Cheshire θ terms, axions, and Yang-Mills

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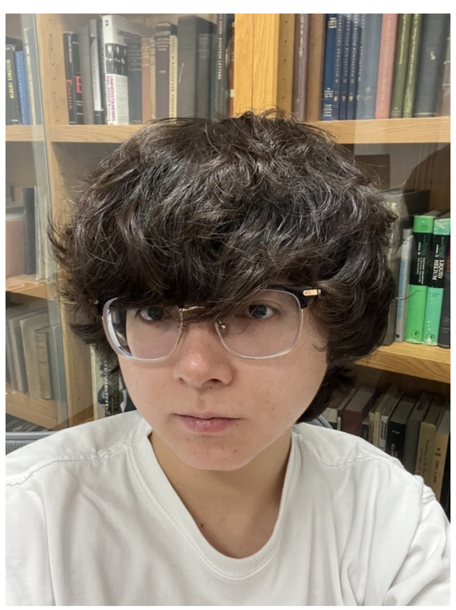
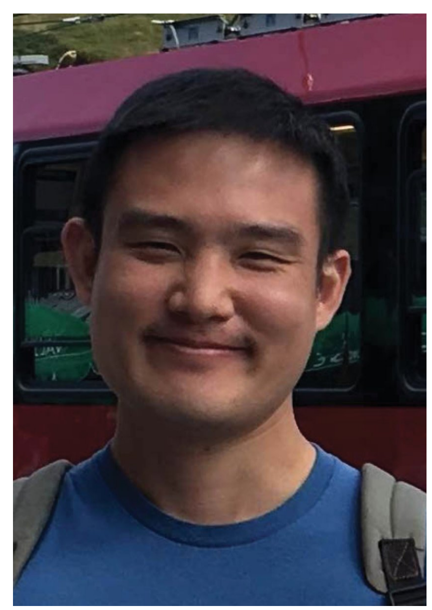
arXiv:2410.23355 + paper to appear soon



Shi Chen



Gongjun Choi

Maria Neuzil

Topic change apology

- I decided to give a talk which isn't about the lattice, different from what I advertised originally.
- My apologies!

- This talk **does** fit "formal developments" and "hadron interactions", but I won't say anything about the lattice.
  - However, there are lattice applications of some things I'll discuss - keep an eye on the arXiv in the spring…

