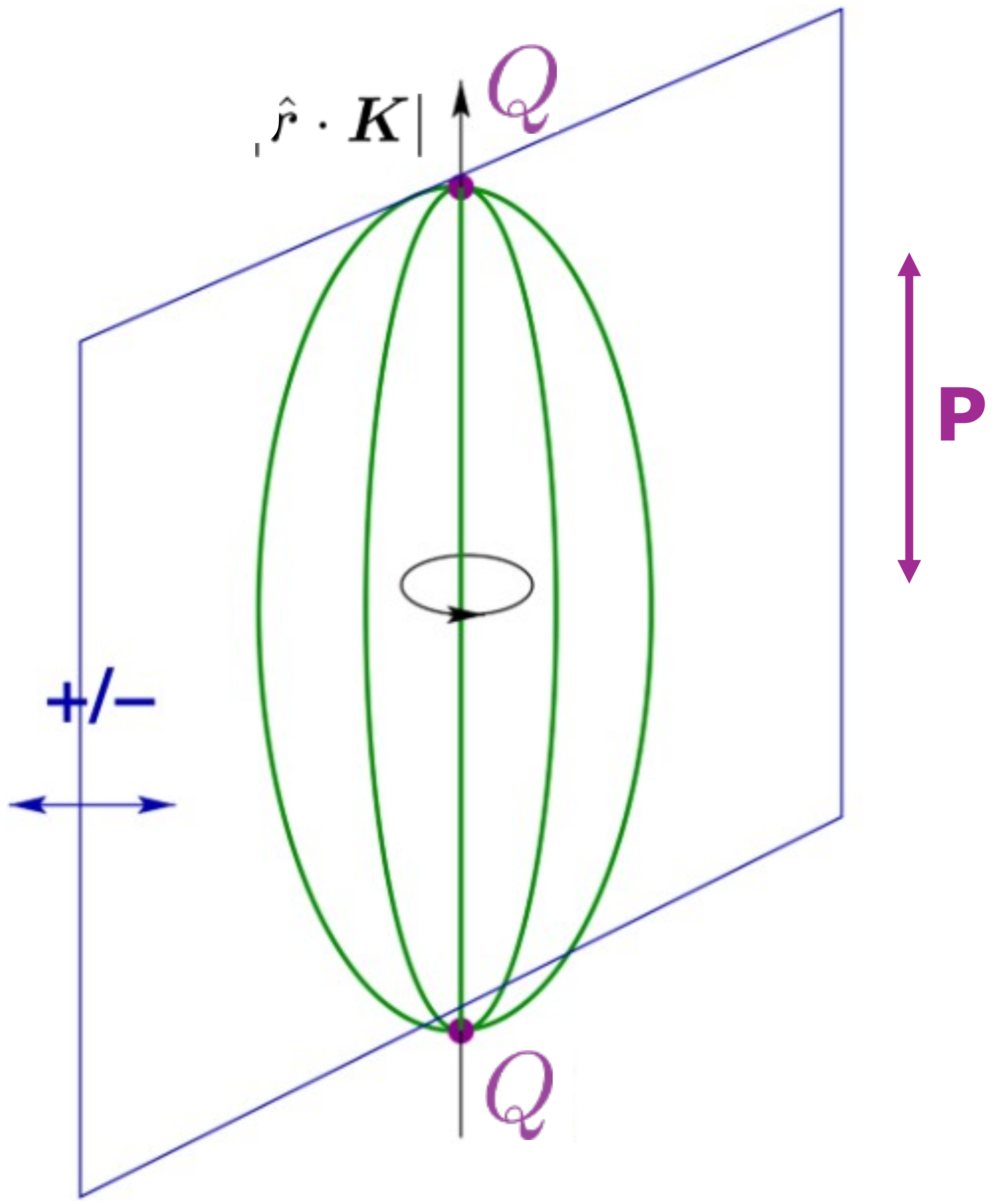

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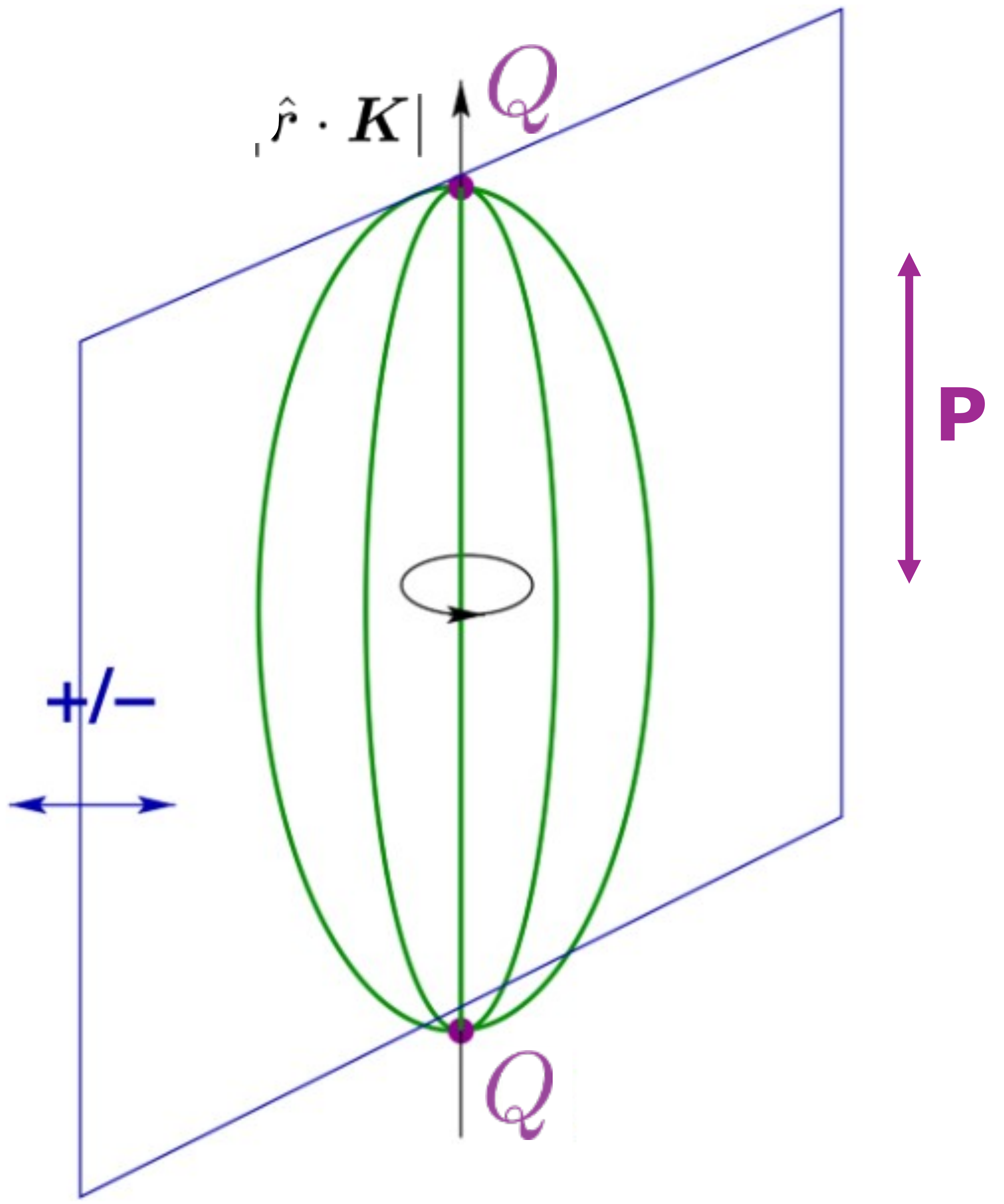

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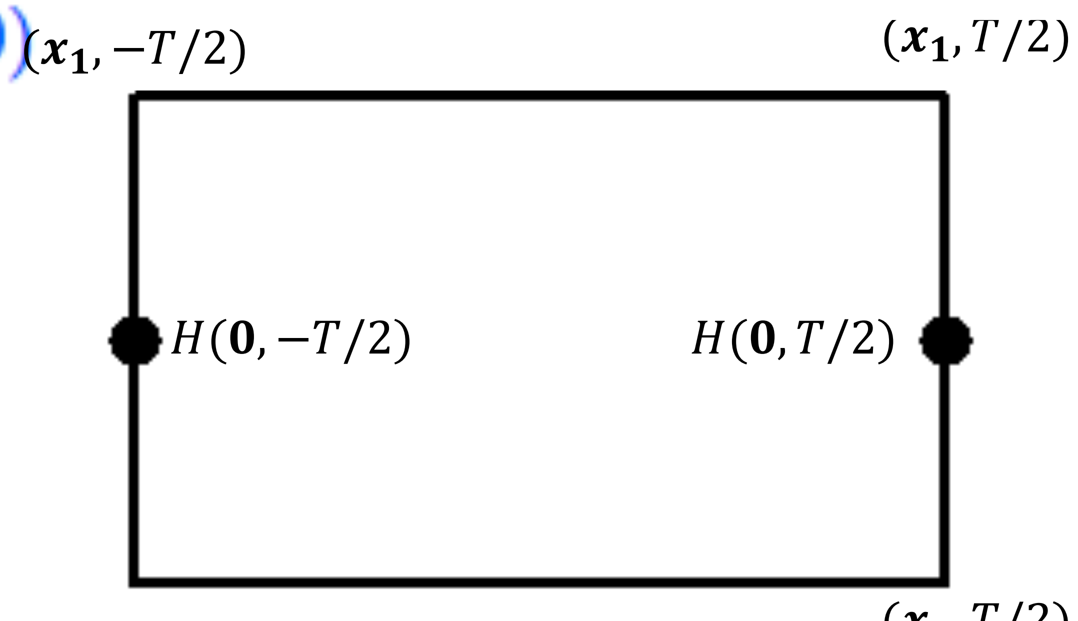

(b)

Phi =Wilson lines and H= gluonic and light quarks

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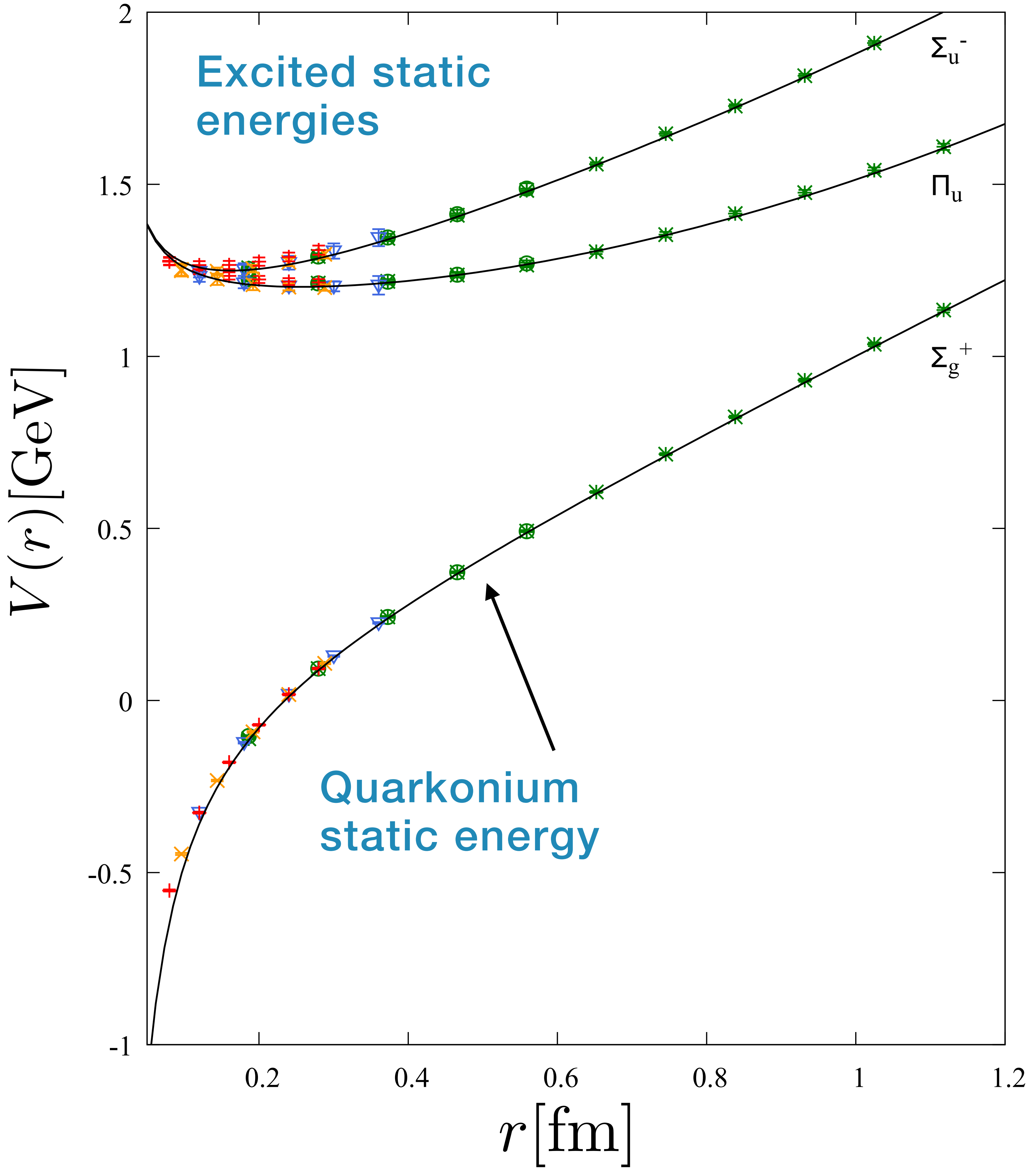

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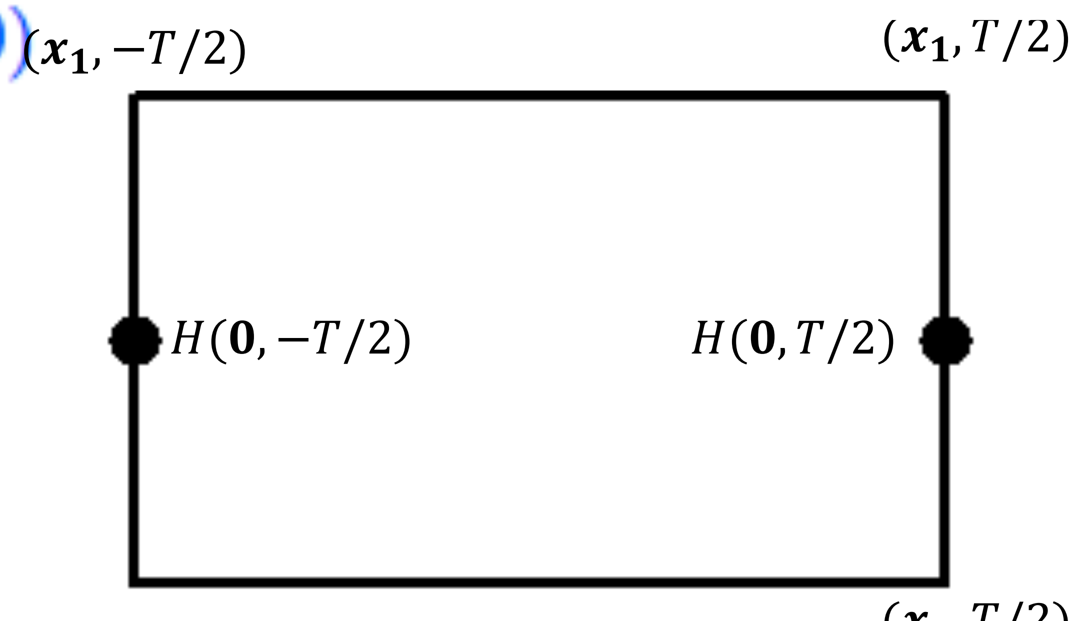

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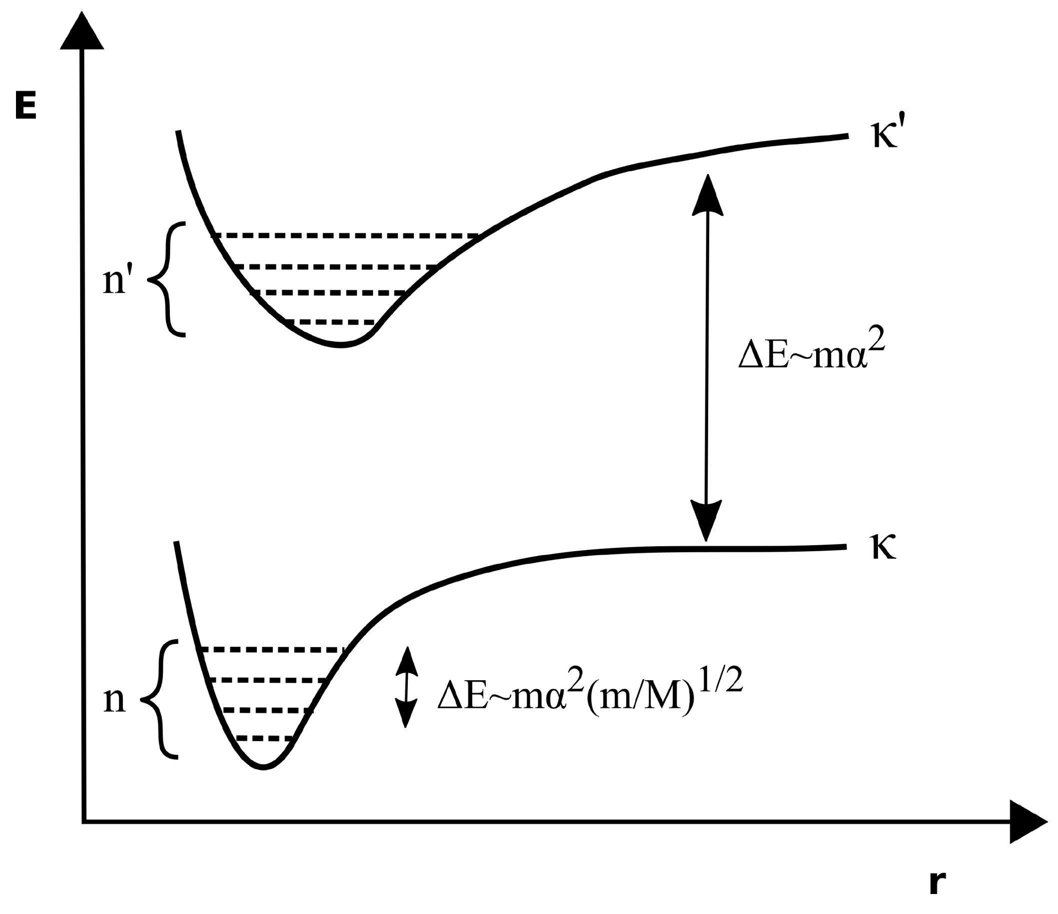


QED —— $\Lambda_{QCD} > mv^2$

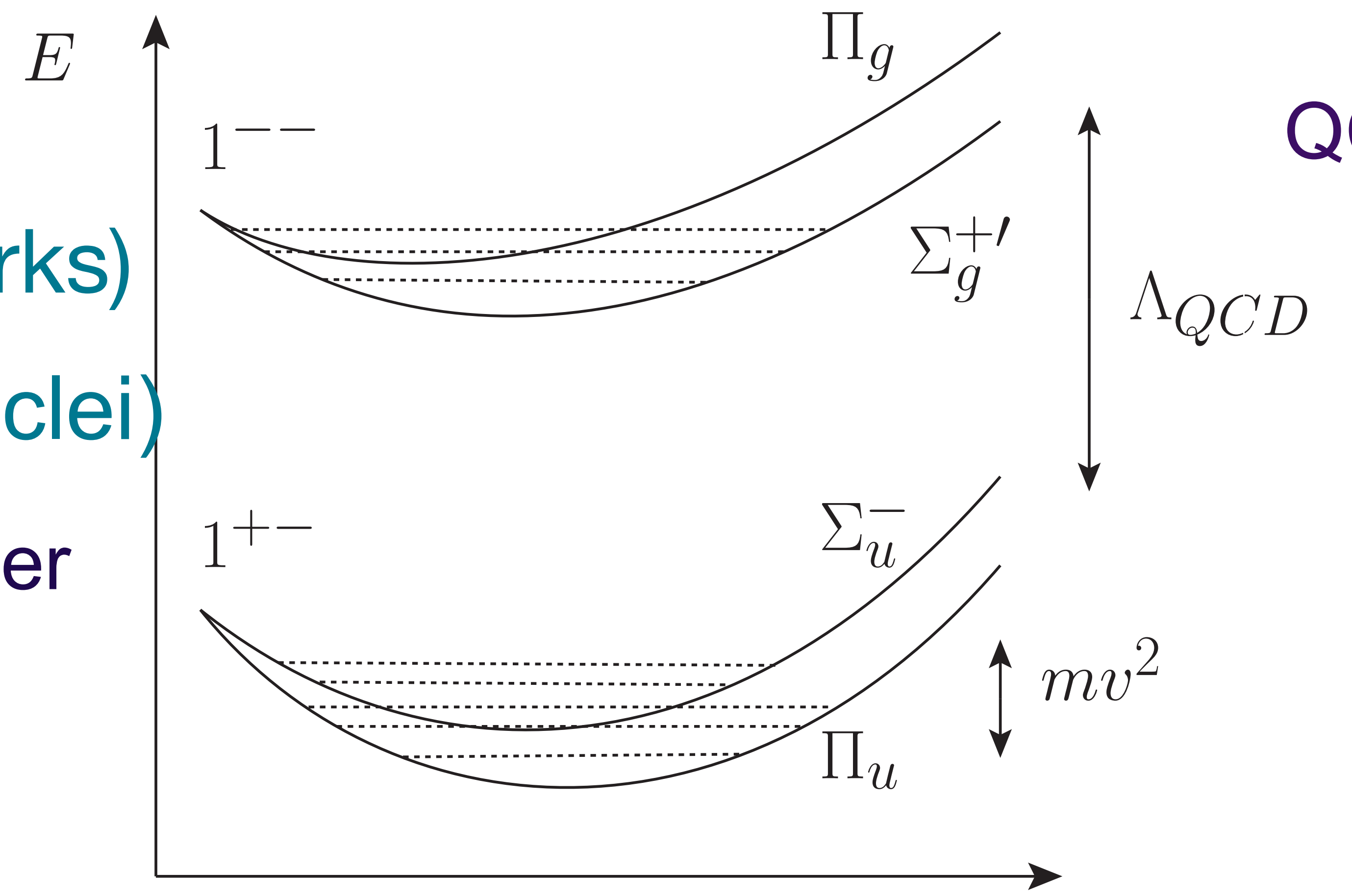
fast (gluons, light quarks) and slow (heavy quarks)

like in molecular physics (fast-electrons, slow nuclei)

Born Oppenheimer Description

Braaten PRL 111 (2013) 162003

Braaten Langmack Smith PRD 90 (2014) 014044



QCD

Higher excitations develop a gap of order Lambda_QCD

Introducing a finite mass m:

- The spectrum of the mv^2 fluctuations around the lowest static energy is the quarkonium spectrum
- The spectrum of the mv^2 fluctuations around the higher excitations is the exotic spectrum (hybrids and tetraquarks)

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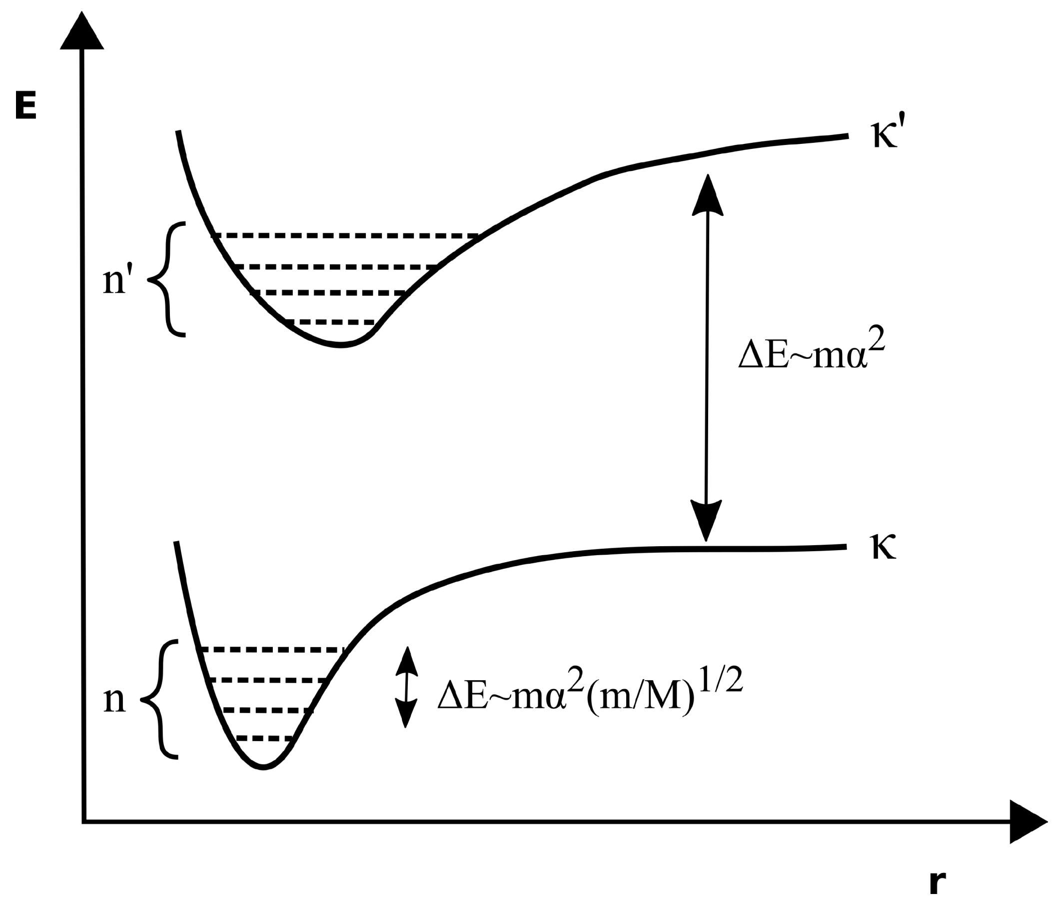

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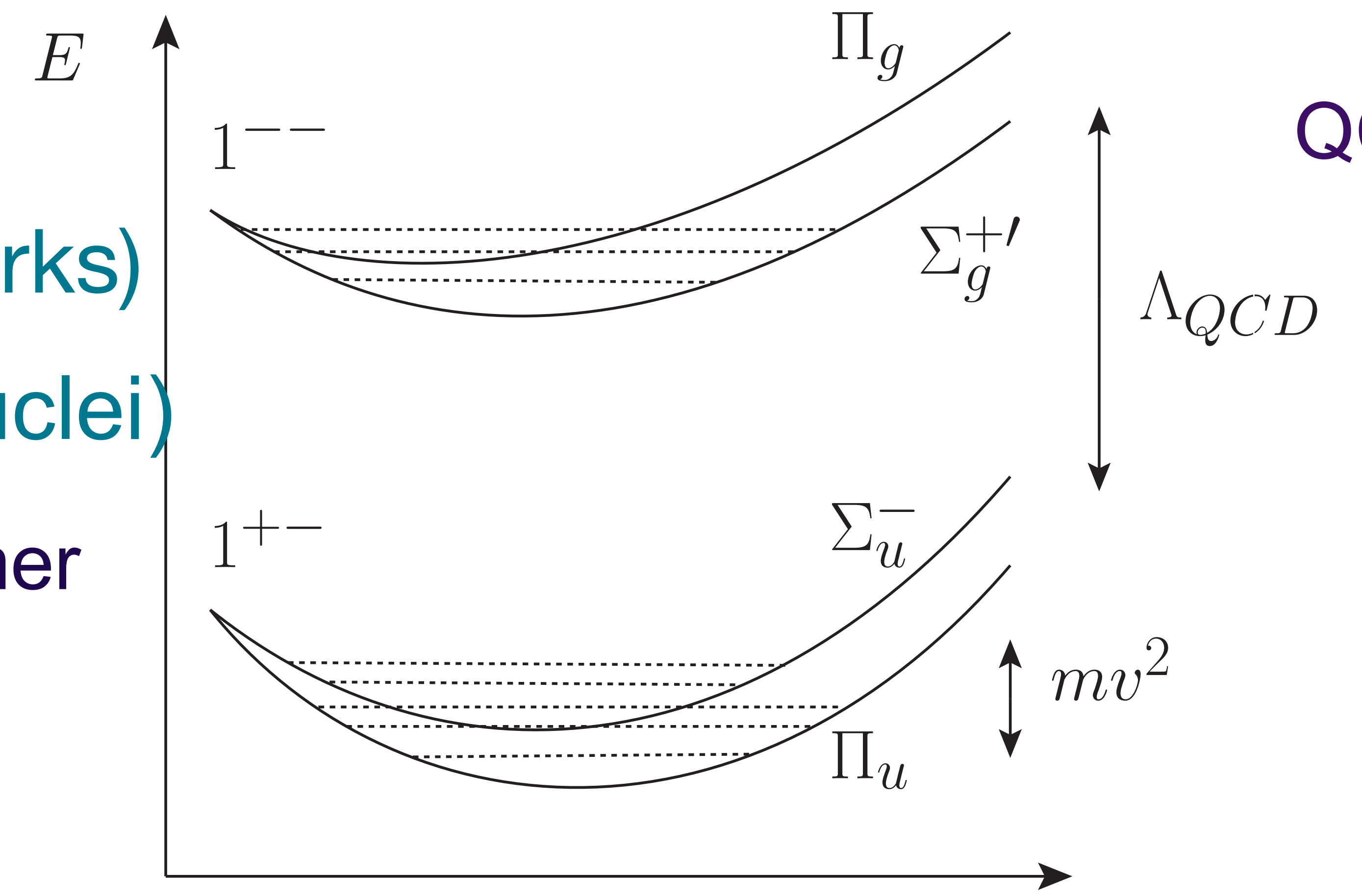

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Nonperturbative matching to the pNREFT

systematically

$$\langle H|\,\mathcal{H}\,|H\rangle = \langle nljs|\frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}}{m^n}|nljs\rangle$$

expand quantomechanically NRQCD states and energies in 1/m around the zero order and identify the QCD potentials

$|\underline{0};\mathbf{x}_1\mathbf{x}_2\rangle -> \ |(Q\bar{Q})_1\rangle \rightarrow$ Quarkonium Singlet

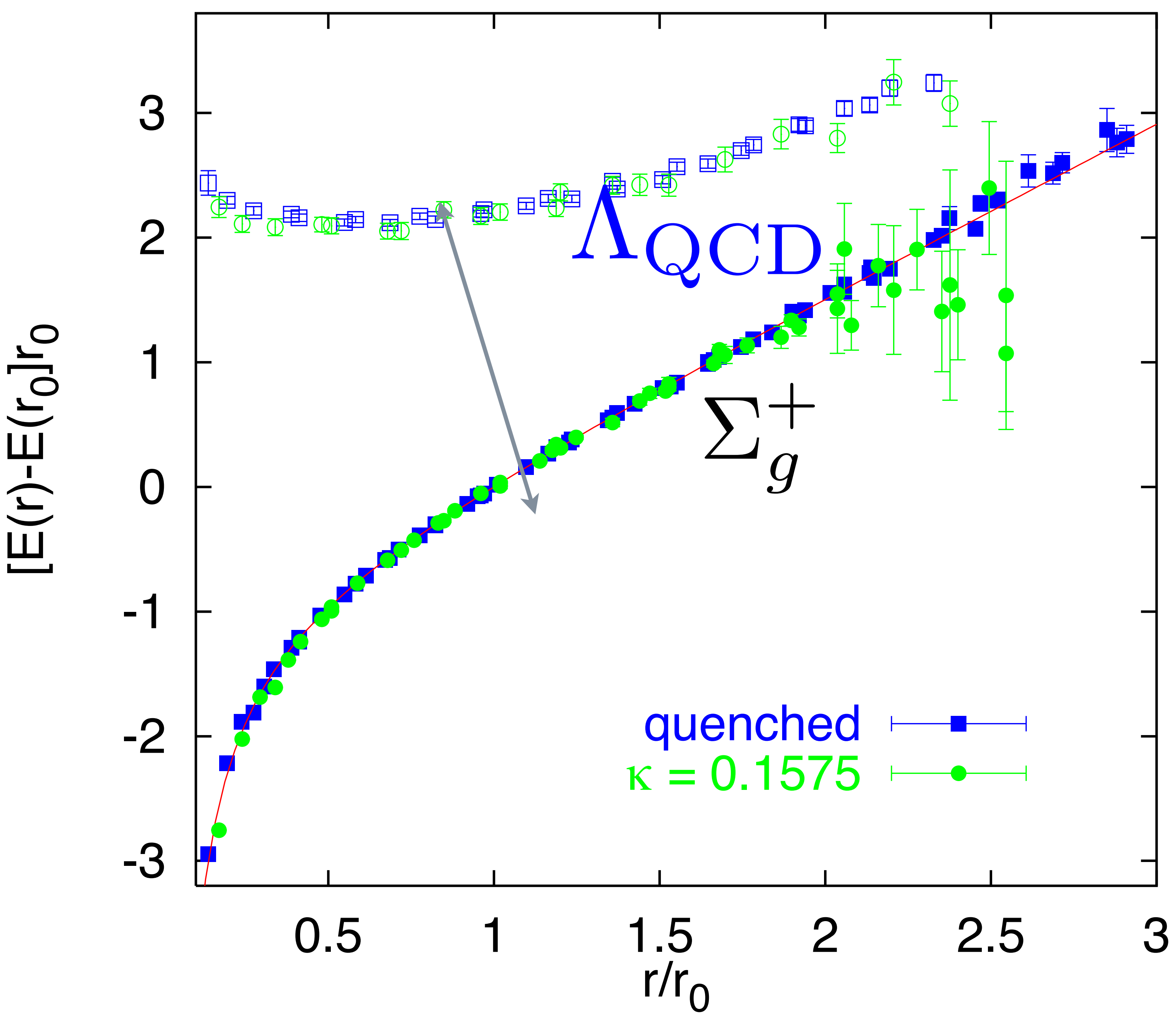
$E_0(r) -> V_0(r)$ pNRQCD

$|\underline{n}>0;\mathbf{x}_1\mathbf{x}_2\rangle -> \ |(Q\bar{Q})g^{(n)}\rangle \rightarrow$ Higher Gluonic Excitations

$|Q\bar{Q}q\bar{q}\rangle$ Tetraquarks

$E_n^{(0)}(r) -> V_n^{(0)}(r)$ BOEFT

Strongly coupled pNRQCD for quarkonium Σ_g^+ k^PC=0^++



Bali et al. 98

the potentials come from integrating out all scales up to mv^2

- gluonic excitations develop a gap $\Lambda_{\rm QCD}$ and are integrated out

$\Rightarrow$ The singlet quarkonium field S of energy mv^2 is the only the degree of freedom of pNRQCD (up to ultrasoft light quarks, e.g. pions).

Brambilla Pineda Soto Vairo 00

Strongly coupled pNRQCD for quarkonium

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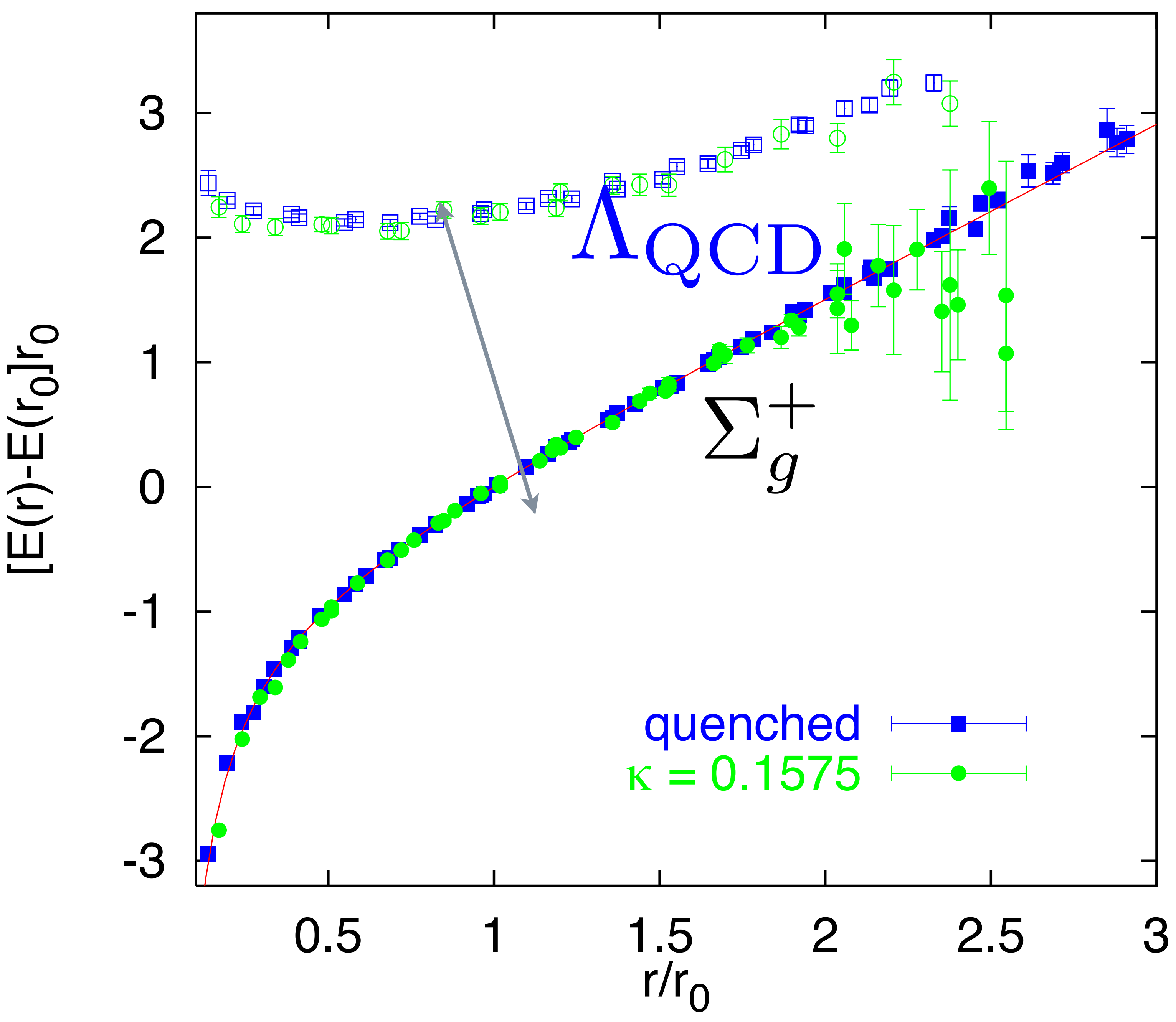

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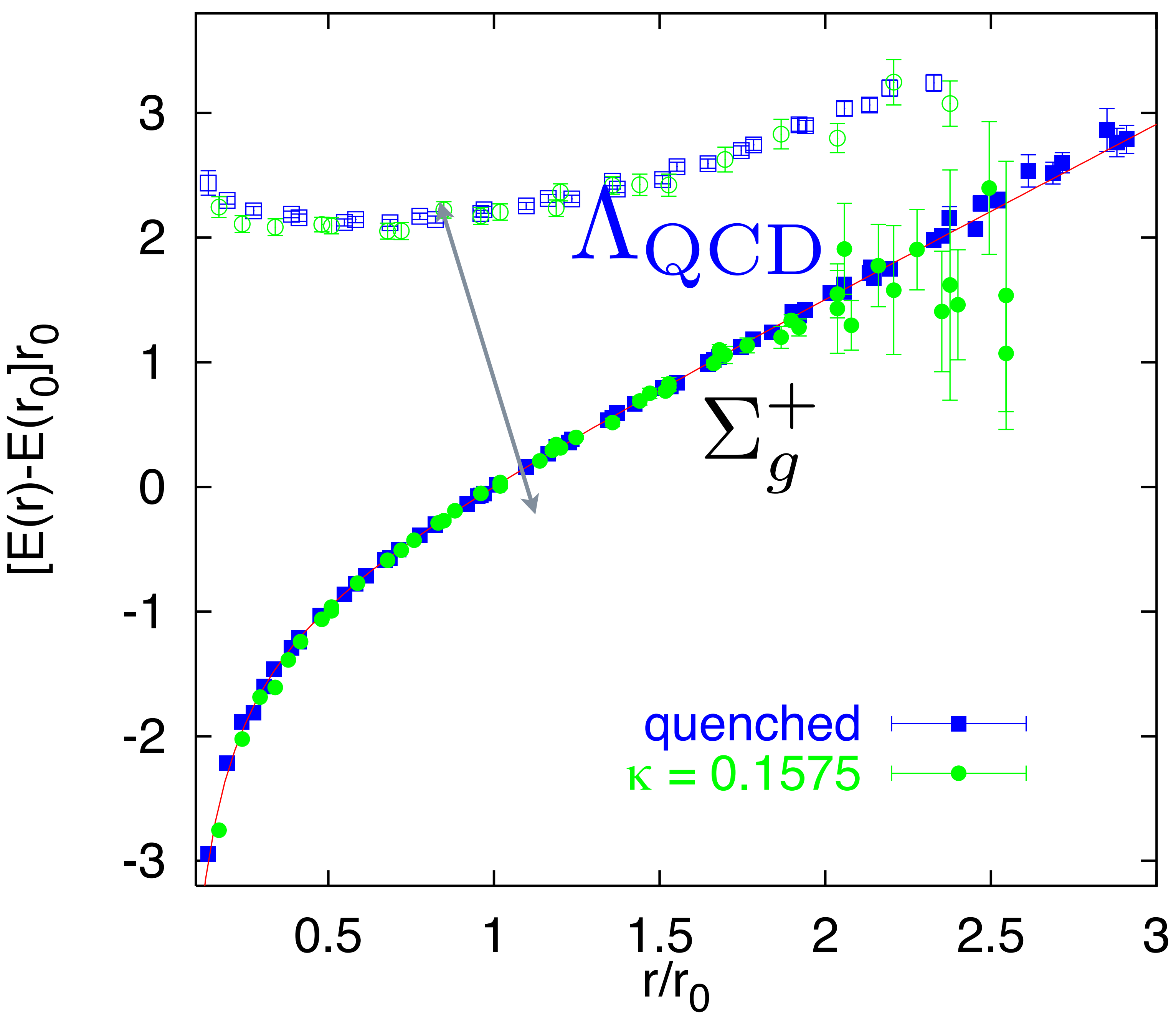

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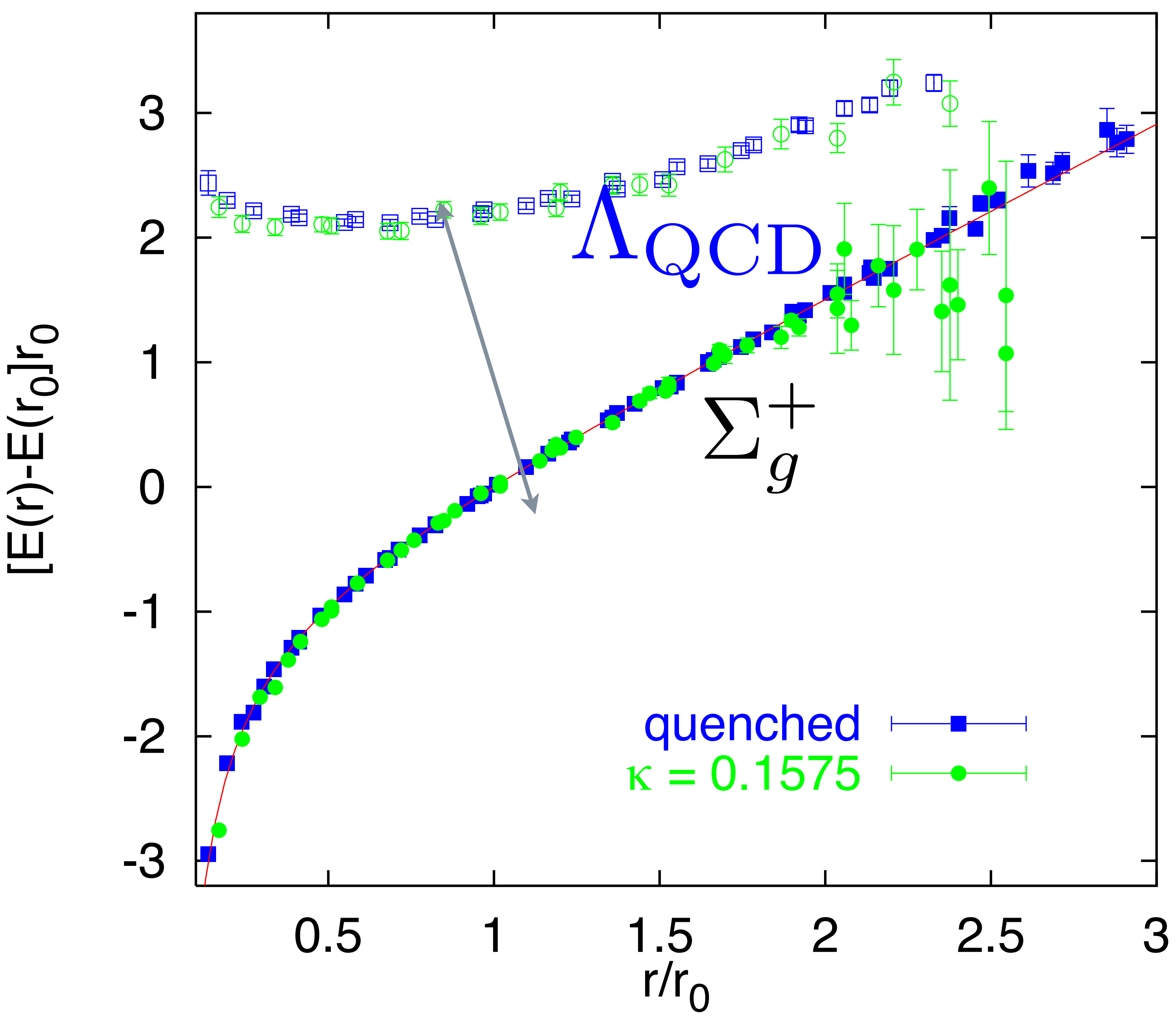

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Applications regard: Spectrum, decays, production at LHC, studies of confinement

The singlet potential has the general structure

$$V = V_0 + \frac{1}{m}V_1 + \frac{1}{m^2}(V_{SD} + V_{VD})$$

static — spin dependent — velocity dependent

the fact that spin dependent corrections appear at order 1/m^2 is called Heavy Quark Spin Symmetry

$$W = \langle \exp\{ig \oint A^\mu dx_\mu\}\rangle$$

E_0(r)= $V_s^{(0)} = \lim_{T\to\infty} \frac{i}{T} \ln\langle W(r \times T)\rangle = \lim_{T\to\infty} \frac{i}{T} \ln \langle \square \rangle$

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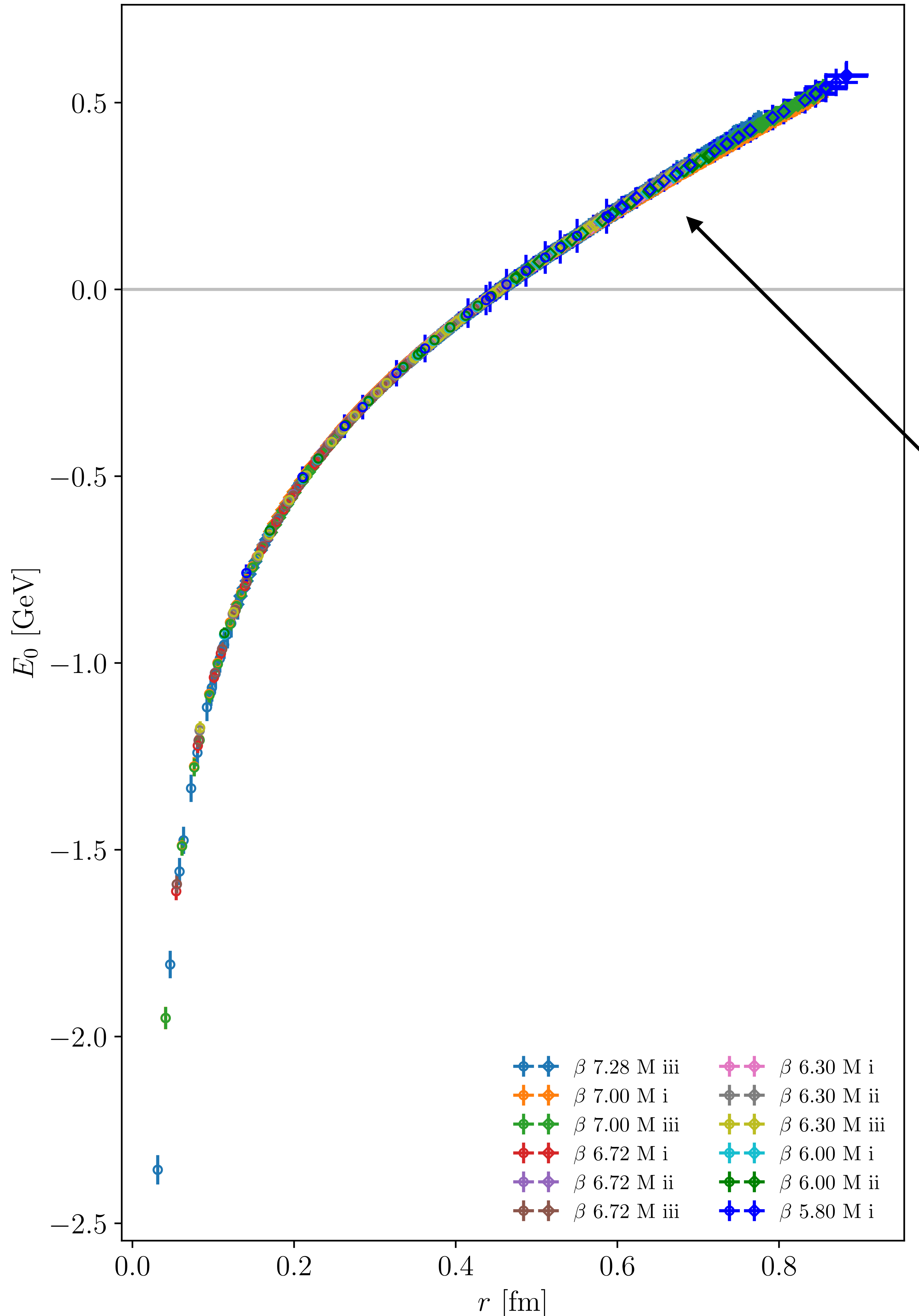
State of the art:

2+1+1 quarkonium

static energy E_0(r)

Σ_g^+



confinement

TUMQCD
N.B., Delgado, Kronfeld, Leino, Petreczky, Steinbeisser, Vairo, Weber *Phys.Rev.D* 107 (2023) 7, 074503 •

Behaviour of the static energy at short distance $r \ll \Lambda_{\rm QCD}^{-1}$

Multipole expansion in r is possible: color singlet and color octet degrees of freedom:
theory is weakly coupled pNRQCD

The gauge fields are multipole expanded:

$$A(R,r,t) = A(R,t) + \mathbf{r}\cdot\boldsymbol{\nabla}A(R,t) + \ldots$$

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LO in r

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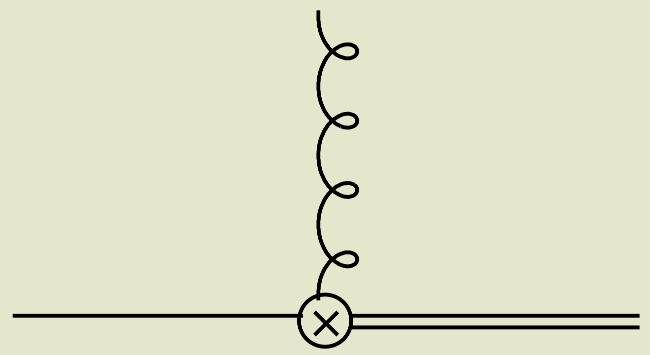
NLO in r

$$-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} + \sum_{i=1}^{n_f}\bar{q}_i\, i\not{D} q_i$$

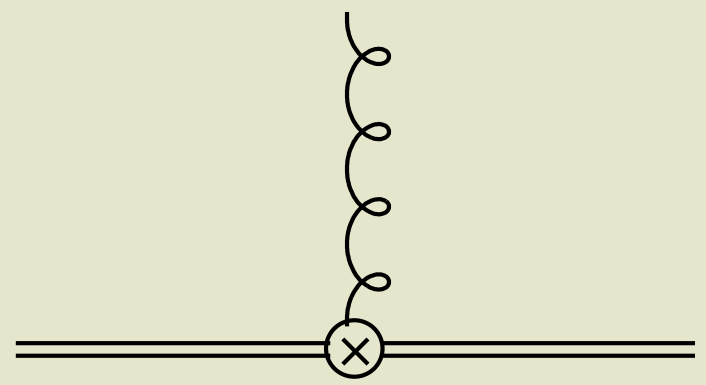
Feynman rules

$\theta(T)\, e^{-iTV_s}$

$\theta(T)\, e^{-iTV_o}\left(e^{-i\int dt\, A^{\rm adj}}\right)$



$O^\dagger \mathbf{r}\cdot g\mathbf{E}\,{\rm S}$



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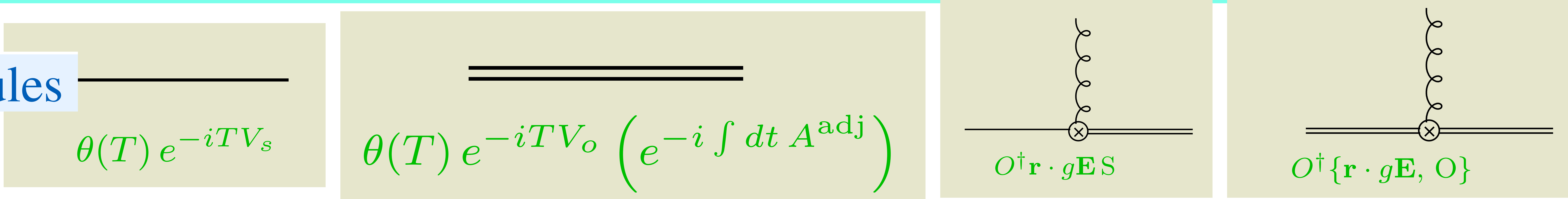
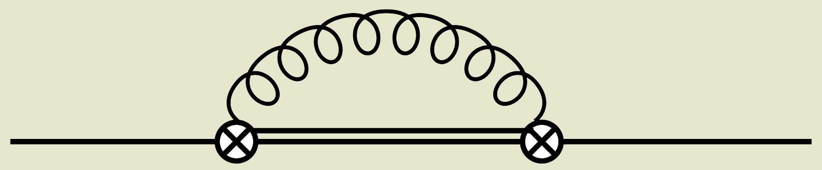


The NRQCD static energy E_0 is calculable in perturbation theory in pNRQCD

$$V^{(0)}(r,\mu') = \lim_{T\to\infty}\frac{i}{T}\ln\langle\,\Box\,\rangle - $$



$$+\cdots$$

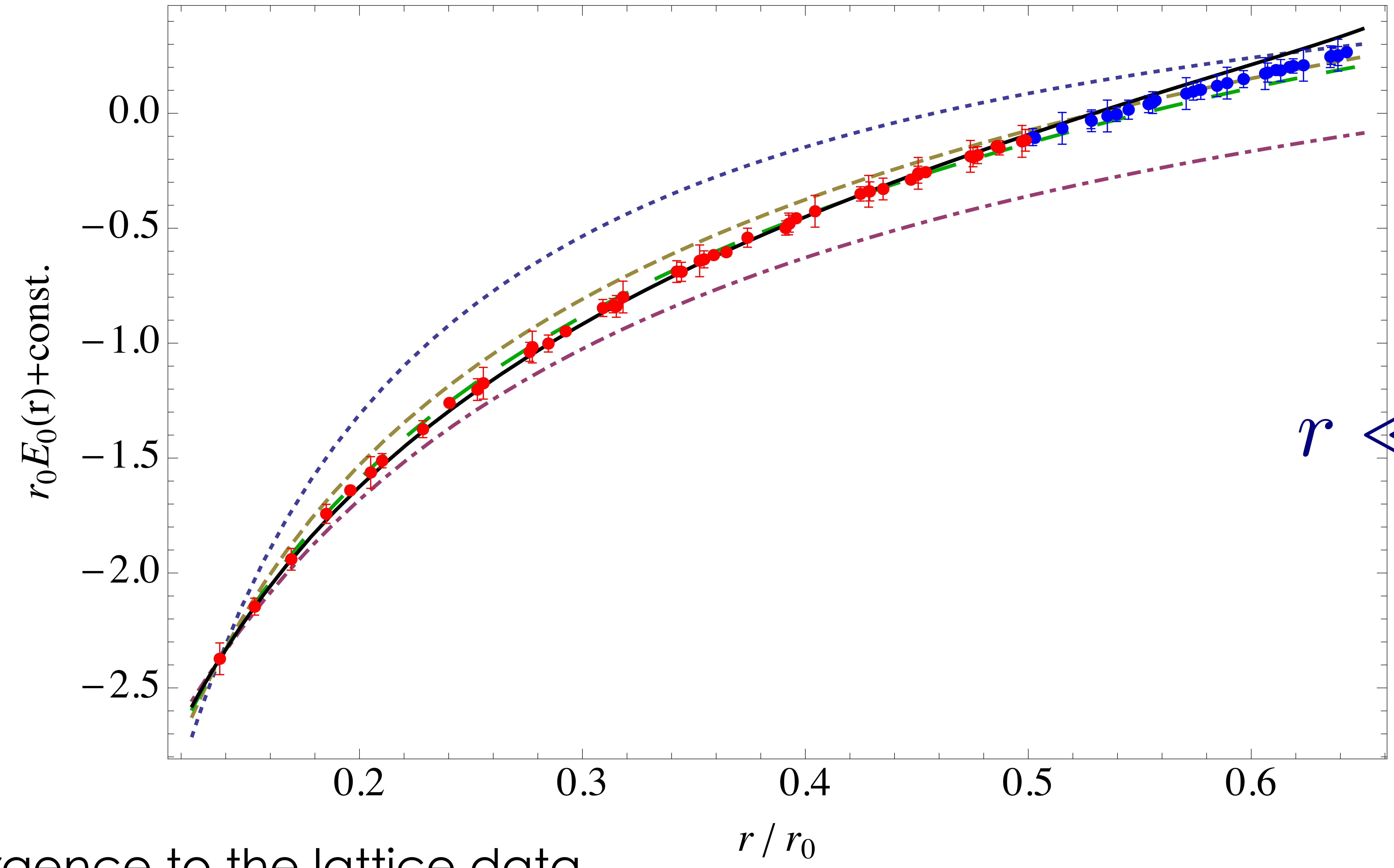
$$= E_0(r) + \frac{i}{N}\int_0^\infty dt\, e^{-it(V_o - V)}\langle{\rm Tr}\,\mathbf{r}\cdot g\mathbf{E}(t)\,\mathbf{r}\cdot g\mathbf{E}(0)\rangle(\mu') + \cdots$$

Full control

at short distance!

QQbar singlet static energy at NNNLL in pNRQCD in comparison with unquenched (n_f=2+1) lattice data (red points,blue points)

Bazanov, N. B., Garcia, Petreczky, Soto, Vairo , 2012, 2014, with Weber 2019



$$r \ll 0.2fm$$

Good convergence to the lattice data

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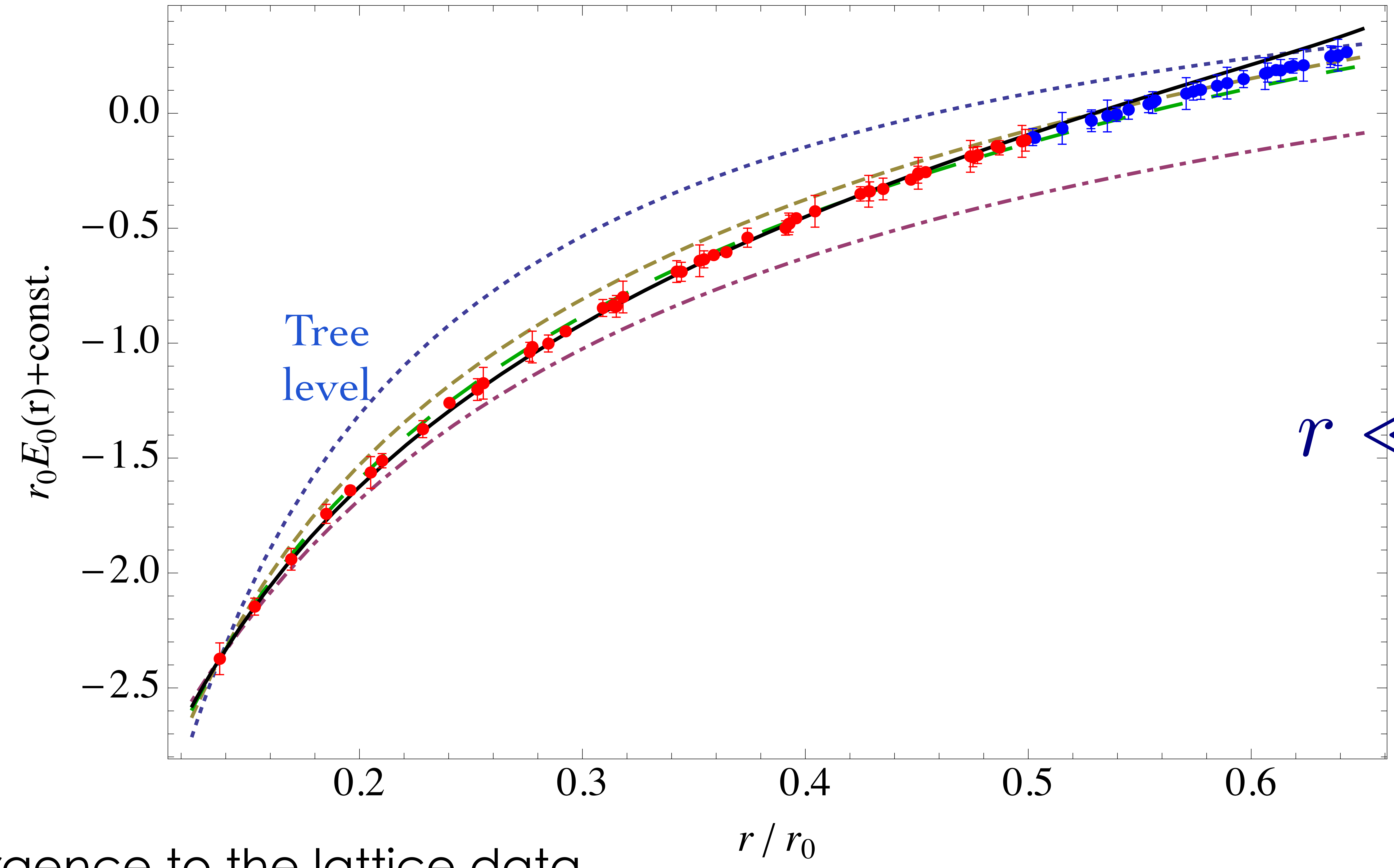
Full control

at short distance!



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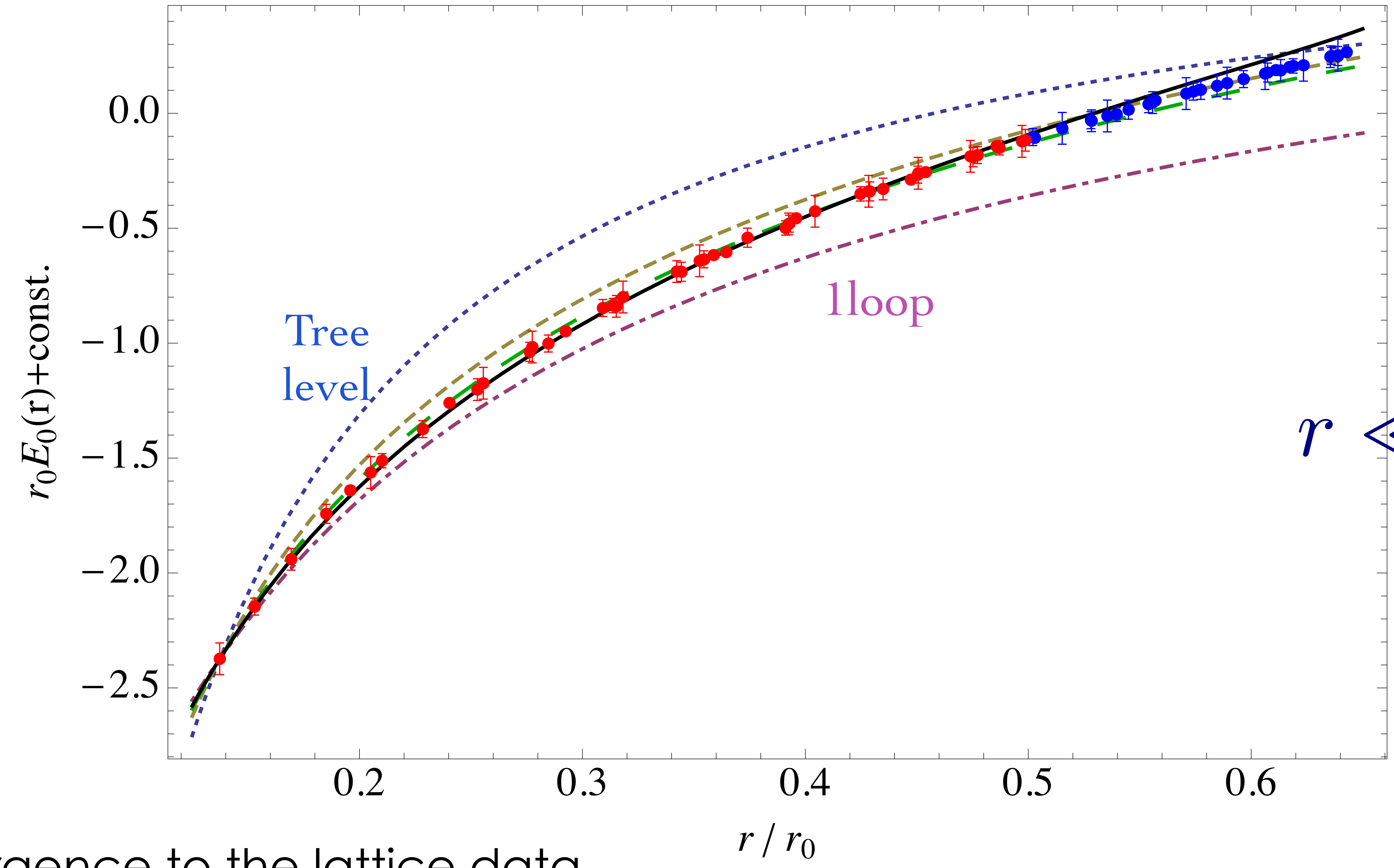

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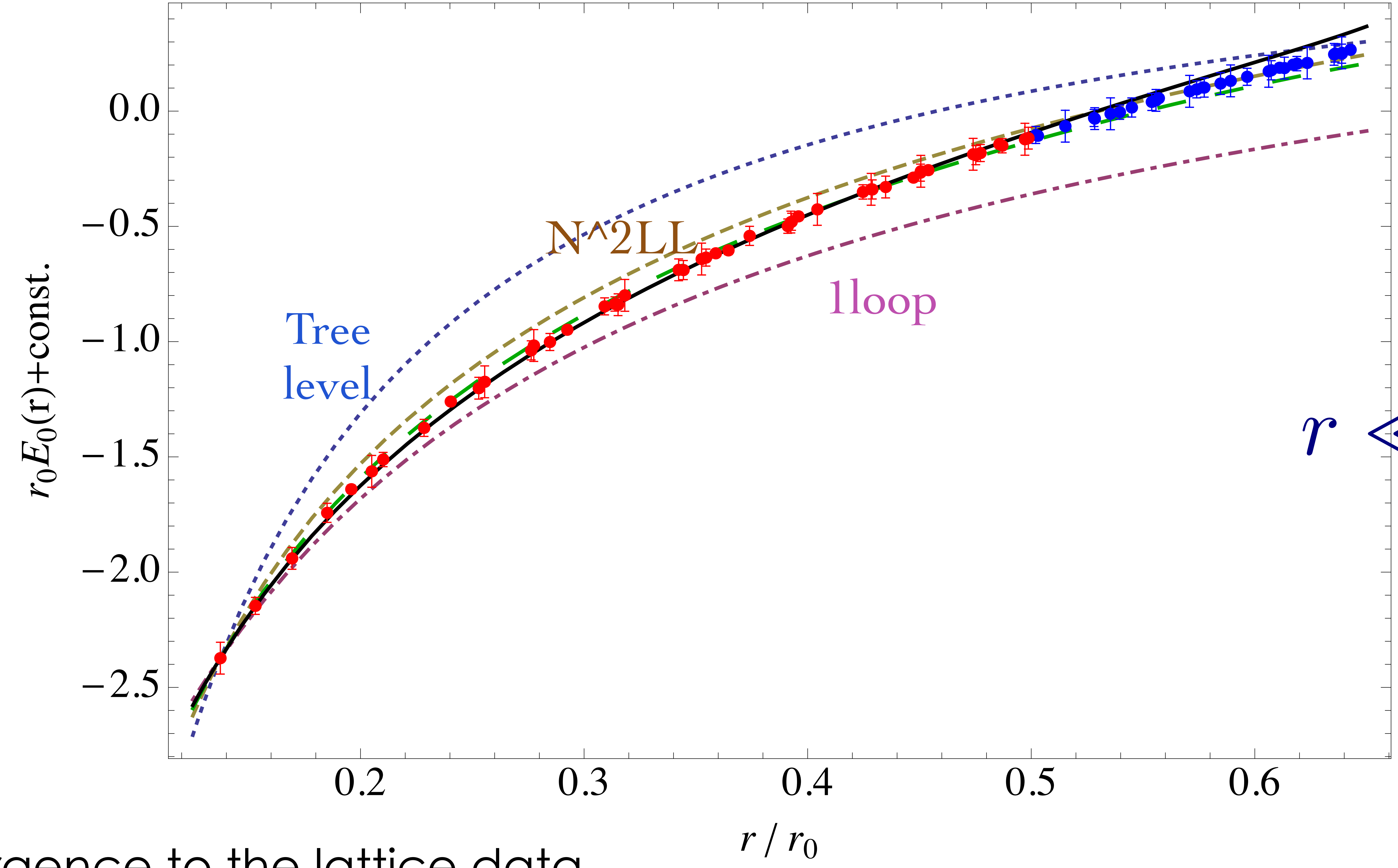

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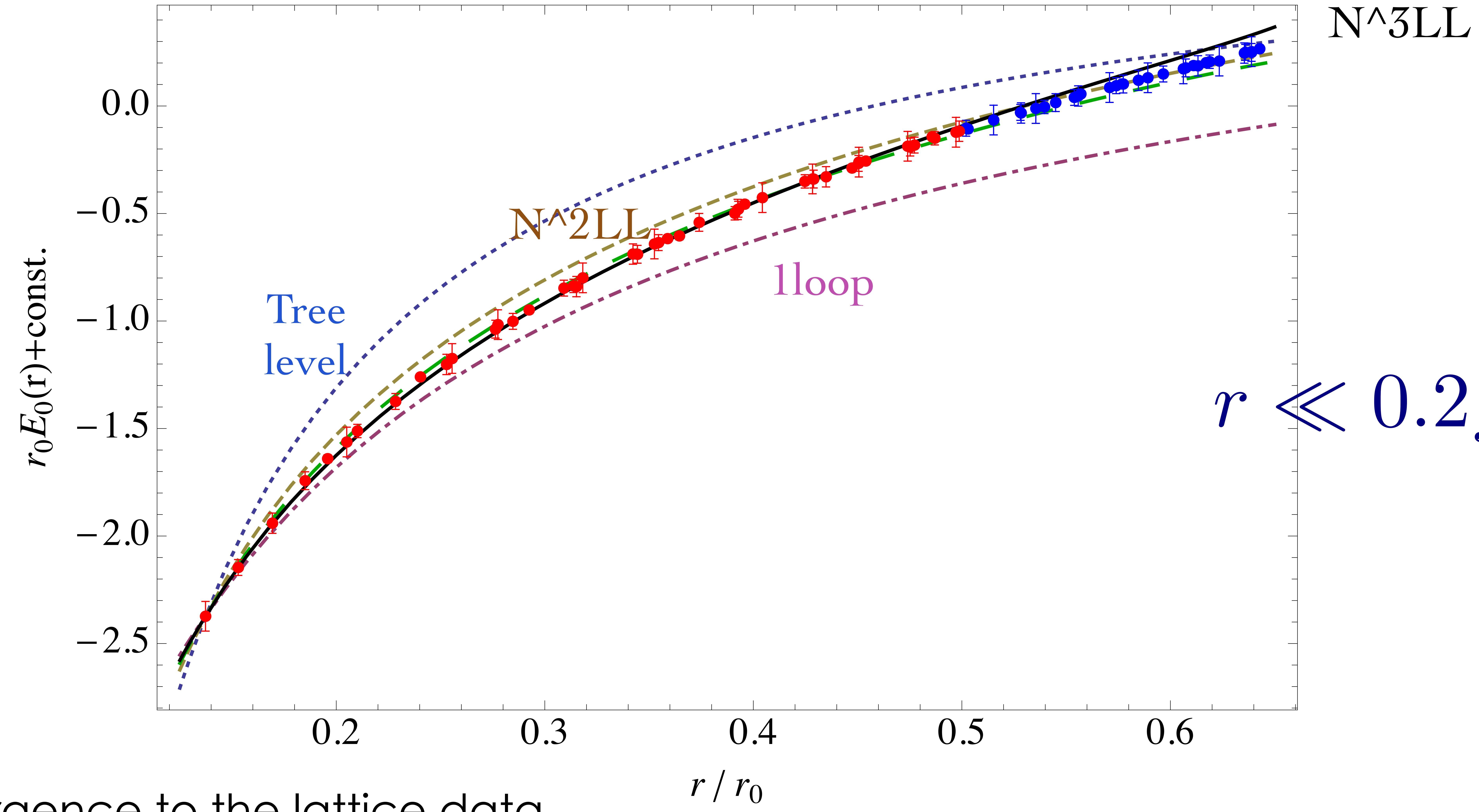

$r \ll 0.2 fm$

Good convergence to the lattice data

The singlet potential has the general structure

$$V = V_0 + \frac{1}{m} V_1 + \frac{1}{m^2}(V_{SD} + V_{VD})$$

static — spin dependent — velocity dependent

the fact that spin dependent corrections appear at order 1/m^2 is called Heavy Quark Spin Symmetry