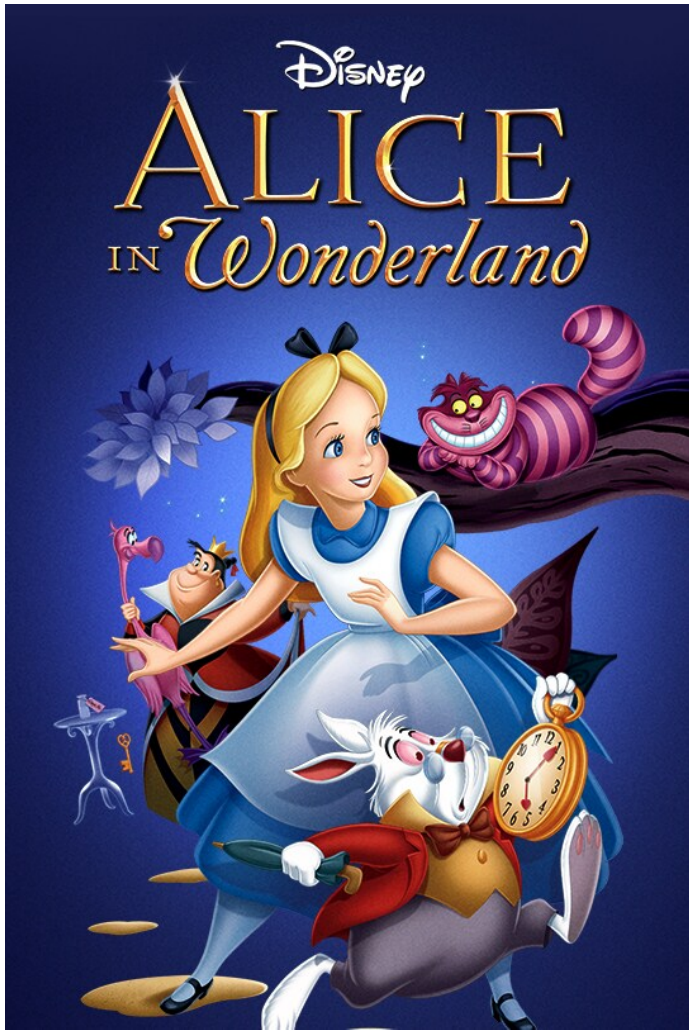
Theta terms

- Many QFTs — including QCD — have continuous 2π-periodic couplings θ, appearing as coefficients of **topological terms** $\int d^D x\, q(x) \in \mathbb{Z}$.
- Are there other ways for periodic couplings to appear?
- **Part I**: bottom-up construction of a new type of θ term, the Cheshire θ term.
  - Might have observable consequences when considering the Standard Model with axions.
- **Part II**: reexamine Witten effect in QED, and use it to inspire the general construction in part III.
- **Part III** of this talk: explain a general way to construct θ terms, which leads to the usual θ terms + new ones.

Part I: Cheshire θ term

Terminology

- "Alice in Wonderland" is a somewhat famous British children's book from 1865. The book tells a story about Alice falling through a rabbit hole into a magical land full of amazing and sometimes creepy creatures.



The Cheshire Cat

- One of them is the Cheshire Cat. It likes to turn invisible. But not completely: the cat's grin remains.

- I'll now introduce a "Cheshire θ term". It's also almost invisible… except for its grin!

Cheshire θ term

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Setup

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- Consider a D-dimensional QFT with
  A. 2π-periodic compact scalar φ, with a $U(1)^{[D-2]}$ 2-form symmetry that counts φ vortices.
  B. $U(1)^{[0]}$ symmetry not involved in any 't Hooft anomalies, with a background gauge field $A^{[0]}$
- **Not** assuming a φ shift symmetry $U(1)^{[0]}_{\text{shift}}$.
- The path integral involves φ and other fields acted on by $U(1)^{[0]}$. Schematically, it looks like

$$Z(A^{[0]}) = \int \mathcal{D}\varphi \, \exp\left[-S_\varphi(\varphi, A^{[0]})\right]$$

- No 't Hooft anomaly ⇔ $Z(A^{[0]} + d\chi) = Z(A^{[0]}), \chi \simeq \chi + 2\pi$

Cheshire $\hat{\theta}$ term

- Let's set $A^{[0]} \to a = \frac{\hat{\theta}}{2\pi} d\varphi$. Sum over φ means that $U(1)^{[0]}$ just got gauged, but only a tiny bit.
  - da is flat: no propagating degrees of freedom.
  - a is a non-generic flat gauge field: tied to φ.
- If $\hat{\theta} \to \hat{\theta} + 2\pi$, $a \to a + d\varphi$, so the path integral doesn't change, and $\hat{\theta}$ is 2π periodic. It is the Cheshire θ parameter.
- Let's make this more concrete!

Cheshire $\hat{\theta}$ term

- $U(1)^{[0]}$ could be e.g. the $U(1)_V$ symmetry of a free Dirac fermion ψ. Then the Cheshire θ term is

$$i\hat{\theta}\int d^D x\,\frac{\partial_\mu\varphi}{2\pi}\bar{\psi}\gamma^\mu\psi$$

- Clearly different from standard θ terms: dimension-5 operator when $[\varphi]=1$ in $D=4$.
- Vanishes after (naive) integration by parts and use of EoM.
- Textbooks say that when building EFTs, we should discard terms that vanish after integrations by parts and/or use of EoMs because they cannot contribute to scattering.
- It is true that $\hat{\theta}$ doesn't change **particle-particle** scattering amplitudes. Like the Cheshire Cat, it's **almost** invisible.