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$$V^{(1)} = -\frac{1}{2}\int_0^\infty dt\, t \,\langle \Box_{E(t)} \rangle$$

$$V^{(2)}_{\rm SD} = -\frac{r^k}{r^2} c_F\, \epsilon^{kij}\, i \int_0^\infty dt\, t \langle \Box^{E(t)}_{B}{}_{i\ j} \rangle\ \mathbf{L}_1 \cdot \mathbf{S}_2 + (1 \leftrightarrow 2)\ |V^{(2)}_{LS}$$

$$-\frac{r^k}{r^2}\left(c_F\, \epsilon^{kij}\, i \int_0^\infty dt\, t\, \langle \Box_{i\ j} \rangle - \frac{2c_F - 1}{2}\nabla^k V^{(0)}\right)\mathbf{L}_1 \cdot \mathbf{S}_1 + (1 \leftrightarrow 2)\ |V^{(1)}_{LS}$$

$$-c_F^2 \hat{r}_i \hat{r}_j i \int_0^\infty dt \left(\langle \Box_{i\ j} \rangle - \frac{\delta_{ij}}{3} \langle \Box \rangle \right)\left(\mathbf{S}_1 \cdot \mathbf{S_2} - 3(\mathbf{S}_1 \cdot \hat{\mathbf{r}})(\mathbf{S_2} \cdot \hat{\mathbf{r}})\right)\ |V_T$$

$$+\left(\frac{2}{3}c_F^2 i \int_0^\infty dt \langle \Box \rangle - 4\left(d_{sv} + \frac{4}{3}d_{vv}\right)\delta^{(3)}(\mathbf{r})\right)\mathbf{S}_1 \cdot \mathbf{S_2}\ |V_S$$

gauge invariant wilson loops can be calculated also in QCD vacuum model and large N

Pineda Vairo PRD 63 (2001) 054007

Brambilla Pineda Soto Vairo PRD 63 (2001) 014023

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$$-\frac{r^k}{r^2}\left(c_F\, \epsilon^{kij}\, i \int_0^\infty dt\, t \,\langle \Box_{i\,j} \rangle - \frac{2c_F - 1}{2} \nabla^k V^{(0)} \right) \mathbf{L}_1 \cdot \mathbf{S}_1 + (1 \leftrightarrow 2) \quad |V^{(1)}_{LS}$$

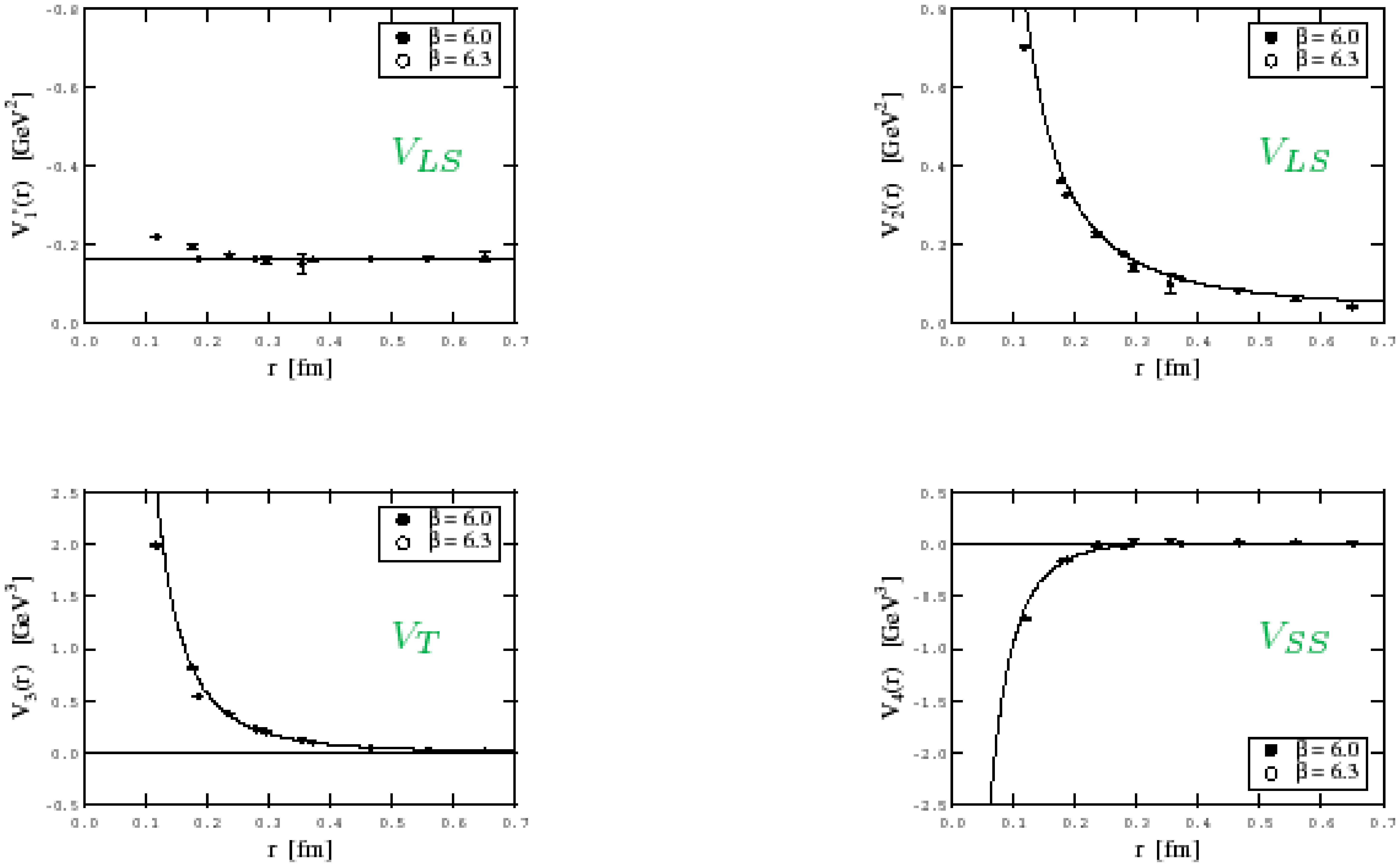
$$-c_F^2 \hat{r}_i \hat{r}_j i \int_0^\infty dt \left(\langle \Box_{i\,j} \rangle - \frac{\delta_{ij}}{3} \langle \Box \rangle \right) \left(\mathbf{S}_1 \cdot \mathbf{S_2} - 3(\mathbf{S}_1 \cdot \hat{\mathbf{r}})(\mathbf{S_2} \cdot \hat{\mathbf{r}}) \right) \quad |V_T$$

$$+\left(\frac{2}{3} c_F^2 i \int_0^\infty dt \langle \Box \rangle - 4\left(d_{sv} + \frac{4}{3} d_{vv} \right) \delta^{(3)}(\mathbf{r}) \right) \mathbf{S}_1 \cdot \mathbf{S_2} \quad |V_S$$

- the potentials contain the contribution of the scale m inherited from NRQCD matching coefficients—> they cancel any QM divergences, good UV behaviour
- the flavour dependent part is extracted in the NRQCD matching coefficients
- the nonperturbative part is factorized and depends only on the glue —> only one lattice calculation to get the dynamics and the observables instead of an ab initio calculation of multiple Green functions

Also the generalised Wilson Loops have been calculated on the latiice

spin dependent potentials



Koma Koma Wittig 05, Koma Koma 06

Terrific advance in the data precision with Lüscher multivel algorithm!

Such data can distinguish different models for the dynamics of low energy QCD e.g. effective string model

N. B., Martinez, Vairo 2014

In the EFT it has been possible to calculate

quarkonium production and in nonequilibrium evolution in

medium which makes this treatment promising for XYZ

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quarkonium production and in nonequilibrium evolution in

medium which makes this treatment promising for XYZ

This can be extended to the XYZ that are however a much more interesting case

- We consider all the NRQCD static energies in presence of glue and light quarks: QQbarg, QQbar qqbar, QQ qbar qbar, QQbar qqq, QQ qqqbar, QQq

- We define the NRQCD static energies via gauge-invariant correlator of appropriate interpolating operators

- We calculate the short distance behaviour: gluelumps, adjoint meson, triplet mesons, sextet mesons ….

- BO quantum number is conserved: BO static energies evolve in heavy-light static energies with the same quantum numbers: allows to understand the form of the fundamental strong force

- We consider separately NRQCD static energies separated by a gap Lambda_QCD

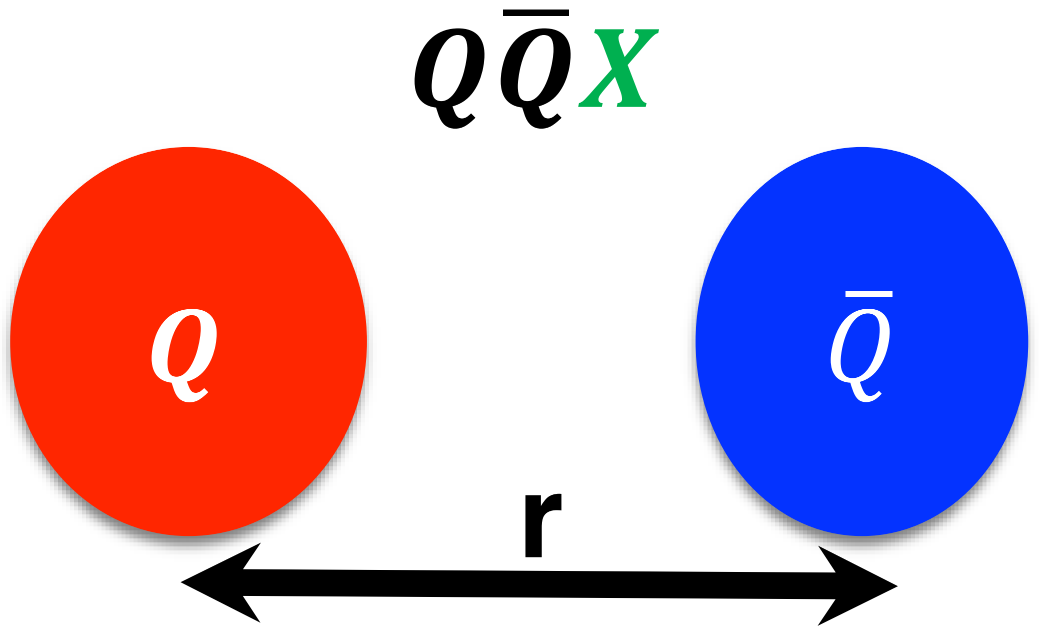
- We match the NRQCD static energies to the corresponding potentials in BOEFT

Mixing appears: at short distance between static energies with same k—> coupled Schr. Eqs.

at large distance between static energies with same BO numbers that get close in energy (avoided level crossing)—> couples Schr. Eqs.

- Use it to understand the X and the Tcc, the hybrids…

Exotic hadrons and corresponding NRQCD static energies



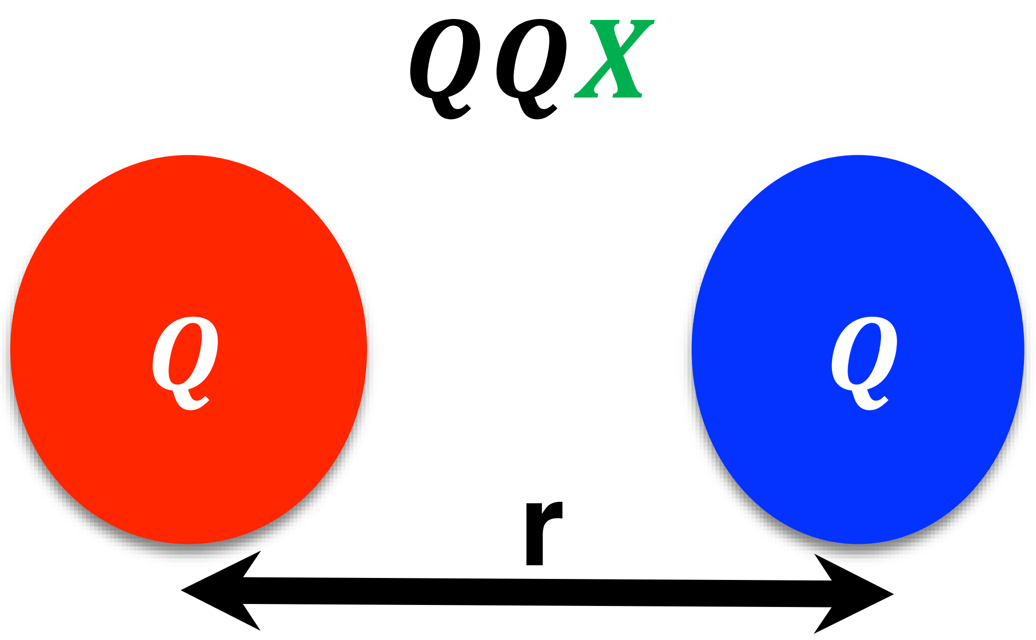
color: $3 \otimes \bar{3} = 1 \oplus 8$

X_8 = gluon → **Hybrid**

$X_8 = q\bar{q}$ → **Tetraquark**

$X_8 = qqq$ → **Pentaquark**

Total angular momentum of **$Q\bar{Q}$X** or **QQX** :
$\mathbf{J} = \mathbf{L}_{Q\bar{Q}} + \mathbf{K} + \mathbf{S}_{Q\bar{Q}}$



color: $3 \otimes \bar{3} = \bar{3} \oplus 6$

X= q → **Double heavy baryon**

$X = \bar{q}\bar{q}$ → **Tetraquark**

$X = q\bar{q}q$ → **Pentaquark** and so on

BOEFT potentials $E^{(0)}_{\kappa,|\lambda|}(\mathbf{r})$: LDF (light quarks, gluons) static energies.
Potential between 2 heavy quarks

BOEFT can address all these states with inputs from Lattice QCD on $E^{(0)}_{\kappa,|\lambda|}$

NRQCD static energies and lattice operators

Berwein, N.B.,
Mohapatra, Vairo
2408.04719

NRQCD operator (gauge invariant) for exotic hadron $Q\bar{Q}X$ or QQX :

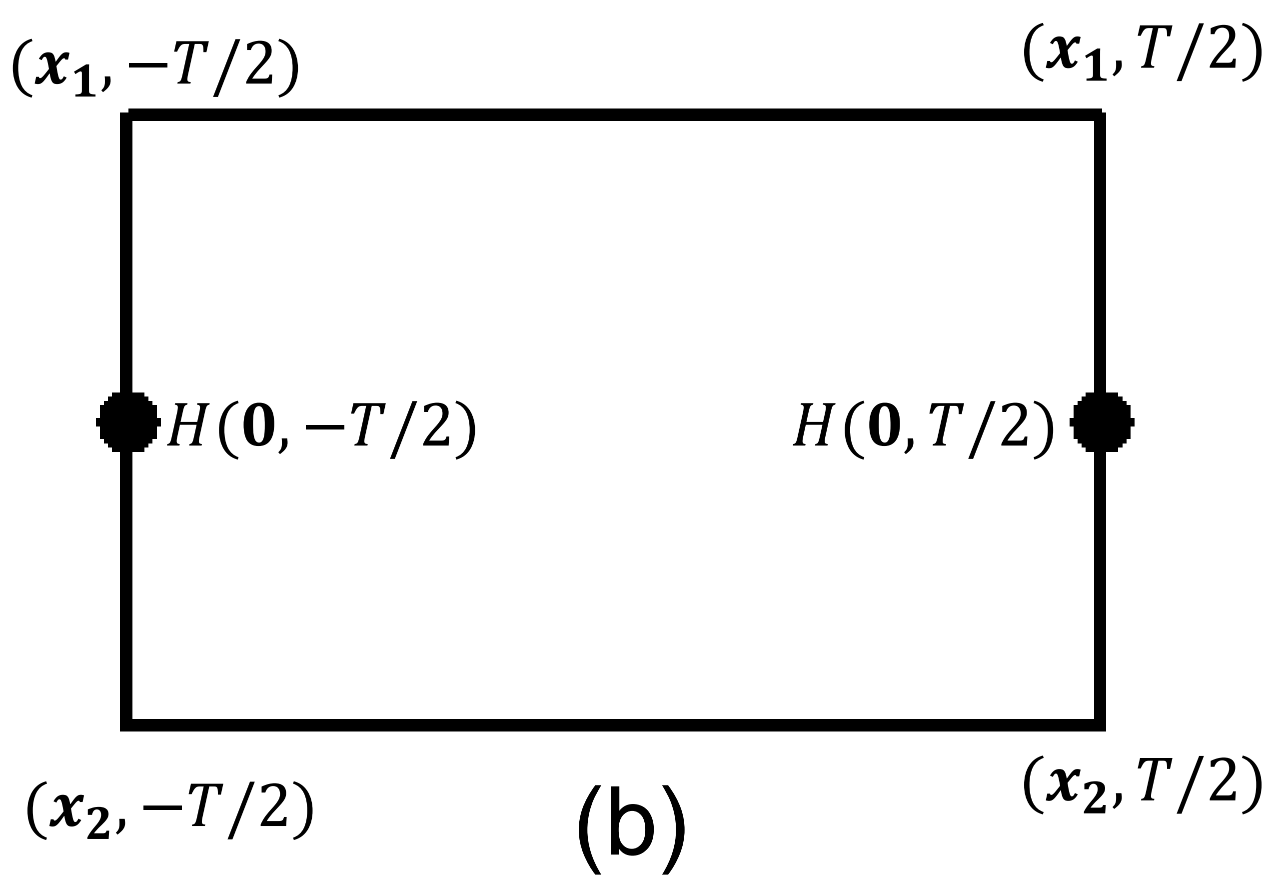
$$\mathcal{O}_{\kappa,\lambda}\left(t,\boldsymbol{r}\right)=\chi^{\dagger}\left(t,\boldsymbol{r}/2\right)\phi\left(t;\boldsymbol{r}/2,\boldsymbol{0}\right)P^{\alpha\dagger}_{\kappa,\lambda}H^{\alpha}_{\kappa}(t,\boldsymbol{0})\,\phi\left(t;\boldsymbol{0},-\boldsymbol{r}/2\right)\psi\left(t,-\boldsymbol{r}/2\right)$$

H^{α}_{κ} : LDF (gluon or light-quarks) operator characterizing X based on quantum # κ (isospin, color etc..)

$P^{\alpha}_{\kappa,\lambda}$: Projection vectors for projecting onto cyclindrical symmetry $D_{\infty h}$ representations.

$$E^{(0)}_{\kappa,|\lambda|}(r)=\lim_{T\to\infty}\frac{i}{T}\log\left[\langle\text{vac}|\mathcal{O}_{\kappa,\lambda}(T/2,\,\boldsymbol{r},\,\boldsymbol{R})\,\mathcal{O}^{\dagger}_{\kappa,\lambda}(-T/2,\,\boldsymbol{r},\,\boldsymbol{R})|\text{vac}\rangle\right]$$

(a)
Quarkonium

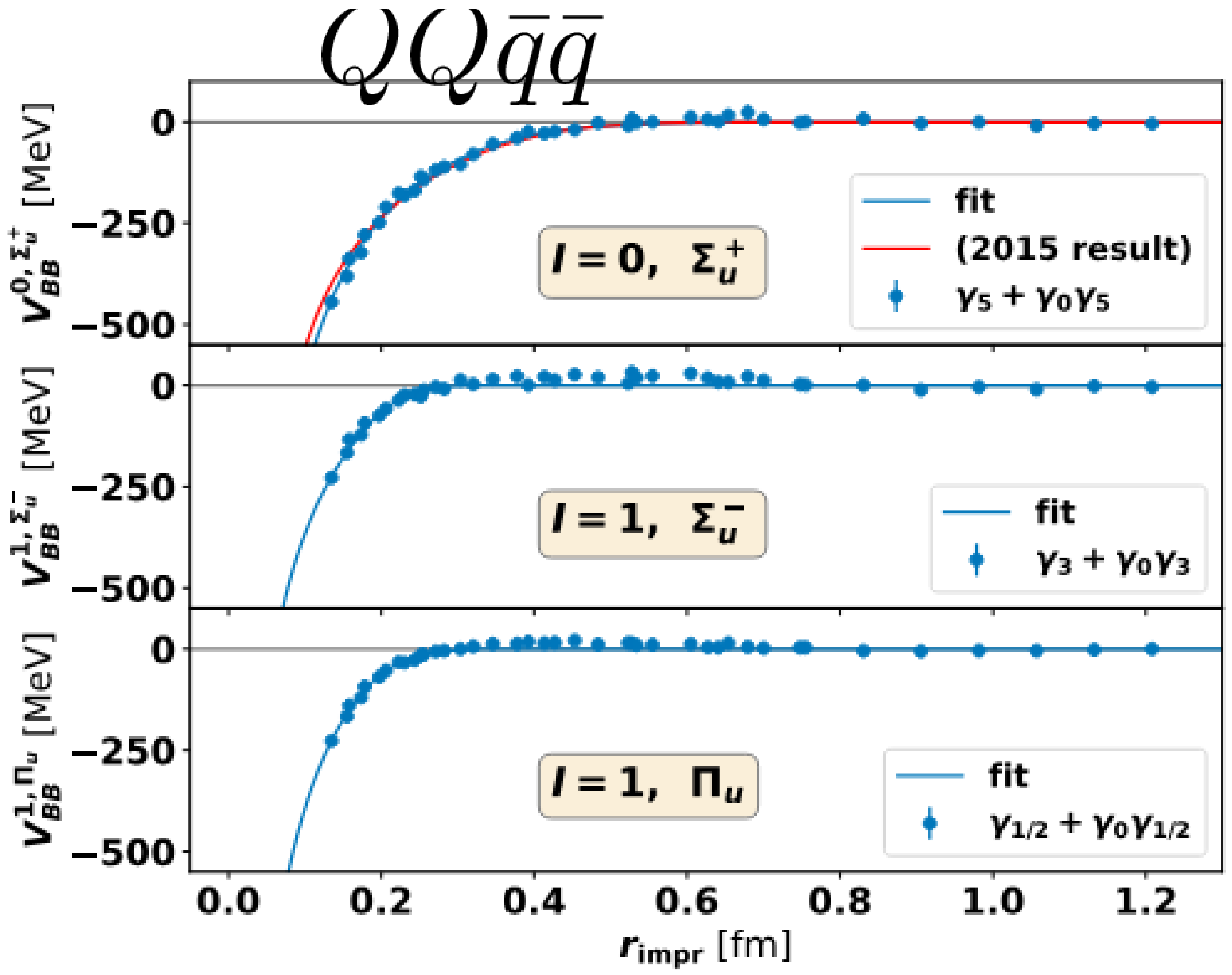


(b)
Wilson loop for exotics

Lattice NRQCD static energies for the QQqbarqbar case

already calculated with other interpolators

Liu Aoki Doi Hatsuda Ikea and Meng   Hal QCD method 2401.13917   heavy-light



Mueller et al, PoS LATTICE2023, 64 (2024)

Bicudo, Cichy, Peters, Wagner, Phys. Rev. D. 93, (2016)

Bicudo, Marinkovic,
Muller Wagner 2409.10786
BO heavy-light operators

$Q\bar{Q}q\bar{q}$: Operator Overlap

Berwein, N.B.,
Mohapatra, Vairo
2408.04719

NRQCD operator (gauge invariant) for exotic hadron: $Q\bar{Q}$ pair in **octet** color

$$\mathcal{O}_K(t,\,\mathbf{r},\,\mathbf{0}) = \chi^\dagger(\mathbf{t},\,\mathbf{r}/2)\phi(\mathbf{t},\,\mathbf{r}/2,\mathbf{0})\mathbf{H_K}(\mathbf{t},\,\mathbf{0})\phi(\mathbf{t},\,\mathbf{0},\,-\mathbf{r}/2)\psi(\mathbf{t},\,-\mathbf{r}/2)$$

$$\mathbf{H}_{\boldsymbol{K}}(t,\boldsymbol{x}) = \left[\bar{q}(t,\boldsymbol{x})\tilde{\boldsymbol{\Gamma}}T^a q(t,\boldsymbol{x})\right]T^a$$

$\tilde{\boldsymbol{\Gamma}}$: Dirac matrices based on quantum #'s

Quarkonium + Pions

Quarkonium state:

$$|\mathcal{Q}\rangle = \mathcal{N}\int d^3\boldsymbol{r}\,\Psi^{(n)}(\boldsymbol{r})\;\psi_b^\dagger(t,-\boldsymbol{r}/2)\,\phi_{bc}(t;-\boldsymbol{r}/2,\boldsymbol{r}/2)\,\chi_c(t,\boldsymbol{r}/2)\,|\Omega\rangle$$

Overlap of our operator on quarkonium + pion:

$$\langle\mathcal{Q}|\mathcal{O}_K^{Q\bar{Q}}(t,\boldsymbol{r})\,|\mathcal{Q}\rangle = 0$$

Meson-antimeson

Meson-antimeson state:

$$\begin{aligned}|M\bar{M}\rangle = \Big[&\mathcal{N}\int d^3\boldsymbol{x}\,\Psi_J(\boldsymbol{x})\\ &\times\int d^3\boldsymbol{y}\,\varphi_{J_1}(\boldsymbol{y}+\boldsymbol{x}/2)\;\psi_c^\dagger(t,-\boldsymbol{x}/2)\,\phi_{cd}(t;-\boldsymbol{x}/2,\boldsymbol{y}+\boldsymbol{x}/2)\,[P_+\,\Gamma_1\,q_d(t,\boldsymbol{y}+\boldsymbol{x}/2)]\\ &\times\int d^3\boldsymbol{z}\,\varphi_{J_2}(\boldsymbol{z}-\boldsymbol{x}/2)\;[\bar{q}_b(t,\boldsymbol{z}-\boldsymbol{x}/2)\,\Gamma_2\,P_-]\,\phi_{be}(t;\boldsymbol{z}-\boldsymbol{x}/2,\boldsymbol{x}/2)\,\chi_e(t,\boldsymbol{x}/2)\Big]|\mathrm{vac}\rangle\end{aligned}$$

Overlap of our operator on meson-antimeson:

$$\langle M\bar{M}|\mathcal{O}_K^{Q\bar{Q}}(t,\boldsymbol{r})\,|M\bar{M}\rangle \neq 0$$

Adjoint operators are good operators for lattice computation for $Q\bar{Q}q\bar{q}$ potentials !!!

Difficulties in QQbarqqbar

Lattice calculation

quarkonium plus pions

the BOEFT operators can solve this

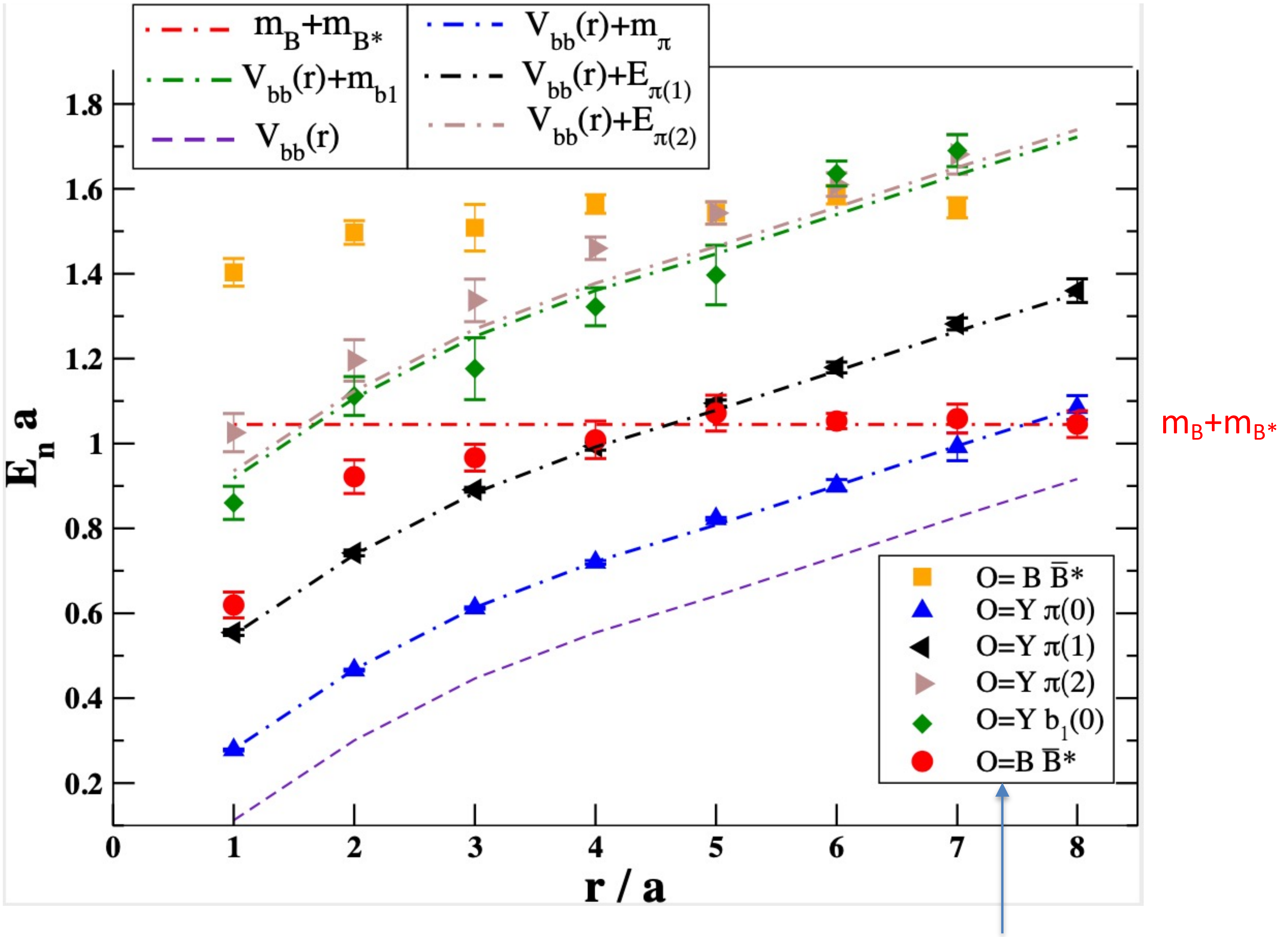
Tetraquark static energies

$\bar{b}\, b\, \bar{d}\, u$

Z_b channe $\vec{S}_h = \vec{S}_b + \vec{S}_{\bar{b}}$

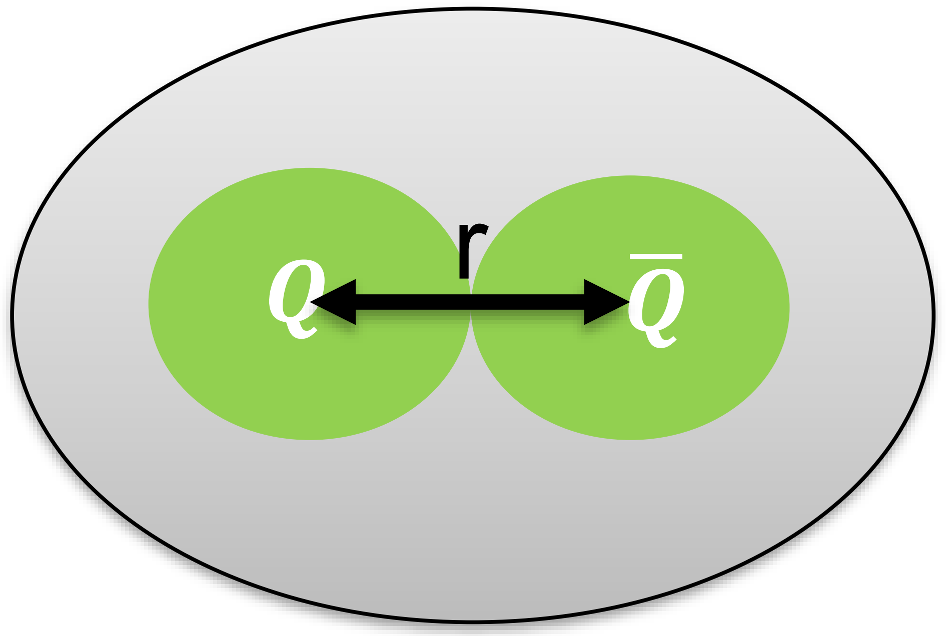
Sadl, Prevlosek, 211014568

Eigen-energies $E_n(r)$: channel S_h=1, CP=-1, ε=-1



Short distance behaviour of the NRQCD static energies

LDF-quantum #: $\kappa = \{K^{PC}, f\}$



Short-distance $(r \to 0)$ $\quad \Lambda_{H_\kappa}$

$Q\bar{Q}$: $$E^{(0)}_{\Sigma_g^+}(r) = V_s(r) + b_{\Sigma_g^+} r^2 + \cdots$$

$Q\bar{Q}X$: $$E^{(0)}_{\Lambda_\eta^\sigma}(r) = V_o(r) + \Lambda_{H_\kappa} + b_{\Lambda_\eta^\sigma} r^2 + \cdots$$

QQX: $$E^{(0)}_{\Lambda_\eta^\sigma}(r) = V_l(r) + \Lambda_{H_\kappa,l} + b_{\kappa\lambda,l}\, r^2 + \cdots \; (l = T, \Sigma)$$

$$V_s(r) = -\frac{4\alpha_s}{3r},\; V_o(r) = \frac{\alpha_s}{6r}$$
$$V_T(r) = -\frac{2\alpha_s}{3r},\; V_\Sigma(r) = \frac{\alpha_s}{3r}$$

Short distance behaviour of the NRQCD static energies

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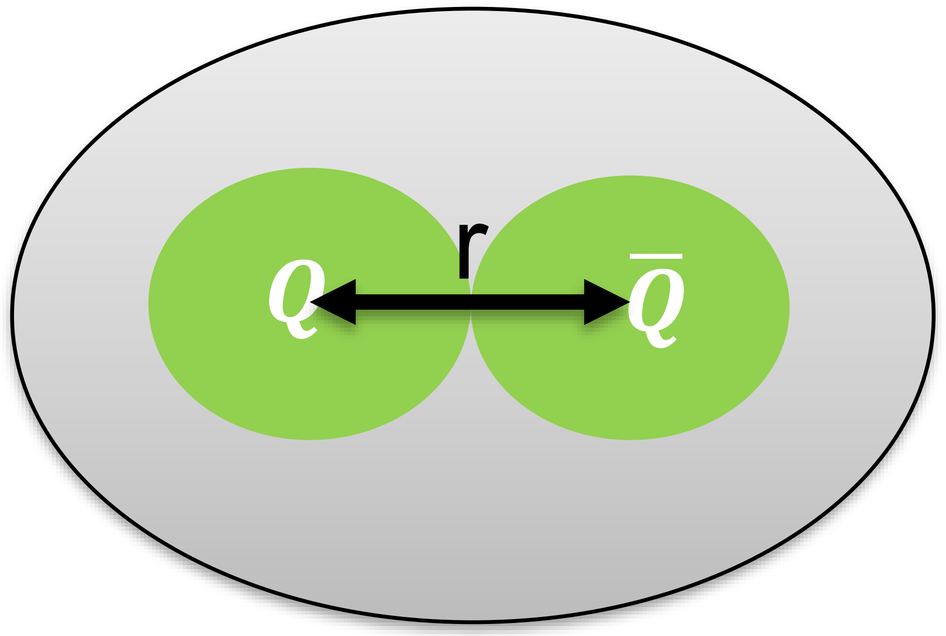


Short-distance $(r \to 0)$

$$\Lambda_{H_\kappa} = \lim_{T\to\infty} \frac{i}{T} \langle \text{vac}| H_\kappa^a(T/2, \boldsymbol{R})\, \phi^{ab}(T/2, -T/2)\, H_\kappa^{a\dagger}(-T/2, \boldsymbol{R}) |\text{vac}\rangle$$

- Gluelump / adjoint meson or baryon mass for $Q\bar{Q}X$ states
- Triplet meson or baryon / Sextet meson or baryon mass for QQX states
- Λ_{H_κ} depends only on κ ->degeneration

$Q\bar{Q}$: $E^{(0)}_{\Sigma_g^+}(r) = V_s(r) + b_{\Sigma_g^+} r^2 + \cdots$

$Q\bar{Q}X$: $E^{(0)}_{\Lambda_\eta^\sigma}(r) = V_o(r) + \Lambda_{H_\kappa} + b_{\Lambda_\eta^\sigma} r^2 + \cdots$

QQX: $E^{(0)}_{\Lambda_\eta^\sigma}(r) = V_l(r) + \Lambda_{H_\kappa, l} + b_{\kappa\lambda,l}\, r^2 + \cdots \quad (l = T, \Sigma)$

$$V_s(r) = -\frac{4\alpha_s}{3r}, \quad V_o(r) = \frac{\alpha_s}{6r}$$

$$V_T(r) = -\frac{2\alpha_s}{3r}, \quad V_\Sigma(r) = \frac{\alpha_s}{3r}$$

Short distance behaviour of the NRQCD static energies

Lowest gluelump 1^{+-}: ≈ 1.150 GeV

Most recent results on gluelump spectrum:
Herr, Schlosser, Wagner Phys. Rev. D 109 (2024)

$$m\left(1^{--}\right) - m\left(1^{+-}\right) \approx 300 \text{ MeV}$$

$$m\left(2^{--}\right) - m\left(1^{+-}\right) \approx 700 \text{ MeV}$$

Gluelump spectrum with 2+1 dynamical light quarks
Marsh, Lewis Phys. Rev. D 89 (2014):

$$m_A\left(1^{--}\right) - m_G\left(1^{+-}\right) = -10(103) \text{ MeV}$$

$$m_A\left(0^{-+}\right) - m_G\left(1^{+-}\right) = 34(161) \text{ MeV}$$

Adjoint meson spectrum (1^{--} & 0^{-+}):
Foster, Michael (UKQCD) Phys. Rev. D 59 (1999)

No results available on adjoint baryon, triplet meson or baryon / sextet meson or baryon masses

Our operator for triplet meson (lowest state) coincide with the operator of the good diquark but it is directly gauge invariant

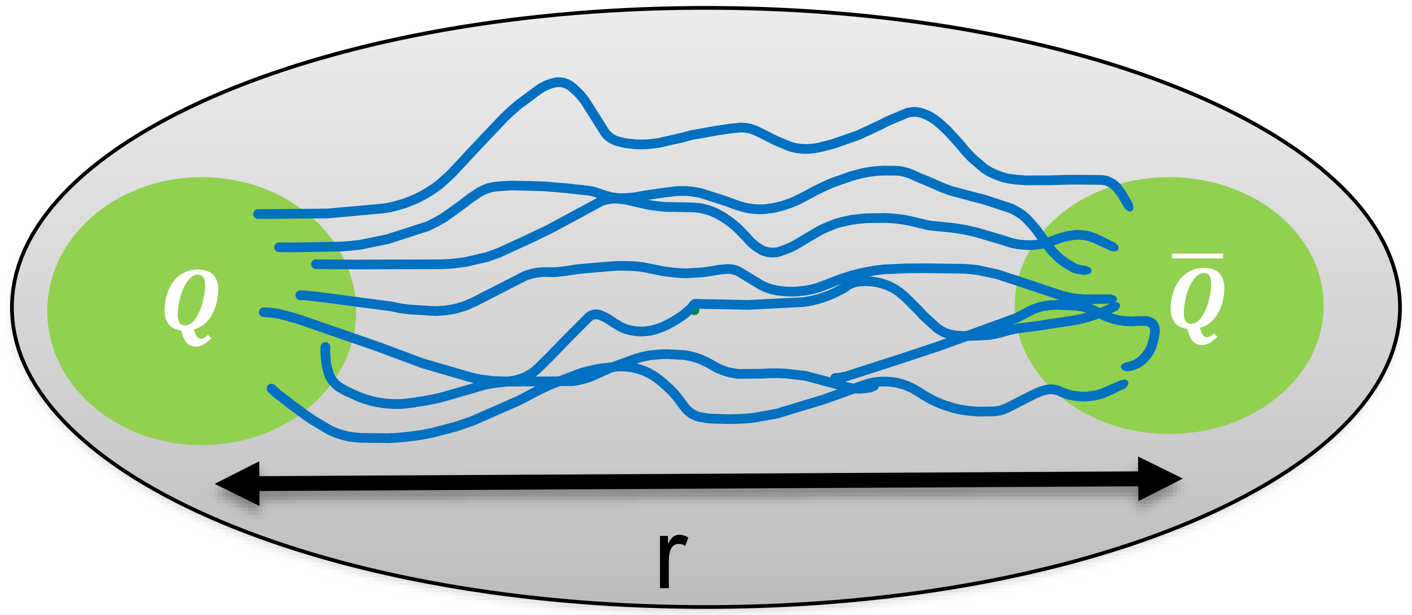
Francis, de Forcrand, Lewis, Maltman ICHEP2022

Behaviour of NRQCD BO static energies

1.BO quantum numbers Λ_η^σ are conserved at all r

2. Different BO quantum numbers can intersect each other

3. Static energies with the same BO quantum number that are close at some distance r show avoided level crossing



Large-distance ($r \to \infty$)

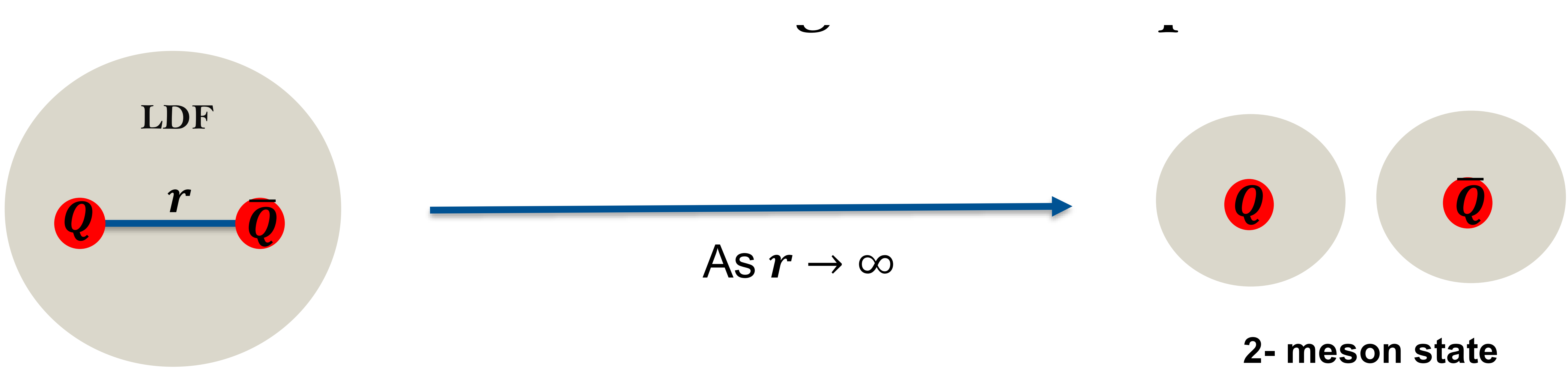
- String behavior (**pure SU(3) gauge**)

$$E_N(r) = \sqrt{\sigma^2 r^2 + 2\pi\sigma\,(N - 1/12)}$$

K. Juge, J. Kuti, C. Morningstar, Phys. Rev. Lett. 90 (2003)

- Mixing with pair of heavy-light states based on BO-quantum number Λ_η^σ representations

Quarkonium and Tetraquarks evolve in heavy-light at large distance



Consider $Q\bar{Q}q\bar{q}$ system:

BO-quantum # Λ_η^σ for adjoint meson:

$Q\bar{Q}$ (color)	Light Spin κ^{PC}	Λ_η^σ $(D_{\infty h})$
Octet	0^{-+}	Σ_u^-
	1^{--}	$\{\Sigma_g^+, \Pi_g\}$

BO-quantum # Λ_η^σ for meson-antimeson

$K_{\bar{q}}^P \otimes K_q^P$	K^{PC}	Static energies $D_{\infty h}$
$(1/2)^- \otimes (1/2)^+$	0^{-+}	$\{\Sigma_u^-\}$
	1^{--}	$\{\Sigma_g^+, \Pi_g\}$

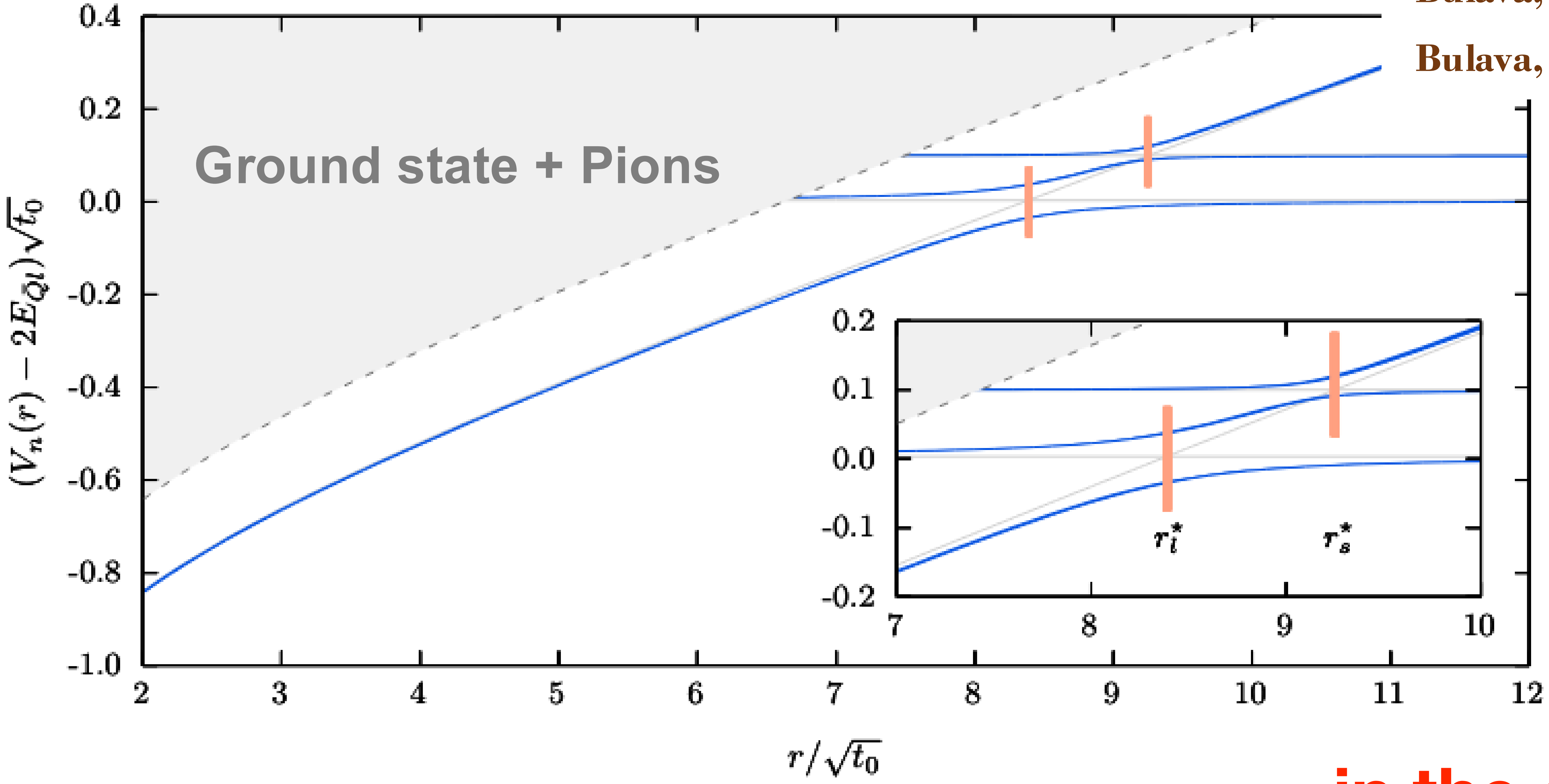
s-wave+s-wave
Ex. $\mathbf{D\bar{D}}$ threshold

Meson-antimeson have same BO-quantum # Λ_η^σ as of adjoint meson !!!

Avoided level crossing between quarkonium $1\Sigma_g$ and tetraquark $2\Sigma_g$

Bulava, Hoerz, Knechtli, Koch, Moir, Morningstar, Peardon, Phys. Lett. B. 793

Bulava, Knechtli, Koch, Morningstar, Peardon, Phys. Lett. B. 854 (2024)



avoided level crossing

In adiabatic repr

in the diabatic reps gives mixing

Model Hamiltonian for determining parameters:

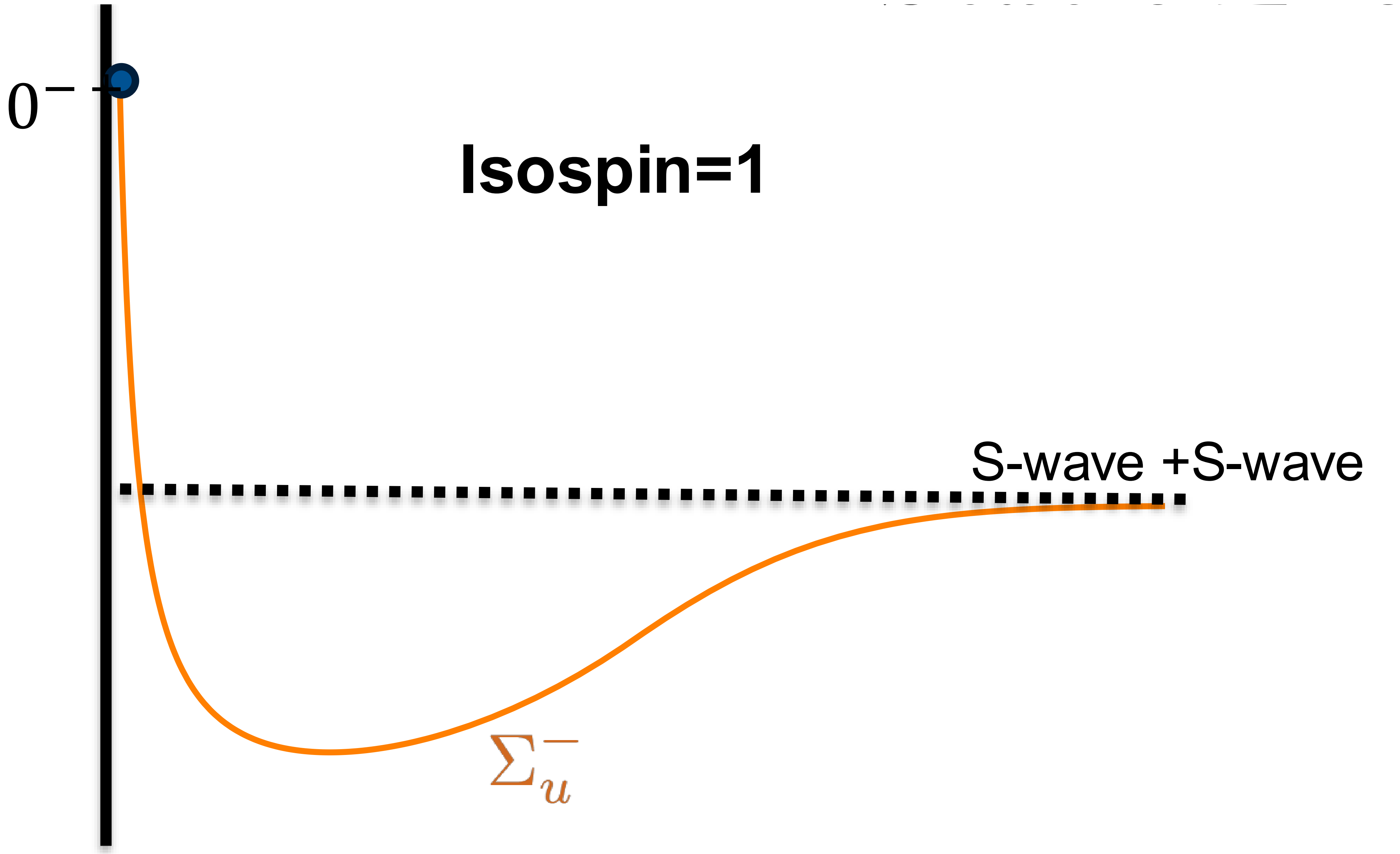
String breaking radius ≈ 1.22 fm

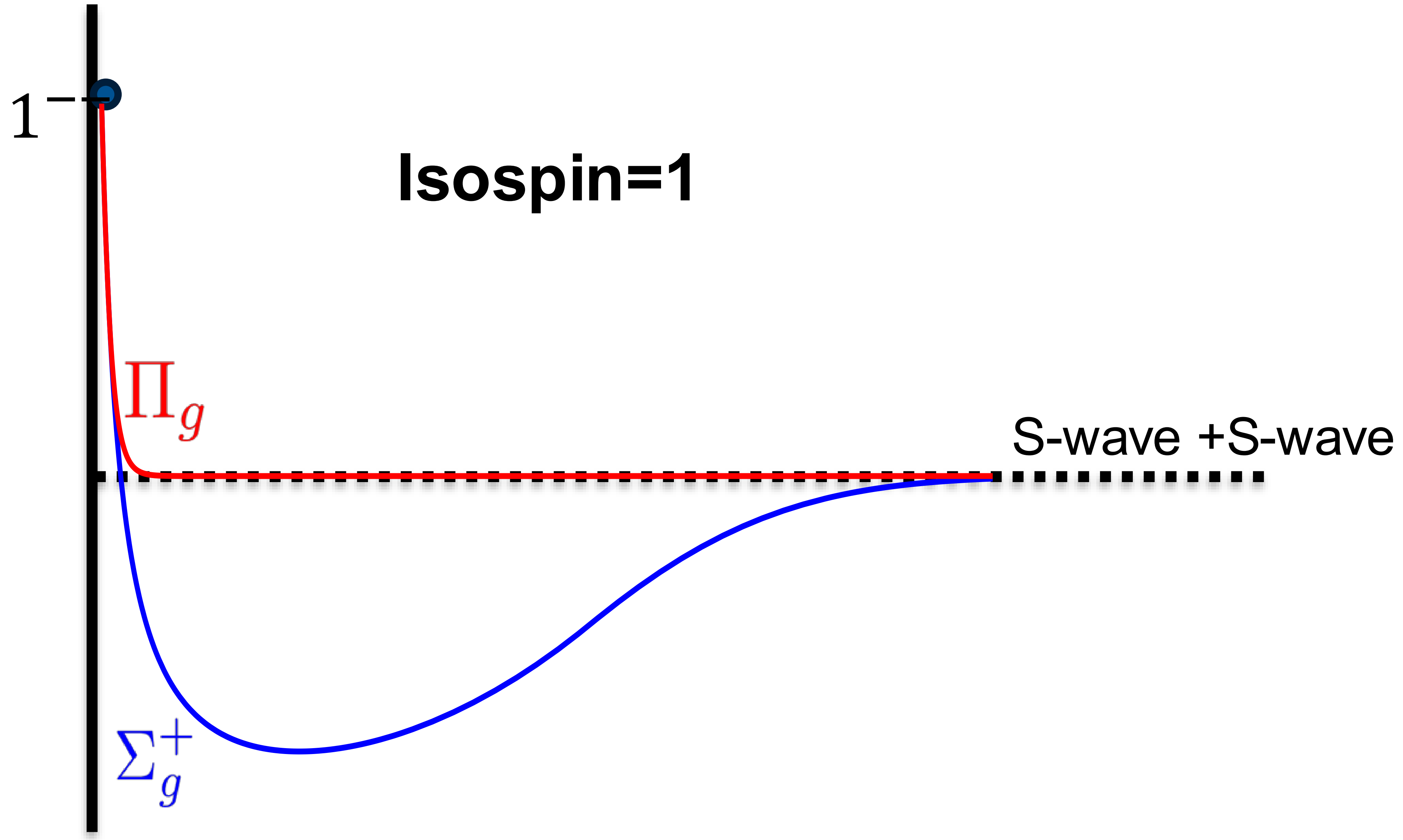
a ≈ 0.063 fm

$$H(r) = \begin{pmatrix} \hat{V}(r) & \sqrt{2}g_l & g_s \\ \sqrt{2}g_l & \hat{E}_1 & 0 \\ g_s & 0 & \hat{E}_2 \end{pmatrix}, \hat{V}(r) = \hat{V}_0 + \sigma r + \gamma/r$$

$$m_\pi \approx 200 - 340 \text{ MeV} \quad m_K \approx 440 - 480 \text{ MeV}$$

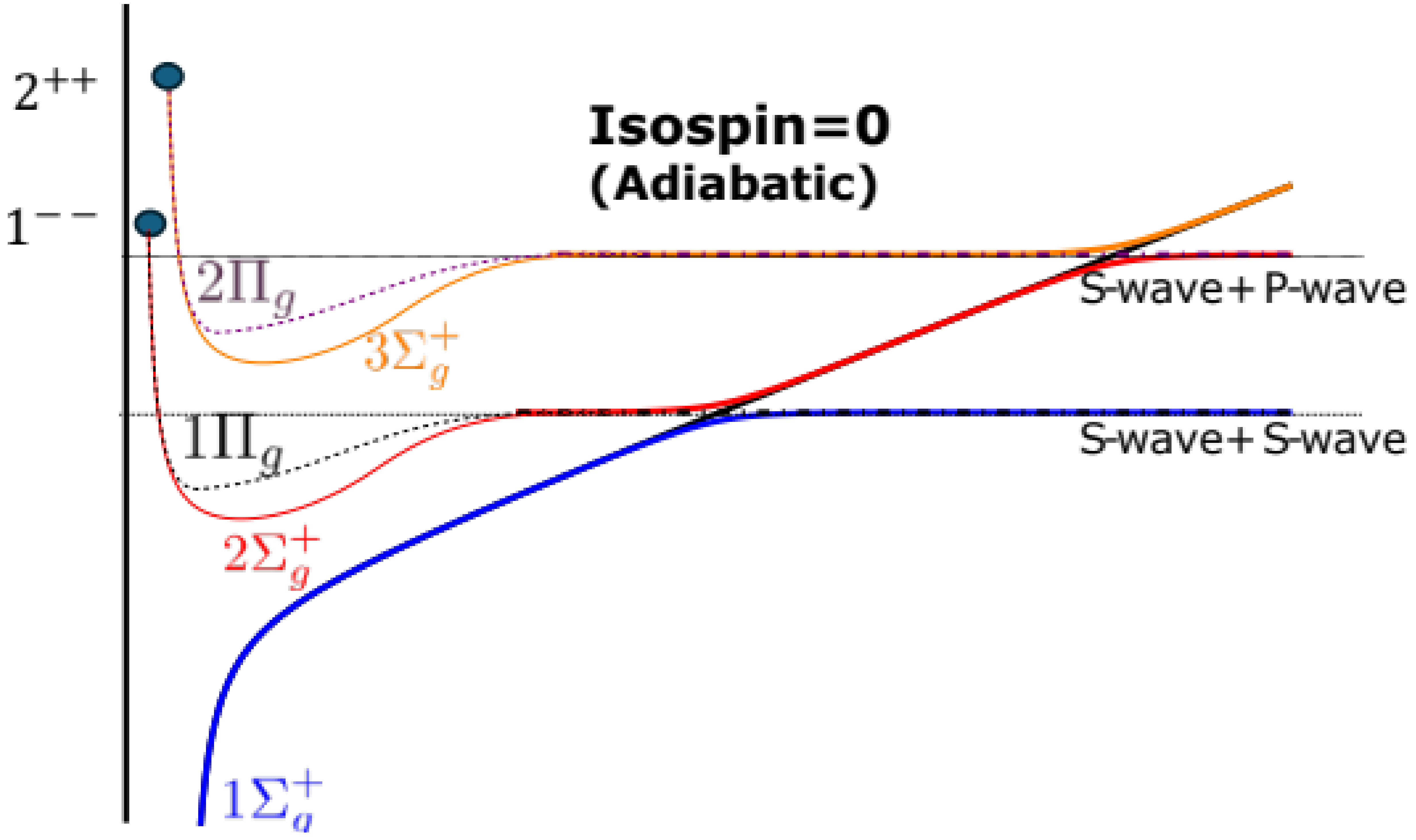
Behaviour static energies quarkonium and tetraquarks





Behavior of tetraquark static energy:

- Adjoint meson behavior at small r $(r \to 0)$
- Heavy meson pair threshold at large r $(r \to \infty)$
- Avoided crossing with quarkonium static energy (Isopsin=0)



BOEFT: mixing between static energies with same kappa

Berwein, N.B., Mohapatra, Vairo
2408.04719

- BOEFT Lagrangian:

$$L_{\rm BOEFT} = \int d^3\boldsymbol{R} \int d^3\boldsymbol{r} \sum_{\kappa\lambda\lambda'} {\rm Tr}\Big\{ \Psi^{\dagger}_{\kappa\lambda}(\mathbf{r},\, \mathbf{R},\, t) \Big[i\partial_t\, \delta_{\lambda\lambda'} - V_{\kappa\lambda\lambda'}(r) + P^{i\dagger}_{\kappa\lambda}(\theta,\phi) \frac{\boldsymbol{\nabla}^2_r}{m_Q} P^{i}_{\kappa\lambda'}(\theta,\phi) \Big] \Psi_{\kappa\lambda'}(\mathbf{r},\, \mathbf{R},\, t) \Big\}$$

LDF-quantum #: $\kappa = \{K^{PC}, f\}$ BO-quantum #: Λ^{σ}_{η} $\lambda = \pm\Lambda$

Projection vectors : $P^{i}_{K\lambda}(\theta,\varphi) = D^{\lambda *}_{Ki}(0,\theta,\varphi)$ Wigner D matrices

BOEFT: mixing between static energies with same kappa

Berwein, N.B., Mohapatra, Vairo 2408.04719

- BOEFT Lagrangian:

$$L_{\mathrm{BOEFT}} = \int d^3\boldsymbol{R} \int d^3\boldsymbol{r} \sum_{\kappa\lambda\lambda'} \mathrm{Tr}\Big\{ \Psi^{\dagger}_{\kappa\lambda}(\mathbf{r},\,\mathbf{R},\,t) \Big[i\partial_t\, \delta_{\lambda\lambda'} - V_{\kappa\lambda\lambda'}(r) + P^{i\dagger}_{\kappa\lambda}(\theta,\phi)\, \frac{\boldsymbol{\nabla}^2_r}{m_Q} P^{i}_{\kappa\lambda'}(\theta,\phi) \Big] \Psi_{\kappa\lambda'}(\mathbf{r},\,\mathbf{R},\,t) \Big\}$$

LDF-quantum #: $\kappa = \{K^{PC}, f\}$ BO-quantum #: Λ^{σ}_{η} $\lambda = \pm\Lambda$

Projection vectors : $P^{i}_{K\lambda}(\theta,\varphi) = D^{\lambda *}_{Ki}(0,\theta,\varphi)$ Wigner D matrices

- **BO potentials: Potential between Q & $\bar{Q}$** due to LDF (light quarks, gluons).

Born-Oppenheimer (BO) potential: $V_{\kappa\lambda\lambda'}(r) = \boxed{E^{(0)}_{\kappa,|\lambda|}(r)}\delta_{\lambda\lambda'} + \boxed{\frac{V^{(1)}_{\kappa\lambda\lambda'}(r)}{m_Q}} + \dots\,,$

Static Energy **Spin-dependent potentials**

Brambilla, Lai, Segovia, Castellà, Phys. Rev. D. 101, (2020)

Brambilla, Lai, Segovia, Castellà, Vairo Phys. Rev. D. 99, (2019)

Soto, Valls, Phys. Rev. D 108 (2023)

Castellà , Soto Phys. Rev. D. 102, (2020)

BOEFT coupled Schrödinger equations

Berwein, N.B.,
Mohapatra, Vairo
2408.04719

Wave-function for Exotic State:

$$|X_N\rangle = \sum_\lambda \int d^3r |\boldsymbol{r}\rangle \otimes |k,\lambda\rangle \phi^{(N)}_{\kappa\lambda}(\boldsymbol{r})$$

$|\boldsymbol{r}\rangle$: Heavy quark pair state separated by position r

$|k,\lambda\rangle$: Light quark or gluon state: Parametrically depends on r

Total orbital momentum for Exotic State:

$$\boldsymbol{L} = \boldsymbol{L}_Q + \boldsymbol{K}$$

$\boldsymbol{K}$: angular-momentum of light d.o.f

$\boldsymbol{L_Q}$: orbital-angular momentum of QQ or $Q\bar{Q}$ pair

Angular wave-function:

$$|l,m;k,\lambda\rangle = \int \frac{d\Omega}{\sqrt{2\pi}} |\theta,\phi\rangle\, |k,\lambda\rangle\, D^{\lambda}_{lm}(\psi,\theta,\varphi)$$

BOEFT coupled Schrödinger equations

Berwein, N.B., Mohapatra, Vairo
2408.04719

- Adiabatic Radial Schrödinger equation:

  Mixing different static energies with same LDF-quantum #: $\kappa = \{K^{PC}, f\}$

$$\sum_{\lambda}\left[-\frac{1}{m_Q r^2}\partial_r\, r^2\partial_r + \frac{1}{m_Q r^2}\boxed{M_{\lambda'\lambda}} + E^{(0)}_{\kappa,|\lambda|}(r)\delta_{\lambda\lambda'}\right]\psi^{(N)}_{\kappa\lambda}(r) = \mathcal{E}_N\,\psi^{(N)}_{\kappa\lambda'}(r)$$

Mixing term from angular momentum piece:

Coupling static energies with different BO-quantum numbers Λ^{σ}_{η}

- General expression of $M_{\lambda'\lambda}$ (matrix in $\lambda' - \lambda$ basis) :

$$\lambda, \lambda' = \pm\Lambda$$

$$\begin{aligned} M_{\lambda'\lambda} &= \langle l, m; k, \lambda'|\boldsymbol{L}_Q^2|l, m; k, \lambda\rangle \\ &= \left(l(l+1) - 2\lambda^2 + k(k+1)\right)\delta^{\lambda'\lambda} - \sqrt{k(k+1) - \lambda(\lambda+1)}\sqrt{l(l+1) - \lambda(\lambda+1)}\,\delta^{\lambda'\lambda+1} \\ &- \sqrt{k(k+1) - \lambda(\lambda-1)}\sqrt{l(l+1) - \lambda(\lambda-1)}\,\delta^{\lambda'\lambda-1} \end{aligned}$$

Coupled Equations for lowest Hybrids ($Q\bar{Q}g$) and Tetraquarks ($QQ\overline{qq}$ or $Q\bar{Q}q\bar{q}$):

Berwein, N.B.,
Mohapatra, Vairo
2408.04719

LDF quantum # K=1

$$\left[-\frac{1}{m_Q r^2}\,\partial_r r^2 \partial_r + \frac{1}{m_Q r^2}\begin{pmatrix} l(l+1)+2 & -2\sqrt{l(l+1)} \\ -2\sqrt{l(l+1)} & l(l+1) \end{pmatrix} + \begin{pmatrix} E_\Sigma & 0 \\ 0 & E_\Pi \end{pmatrix}\right]\begin{pmatrix} \psi^{(N)}_{\Sigma,\sigma_p} \\ \psi^{(N)}_{\Pi,\sigma_P} \end{pmatrix} = \mathcal{E}_N \begin{pmatrix} \psi^{(N)}_{\Sigma,\sigma_P} \\ \psi^{(N)}_{\Pi,\sigma_P} \end{pmatrix}$$

$$\left[-\frac{1}{m_Q r^2}\,\partial_r\, r^2\, \partial_r + \frac{l(l+1)}{m_Q r^2} + E_\Pi\right]\psi^{(N)}_{\Pi,-\sigma_P} = \mathcal{E}_N\, \psi^{(N)}_{\Pi,-\sigma_P}$$

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92 (2015)

LDF quantum # K=2

$$\left[-\frac{1}{m_Q r^2}\,\partial_r r^2 \partial_r + \frac{1}{m_Q r^2}\begin{pmatrix} l(l+1)+6 & -2\sqrt{3l(l+1)} & 0 \\ -2\sqrt{3l(l+1)} & l(l+1)+4 & -2\sqrt{(l-1)(l+2)} \\ 0 & -2\sqrt{(l-1)(l+2)} & (l-1)(l+2) \end{pmatrix} + \begin{pmatrix} E_\Sigma & 0 & 0 \\ 0 & E_\Pi & 0 \\ 0 & 0 & E_\Delta \end{pmatrix}\right]\begin{pmatrix} \psi^{(N)}_{\Sigma,\sigma_p} \\ \psi^{(N)}_{\Pi,\sigma_P} \\ \psi^{(N)}_{\Delta,\sigma_p} \end{pmatrix} = \mathcal{E}_N \begin{pmatrix} \psi^{(N)}_{\Sigma,\sigma_P} \\ \psi^{(N)}_{\Pi,\sigma_P} \\ \psi^{(N)}_{\Delta,\sigma_p} \end{pmatrix}$$

$$\left[-\frac{1}{m_Q r^2}\,\partial_r r^2 \partial_r + \frac{1}{m_Q r^2}\begin{pmatrix} l(l+1)+4 & -2\sqrt{(l-1)(l+2)} \\ -2\sqrt{(l-1)(l+2)} & (l-1)(l+2) \end{pmatrix} + \begin{pmatrix} E_\Pi & 0 \\ 0 & E_\Delta \end{pmatrix}\right]\begin{pmatrix} \psi^{(N)}_{\Pi,-\sigma_P} \\ \psi^{(N)}_{\Delta,-\sigma_p} \end{pmatrix} = \mathcal{E}_N \begin{pmatrix} \psi^{(N)}_{\Pi,-\sigma_P} \\ \psi^{(N)}_{\Delta,-\sigma_p} \end{pmatrix}$$

Coupled Equations for Doubly Heavy Baryons (QQq) and Pentaquarks (QQq̄qq or QQ̄qqq):

Berwein, N.B., Mohapatra, Vairo 2408.04719

LDF quantum # K=1/2

$$\left[-\frac{1}{m_Q r^2}\,\partial_r\, r^2\,\partial_r + \frac{(l-1/2)(l+1/2)}{m_Q r^2} + E_{K_\eta}\right]\psi^{(N)}_{K_\eta,\sigma_P} = \mathcal{E}_N\,\psi^{(N)}_{K_\eta,\sigma_P}$$

LDF quantum # K=3/2

$$\left[-\frac{1}{m_Q r^2}\partial_r r^2 \partial_r + \frac{1}{m_Q r^2}\begin{pmatrix} l(l-1)-\frac{9}{4} & -\sqrt{3l(l+1)-\frac{9}{4}} \\ -\sqrt{3l(l+1)-\frac{9}{4}} & l(l+1)-\frac{3}{4}\end{pmatrix} + \begin{pmatrix} E_{(1/2)_u} & 0 \\ 0 & E_{(3/2)_u}\end{pmatrix}\right]\begin{pmatrix}\psi^{(N)}_{1/2,\sigma_P} \\ \psi^{(N)}_{3/2,\sigma_P}\end{pmatrix} = \mathcal{E}_n\begin{pmatrix}\psi^{(N)}_{1/2,\sigma_P} \\ \psi^{(N)}_{3/2,\sigma_P}\end{pmatrix}$$

$$\left[-\frac{1}{m_Q r^2}\partial_r r^2 \partial_r + \frac{1}{m_Q r^2}\begin{pmatrix} l(l+3)+\frac{17}{4} & -\sqrt{3l(l+1)-\frac{9}{4}} \\ -\sqrt{3l(l+1)-\frac{9}{4}} & l(l+1)-\frac{3}{4}\end{pmatrix} + \begin{pmatrix} E_{(1/2)_u} & 0 \\ 0 & E_{(3/2)_u}(r)\end{pmatrix}\right]\begin{pmatrix}\psi^{(N)}_{1/2,-\sigma_P} \\ \psi^{(N)}_{3/2,-\sigma_P}\end{pmatrix} = \mathcal{E}_n\begin{pmatrix}\psi^{(N)}_{1/2,-\sigma_P} \\ \psi^{(N)}_{3/2,-\sigma_P}\end{pmatrix}$$

Castellà , Soto Phys. Rev. D. 104, 074027 (2021)

Castellà , Soto Phys. Rev. D. 102, 014013 (2020)

$X(3872)$ & $T_{cc}^{+}(3875)$

$Q\bar{Q}q\bar{q}$

Berwein, N.B., Mohapatra, Vairo 2408.04719

$Q\bar{Q}$ color state	Light spin	Static energies	l	J^{PC} $\{S_Q=0, S_Q=1\}$	Multiplets
Octet	0^{-+}	$\{\Sigma_u^-\}$	0	$\{0^{++}, 1^{+-}\}$	T_1^0
			1	$\{1^{--}, (0,1,2)^{-+}\}$	T_2^0
			2	$\{2^{++}, (1,2,3)^{+-}\}$	T_3^0
	1^{--}	$\{\Sigma_g^{+\prime}, \Pi_g\}$	1	$\{1^{+-}, (0,1,2)^{++}\}$	T_1^1
		$\{\Sigma_g^{+\prime}\}$	0	$\{0^{-+}, 1^{--}\}$	T_2^1
		$\{\Pi_g\}$	1	$\{1^{-+}, (0,1,2)^{--}\}$	T_3^1
		$\{\Sigma_g^{+\prime}, \Pi_g\}$	2	$\{2^{-+}, (1,2,3)^{--}\}$	T_4^1

Isospin-1 channel:
$Z_c(3900), Z_c(4200), Z_b(10610)$, $Z_b(10610)$**states:**

Mixing between $K^{PC} = 0^{-+}$ and $K^{PC} = 1^{--}$
Light-quark spin-symmetry !!