Aharonov-Bohm effect

- Consider the integral of $a = \frac{\hat{\theta}}{2\pi} d\varphi$:

$$\int_C \frac{\hat{\theta}}{2\pi} d\varphi = \frac{\hat{\theta}}{2\pi} 2\pi n = n\hat{\theta}$$

assuming C goes around a winding-n vortex

- a is flat, but it has non-trivial holonomies around vortices!
- Particles with $U(1)^{[0]}$ charge q get an AB-like phase $e^{iqn\hat{\theta}}$!
- For experts: this is not an operator-operator braiding phase. Point and vortex operators are both in the vacuum sector of the theory at any $\hat{\theta}$.

More on AB phases

- The AB effect induced by $\hat{\theta}$ is quite robust.
  - Shift symmetry for φ isn't required.
  - If we impose a symmetry like $\varphi \to -\varphi$, it quantizes but does not kill the $\hat{\theta}$ term: $\hat{\theta} = 0, \pi$ both allowed.
  - If we break $U(1)^{[0]} \to \mathbb{Z}_K$, $A^{[0]}$ should be a $\mathbb{Z}_K$ gauge field, so that it satisfies $KA = d\alpha, \alpha \simeq \alpha + 2\pi$.
    - Comparing to $a = \dfrac{\hat{\theta}}{2\pi} d\varphi$, we learn that $\hat{\theta}$ gets quantized: $\hat{\theta} = 0, 2\pi/K, 4\pi/K, \ \ldots$
- Soon: will show that $\hat{\theta}$ induces a Witten effect.
- First I want to address the "who cares?" question, so let's talk about axions.

Axions and the Standard Model

- One of the most popular approaches to the Standard Model and the strong CP problem is to couple QCD to a P and CP odd 2π periodic `axion' scalar $\varphi$, which is the (pseudo) NG boson of a spontaneously broken $U(1)_{\rm PQ}$ `Peccei-Quinn' symmetry:

$$\frac{i}{32\pi^2}\int d^4x\,\varphi\,{\rm tr}\,f_{\mu\nu}\tilde{f}^{\mu\nu}+\int d^4x\,\frac{f_{\rm PQ}^2}{2}(\partial_\mu\varphi)^2$$

- The QCD θ term can be absorbed by a shift of φ, and θ is replaced by $\varphi_{\rm min}$, the value of φ at minimum of $V(\varphi)$
- If QCD is only source of explicit $U(1)_{\rm PQ}$ breaking (to a high-enough accuracy), minimum of $V(\varphi)$ is at 0, solving the strong CP problem.

Cheshire $\hat{\theta}$ term

- Now for a little bit of (hopefully productive) trolling. The point of coupling QCD to an axion is to get rid of a 2π periodic continuous periodic coupling…

Cheshire $\hat{\theta}$ term

- Now for a little bit of (hopefully productive) trolling. The point of coupling QCD to an axion is to get rid of a 2π periodic continuous periodic coupling…
- But as soon as you do this, you can write another 2π periodic continuous coupling:

$$i\,\hat{\theta}\int d^4x\,\frac{\partial_\mu\varphi}{2\pi}\,j^\mu_{B-L}\,,$$

- C or P symmetry $\Rightarrow \hat{\theta} = 0$ or π. But C and P are violated in SM. Any value of $\hat{\theta}$ is allowed.
    - Take $f_{\rm PQ}, \hat{\theta}$ as free parameters, to be constrained by data.
- Can also add $\hat{\theta}_{B+L} \in \mathbb{Z}_3$ coupling for $(\mathbb{Z}_3)_{B+L}$ symmetry

Effect of Cheshire $\hat{\theta}$ term

- The Cheshire $\hat{\theta}$ terms induces AB phases for leptons and quarks around axion strings:

leptons: $e^{i\hat{\theta}}$, **quarks:** $e^{i\hat{\theta}/3}$

- Aharonov-Bohm phases imply Aharonov-Bohm scattering!
- If you have e.g. a straight Abrikosov-Nielsen-Olesen string carrying gauge flux Φ (in units of the flux quantum), then particles with gauge charge q have a cross-section σ