Voloshin, Phys. Rev. D. 93, 074011 (2016)

Braaten, Bruschini arXiv 2409.08002

Isospin-0 channel:
$X(3872)$

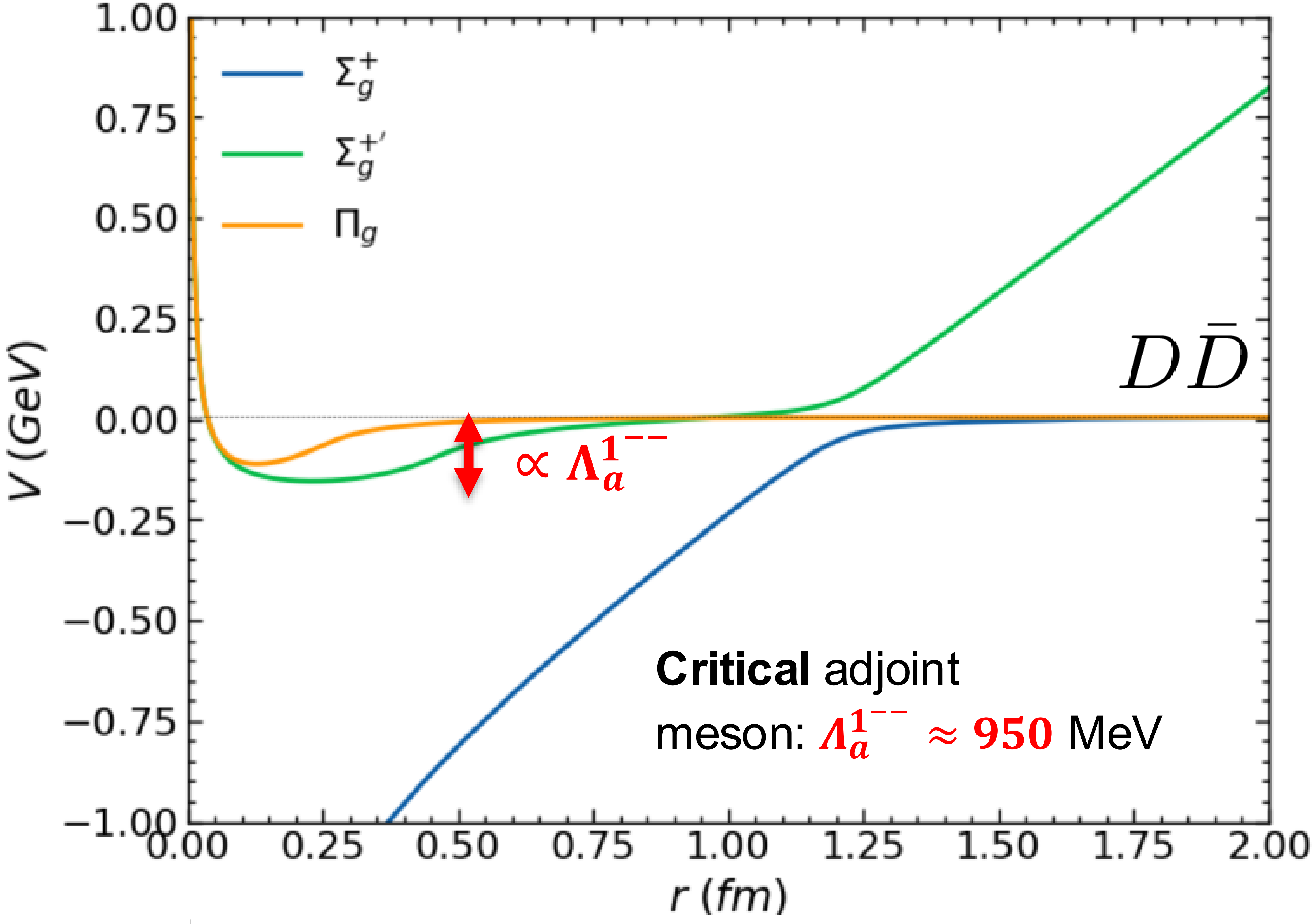
Brambilla, AM, Scirpa, Vairo 2411.xxxx

Until when we introduce spin all states in the X multiplet are degenerate

X(3872)

Berwein, N.B.,Mohapatra, Vairo 2408.04719

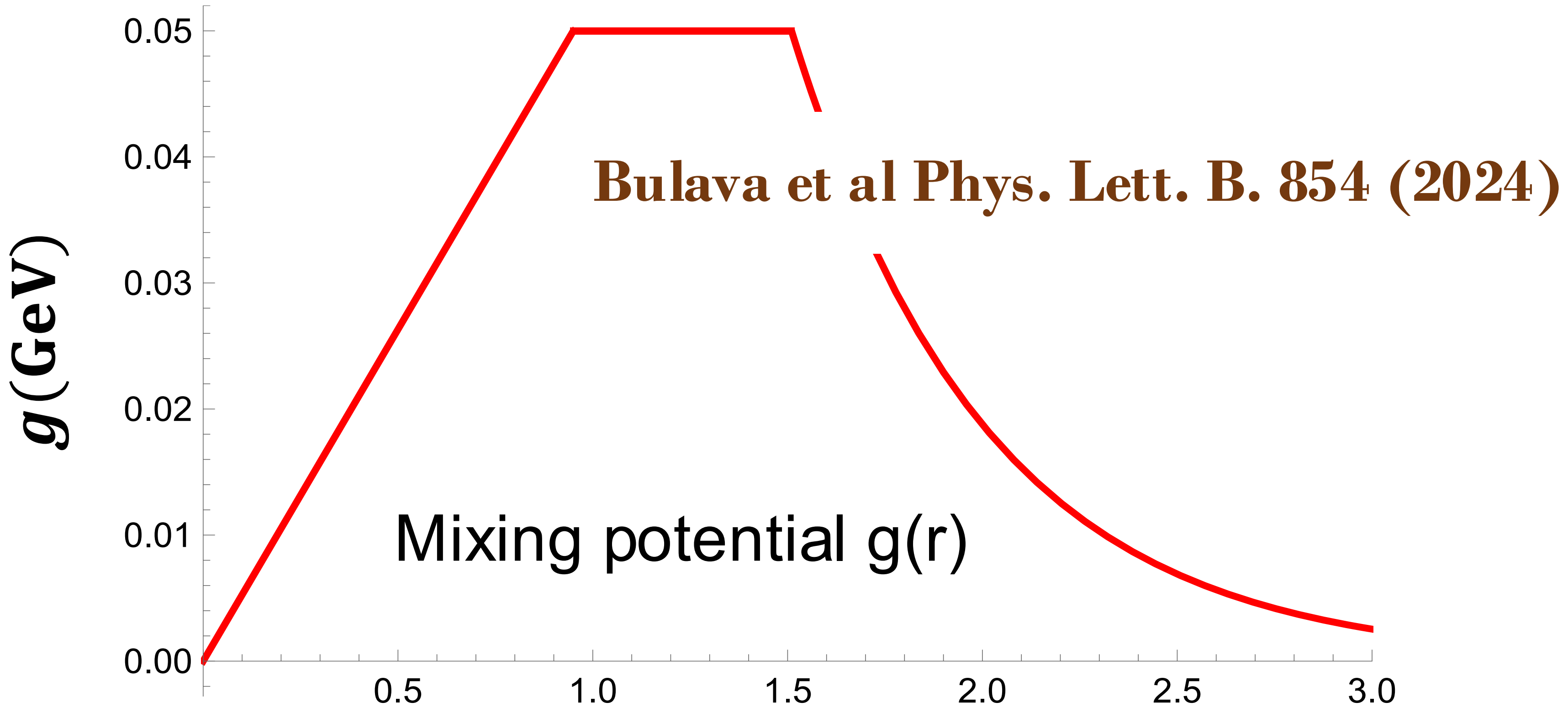
N.B. Mohapatra, Scirpa, Vairo 2411.xxxx



Coupled-channel Equations:

$$\left[-\frac{1}{m_Q r^2}\partial_r r^2 \partial_r + \frac{1}{m_Q r^2}\begin{pmatrix} l\,(l+1) & 0 & 0 \\ 0 & l(l+1)+2 & -2\sqrt{l(l+1)} \\ 0 & -2\sqrt{l(l+1)} & l(l+1) \end{pmatrix} + \begin{pmatrix} E_{\Sigma_g^+}(r) & g(r) & 0 \\ g(r) & E_{\Sigma_g^{+\prime}}(r) & 0 \\ 0 & 0 & E_{\Pi_g}(r) \end{pmatrix}\right]\begin{pmatrix}\psi_\Sigma \\ \psi_{\Sigma'} \\ \psi_\Pi\end{pmatrix} = \mathcal{E}\begin{pmatrix}\psi_\Sigma \\ \psi_{\Sigma'} \\ \psi_\Pi\end{pmatrix}$$

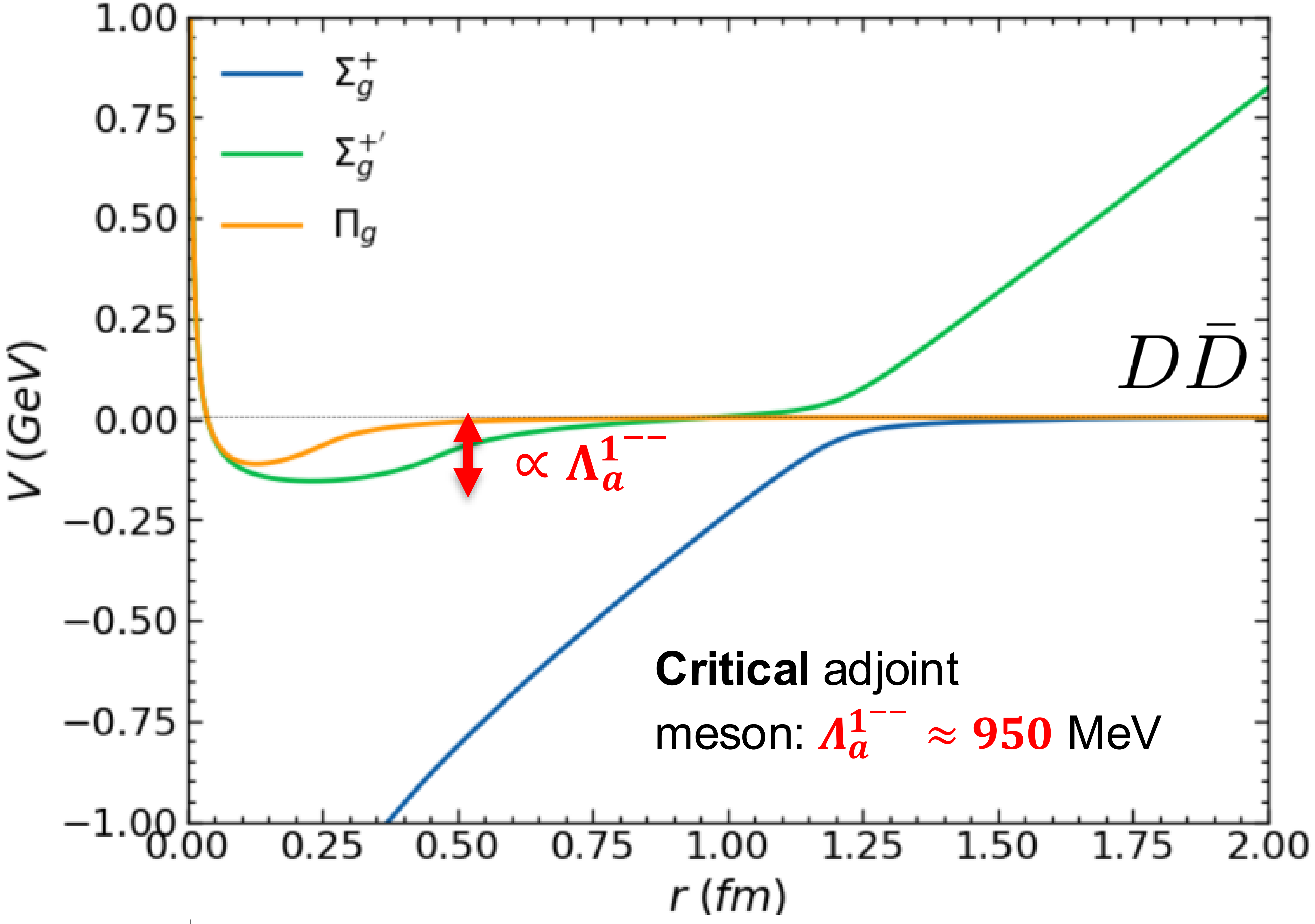
$l = 1$



X(3872)

Berwein, N.B.,Mohapatra, Vairo 2408.04719

N.B. Mohapatra, Scirpa, Vairo 2411.xxxx



Coupled-channel Equations:

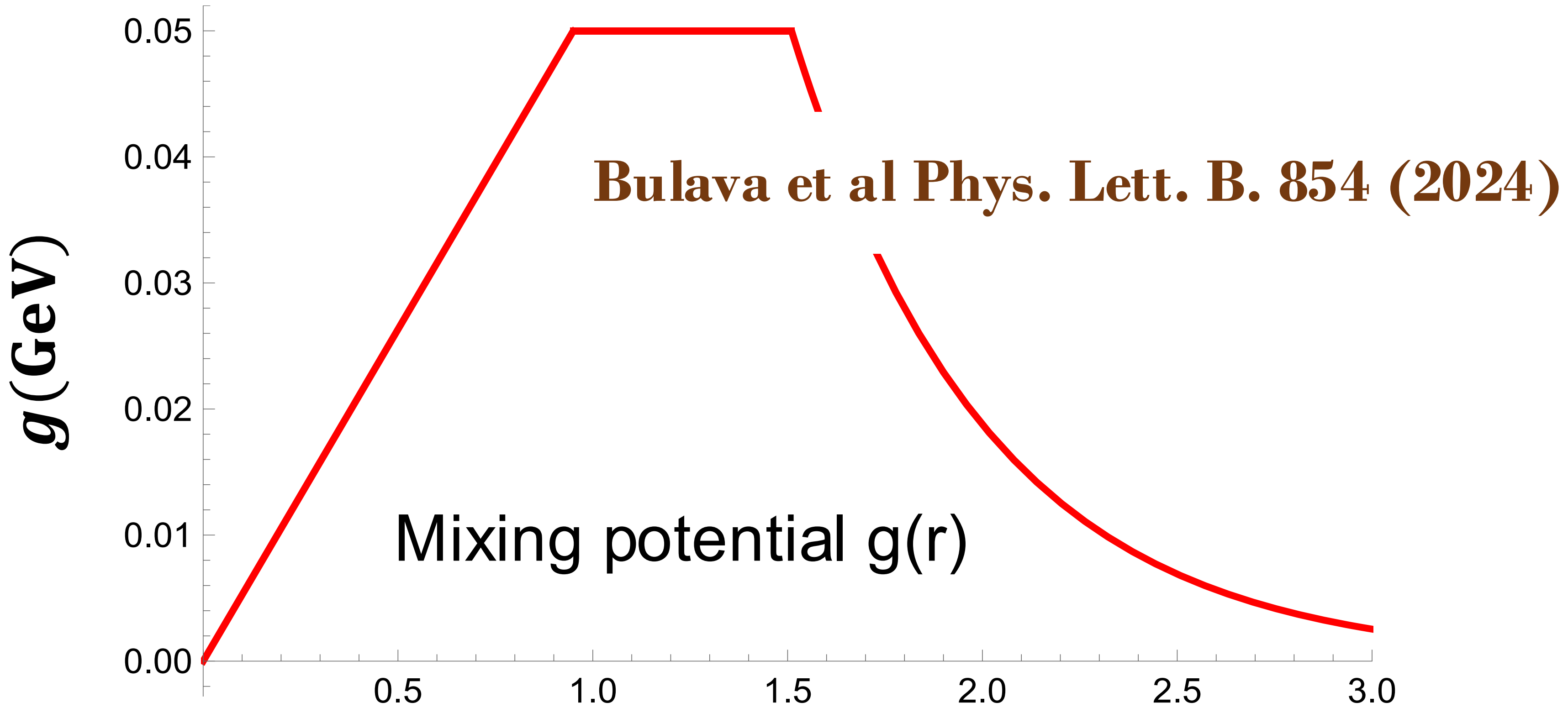
$$\left[-\frac{1}{m_Q r^2}\partial_r r^2 \partial_r + \frac{1}{m_Q r^2}\begin{pmatrix} l(l+1) & 0 & 0 \\ 0 & l(l+1)+2 & -2\sqrt{l(l+1)} \\ 0 & -2\sqrt{l(l+1)} & l(l+1) \end{pmatrix} + \begin{pmatrix} E_{\Sigma_g^+}(r) & g(r) & 0 \\ g(r) & E_{\Sigma_g^{+'}}(r) & 0 \\ 0 & 0 & E_{\Pi_g}(r) \end{pmatrix}\right]\begin{pmatrix}\psi_\Sigma \\ \psi_{\Sigma'} \\ \psi_\Pi\end{pmatrix} = \mathcal{E}\begin{pmatrix}\psi_\Sigma \\ \psi_{\Sigma'} \\ \psi_\Pi\end{pmatrix}$$

$l = 1$

1) Quarkonium percentage: $|\psi_\Sigma|^2 \sim 6$ %
2) Tetraquark percentage: $|\psi_{\Sigma'}|^2 \sim 35$ %, $|\psi_\Pi|^2 \sim 59$ %
3) Radius $\sim$ 14 fm. (and a)
4) Deeper bound state in bottom sector: 5 MeV below spin-isospin averaged $B\bar{B}$ threshold.

clear prediction of BOEFT-> the adjoint does not depend on the flavor

5) we found also a deeper bound state 400 MeV below DDbar_> spin average 1P quarkonium (3529 MeV)



Critical adjoint meson: $\Lambda_a^{1^{--}} \approx 950$ MeV:
No other bound states in higher multiplets T_2^1, T_3^1, T_4^1 ….

X(3872)

Berwein, N.B.,Mohapatra, Vairo 2408.04719

N.B. Mohapatra, Scirpa, Vairo 2411.xxxx

Radiative decays

$$\mathcal{R}_{\gamma\psi} = \frac{\Gamma_{\chi_{c1}(3872)\to\gamma\psi(2s)}}{\Gamma_{\chi_{c1}(3872)\to\gamma J\psi}},$$

We find: $R_{\gamma\psi} = 2.95 \pm 2.28$

In agreement within errors with LHCb

This is thanks to the quarkonium component-> may work also for production, gives the correct order of magnitude for compositeness

X(3872)

Berwein, N.B.,Mohapatra, Vairo 2408.04719
N.B. Mohapatra, Scirpa, Vairo 2411.xxxx

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Multiplet T_1^1: $\{1^{+-}, (0,1,2)^{++}\}$

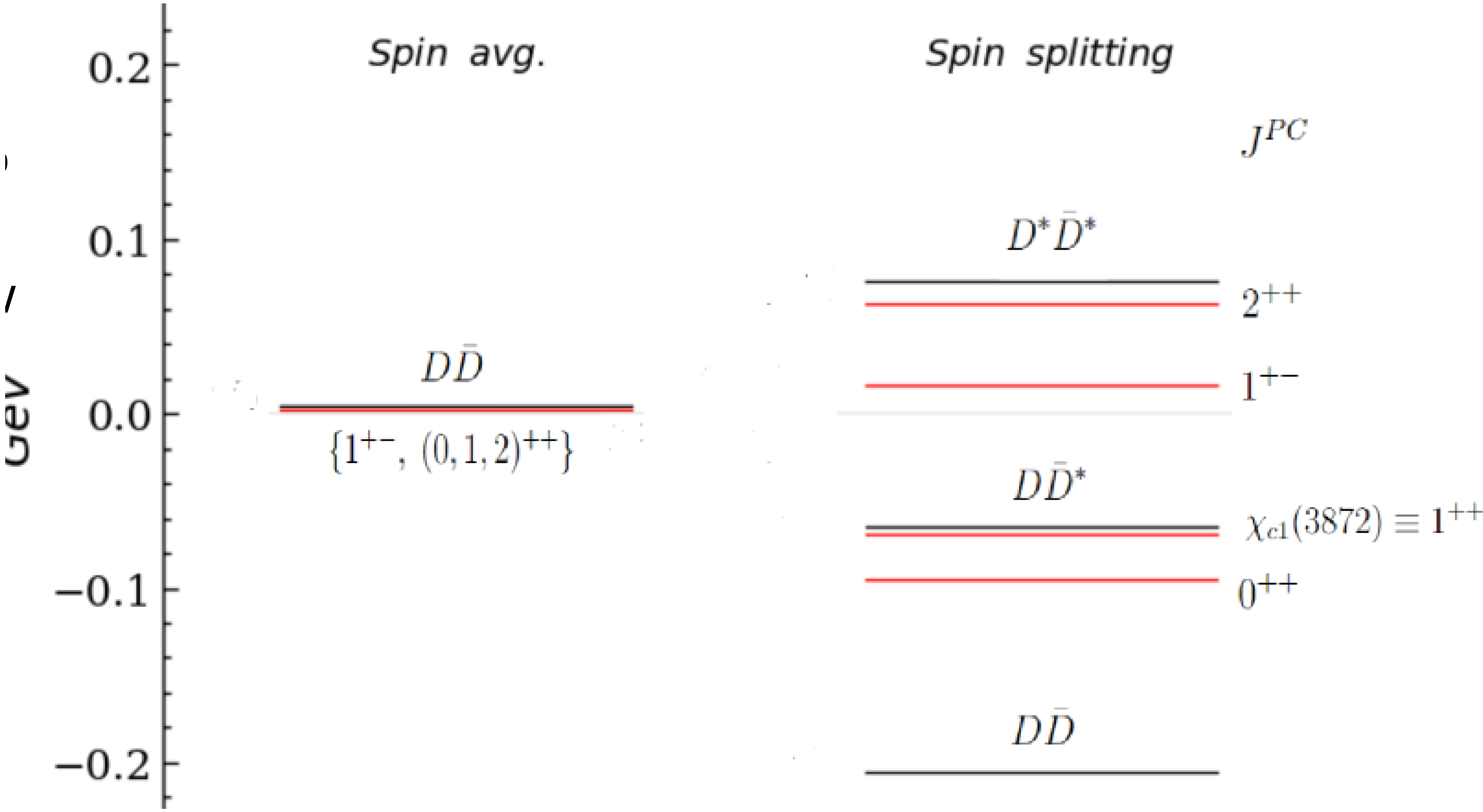
1^{++} state: Identified with $\chi_{c1}(3872)$

1^{+-} state: Mass around 3.956 (11) GeV.

2^{++} state: Mass around 3.996 (11) GeV.

0^{++} state: Mass around 3.838 (11) GeV.

Identified with X(3940) ?



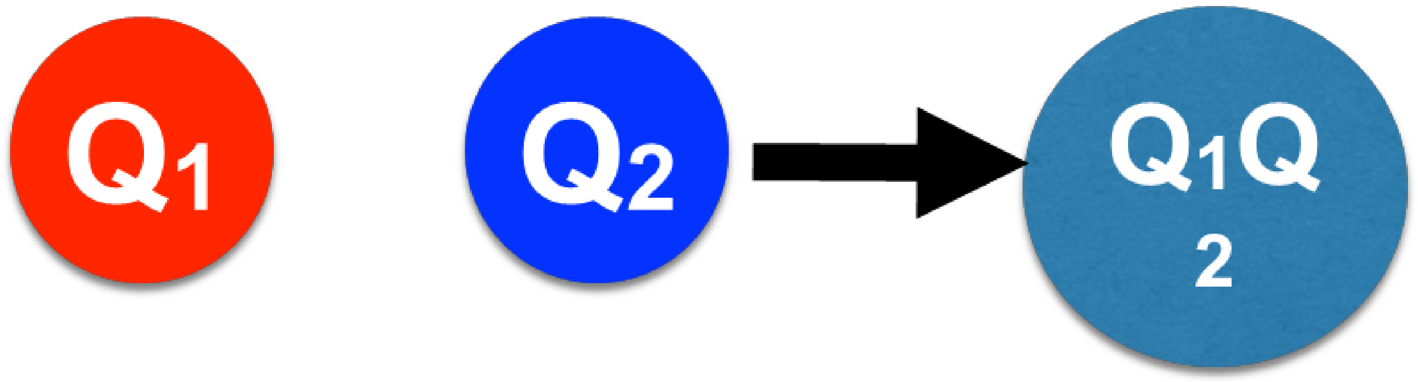
Also indicated in the lattice calculations: Prelovsek et al JHEP 06 (2021) 035.

BOEFT: $QQ\bar{q}\bar{q}$ multiplets

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

doubly heavy core

spin: $1/2 \otimes 1/2 = 0 \oplus 1$



color: $3 \otimes 3 = 6 \oplus 3^*$

flavor: $\{Q_1Q_2\}$ $[Q_1Q_2]$

J^P: 1^+ 0^+

light antiquarks

q q'

$\{qq'\}$, 1^+ $[qq']$, 0^+

Defines the Born-Oppenheimer static potentials $\Sigma_g^+, \{\Sigma_g^-, \Pi_g\}$

doubly heavy tetraquarks

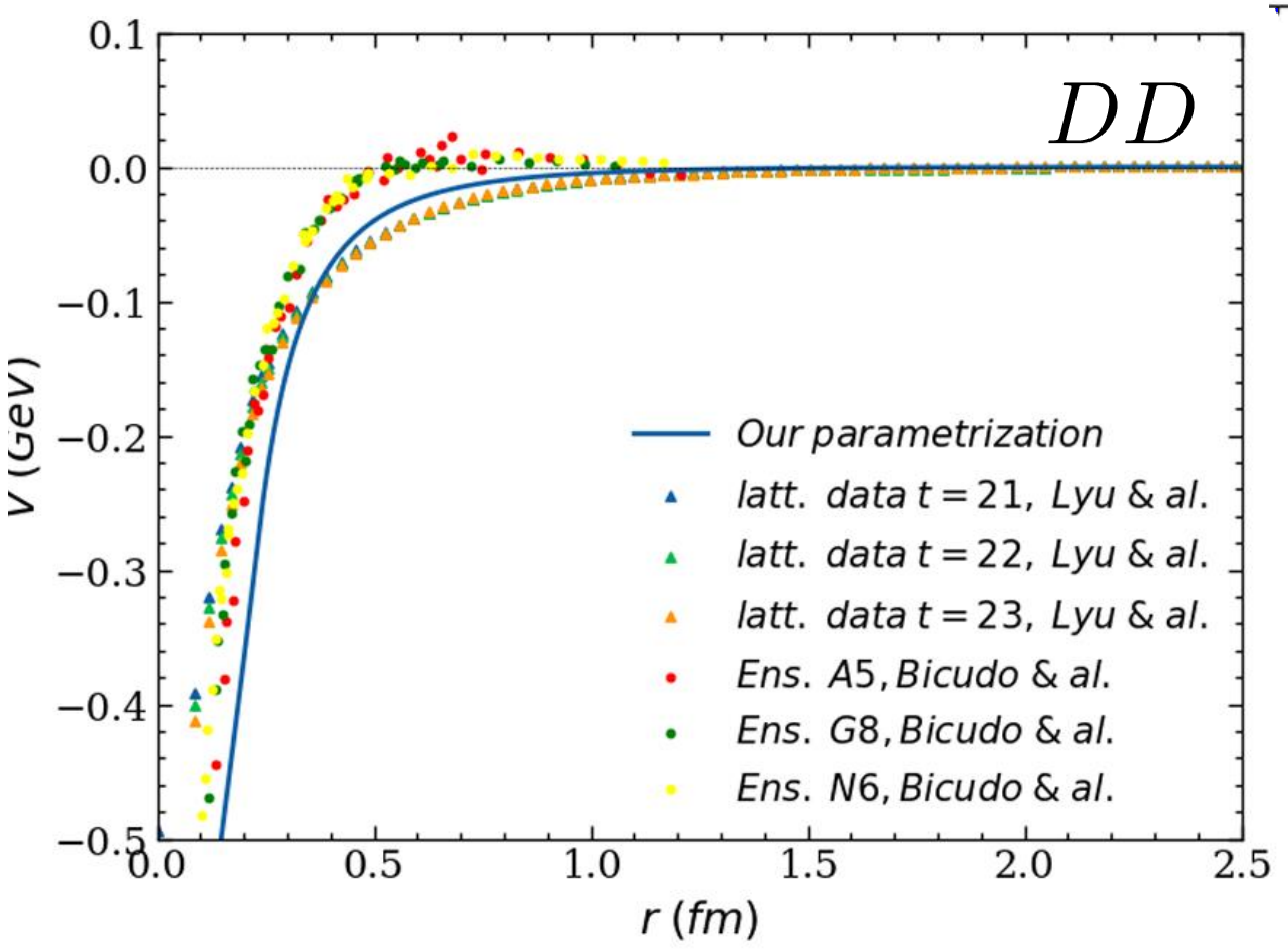
QQ color state	Light spin K^{PC}	Static energies	Isospin I	l	J^P $S_Q=0$	J^P $S_Q=1$
anti-triplet $\bar{3}$	0^+	$\{\Sigma_g^+\}$	0	0	—	1^+
				1	1^-	—
	1^+	$\{\Sigma_g^-, \Pi_g\}$	1	0	0^-	—
				1	1^-	$(0,1,2)^+$

J^P for T_{cc}^+

$T_{cc}^{+}(3875)$

Berwein, N.B.,Mohapatra, Vairo 2408.04719

N.B. Mohapatra, Scirpa, Vairo 2411.xxxx



Critical triplet meson: $\Lambda_t^{0^+} \approx 650$ MeV

Lyu, Aoki, Doi, Hatsuda, Ikeda, Meng, Phys. Rev. Lett. 131, 161901 (2023)

Bicudo, Marinkovic, Mueller, Wagner, arXiv 2409.10786

Schrödinger equation

$$\left[-\frac{1}{m_Q r^2}\,\partial_r\, r^2\,\partial_r + \frac{l(l+1)}{m_Q r^2} + V_{\Sigma_g^+}\right]\psi_{\Sigma_g^+} = \mathcal{E}_N\,\psi_{\Sigma_g^+}\,.$$

$l = 0$

Preliminary results:

1) T_{cc} state : 320 keV below DD threshold
2) Radius ~ 8 fm or larger.
3) Deeper bound state in bb sector: T_{bb} 110 MeV below DD threshold. r=0.29 fm
4) Deeper bound state in bc sector: T_{bc} 20 MeV below DD threshold. Both 0+ 1+ degenerate r=0.79 fm

$$a = \hbar/\sqrt{m_c E_b} = 7.95\ \text{fm},$$

HALQCD collaboration: pion mass 146 MeV: T_{cc} a virtual state. $E_{\text{pole}} = -59\,^{+53\,+2}_{-99\,-67}$ keV

physical pion mass 135 MeV: T_{cc} a bound state

Lyu et al, Phys. Rev. Lett. 131, 161901 (2023)

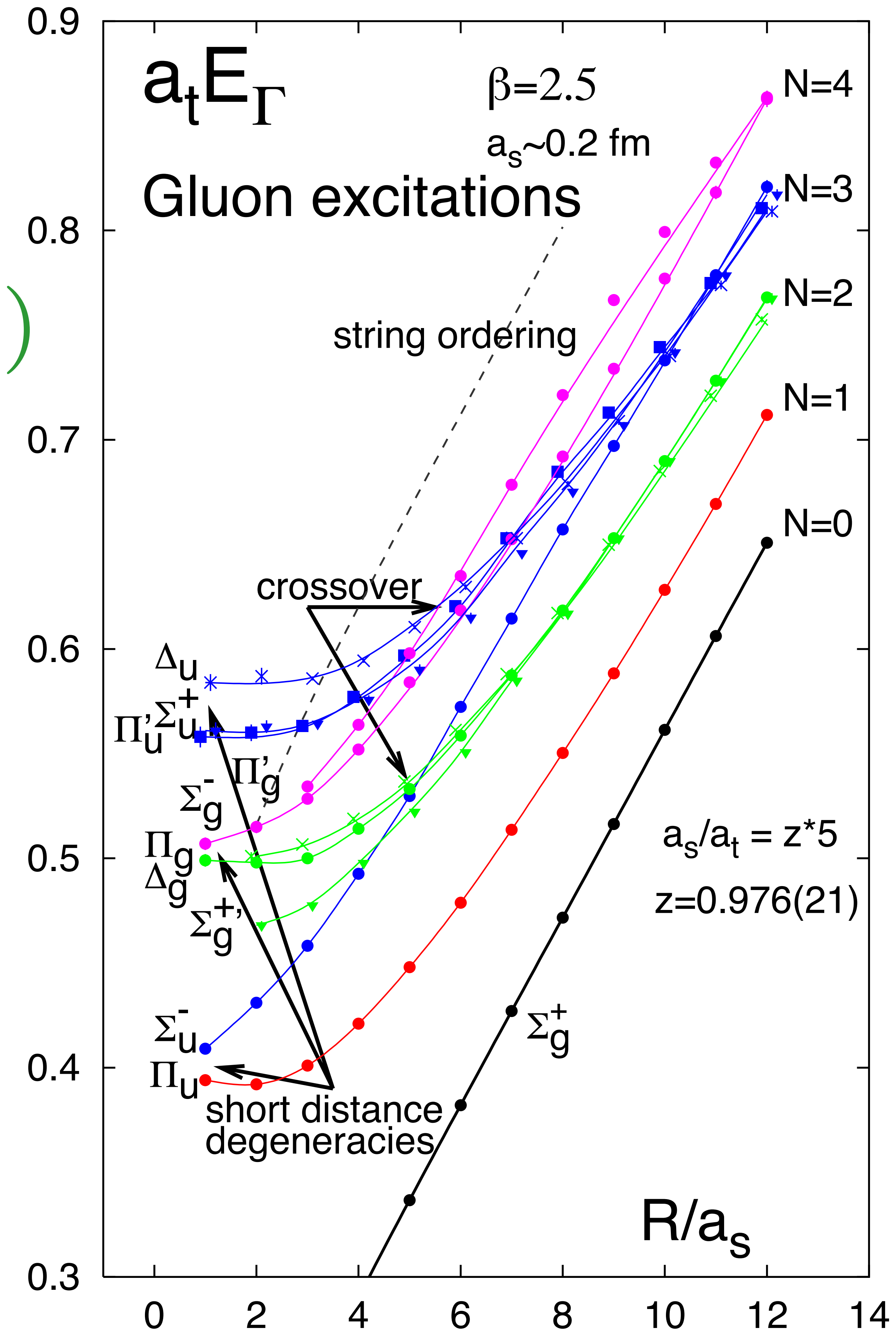
Hybrids

Exotics: Hybrids

Exotics: Hybrids

Lattice Spectrum of NRQCD
hybrid static energies E^0_n

$E_n^{(0)}$



$r = |\mathbf{x}_1 - \mathbf{x}_2|$

Juge Kuti Mornigstar 98-06

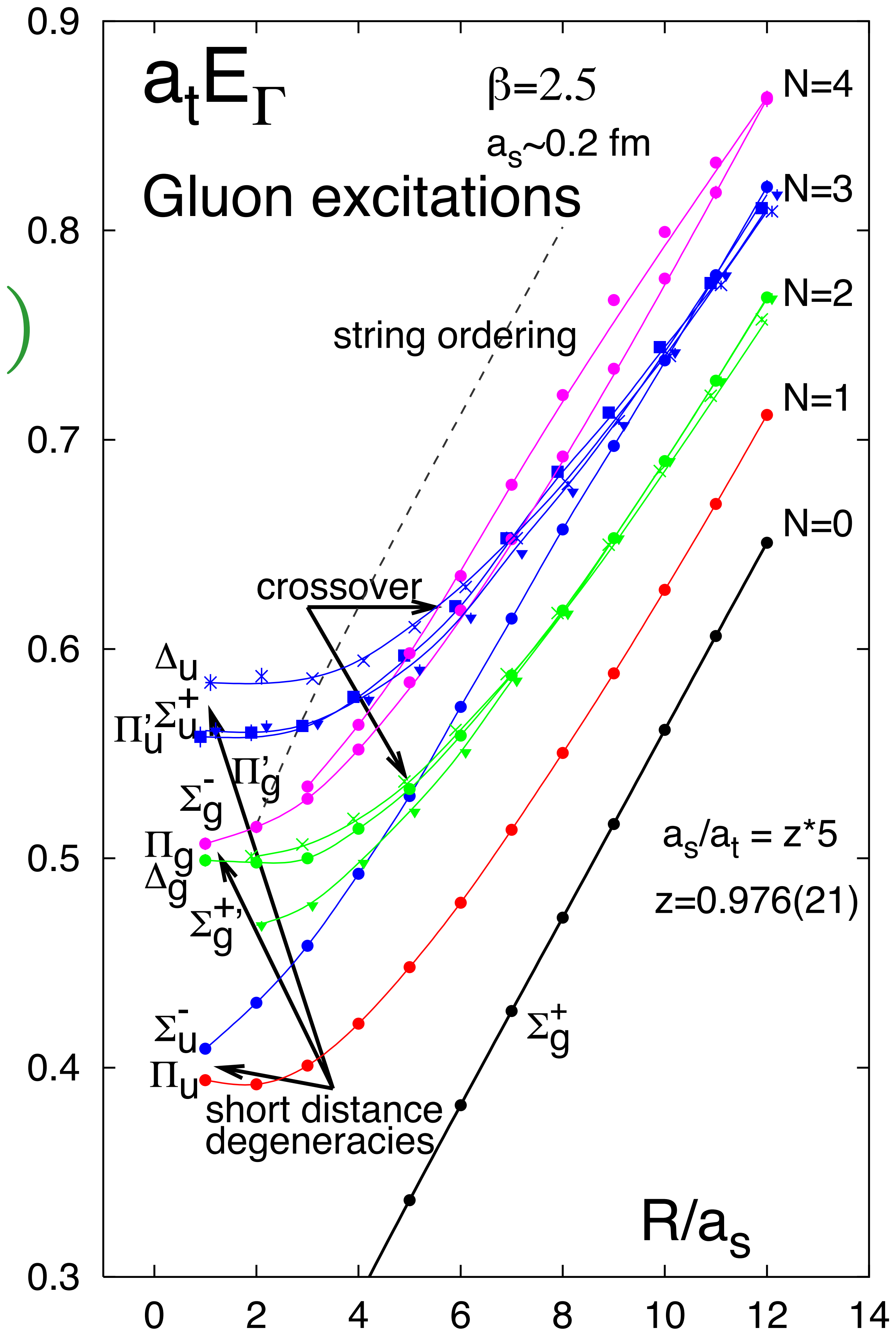
Schlosser, Wagner 2111.00741, Bali Pineda 2004

○ Capitani Philipsen Reisinger Riehl Wagner PRD 99 (2019) 034502

Exotics: Hybrids

Lattice Spectrum of NRQCD
hybrid static energies E^0_n

$E_n^{(0)}$



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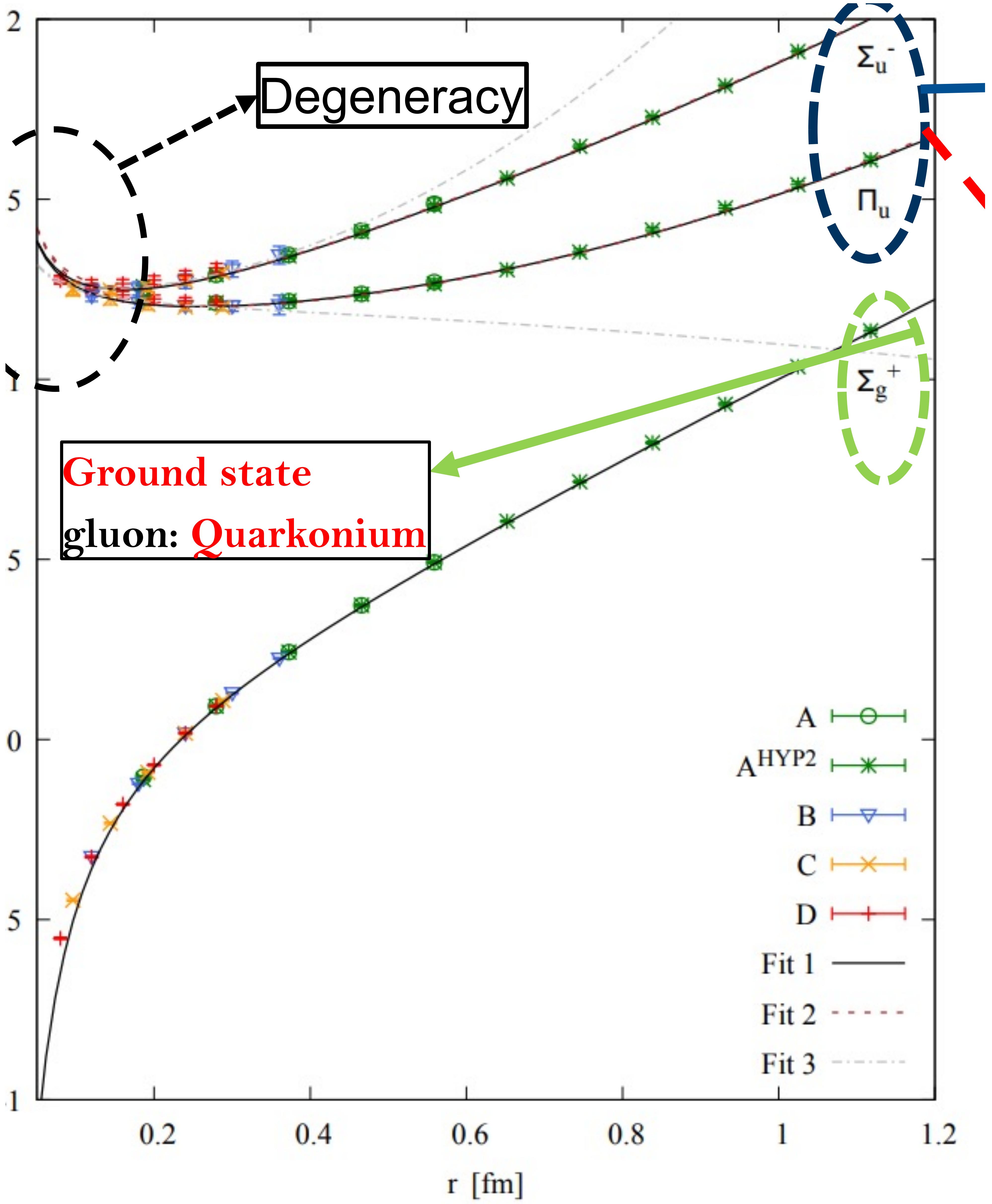
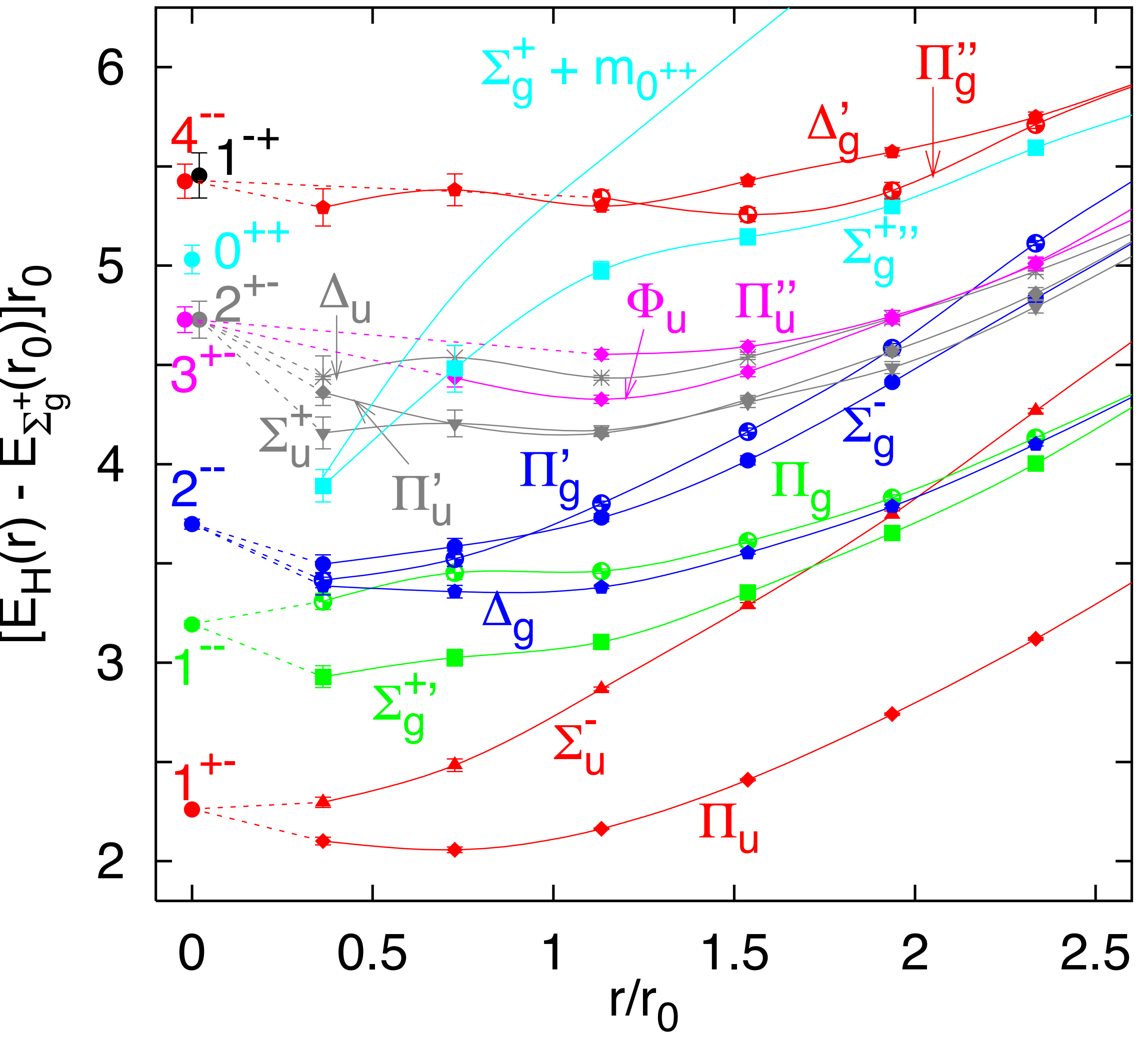


Schlosser and Wagner Phys. Rev. D. 105, (2022)

Hybrids static energies at short distances



The BOEFT characterises the hybrids static energy for short distance

In the short-range hybrids become gluelumps, i.e., quark-antiquark octets, O^a, in the presence of a gluonic field, H^a: $H(R,r,t) = H^a(R,t)O^a(R,r,t)$.

the hybrid static energy can be written as a (multipole) expansion in r:

octet potential → non perturbative coefficient

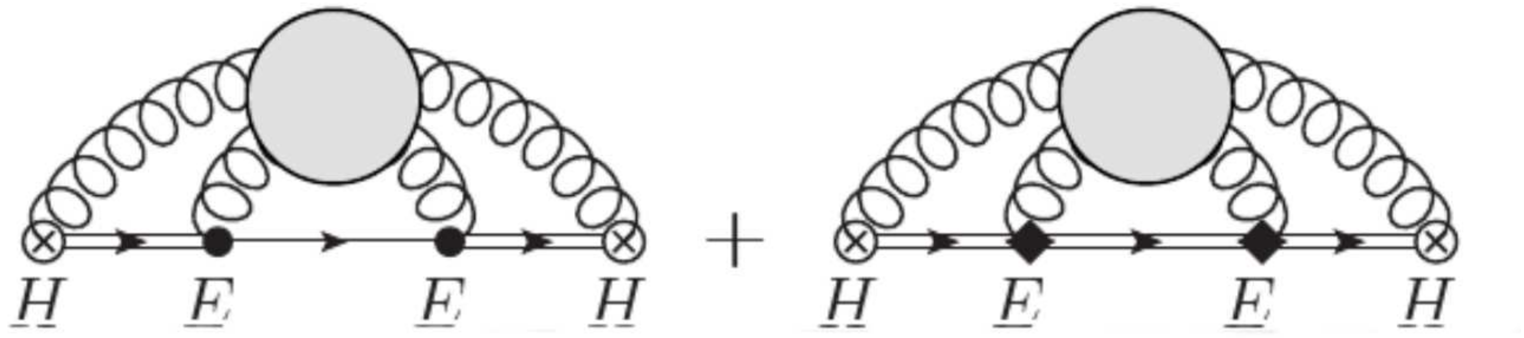
$$E_g = \frac{\alpha_s}{6r} + \Lambda_g + a_g r^2 + \ldots$$

Λ_g is the gluelump mass: $\Lambda_g = \lim_{T\to\infty} \frac{i}{T} \ln\langle H^a(T/2)\phi_{ab}^{\rm adj}(T/2,-T/2)H^b(-T/2)\rangle$

calculated on the lattice

Foster Michael PRD 59 (1999) 094509
Bali Pineda PRD 69 (2004) 094001
Lewis Marsh PRD 89 (2014) 014502

a_g can be expressed as field correlators (single line = singlet, double line = octet), e.g.,



In the limit $r \to 0$ more symmetry: $D_{\infty h} \to O(3)\times C$

- Several Λ_η^σ representations contained in one J^{PC} representation:
- Static energies in these multiplets have same $r \to 0$ limit.

The gluelump multiplets Σ_u^-, Π_u; $\Sigma_g^{+\prime}$, Π_g; Σ_g^-, Π_g', Δ_g; Σ_u^+, Π_u', Δ_u are degenerate.

Gluonic excitation operators up to dim 3

Λ_η^σ	K^{PC}	H^a
Σ_u^-	1^{+-}	$\mathbf{r}\cdot\mathbf{B}$, $\mathbf{r}\cdot(\mathbf{D}\times\mathbf{E})$
Π_u	1^{+-}	$\mathbf{r}\times\mathbf{B}$, $\mathbf{r}\times(\mathbf{D}\times\mathbf{E})$
$\Sigma_g^{+\prime}$	1^{--}	$\mathbf{r}\cdot\mathbf{E}$, $\mathbf{r}\cdot(\mathbf{D}\times\mathbf{B})$
Π_g	1^{--}	$\mathbf{r}\times\mathbf{E}$, $\mathbf{r}\times(\mathbf{D}\times\mathbf{B})$
Σ_g^-	2^{--}	$(\mathbf{r}\cdot\mathbf{D})(\mathbf{r}\cdot\mathbf{B})$
Π_g'	2^{--}	$\mathbf{r}\times((\mathbf{r}\cdot\mathbf{D})\mathbf{B}+\mathbf{D}(\mathbf{r}\cdot\mathbf{B}))$
Δ_g	2^{--}	$(\mathbf{r}\times\mathbf{D})^i(\mathbf{r}\times\mathbf{B})^j+(\mathbf{r}\times\mathbf{D})^j(\mathbf{r}\times\mathbf{B})^i$
Σ_u^+	2^{+-}	$(\mathbf{r}\cdot\mathbf{D})(\mathbf{r}\cdot\mathbf{E})$
Π_u'	2^{+-}	$\mathbf{r}\times((\mathbf{r}\cdot\mathbf{D})\mathbf{E}+\mathbf{D}(\mathbf{r}\cdot\mathbf{E}))$
Δ_u	2^{+-}	$(\mathbf{r}\times\mathbf{D})^i(\mathbf{r}\times\mathbf{E})^j+(\mathbf{r}\times\mathbf{D})^j(\mathbf{r}\times\mathbf{E})^i$

The first hybrid static energy excitation

BOEFT for E_{Π_u} and $E_{\Sigma_u^-}$ hybrids

- Berwein Brambilla Tarrus Vairo PRD 92 (2015) 114019
  Oncala Soto PRD 96 (2017) 014004
  Brambilla Krein Tarrus Vairo PRD 97 (2018) 016016

$$P^{i\dagger}_{\kappa\lambda} O^a(\mathbf{r},\mathbf{R},t) H^{ia}_{\kappa}(\mathbf{R},t) = Z_\kappa \Psi_{\kappa\lambda}(\mathbf{r},\mathbf{R},t)$$

$$\mathcal{L}_{\text{BOEFT for } 1+-} = \int d^3r \sum_{\lambda\lambda'} \text{Tr}\left\{ \Psi^\dagger_{1+-\lambda} \left(i\partial_0 - V_{1+-\lambda\lambda'}(r) + \hat{r}^{i\dagger}_{\lambda} \frac{\boldsymbol{\nabla}^2_r}{m} \hat{r}^i_{\lambda'} \right) \Psi_{1+-\lambda'} \right\}$$

- $\lambda = \pm 1, 0;\quad \hat{r}^i_0 = \hat{r}^i$ and $\hat{r}^i_{\pm 1} = \mp \left(\hat{\theta}^i \pm i\hat{\phi}^i\right)/\sqrt{2}$.
- $V_{1+-\lambda\lambda'} = V^{(0)}_{1+-\lambda\lambda'} + \dfrac{V^{(1)}_{1+-\lambda\lambda'}}{m} + \dfrac{V^{(2)}_{1+-\lambda\lambda'}}{m^2} + \cdots$
- For the static potential: $V^{(0)}_{1+-\lambda\lambda'} = \delta_{\lambda\lambda'} V^{(0)}_{1+-\lambda}$, with $V^{(0)}_{1+-0} = E_{\Sigma_u^-}$, $V^{(0)}_{1+-\pm 1} = E_{\Pi_u}$.

→ fitted from the lattice hybrids static energies

BOEFT for E_{Π_u} and $E_{\Sigma_u^-}$ hybrids

o Berwein Brambilla Tarrus Vairo PRD 92 (2015) 114019
Oncala Soto PRD 96 (2017) 014004
Brambilla Krein Tarrus Vairo PRD 97 (2018) 016016

$$P^{i\dagger}_{\kappa\lambda} O^a(\mathbf{r},\mathbf{R},t) H^{ia}_\kappa(\mathbf{R},t) = Z_\kappa \Psi_{\kappa\lambda}(\mathbf{r},\mathbf{R},t)$$

$$\mathcal{L}_{\text{BOEFT for } 1+-} = \int d^3r \sum_{\lambda\lambda'} \text{Tr}\left\{ \Psi^\dagger_{1+-\lambda} \left(i\partial_0 - V_{1+-\lambda\lambda'}(r) + \hat{r}^{i\dagger}_\lambda \frac{\boldsymbol{\nabla}^2_r}{m} \hat{r}^i_{\lambda'} \right) \Psi_{1+-\lambda'} \right\}$$

- $\lambda = \pm 1, 0$; $\quad \hat{r}^i_0 = \hat{r}^i$ and $\hat{r}^i_{\pm 1} = \mp \left(\hat{\theta}^i \pm i\hat{\phi}^i\right)/\sqrt{2}$.
- $V_{1+-\lambda\lambda'} = V^{(0)}_{1+-\lambda\lambda'} + \frac{V^{(1)}_{1+-\lambda\lambda'}}{m} + \frac{V^{(2)}_{1+-\lambda\lambda'}}{m^2} + \cdots$
- For the static potential: $V^{(0)}_{1+-\lambda\lambda'} = \delta_{\lambda\lambda'} V^{(0)}_{1+-\lambda}$, with $V^{(0)}_{1+-0} = E_{\Sigma_u^-}$, $V^{(0)}_{1+-\pm 1} = E_{\Pi_u}$.

fitted from the lattice hybrids static energies

The LO e.o.m. for the fields $\Psi^\dagger_{1+-\lambda}$ are a set of coupled Schrödinger equations:

$$i\partial_0 \Psi_{1+-\lambda} = \left[\left(-\frac{\boldsymbol{\nabla}^2_r}{m} + V^{(0)}_{1+-\lambda} \right) \delta_{\lambda\lambda'} - \sum_{\lambda'} C^{\text{nad}}_{1+-\lambda\lambda'} \right] \Psi_{\kappa\lambda'}$$

The eigenvalues $\mathcal{E}_N$ give the masses M_N of the states as $M_N = 2m + \mathcal{E}_N$.

$$\hat{r}^{i\dagger}_\lambda \left(\frac{\boldsymbol{\nabla}^2_r}{m} \right) \hat{r}^i_{\lambda'} = \delta_{\lambda\lambda'} \frac{\boldsymbol{\nabla}^2_r}{m} + C^{\text{nad}}_{1+-\lambda\lambda'}$$

with $C^{\text{nad}}_{1+-\lambda\lambda'} = \hat{r}^{i\dagger}_\lambda \left[\frac{\boldsymbol{\nabla}^2_r}{m}, \hat{r}^i_{\lambda'} \right]$ called the nonadiabatic coupling.

BOEFT for E_{Π_u} and $E_{\Sigma_u^-}$ hybrids

○ Berwein Brambilla Tarrus Vairo PRD 92 (2015) 114019
Oncala Soto PRD 96 (2017) 014004
Brambilla Krein Tarrus Vairo PRD 97 (2018) 016016

$$\mathcal{L}_{\text{BOEFT for } 1^{+-}} = \int d^3r \sum_{\lambda\lambda'} \mathrm{Tr}\left\{ \Psi^{\dagger}_{1^{+-}\lambda} \left(i\partial_0 - V_{1^{+-}\lambda\lambda'}(r) + \hat{r}^{i\dagger}_{\lambda} \frac{\boldsymbol{\nabla}^2_r}{m} \hat{r}^i_{\lambda'} \right) \Psi_{1^{+-}\lambda'} \right\}$$

- $\lambda = \pm 1, 0; \quad \hat{r}^i_0 = \hat{r}^i$ and $\hat{r}^i_{\pm 1} = \mp \left(\hat{\theta}^i \pm i\hat{\phi}^i\right)/\sqrt{2}$.
- $V_{1^{+-}\lambda\lambda'} = V^{(0)}_{1^{+-}\lambda\lambda'} + \frac{V^{(1)}_{1^{+-}\lambda\lambda'}}{m} + \frac{V^{(2)}_{1^{+-}\lambda\lambda'}}{m^2} + \cdots$
- For the static potential: $V^{(0)}_{1^{+-}\lambda\lambda'} = \delta_{\lambda\lambda'} V^{(0)}_{1^{+-}\lambda}$, with $V^{(0)}_{1^{+-}0} = E_{\Sigma_u^-}$, $V^{(0)}_{1^{+-}\pm 1} = E_{\Pi_u}$.

fitted from the lattice hybrids static energies

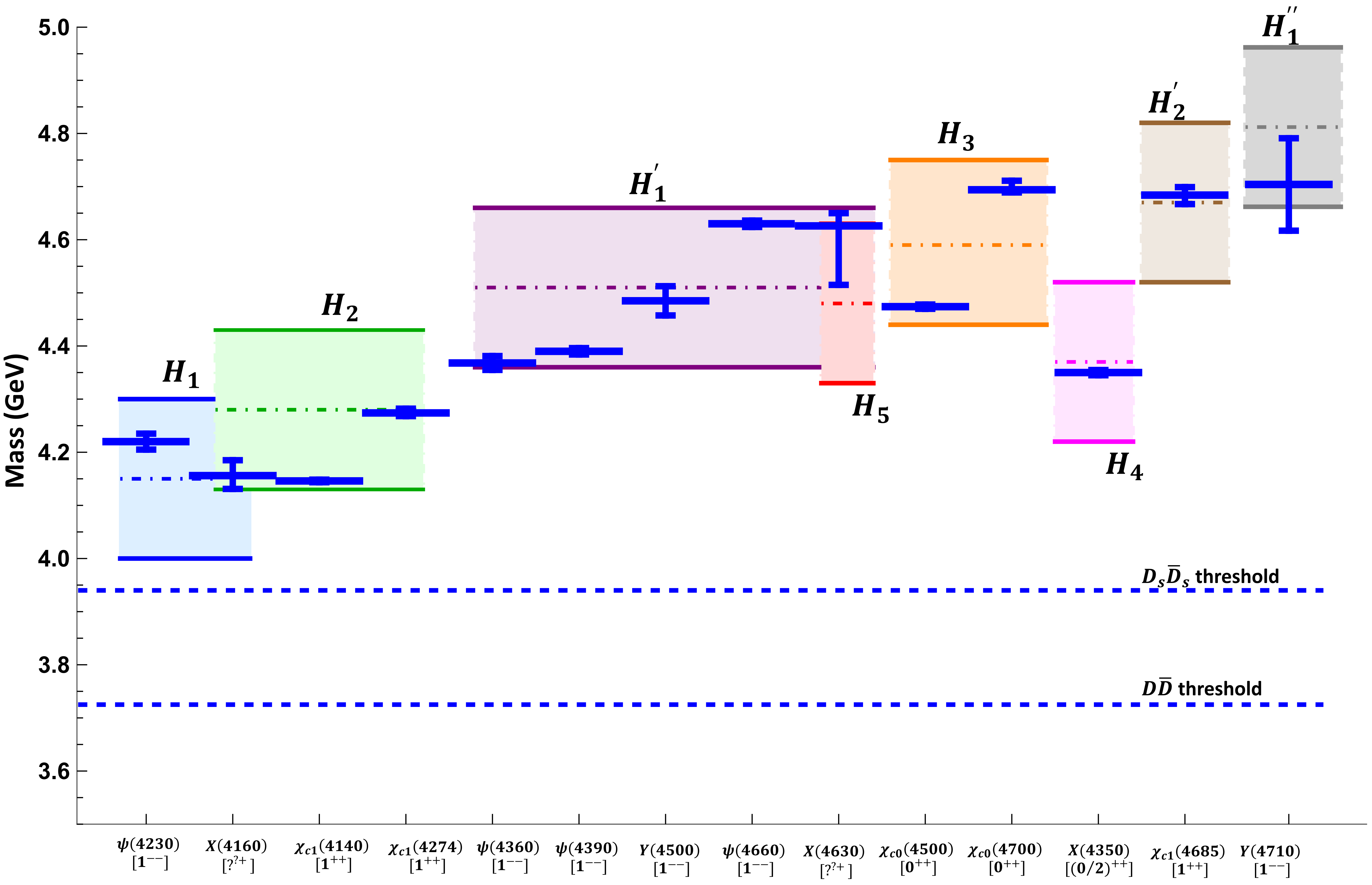
$$\left[-\frac{1}{mr^2}\,\partial_r r^2 \partial_r + \frac{1}{mr^2} \begin{pmatrix} l(l+1)+2 & 2\sqrt{l(l+1)} \\ 2\sqrt{l(l+1)} & l(l+1) \end{pmatrix} + \begin{pmatrix} E^{(0)}_{\Sigma} & 0 \\ 0 & E^{(0)}_{\Pi} \end{pmatrix} \right] \begin{pmatrix} \psi^{(N)}_{\Sigma} \\ \psi^{(N)}_{-\Pi} \end{pmatrix} = \mathcal{E}_N \begin{pmatrix} \psi^{(N)}_{\Sigma} \\ \psi^{(N)}_{-\Pi} \end{pmatrix}$$

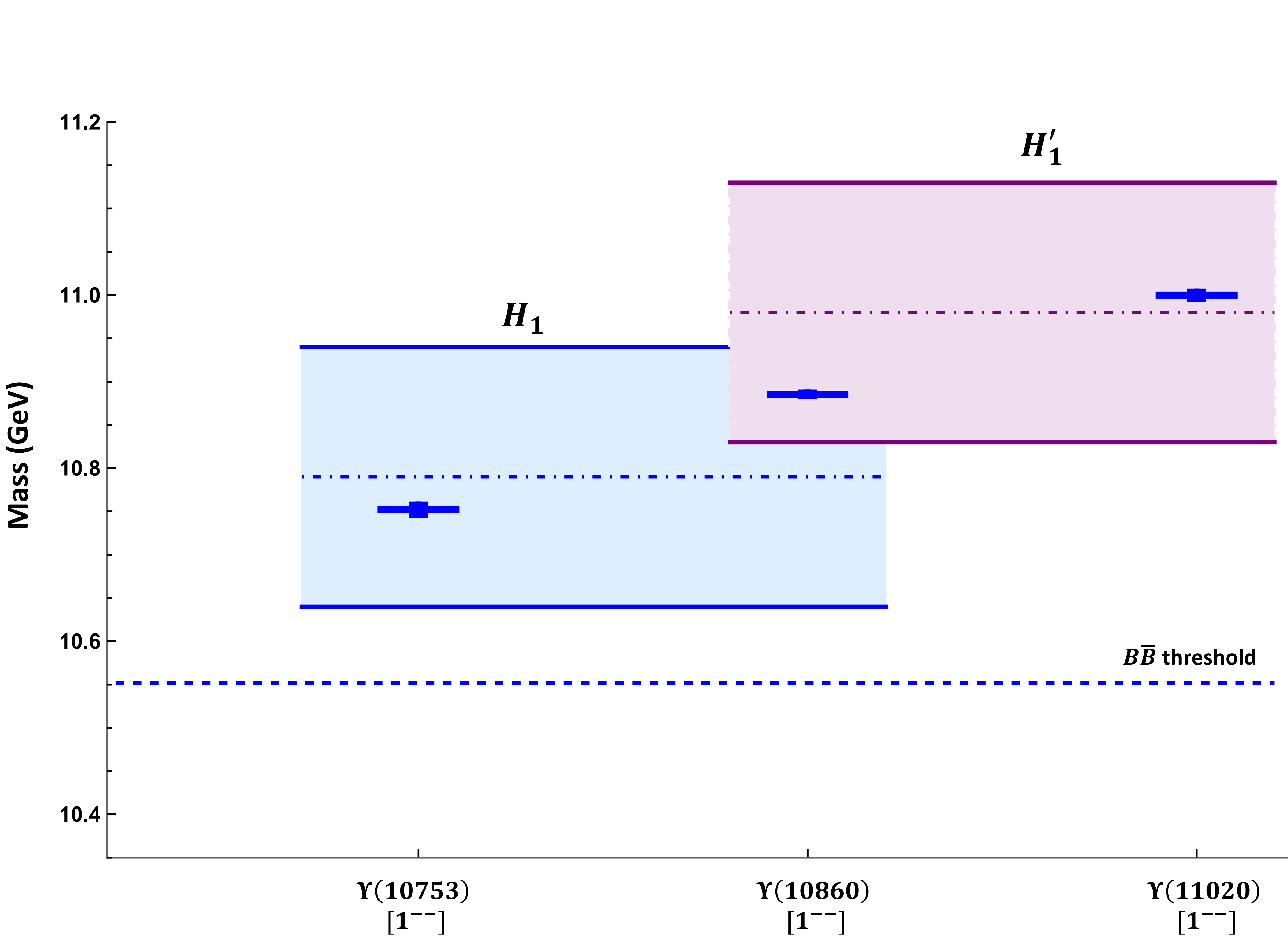
$$\left[-\frac{1}{mr^2}\,\partial_r\, r^2\, \partial_r + \frac{l(l+1)}{mr^2} + E^{(0)}_{\Pi} \right] \psi^{(N)}_{+\Pi} = \mathcal{E}_N\, \psi^{(N)}_{+\Pi}$$

Mixing remove the degeneration among opposite parity states: ->Lambda doubling existing also in molecular physics

- $l(l+1)$ is the eigenvalue of angular momentum $\boldsymbol{L}^2 = \left(\boldsymbol{L}_{Q\bar{Q}} + \boldsymbol{L}_g\right)^2$
- the two solutions correspond to **opposite parity** states: $(-1)^l$ and $(-1)^{l+1}$
- corresponding eigenvalues under charge conjugation: $(-1)^{l+s}$ and $(-1)^{l+s+1}$

Hybrid multiplets as predicted by BOEFT (coloured rectangles) compared to the neutral isoscalar states observed in charmonium/bottomonium sector (crosses)





Note: Band in the mass value for each multiplet is due to the error (150 Mev) on the gluelump mass measured on the lattice

Widths of the semi inclusive decays to quarkonium have also been calculated

N. B. A- Mohapatra, A. Vairo 2212.09187

Multiplet	T	$J^{PC}(S=0)$	$J^{PC}(S=1)$	E_Γ
H_1	1	1^{--}	$(0,1,2)^{-+}$	$E_{\Sigma_u^-}, E_{\Pi_u}$
H_2	1	1^{++}	$(0,1,2)^{+-}$	E_{Π_u}
H_3	0	0^{++}	1^{+-}	$E_{\Sigma_u^-}$
H_4	2	2^{++}	$(1,2,3)^{+-}$	$E_{\Sigma_u^-}, E_{\Pi_u}$

The **BOEFT** gives a prescription to calculate the **hybrids spin dependent potentials at order 1/m and 1/m^2**

1/m

$$V^{(1)}_{1^{+-}\lambda\lambda'\,\mathrm{SD}}(\boldsymbol{r}) = V_{SK}(r)\left(\hat{r}^{i\dagger}_{\lambda}\boldsymbol{K}^{ij}\hat{r}^{j}_{\lambda'}\right)\cdot\boldsymbol{S} + V_{SK\,b}(r)\left[\left(\boldsymbol{r}\cdot\hat{\boldsymbol{r}}^{\dagger}_{\lambda}\right)\left(r^{i}\boldsymbol{K}^{ij}\hat{r}^{j}_{\lambda'}\right)\cdot\boldsymbol{S} + \left(r^{i}\boldsymbol{K}^{ij}\hat{r}^{j\dagger}_{\lambda}\right)\cdot\boldsymbol{S}\left(\boldsymbol{r}\cdot\hat{\boldsymbol{r}}_{\lambda'}\right)\right]$$

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$$

$$S_{12} = 12(\mathbf{S}_1\cdot\hat{\mathbf{r}})(\mathbf{S}_2\cdot\hat{\mathbf{r}}) - 4(\mathbf{S}_1\cdot\mathbf{S}_2)$$

1/m^2

$$V^{(2)}_{1^{+-}\lambda\lambda'\,\mathrm{SD}}(\boldsymbol{r}) = V^{(2)}_{LS\,a}(r)\left(\hat{r}^{i\dagger}_{\lambda}\boldsymbol{L}\,\hat{r}^{i}_{\lambda'}\right)\cdot\boldsymbol{S} + V^{(2)}_{LS\,b}(r)\hat{r}^{i\dagger}_{\lambda}\left(L^{i}S^{j} + S^{i}L^{j}\right)\hat{r}^{j}_{\lambda'} + V^{(2)}_{S^2}(r)\boldsymbol{S}^2\delta_{\lambda\lambda'} + V^{(2)}_{S_{12}\,a}(r)S_{12}\delta_{\lambda\lambda'} + V^{(2)}_{S_{12}\,b}(r)\hat{r}^{i\dagger}_{\lambda}\hat{r}^{j}_{\lambda'}\left(S^{i}_{1}S^{j}_{2} + S^{i}_{2}S^{j}_{1}\right)$$

$\left(K^{ij}\right)^{k} = i\epsilon^{ikj}$ is the angular momentum of the spin one gluons

$\boldsymbol{L}$ is the orbital angular momentum of the heavy-quark-antiquark pair.

The **BOEFT** gives a prescription to calculate the **hybrids spin dependent potentials at order 1/m and 1/m^2**

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$$V^{(1)}_{1+-\lambda\lambda'\,\mathrm{SD}}(\boldsymbol{r}) = V_{SK}(r)\left(\hat{r}^{i\dagger}_{\lambda}\boldsymbol{K}^{ij}\hat{r}^{j}_{\lambda'}\right)\cdot\boldsymbol{S} + V_{SK\,b}(r)\left[\left(\boldsymbol{r}\cdot\hat{\boldsymbol{r}}^{\dagger}_{\lambda}\right)\left(r^{i}\boldsymbol{K}^{ij}\hat{r}^{j}_{\lambda'}\right)\cdot\boldsymbol{S} + \left(r^{i}\boldsymbol{K}^{ij}\hat{r}^{j\dagger}_{\lambda}\right)\cdot\boldsymbol{S}\left(\boldsymbol{r}\cdot\hat{\boldsymbol{r}}_{\lambda'}\right)\right]$$

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$\left(K^{ij}\right)^{k} = i\epsilon^{ikj}$ is the angular momentum of the spin one gluons

$\boldsymbol{L}$ is the orbital angular momentum of the heavy-quark-antiquark pair.

Features:

- New spin structures with respect to the quarkonium case: all terms at order 1/m and two terms at order 1/m^2

Differently from the quarkonium case, the hybrid potential gets a first contribution already at order $\Lambda^2_{\mathrm{QCD}}/m_h$. The corresponding operator does not contribute at LO to matrix elements of quarkonium states as its projection on quark-antiquark color singlet states vanishes. Hence, spin splittings are remarkably less suppressed in heavy quarkonium hybrids than in heavy quarkonia.