$$\frac{d\sigma}{d\alpha} = \frac{\sin^2(q\Phi/2)}{2\pi p \sin^2(\alpha/2)}$$

$\alpha, p =$ scattering angle and momentum

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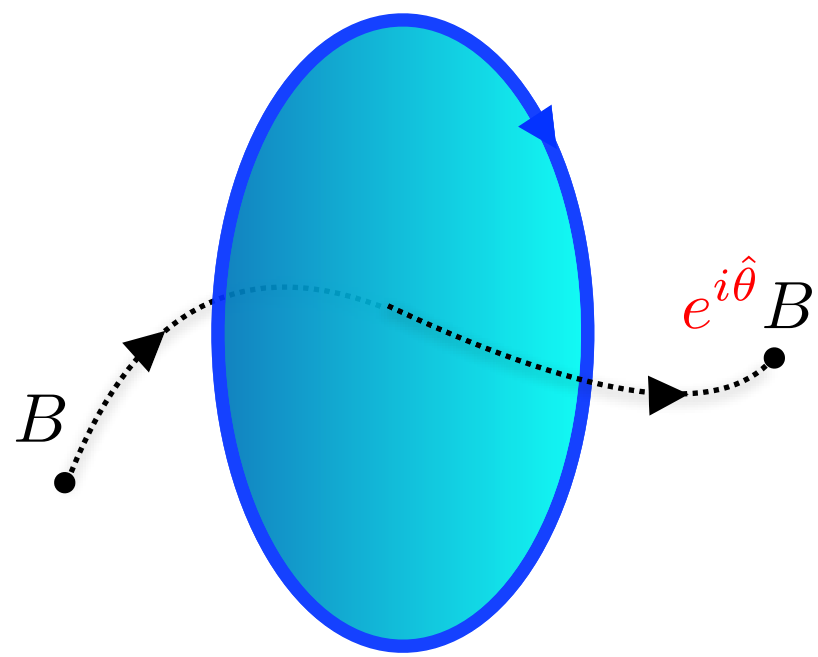

- The AB phase is present even if axion strings are attached to domain walls, since breaking of $U(1)^{[0]}_{\text{shift}}$ is irrelevant.

Effect of Cheshire $\hat{\theta}$ term

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leptons: $e^{i\hat{\theta}}$, quarks: $e^{i\hat{\theta}/3}$

- Aharonov-Bohm phases imply Aharonov-Bohm scattering!
- If you have e.g. a straight axion string in a model with $\hat{\theta} \neq 0$, then B-L charge q particles scatter on it with cross-section