Hybrid spin dependent potentials at order 1/m and 1/m^2

1/m

$$V^{(1)}_{1+-\lambda\lambda'\,\mathrm{SD}}(\boldsymbol{r}) = V_{SK}(r)\left(\hat{r}^{i\dagger}_{\lambda}\boldsymbol{K}^{ij}\hat{r}^{j}_{\lambda'}\right)\cdot\boldsymbol{S} + V_{SK\,b}(r)\left[\left(\boldsymbol{r}\cdot\hat{\boldsymbol{r}}^{\dagger}_{\lambda}\right)\left(r^{i}\boldsymbol{K}^{ij}\hat{r}^{j}_{\lambda'}\right)\cdot\boldsymbol{S} + \left(r^{i}\boldsymbol{K}^{ij}\hat{r}^{j\dagger}_{\lambda}\right)\cdot\boldsymbol{S}\left(\boldsymbol{r}\cdot\hat{\boldsymbol{r}}_{\lambda'}\right)\right]$$

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Features:

- The nonperturbative part in V_i (r) depend on nonperturbative gluonic correlators non local in time not yet calculated on the lattice: six unknowns, the octet perturbative part can be calculated in perturbation theory
- The only flavor dependence is carried by the perturbative NRQCD matching coefficients

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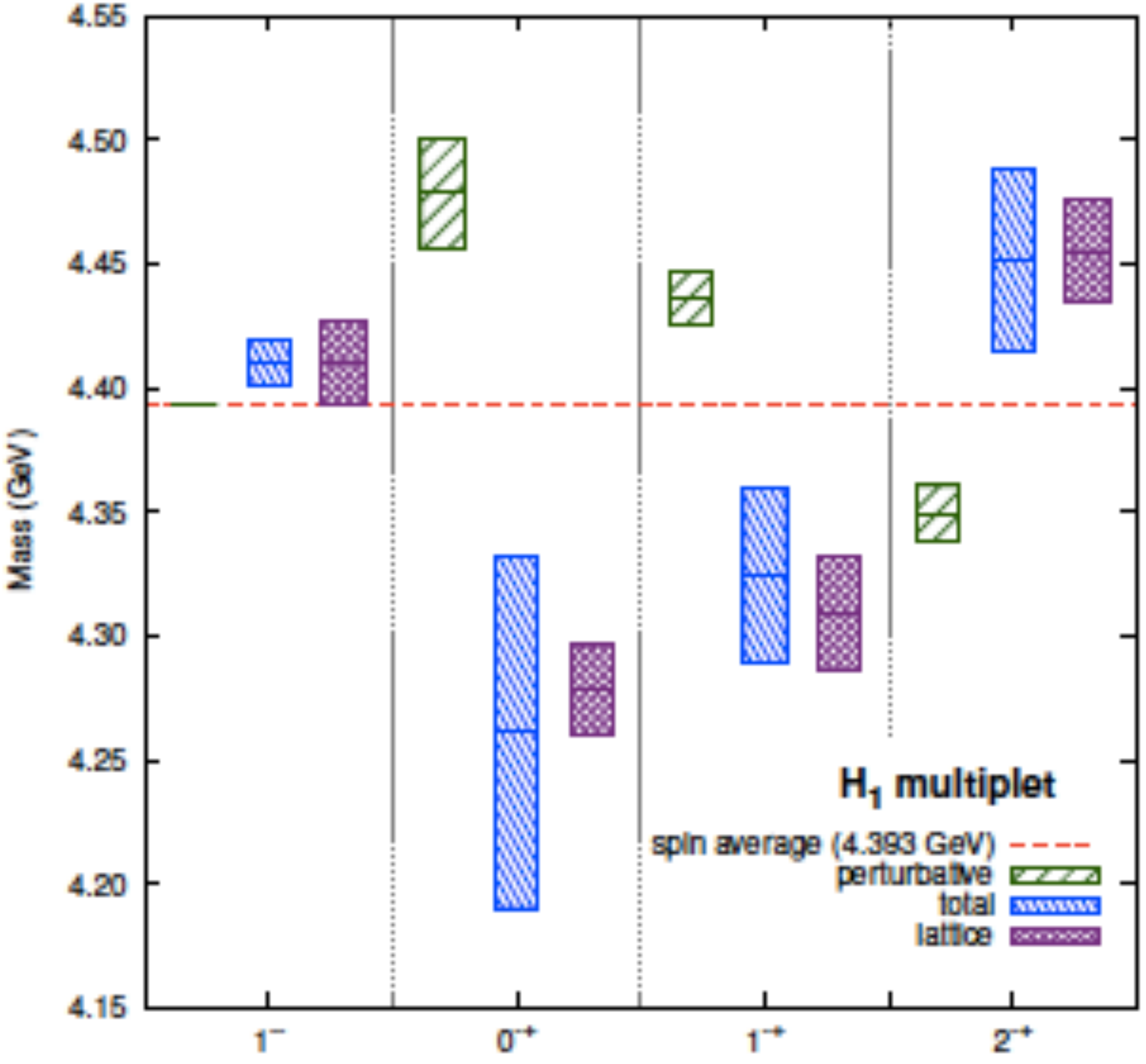
USE LATTICE CALCULATION OF THE CHARMONIUM SPIN MULTIPLETS TO EXTRACT the 6 UNKNOWNs and PREDICT THE BOTTOMONIUM SPIN MULTIPLETS, learn also about the DYNAMICS

Charmonium Hybrids Multiplets H_1

lattice data from (violet) from

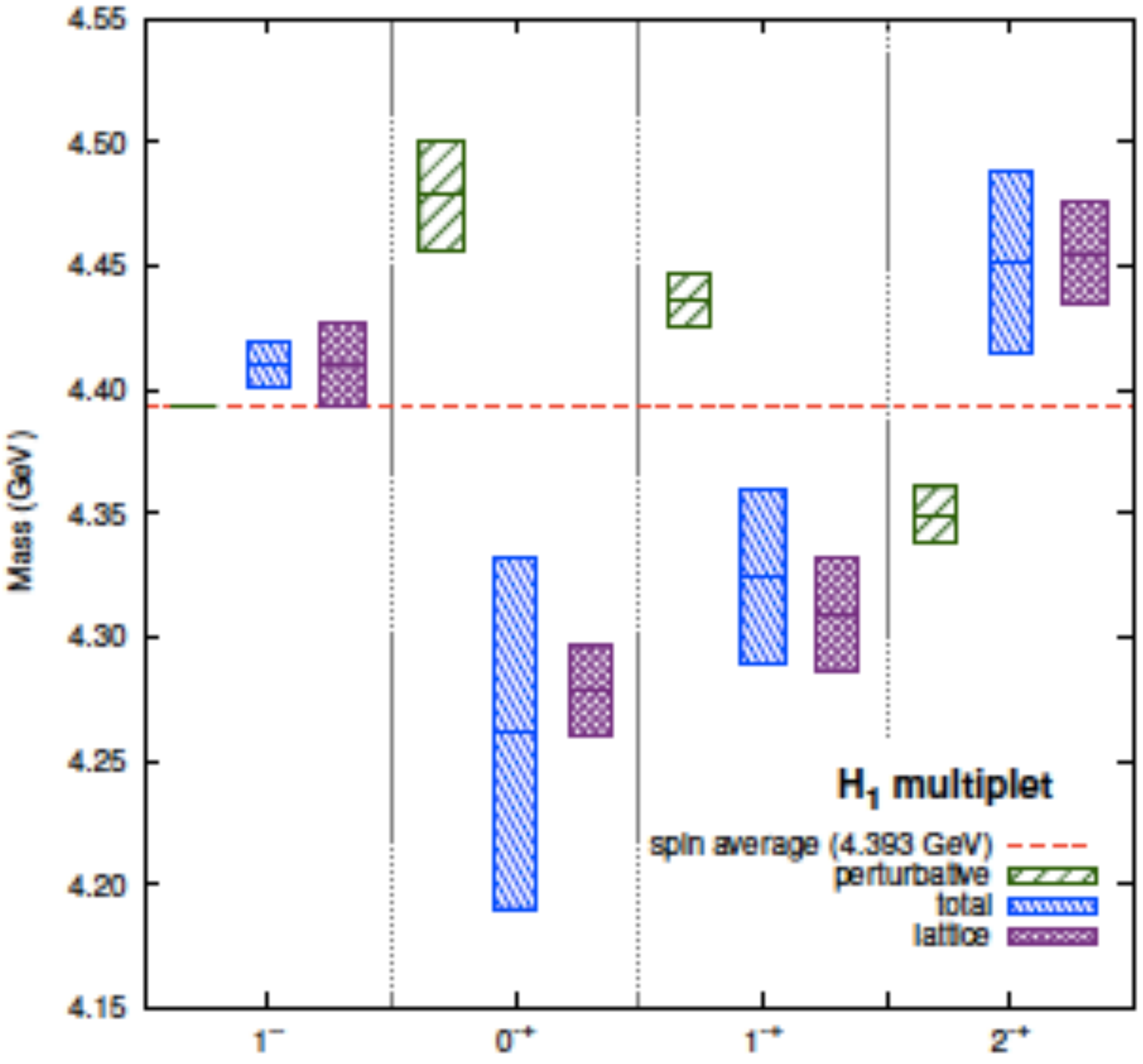
G. K. C. Cheung, C. O'Hara, G. Moir, M. Peardon, S. M. Ryan, C. E. Thomas, and D. Tims (Hadron Spectrum), JHEP **12**, 089 (2016), arXiv:1610.01073 [hep-lat].

with a pion of about 240 MeV



- Brambilla Lai Segovia Tarrus Vairo PRD 99 (2019) 014017

Charmonium Hybrids Multiplets H_1



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height of the boxes is an estimate of the uncertainty:
estimated by the parametric size of higher order corrections, m alpha_s^5 for the perturbative part, powers of Lambda_qcd/m for the nonperturbative part, plus the statistical error on the fit

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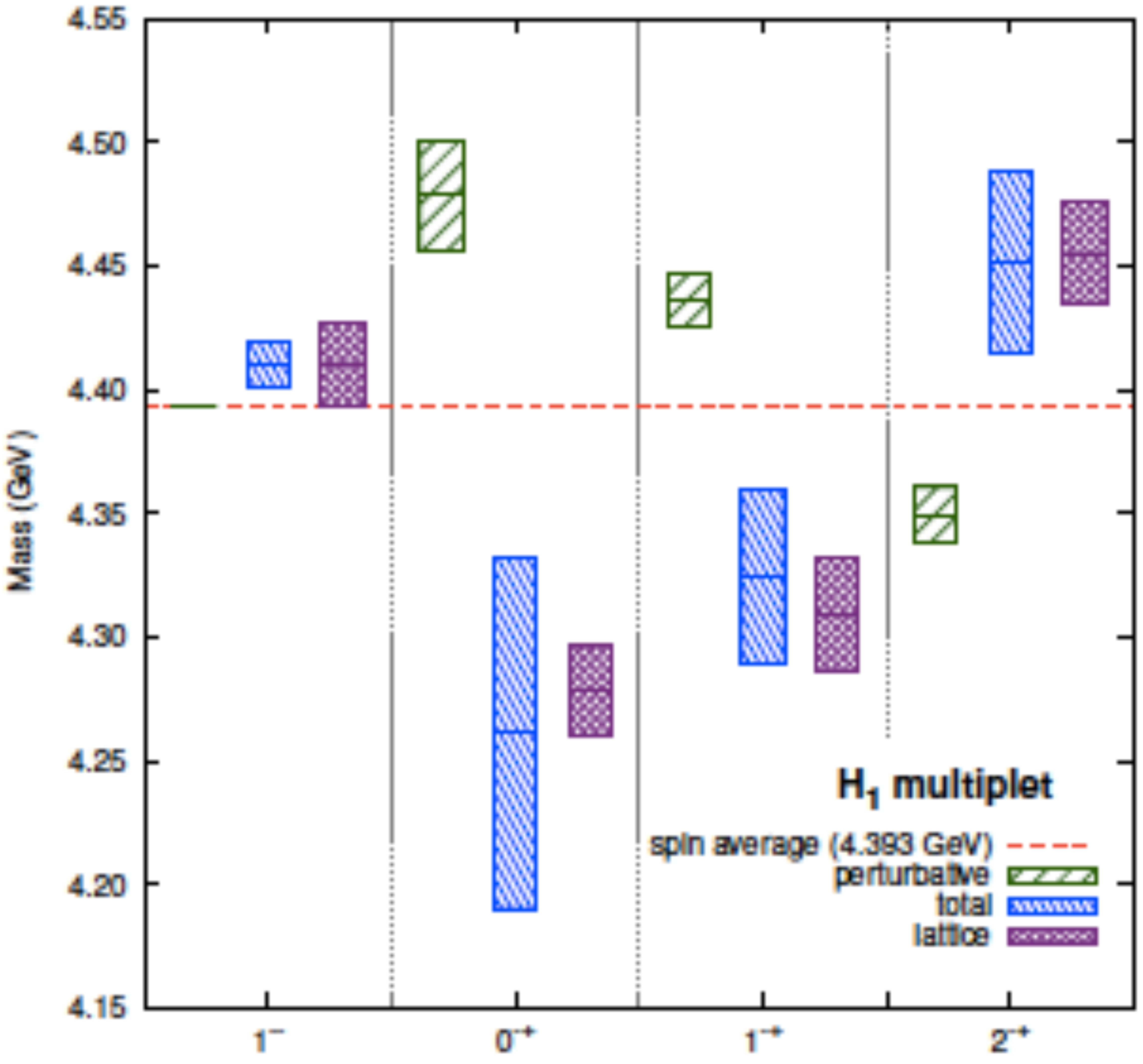
Charmonium Hybrids Multiplets H_1



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height of the boxes is an estimate of the uncertainty:
estimated by the parametric size of higher order corrections, m alpha_s^5 for the perturbative part, powers of Lambda_qcd/m for the nonperturbative part, plus the statistical error on the fit

the perturbative part produces a pattern opposite to the lattice and to ordinary quarkonia —> discrepancy can be reconciled thanks to the nonperturbative parts, especially the one at order 1/m which goes like Lambda^2/m and is parametrically larger than the perturbative contribution at order m v^4

which is interesting as some models are taking the spin interaction from perturbation theory with a constituent gluon

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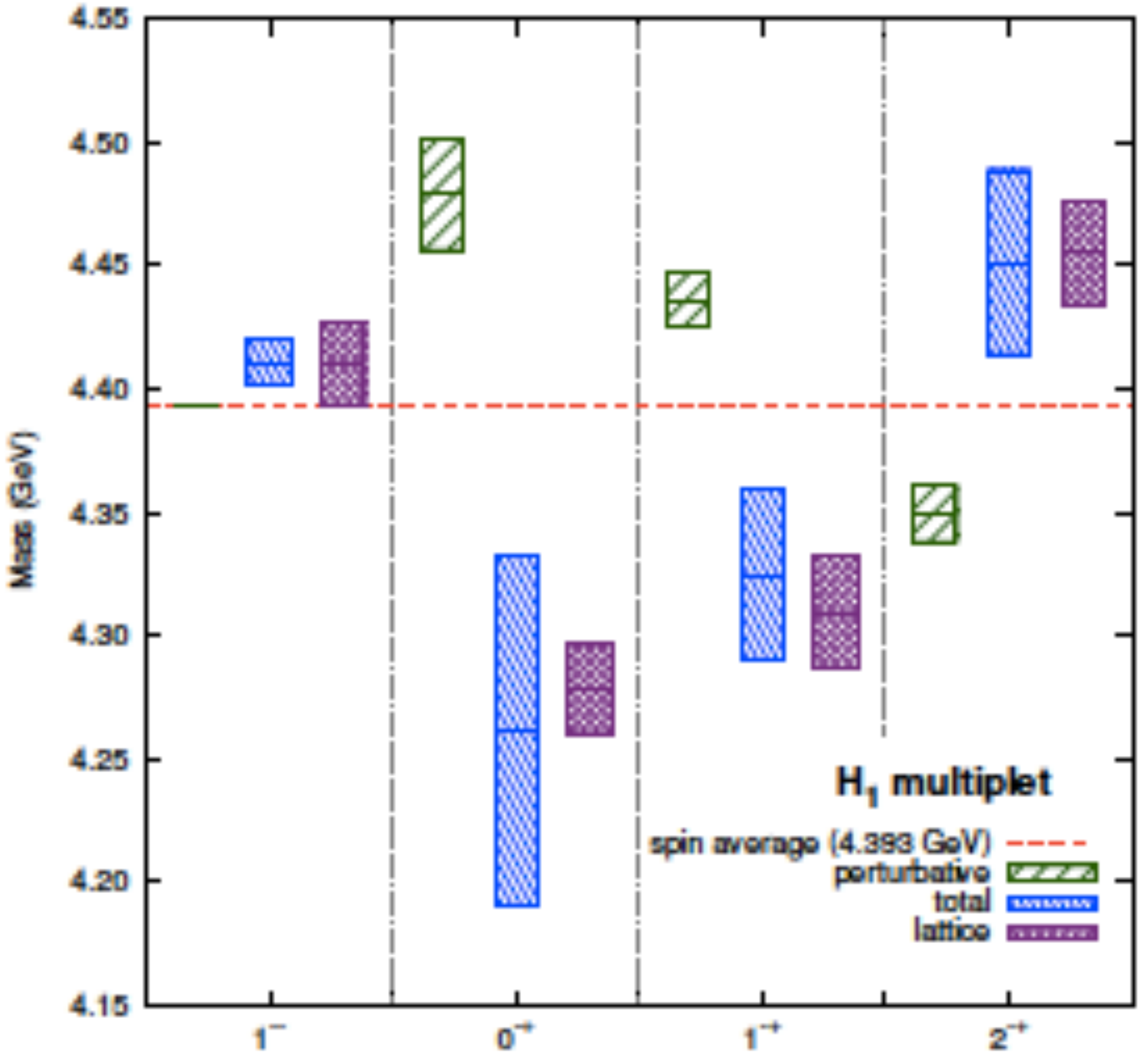
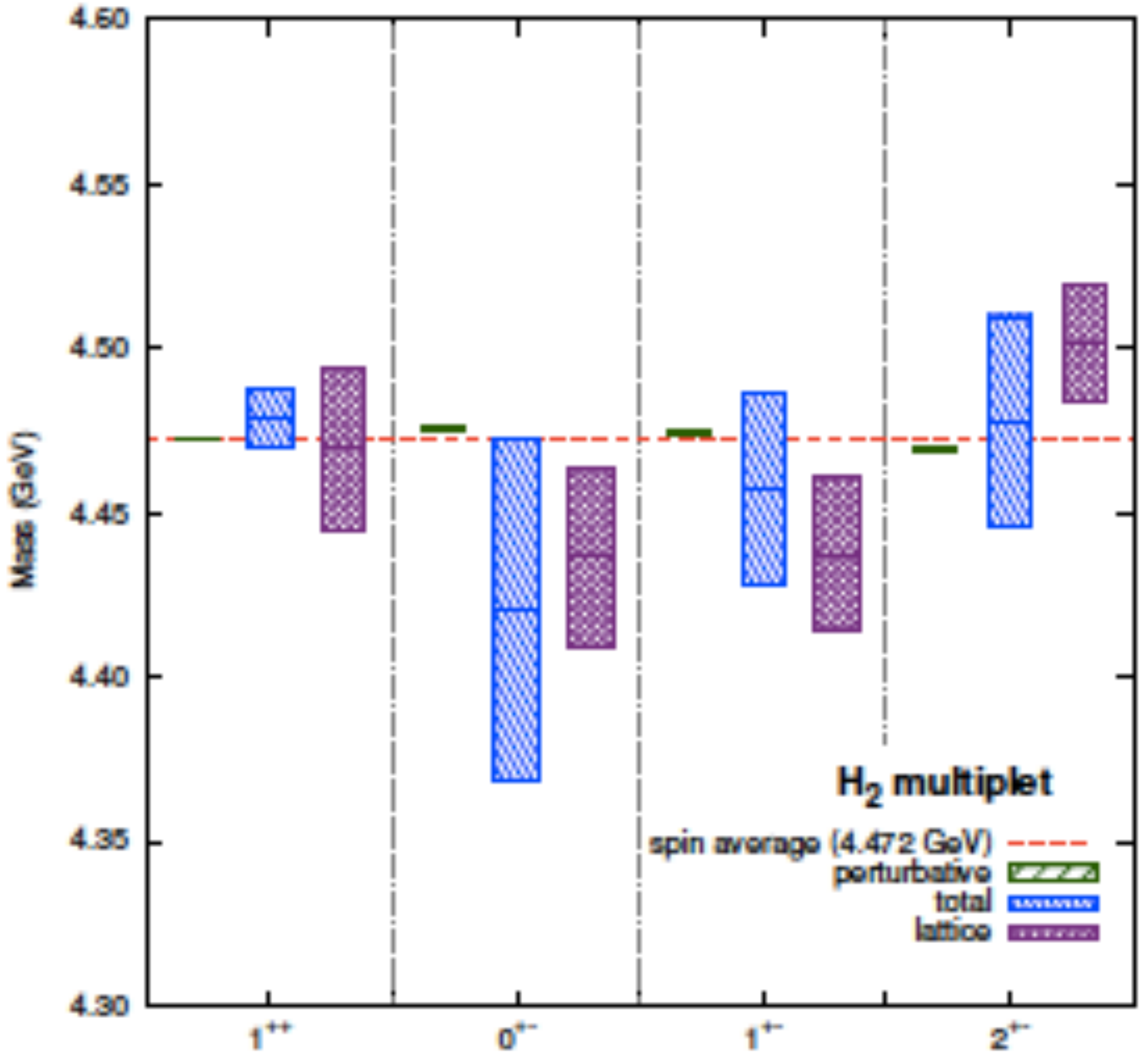
Charmonium Hybrids Multiplets H_1 and H_2





H_1 and H_2 corresponds to l=1 and are negative and positive parity resp. The mass splitting between H_1 and H_2 is a result of lambda-doubling

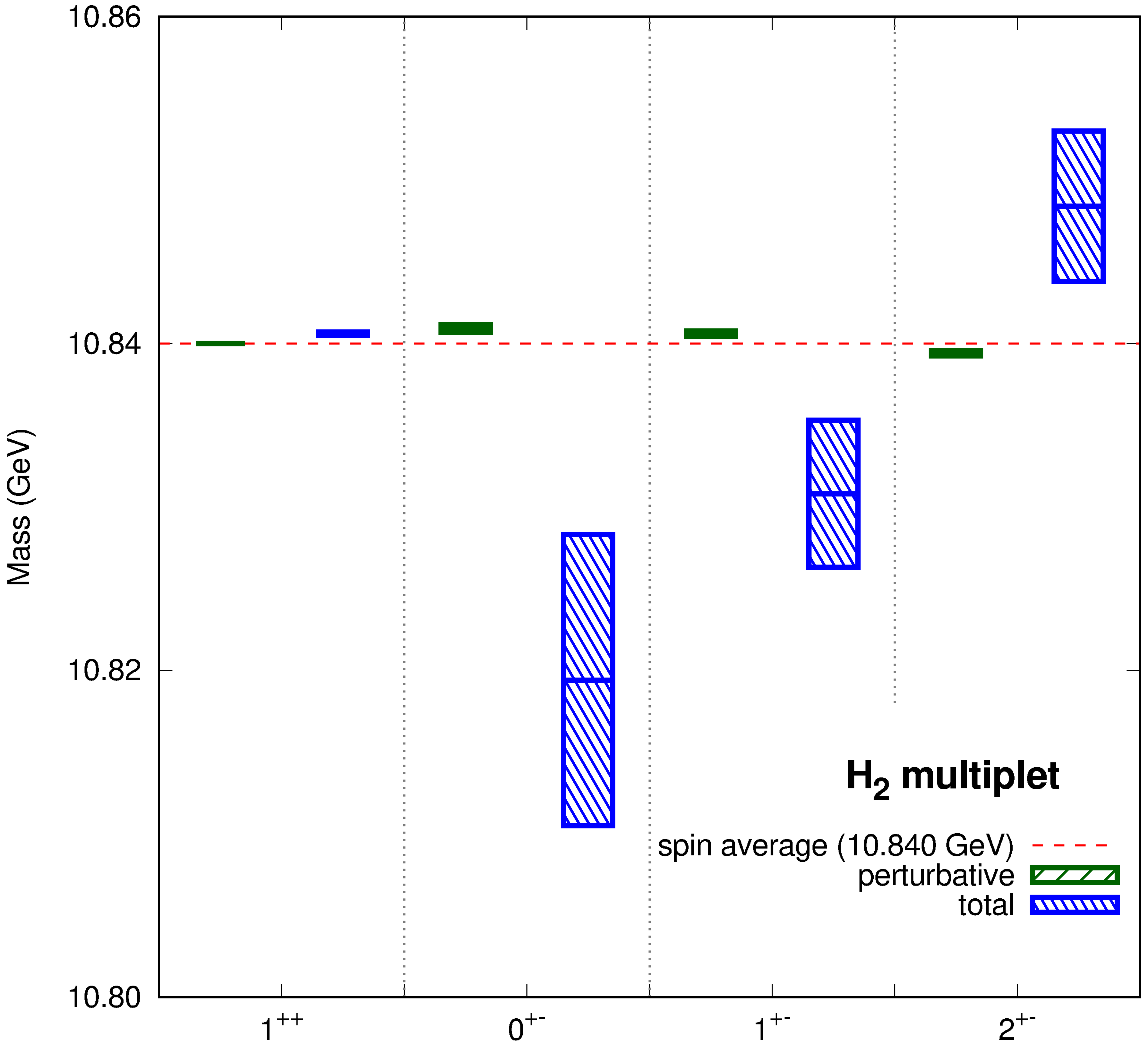
H_3 and H_4 are also calculated

- Brambilla Lai Segovia Tarrus Vairo PRD 99 (2019) 014017

here you find predictions for all H multiplets

Bottomonium hybrid spin splittings

thanks to the BOEFT factorizatio we can fix the nonperturbative unknowns from a charmonium hybrid calculationthe nonperturbative low energy unknownsdo not depend on the flavor: we can predict the bottomonium hybrids splin splittings



and also the other H multiplets

- Brambilla Lai Segovia Tarrus Vairo PRD 99 (2019) 014017

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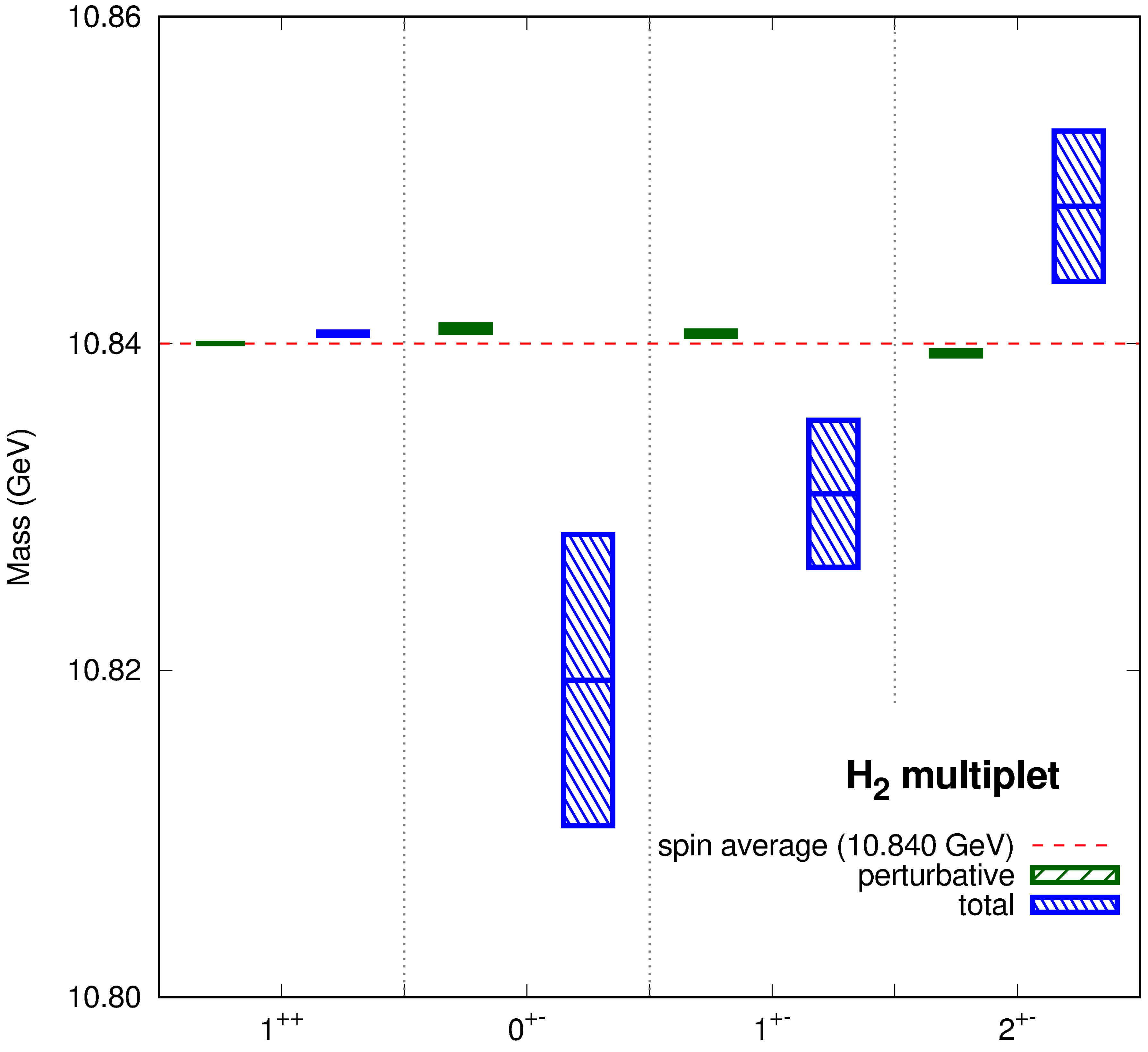

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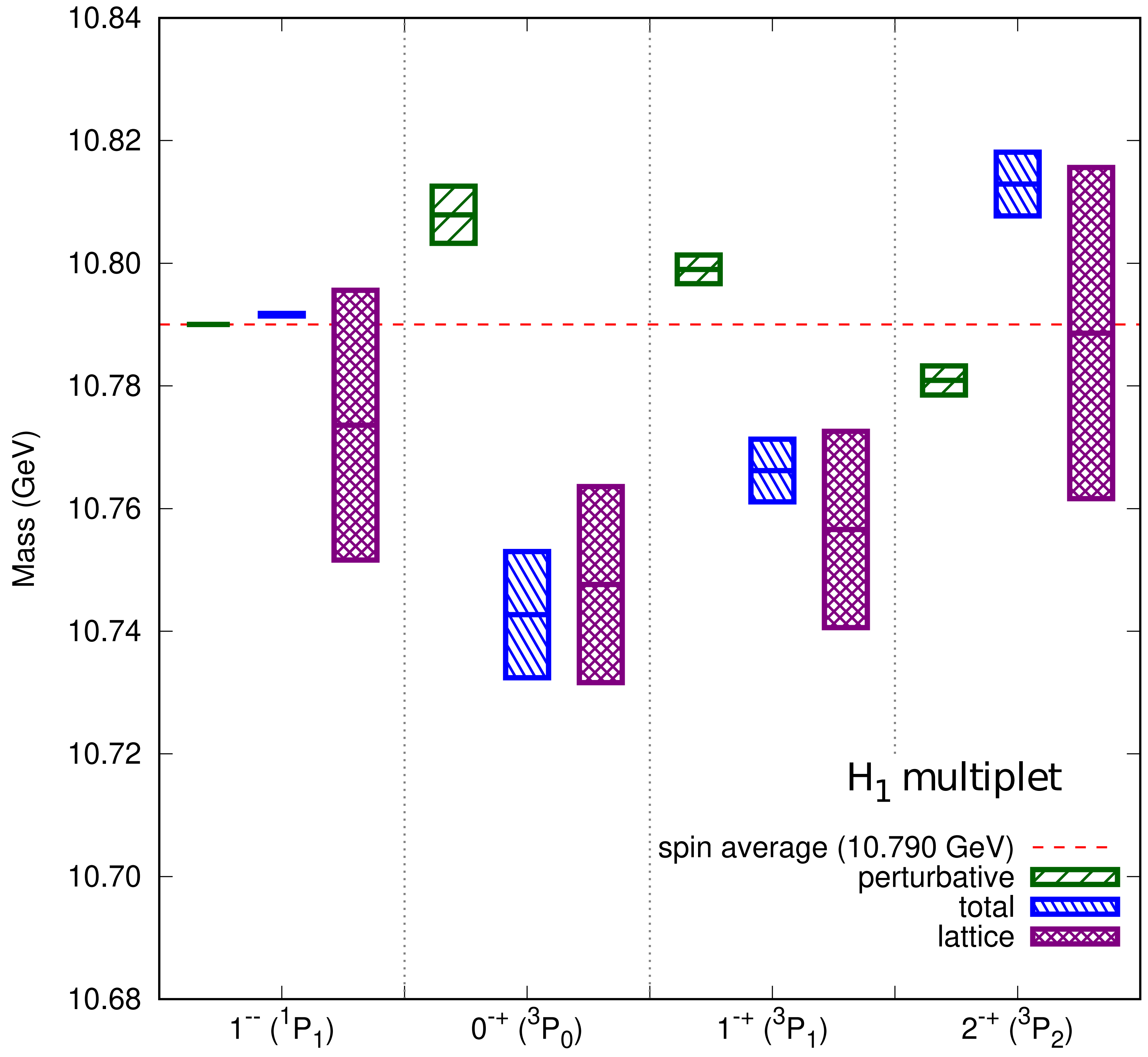
○ Brambilla Lai Segovia Tarrus Vairo PRD 99 (2019) 014017

Comparison of our prediction to the existing lattice data on H1

Bottomonium H_1 hybrid spin splittings



blue BOEFT predictions (more precise), violet actual lattice calculation

○ Ryan et al arXiv:2008.02656 [2+1 flavors, $m_\pi = 400$ MeV] unpublished plot by J. Segovia and J. Tarrus

BOEFT calculation of semi inclusive hybrids decays to quarkonium: H —> Quarkonium (S) plus X (anything light)

$$\Gamma_{H\to S} = -2\,\langle H|\mathrm{Im}\Delta V|H\rangle.$$

we calculated spin conserving and spin flipping decays
they are same size

Decay to open threshold states not accounted

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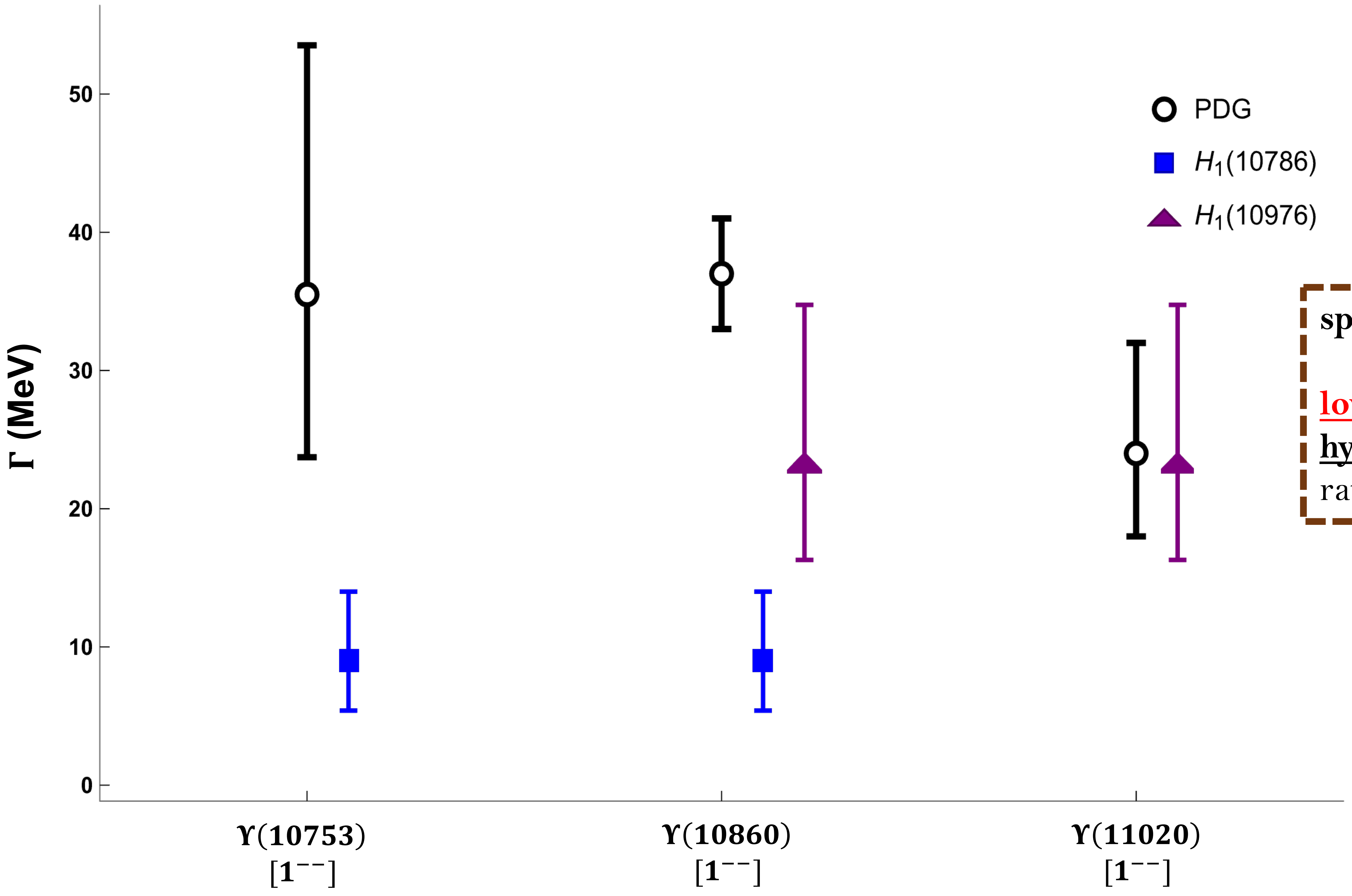
Decay to open threshold states not accounted

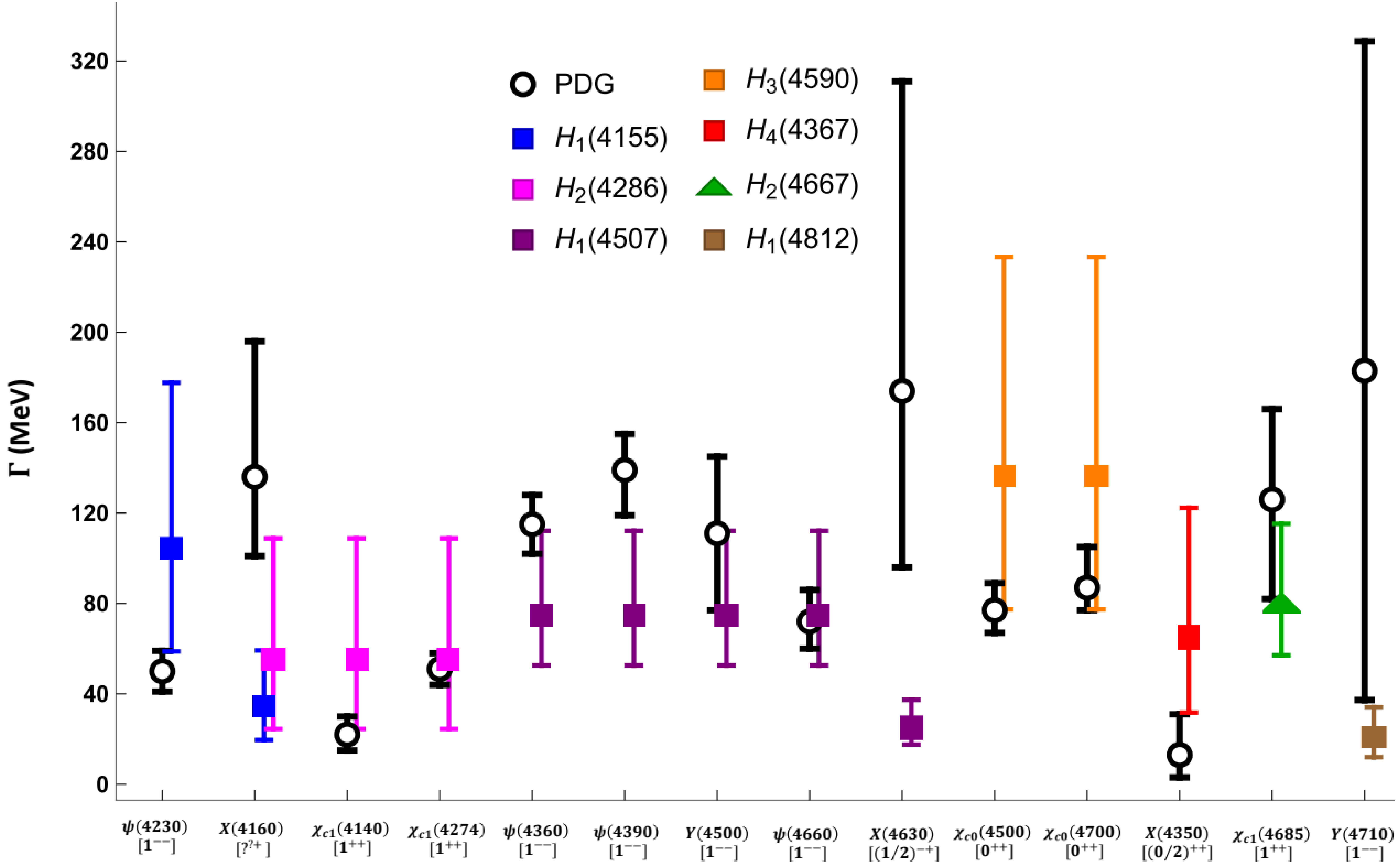
- Comparison: bottom exotic states with corresponding bottomonium hybrid state:



spin-conserving + spin-flipping decays
=
lower bound on the total decay widths of hybrids which is compared with inclusive rate of physical states in PDG.

BOEFT calculation of semi inclusive hybrids decays to quarkonium

Comparison: charm exotic states with corresponding charmonium hybrid state:



$$H_m \to Q_n + X$$

spin-conserving + spin-flipping decays

=

lower bound on the total decay widths of hybrids which is compared with inclusive rate of physical states in PDG.