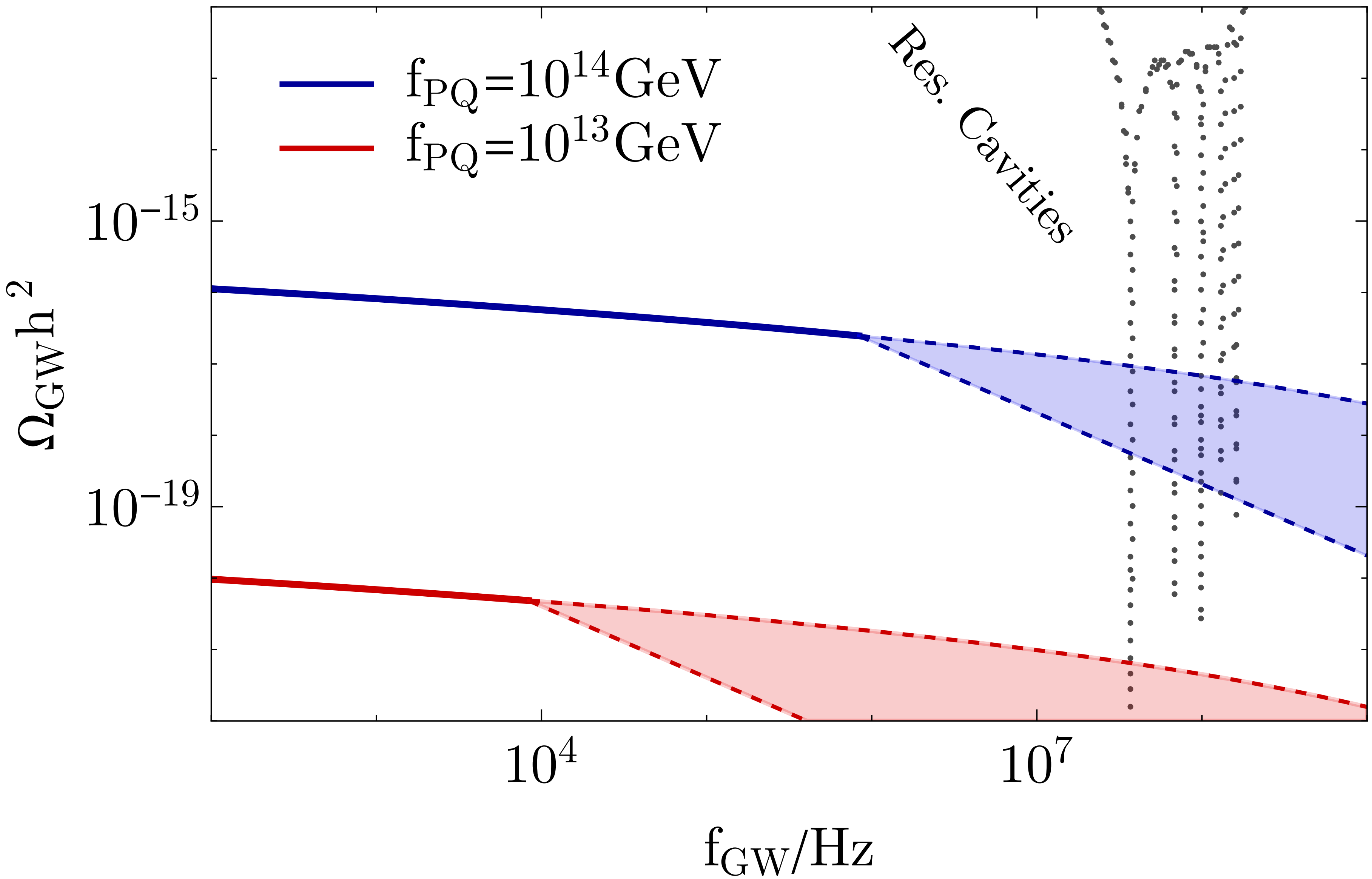
$$\frac{d\sigma}{d\alpha} = \frac{\sin^2(q\hat{\theta}/2)}{2\pi p \sin^2(\alpha/2)}$$

$\alpha, p =$ scattering angle and momentum

Axion strings in cosmology

- Suppose that PQ symmetry is restored after inflation ends. (True in many models.) Then as universe expands, T drops, and $U(1)_{\mathrm{PQ}}$ breaks, producing axion strings.
- As universe expands and cools, strings **move around** and merge to form loops.
- Loops wiggle and emit axions and gravity waves.
- Eventually they collapse completely.
- For ANO-type local strings, it has long been known that AB scattering on SM plasma suppresses high-frequency GW emission. (Vilenkin 1991.) **Cheshire $\hat{\theta}$ terms produce the same effect for axion strings!**
- Value of $\hat{\theta}$ in axion-SM EFT might be constrained by future GW observation experiments.

Axion string stochastic GW background



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- Solid lines = curves for $T < T_{\rm fric}$ not affected by $\hat{\theta}$. Curves should be suppressed to lie somewhere in colored regions due to θ.
- "Resonant cavities" refers to sensitivity of a proposed GW experiment by Herman, Lehoucq, and Fuzfa, 2021 and 2023.

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Neuzil,
coming soon

Part II: Reexamining the Witten effect

Witten effect

- Back to more formal matters.
- Standard Witten Effect (1979): Suppose we add a θ term to Maxwell $U(1)$ gauge theory:

$$\frac{i\theta}{8\pi^2}\int_{M_4}\frac{da}{2\pi}\wedge\frac{da}{2\pi}$$

- Witten noticed this gives magnetic monopoles with magnetic charge q_m an electric charge of $q_e = -\frac{\theta q_m}{2\pi}$.
- Monopoles become dyons.