Hybrid: Summary

Brambilla, Lai, AM, Vairo arXiv:2212.09187

- Hybrids ($Q\bar{Q}g$): Color singlet state of color octet $Q\bar{Q}$ + gluon. (Q = c, b)

✓ **Isoscalar neutral mesons (Isospin=0)**

✓ Candidates for hybrids based on **mass, quantum numbers,** and **decays** to **quarkonium**:

Charm sector:

- $X(4160)$: could be **charm hybrid** $H_1[2^{-+}](4155)$.
- $X(4630)$: could be **charm hybrid** $H_1[(1/2^{-+})](4507)$.
- $\psi(4390)$: could be **charm hybrid** $H_1[1^{--}](4507)$.
- $\psi(4710)$: could be **charm hybrid** $H_1[(1^{--})](4812)$.
- $X(4630)$: could be **charm hybrid** $H_1[(1/2^{-+})](4507)$.
- $\chi_{c1}(4685)$: could be **charm hybrid** $H_2[(1^{++})](4667)$.

Bottom sector:

- $\Upsilon(10753)$: could be **bottom hybrid** $H_1[(1^{--})](10786)$.

DISCLAIMER!!!
All the above interpretation can differ accounting for decays to meson-pair threshold states and hybrid-quarkonium mixing.

Hybrid: Mixing with heavy-light

- **Hybrid decays to s-wave + s-wave meson pairs:**

Conventional Wisdom: Hybrid decays to two S-wave mesons forbidden!

Kou & Pene, Phys Lett B 631 (2005) Page, Phys Lett B 407 (1997) Farina, Tecocoatzi, Giachino, Santopinto & Swanson, Phys Rev D 102 (2020)

Decay allowed based on BO-quantum #

Bruschini Phys. Rev. D 109 L031501 (2024)
J. Castella JHEP 06, 107 (2024)

Hybrid

Light spin K^{PC}	Static energies $D_{\infty h}$	l	J^{PC} $\{S_Q=0, S_Q=1\}$	Multiplets
1^{+-}	$\{\Sigma_u^-, \Pi_u\}$	1	$\{1^{--},(0,1,2)^{-+}\}$	H_1
	$\{\Pi_u\}$	1	$\{1^{++},(0,1,2)^{+-}\}$	H_2
	$\{\Sigma_u^-\}$	0	$\{0^{++},1^{+-}\}$	H_3
	$\{\Sigma_u^-, \Pi_u\}$	2	$\{2^{++},(1,2,3)^{+-}\}$	H_4
	$\{\Pi_u\}$	2	$\{2^{--},(1,2,3)^{-+}\}$	H_5

BO-quantum # Λ_η^σ for threshold

$K_{\bar{q}}^P \otimes K_q^P$	K^{PC}	Static energies $D_{\infty h}$
$(1/2)^- \otimes (1/2)^+$	0^{-+}	$\{\Sigma_u^-\}$
	1^{--}	$\{\Sigma_g^+, \Pi_g\}$

s-wave+s-wave
Ex. $\mathbf{D\bar{D}}$ threshold

Σ_u^- component in hybrids couple with Σ_u^- component in s-wave+s-wave !!!!

Recent lattice computation for $c\bar{c}$ hybrid 1^{-+} **decay** to

$$D_1\bar{D} : 258(133)\ \text{MeV}$$

$$D^*\bar{D} : 88(18)\ \text{MeV}$$

$$D^*\bar{D}^* : 150(118)\ \text{MeV}$$

Shi et al. Phys. Rev. D 109, 094513 (2024)

Outlook

- BOEFT aims at describing all exotics containing two heavy quarks in a QCD derived controlled framework

- BOEFT is based on symmetry and scales factorization

  at the short distance scale we have control of the perturbative calculation

  at the large distance scale we need lattice calculations of few gauge invariant universal correlators

  still the structure of the EFT allows for model independent predictions

- Once the lattice input is there the BOEF allows applications to domain in general not directly accessible to a lattice calculation (decay, production, medium propagation)

- It is important to develop techniques for the calculation of the low energy correlators (gradient flow) and the interface between perturbation theory and lattice

- The results obtained on the X and the Tcc gives is an idea of their nature beyond the models and redefine our knowledge of the strong force

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- The results obtained on the X and the Tcc gives is an idea of their nature beyond the models and redefine our knowledge of the strong force

This picture has the possibility to give a unified description to exotics and to leave the dynamics decide which configuration used by models will dominate in a given range

Outlook

- BOEFT aims at describing all exotics containing two heavy quarks in a QCD derived controlled framework

- BOEFT is based on symmetry and scales factorization
  at the short distance scale we have control of the perturbative calculation

  at the large distance scale we need lattice calculations of few gauge invariant universal correlators

  still the structure of the EFT allows for model independent predictions

- Once the lattice input is there the BOEF allows applications to domain in general not directly accessible to a lattice calculation (decay, production, medium propagation)

- It is important to develop techniques for the calculation of the low energy correlators (gradient flow) and the interface between perturbation theory and lattice

- The results obtained on the X and the Tcc gives is an idea of their nature beyond the models and redefine our knowledge of the strong force

This picture has the possibility to give a unified description to exotics and to leave the dynamics decide which configuration used by models will dominate in a given range

Combining BOEFT + open quantum systems one can attempt to study the X Y Z in heavy ion collisions

Backup

Strongly coupled pNRQCD: quantum mechanical matching

NRQCD

$$\langle H|\,\mathcal{H}\,|H\rangle = \langle nljs|\,\frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}}{m^n}\,|nljs\rangle$$

the matching conditions are :

$$|\underline{0};\mathbf{x}_1\mathbf{x}_2\rangle - > S^\dagger(\mathbf{x}_1\mathbf{x}_2)|\text{vac}\rangle$$

expand quantomechanically NRQCD states and energies in 1/m around the zero order and identify the QCD potentials

NRQCD

$$\mathcal{H} = \mathcal{H}^{(0)} + \frac{\delta\mathcal{H}^{(1)}}{m} + \frac{\delta\mathcal{H}^{(2)}}{m^2} + \frac{\delta\mathcal{H}^{(3)}}{m^3} + \frac{\delta\mathcal{H}^{(4)}}{m^4} + ..$$

$$\mathcal{H}^{(0)} = \int d^3\mathbf{x}\,\frac{1}{2}\left(\mathbf{\Pi}^a\mathbf{\Pi}^a + \mathbf{B}^a\mathbf{B}^a\right) - \sum^{n_f} \bar{q}\,i\mathbf{D}\cdot\boldsymbol{\gamma}\,q$$

$$\delta\mathcal{H}^{(1)} = -\int d^3\mathbf{x}\,\psi^\dagger\left(\frac{\mathbf{D}^2}{2} + c_F\,g\,\mathbf{S}\cdot\mathbf{B}\right)\psi + \text{antip.}$$

$$|H\rangle \longrightarrow |\underline{0};\mathbf{x}_1,\mathbf{x}_2\rangle \otimes |nljs\rangle$$

$$|\underline{0};\mathbf{x}_1,\mathbf{x}_2\rangle = |\underline{0};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)} + \sum_{n\neq 0}\int d^3z_1\,d^3z_2|\underline{n};\mathbf{z}_1,\mathbf{z}_2\rangle^{(0)}$$

$$\times \frac{{}^{(0)}\langle\underline{n};\mathbf{z}_1,\mathbf{z}_2|\delta\mathcal{H}^{(1)}|\underline{0};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}}{E_0^{(0)}(z) - E_n^{(0)}(x)} + \ldots$$

Strongly coupled pNRQCD: quantum mechanical matching

the matching conditions are :

NRQCD

$$\langle H|\mathcal{H}|H\rangle = \langle nljs|\frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}}{m^n}|nljs\rangle$$

$$|\underline{0};\mathbf{x}_1\mathbf{x}_2\rangle -> S^\dagger(\mathbf{x}_1\mathbf{x}_2)|\mathrm{vac}\rangle$$

expand quantomechanically NRQCD states and energies in 1/m around the zero order and identify the QCD potentials

NRQCD

$$\mathcal{H} = \mathcal{H}^{(0)} + \frac{\delta\mathcal{H}^{(1)}}{m} + \frac{\delta\mathcal{H}^{(2)}}{m^2} + \frac{\delta\mathcal{H}^{(3)}}{m^3} + \frac{\delta\mathcal{H}^{(4)}}{m^4} + ..$$

$$\mathcal{H}^{(0)} = \int d^3\mathbf{x}\,\frac{1}{2}\left(\mathbf{\Pi}^a\mathbf{\Pi}^a + \mathbf{B}^a\mathbf{B}^a\right) - \sum^{n_f} \bar{q}\,i\mathbf{D}\cdot\boldsymbol{\gamma}\,q$$

$$\delta\mathcal{H}^{(1)} = -\int d^3\mathbf{x}\,\psi^\dagger\left(\frac{\mathbf{D}^2}{2} + c_F\,g\,\mathbf{S}\cdot\mathbf{B}\right)\psi + \text{antip.}$$

$$|H\rangle \to |\underline{0};\mathbf{x}_1,\mathbf{x}_2\rangle \otimes |nljs\rangle$$

$$|\underline{0};\mathbf{x}_1,\mathbf{x}_2\rangle = |\underline{0};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)} + \sum_{n\neq 0}\int d^3z_1\,d^3z_2|\underline{n};\mathbf{z}_1,\mathbf{z}_2\rangle^{(0)}$$

$$\times\frac{^{(0)}\langle\underline{n};\mathbf{z}_1,\mathbf{z}_2|\delta\mathcal{H}^{(1)}|\underline{0};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}}{E_0^{(0)}(z) - E_n^{(0)}(x)} + \ldots$$

$$V = V^{(0)} + \frac{V^{(1)}}{m} + \frac{V^{(2)}}{m^2} + \ldots$$

$$V^{(0)}(r) = \lim_{T\to\infty}\frac{i}{T}\ln\langle\,\square\,\rangle$$

given in terms of gauge invariant generalised Wilson loops

$$\square = \exp\left\{ig\oint_{r\times T} dz^\mu A_\mu\right\}$$

BOEFT: Lattice Operators

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

Hybrids $\mathbf{Q\bar{Q}g}$

Λ_η^σ	k^{PC}	Representation	Operator Examples $H_{8,\kappa}^{\alpha,a}T^a$	Projectors $P_{\kappa\lambda}^\alpha$
Σ_g^+	0^{++}	scalar	$\mathbb{1}$[a]	1
Σ_u^+	0^{+-}	scalar	$\boldsymbol{D}\cdot\boldsymbol{E}$	1
Σ_g^-	0^{--}	pseudoscalar	$[\boldsymbol{E}\cdot,\boldsymbol{B}]$	1
Σ_u^-	0^{-+}	pseudoscalar	$\{\boldsymbol{E}\cdot,\boldsymbol{B}\}$	1
$\{\Sigma_g^+,\Pi_g\}$	1^{--}	vector	E^i	$\{\hat{r}^i,\hat{r}_\pm^i\}$
$\{\Sigma_u^+,\Pi_u\}$	1^{-+}	vector	$([\boldsymbol{E}\times,\boldsymbol{B}])^i$	$\{\hat{r}^i,\hat{r}_\pm^i\}$
$\{\Sigma_g^-,\Pi_g\}$	1^{++}	pseudovector	$(\boldsymbol{D}\times[\boldsymbol{E}\times,\boldsymbol{B}])^i$	$\{\hat{r}^i,\hat{r}_\pm^i\}$
$\{\Sigma_u^-,\Pi_u\}$	1^{+-}	pseudovector	B^i	$\{\hat{r}^i,\hat{r}_\pm^i\}$

Quarkonium tetraquarks $\mathbf{Q\bar{Q}q\bar{q}}$ (I=0)

Λ_η^σ	k^{PC}	Representation	Operator Examples $H_{8,\kappa}^{\alpha,a}\,(I=0)$	Projectors $P_{\kappa\lambda}^\alpha$
Σ_g^+	0^{++}	scalar	$\bar{q}T^aq$	1
Σ_u^-	0^{-+}	pseudoscalar	$\bar{q}\gamma^5T^aq$	1
$\{\Sigma_g^+,\Pi_g\}$	1^{--}	vector	$\bar{q}\,\gamma^i\,T^aq$	$\{\hat{r}^i,\hat{r}_\pm^i\}$
$\{\Sigma_g^-,\Pi_g\}$	1^{++}	pseudovector	$\bar{q}\,\gamma^i\,\gamma^5\,T^aq$	$\{\hat{r}^i,\hat{r}_\pm^i\}$
$\{\Sigma_u^-,\Pi_u\}$	1^{+-}	pseudovector	$\bar{q}\,(\boldsymbol{\gamma}\times\boldsymbol{\gamma})^i\,\gamma^5\,T^aq$	$\{\hat{r}^i,\hat{r}_\pm^i\}$

I=1 operator: Insert $\boldsymbol{e}_{I_3}\cdot\boldsymbol{\tau}$ between light quarks

Quarkonium pentaquarks $\mathbf{Q\bar{Q}qqq}$

$$
\begin{aligned}
H^{\alpha,a}_{8,I_3=\pm1/2,(1/2)^+}(t,\boldsymbol{x}) = \Big[& \left(\delta_{\alpha\beta_1}\sigma^2_{\beta_2\beta_3}+\delta_{\alpha\beta_2}\sigma^2_{\beta_1\beta_3}+\delta_{\alpha\beta_3}\sigma^2_{\beta_1\beta_2}\right)\left(\delta_{I_3f_1}\tau^2_{f_2f_3}+\delta_{I_3f_2}\tau^2_{f_1f_3}+\delta_{I_3f_3}\tau^2_{f_1f_2}\right)(T_2)^a_{l_1,l_2,l_3} \\
& + \left(\delta_{\alpha\beta_1}\sigma^2_{\beta_2\beta_3}+\delta_{\alpha\beta_2}\sigma^2_{\beta_3\beta_1}+\delta_{\alpha\beta_3}\sigma^2_{\beta_2\beta_1}\right)\left(\delta_{I_3f_1}\tau^2_{f_2f_3}+\delta_{I_3f_2}\tau^2_{f_3f_1}+\delta_{I_3f_3}\tau^2_{f_2f_1}\right)(T_3)^a_{l_1,l_2,l_3} \\
& + \left(\delta_{\alpha\beta_1}\sigma^2_{\beta_3\beta_2}+\delta_{\alpha\beta_2}\sigma^2_{\beta_3\beta_1}+\delta_{\alpha\beta_3}\sigma^2_{\beta_1\beta_2}\right)\left(\delta_{I_3f_1}\tau^2_{f_3f_2}+\delta_{I_3f_2}\tau^2_{f_3f_1}+\delta_{I_3f_3}\tau^2_{f_1f_2}\right)(T_1)^a_{l_1,l_2,l_3}\Big] \\
& \left(P_+q_{l_1f_1}(t,\boldsymbol{x})\right)^{\beta_1}\left(P_+q_{l_2f_2}(t,\boldsymbol{x})\right)^{\beta_2}\left(P_+q_{l_3f_3}(t,\boldsymbol{x})\right)^{\beta_3}
\end{aligned}
$$

Castellà , Soto Phys. Rev. D. 102, 014012 (2020)

BOEFT: Lattice Operators

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

Doubly heavy baryons **QQq**

BO quantum # $D_{\infty h}$	k^P	$(k-1/2)$ Representation	Operator Examples $H^{\alpha,\ell}_{3,\kappa}$	Projectors $P^{\alpha}_{\kappa\lambda}$
$(1/2)_g$	$(1/2)^+$	scalar	$[P_+ q^a]^\alpha$	$P^\alpha_{1/2,\pm1/2}$
$(1/2)'_u$	$(1/2)^-$	pseudoscalar	$[P_+ \gamma^5 q^a]^\alpha$	$P^\alpha_{1/2,\pm1/2}$
$\{(1/2)_u, (3/2)_u\}$	$(3/2)^-$	vector	$\mathcal{C}^{3/2\,\alpha}_{1\,m\,1/2\,\beta}\left[(\boldsymbol{e}_m\cdot\boldsymbol{D})\,(P_+ q^a)^\beta\right]$	$\{P^\alpha_{3/2,\pm1/2}, P^\alpha_{3/2,\pm3/2}\}$

Castellà , Soto
Phys. Rev. D. 102, 014012 (2020)

Doubly heavy tetraquarks **$QQ\bar{q}\bar{q}$** (I=0)

Λ^σ_η	k^P	Representation	Operator Examples $H^{\alpha,\ell}_{3,\kappa}\,(I=0)$	Operator Examples $H^{\alpha,\sigma}_{6,\kappa}\,(I=0)$	Projectors $P^{\alpha}_{\kappa\lambda}$
Σ^+_g	0^+	scalar	$\bar{q}\gamma^5\gamma^2\tau^2\,\underline{T}^a\,q^*$	–	1
Σ^-_u	0^-	pseudoscalar	–	$\bar{q}\gamma^2\tau^2\,\underline{\Sigma}^a\,q^*$	1
$\{\Sigma^+_u, \Pi_u\}$	1^-	vector	$\bar{q}\gamma^i\,\gamma^5\gamma^2\tau^2\,\underline{T}^a\,q^*$	$\bar{q}\gamma^i\,\gamma^5\gamma^2\tau^2\,\underline{\Sigma}^a\,q^*$	$\{\hat{r}^i, \hat{r}^i_\pm\}$
$\{\Sigma^-_g, \Pi_g\}$	1^+	pseudovector	–	$\bar{q}\gamma^i\,\gamma^2\tau^2\,\underline{\Sigma}^a\,q^*$	$\{\hat{r}^i, \hat{r}^i_\pm\}$

Doubly heavy tetraquarks **$QQ\bar{q}\bar{q}$** (I=1)

Λ^σ_η	k^P	Representation	Operator Examples $H^{\alpha,\ell}_{3,\kappa}\,(I=1)$	Operator Examples $H^{\alpha,\sigma}_{6,\kappa}\,(I=1)$	Projectors $P^{\alpha}_{\kappa\lambda}$
Σ^+_g	0^+	scalar	–	$\bar{q}\gamma^5\gamma^2\boldsymbol{e}_{I_3}\cdot(\tau^2\boldsymbol{\tau})\,\underline{\Sigma}^a\,q^*$	1
Σ^-_u	0^-	pseudoscalar	$\bar{q}\gamma^2\boldsymbol{e}_{I_3}\cdot(\tau^2\boldsymbol{\tau})\,\underline{T}^a\,q^*$	–	1
$\{\Sigma^+_u, \Pi_u\}$	1^-	vector	$\bar{q}\gamma^i\,\gamma^5\gamma^2\boldsymbol{e}_{I_3}\cdot(\tau^2\boldsymbol{\tau})\,\underline{T}^a\,q^*$	$\bar{q}\gamma^i\,\gamma^5\gamma^2\boldsymbol{e}_{I_3}\cdot(\tau^2\boldsymbol{\tau})\,\underline{\Sigma}^a\,q^*$	$\{\hat{r}^i, \hat{r}^i_\pm\}$
$\{\Sigma^-_g, \Pi_g\}$	1^+	pseudovector	$\bar{q}\gamma^i\,\gamma^2\boldsymbol{e}_{I_3}\cdot(\tau^2\boldsymbol{\tau})\,\underline{T}^a\,q^*$	–	$\{\hat{r}^i, \hat{r}^i_\pm\}$

Doubly heavy pentaquark **$QQqq\bar{q}$**

$$H^{\alpha,\ell}_{3,I_3=\pm1/2,(1/2)^+}(t,\boldsymbol{x}) = \left[\left(\delta_{\alpha\beta_1}\sigma^2_{\beta_2\beta_3} + \delta_{\alpha\beta_2}\sigma^2_{\beta_3\beta_1} + \delta_{\alpha\beta_3}\sigma^2_{\beta_2\beta_1}\right)\left(\delta_{I_3 f_1}\tau^2_{f_2 f_3} + \delta_{I_3 f_2}\tau^2_{f_3 f_1} + \delta_{I_3 f_3}\tau^2_{f_2 f_1}\right)\underline{T}^i_{l_1,l_2}\underline{T}^\ell_{i,l_3}\right]$$

$$\left(P_+ q_{l_1 f_1}(t,\boldsymbol{x})\right)^{\beta_1}\left(P_+ q_{l_2 f_2}(t,\boldsymbol{x})\right)^{\beta_2}\left(\bar{q}_{l_3 f_3}(t,\boldsymbol{x})P_-\right)^{\beta_3},$$

Tcc

For illustration purpose, we model $V_{\Sigma_g^+}$ considering short-distance behavior from [75] and long-distance behavior with a two-pion exchange potential [76]

$$V_{\Sigma_g^+} = \begin{cases} \frac{\kappa_3}{r} + E_{0^+} + A_{\Sigma_g^+}\, r^2 & r < R_{\Sigma_g^+} \\ F_{\Sigma_g^+}\, e^{-r/d}/r^2 & r > R_{\Sigma_g^+}. \end{cases} \tag{7}$$

where $\kappa_3 = -0.120$ and $A_{\Sigma_g^+} = 0.197\,\mathrm{GeV}^3$ [75], the parameters $F_{\Sigma_g^+}$ and $R_{\Sigma_g^+}$ are determined by imposing continuity up to first derivatives. We treat 0^+ triplet meson energy E_{0^+} as free parameter to obtain $T_{cc}^+(3875)$ state.

X

We use the lattice parametrization (where energy levels are normalized with respect to twice the energy of the static heavy-light pair $E_{Q\bar{l}}$) in [71] for $V_{\Sigma_g^+}$ across all r. For $V_{\Sigma_g^{+\prime}}$ and V_{Π_g}, we model the short-distance behavior using the quenched BO-potential parametrization from [75] due to lack of lattice computation, long-distance behavior with a two-pion exchange potential [76], and the asymptotic limit ($r \to \infty$) with a constant $E_1 = 0.005$ GeV as in [71]:

$$V_{\Sigma_g^+}(r) = V_0 + \frac{\gamma}{r} + \sigma r, \tag{2}$$

$$V_\Lambda(r) = \begin{cases} \frac{\kappa_8}{r} + E_{1--} + A_\Lambda\, r^2 + B_\Lambda\, r^4 & r < R_\Lambda \\ F_\Lambda\, e^{-r/d}/r^2 + E_1 & r > R_\Lambda. \end{cases} \tag{3}$$

where $\Lambda \equiv \{\Sigma_g^{+\prime}, \Pi_g\}$, $\gamma = -0.434$, $\sigma = 0.198\,\mathrm{GeV}^2$, $\kappa_8 = 0.037$, $A_{\Sigma_g^{+\prime}} = 0.0065\,\mathrm{GeV}^3$, $B_{\Sigma_g^{+\prime}} = 0.0018\,\mathrm{GeV}^5$, $A_{\Pi_g} = 0.0726\,\mathrm{GeV}^3$, $B_{\Pi_g} = -0.0051\,\mathrm{GeV}^5$, $d \sim 1/(2m_\pi) \sim 1/0.3\,\mathrm{GeV}^{-1} \sim 0.65\,\mathrm{fm}$ and parameters F_Λ and R_Λ are determined by imposing continuity up to first derivatives. The constant $V_0 = -1.142$ GeV is interpreted as $-2E_{Q\bar{l}}$. For $V_{\Sigma_g^+ - \Sigma_g^{+\prime}}$, it must vanish as $r \to 0$ based on pNRQCD [77], and approach zero asymptotically as $r \to \infty$, with a peak near the string-breaking region[4]. Hence, we parametrize $V_{\Sigma_g^+ - \Sigma_g^{+\prime}}$ as

$$V_{\Sigma_g^+ - \Sigma_g^{+\prime}} = \begin{cases} \frac{g}{r_1}\, r & r < r_1 \\ g & r_1 \le r \le r_2 \\ A\, \exp(-r/r_0) & r > r_2, \end{cases} \tag{4}$$

where the parameters $g = 0.05$ GeV, $r_1 = 0.95$ fm, and $r_2 = 1.51$ fm are fixed considering the lattice data in [71], $r_0 = 0.5$ fm is the Sommer scale, $A = 1.02$ GeV has been fixed by demanding the continuity of the potential at r_2.

BOEFT: Pentaquark multiplets

Berwein, Brambilla, AM, Vairo,
arXiv 2408.04719

$Q\bar{Q}qqq$

$Q\bar{Q}$ color state	Light spin $\boldsymbol{K}^P$	Static energies	l	J^P $\{S_Q = 0, S_Q = 1\}$
Octet	$(1/2)^+$	$(1/2)_g$	$1/2$	$\{1/2^-, (1/2, 3/2)^-\}$
	$(3/2)^+$	$(3/2)_g$	$3/2$	$\{3/2^-, (1/2, 3/2, 5/2)^-\}$

No lattice inputs available on Born-Oppenheimer static potentials for pentaquarks

$QQqq\bar{q}$

QQ color state	Light spin $\boldsymbol{K}^P$	heavy spin $S_Q = 0$	heavy spin $S_Q = 1$
sextet	$(1/2)^-$	$\{(1/2)^-\}$	$\{(1/2, 3/2)^+, (1/2, 3/2, 5/2)^+\}$
	$(3/2)^-$	$\{(3/2)^-\}$	$\{(1/2, 3/2)^+\}, \{(1/2, 3/2, 5/2)^+\}, \{(3/2, 5/2, 7/2)^+\}$
antitriplet	$(1/2)^-$	$\{(1/2)^+, (3/2)^+\}$	$\{(1/2, 3/2)^-\}$
	$(3/2)^-$	$\{(1/2)^+\}, \{(3/2)^+\}, \{(5/2)^+\}$	$\{(1/2, 3/2, 5/2)^-\}$

Ball, Neff, Duessel, Lippert, Schilling
(SESAM collaboration)
Phys. Rev. D. 71, (2005)

Static Energies: Avoided crossing

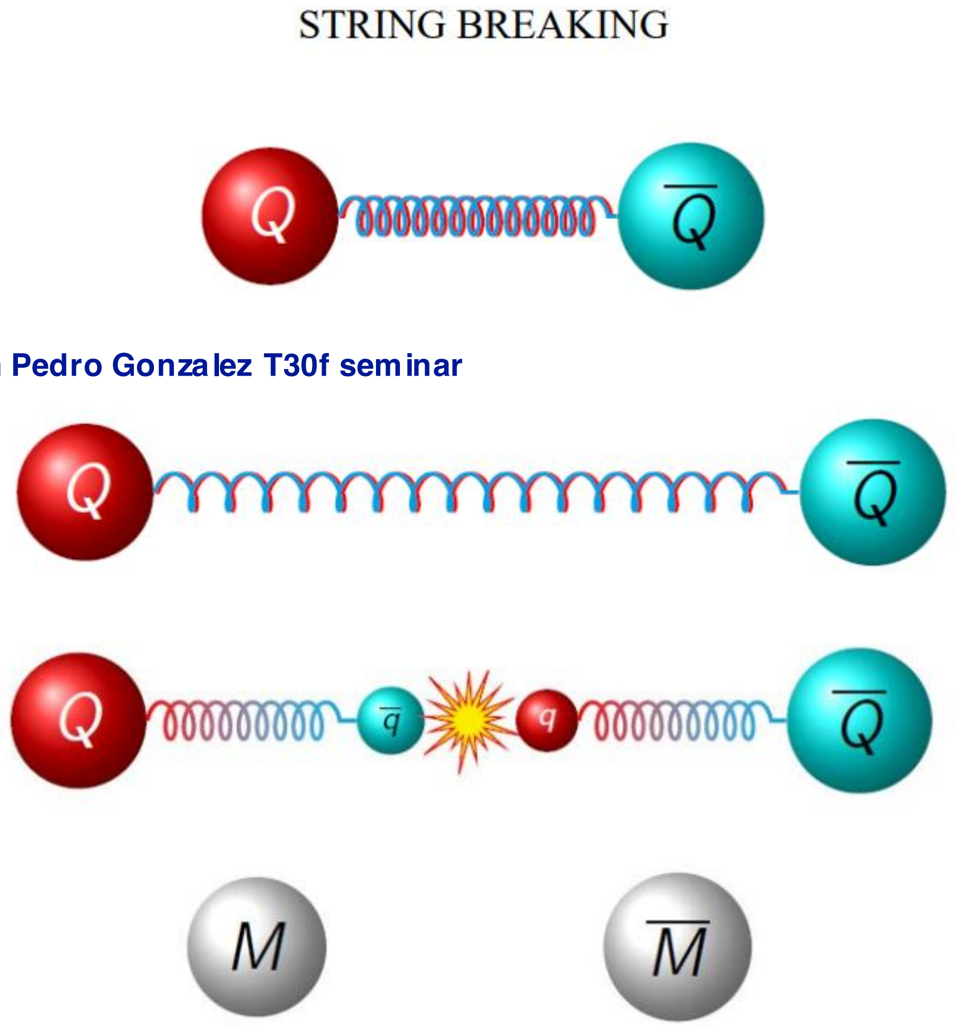
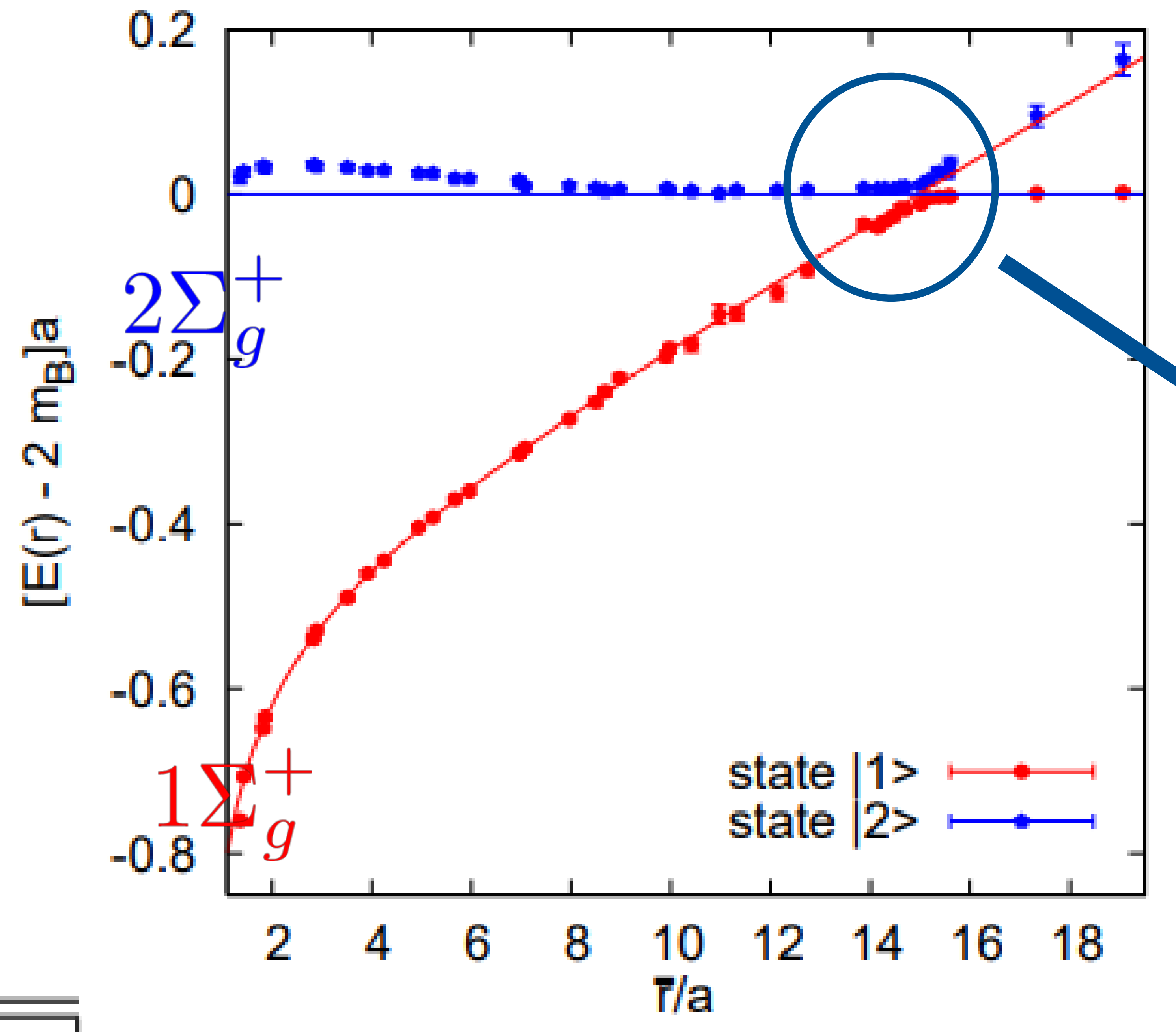


Figure from Pedro Gonzalez T30f seminar

Meson-antimeson threshold

$K_{\bar{q}}^P \otimes K_q^P$	K^{PC}	Static energies $D_{\infty h}$
$(1/2)^- \otimes (1/2)^+$	0^{-+}	$\{\Sigma_u^-\}$
	1^{--}	$\{\Sigma_g^+, \Pi_g\}$
$(1/2)^- \otimes (1/2)^-$	0^{++}	$\{\Sigma_g^+\}$
	1^{+-}	$\{\Sigma_u^-, \Pi_u\}$
$(1/2)^- \otimes (3/2)^-$	1^{+-}	$\{\Sigma_u^-, \Pi_u\}$
	2^{++}	$\{\Sigma_g^+, \Pi_g, \Delta_g\}$

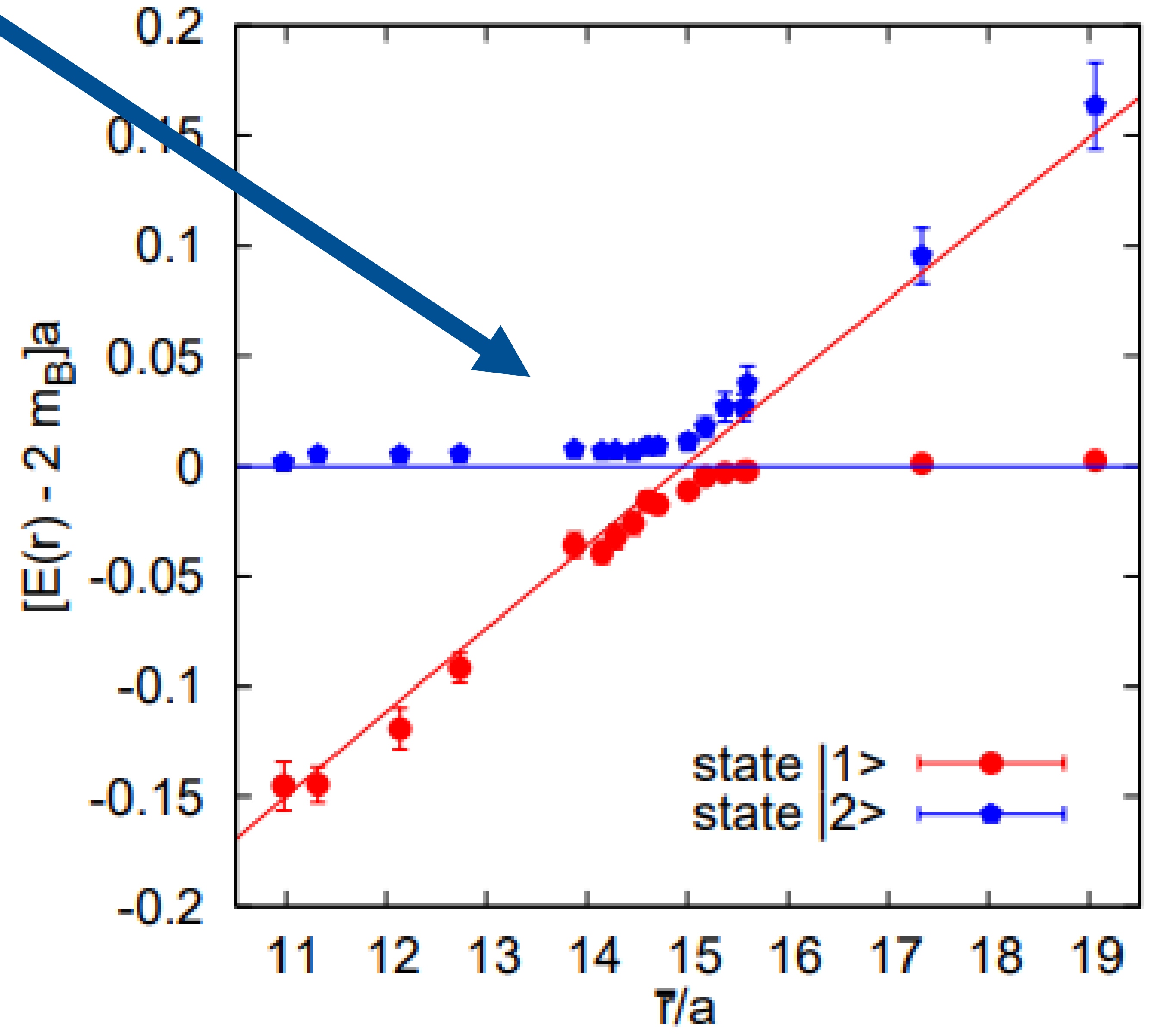
s-wave+s-wave
Ex. $D\bar{D}$ threshold



$m_\pi \approx 650$ MeV

$V_\Psi(r) = -\frac{4}{3}\frac{\alpha_s}{r} + \sigma r$

$m_M + m_{\bar{M}}$



String breaking radius ≈ 1.25 fm

a ≈ 0.083 fm

BO-quantum # Σ_g^+ mix: avoided crossing between $Q\bar{Q}$ & $M\bar{M}$

Hybrid static energies: $(\boldsymbol{\Sigma_u^-}, \boldsymbol{\Pi_u})$

1) **Avoided crossing** with **s-wave + p-wave threshold.** No lattice results available on this till now !!

2) $\boldsymbol{\Sigma_u^-}$ **component mixing** with **s-wave + s-wave threshold** (significant effects only if the energy gap less than Λ_{QCD} scale).

Bruschini Phys. Rev. D 109 L031501 (2024) **J. Castella JHEP 06, 107 (2024)**