Cheshire Witten effect

- Suppose there's a $U(1)^{[0]}_{\text{shift}}$ shift symmetry for φ with a background field $A^{[0]}_{\text{shift}}$

$$i\hat{\theta}\int d^D x\, \frac{d\varphi - A^{[0]}_{\text{shift}}}{2\pi} \wedge \star j$$

- So particles with charge q under $U(1)^{[0]}$ get a $U(1)^{[0]}_{\text{shift}}$ charge $q_{\text{shift}} = -\dfrac{\theta}{2\pi}q$. This is a Witten effect!
- If $U(1)^{[0]}_{\text{shift}}$ is broken, this Witten effect disappears, but the AB effect persists as Witten effect's Cheshire grin.
- What about the ``Witten effect'' in U(1) gauge theory?
  - As defined by Witten, it requires a $U(1)^{[1]}_e$ symmetry. Does part of it somehow survive if $U(1)^{[1]}_e$ is broken?

Twisted sector shuffle

- 4d U(1) Maxwell theory has 1-form $U(1)_e^{[1]}$ and $U(1)_m^{[1]}$ symmetries. Call their 2-form background fields A_e, A_m .
- If we place the model on $X = S^2 \times T^2$ with A_e, A_m holonomies α_e, α_m on T^2, the partition function is

$$Z(\theta, X) = \mathcal{P}h(X) \sum_{\substack{m \in \mathbb{Z} \\ e \in \mathbb{Z}}} \exp \left\{ -\frac{L_t L_s}{2A_{S^2}} \left[\frac{4\pi^2}{g^2} m^2 + g^2 \left(e - \frac{\theta m}{2\pi} \right)^2 \right] + i\alpha_m m + i\alpha_e e \right\}.$$

- If we also turn on an S^2 holonomy χ for A_m, we push the system into a \`twisted sector'.
- Twisted sector operators = operators attached to topological lines/surfaces/etc.

Twisted sector shuffle

- 4d U(1) Maxwell theory has 1-form $U(1)_e^{[1]}$ and $U(1)_m^{[1]}$ symmetries. Suppose A_m is 2-form background gauge field for $U(1)_m^{[1]}$
- Let's put this QFT on $M = S^2 \times T^2$, and turn on an S^2 holonomy for A_m: $\exp(i\int_{S^2} A_m) = e^{i\chi}$. This defines a ``twisted partition function":

$$Z(\theta,\chi,M) = \mathcal{P}h(M) \sum_{\substack{m\in\mathbb{Z} \\ e\in\mathbb{Z}-\frac{\chi}{2\pi}}} \exp\left\{-\frac{L_t L_s}{2A_{S^2}}\left[\frac{4\pi^2}{g^2}m^2 + g^2\left(e-\frac{\theta m}{2\pi}\right)^2\right]\right\}.$$

- Fractionalization of $U(1)_e$ charge is induced by mixed 't Hooft anomaly of $U(1)_e^{[1]}$ and $U(1)_m^{[1]}$.

Twisted sector shuffle

- Staring at the twisted partition function for a while, we discover that twist-χ charge-q sector Hilbert spaces satisfy

$$H_m^{\chi}\left(\theta,\ S^2 \times S^1_{L_s}\right) = H_m^{\chi+\theta m}\left(0,\ S^2 \times S^1_{L_s}\right)$$

- θ shuffles twisted and vacuum sectors with fixed $U(1)_m^{[1]}$ charge.
- Example: $\theta > 0$ vacuum sector ($\chi = 0$) ⇔ $\theta = 0$ twist-θm sector.
- Therefore θ has **two** effects:
  1. Reshuffle of $U(1)_m^{[1]}$ twisted and vacuum sectors
     - Not discussed much…
  2. Charge fractionalization
     - Discussed all the time

Primary and secondary Witten effects

- The twisted-vacuum sector shuffle driven by θ only cares about $U(1)_m^{[1]}$.
  - Similar shuffle with our Cheshire $\hat{\theta}$ term from Part I.
- Usually charge fractionalization is taken as the *definition* of the Witten effect. But it requires the twisted-sector-shuffle along with an $U(1)_e$ symmetry with the right 't Hooft anomaly.
  - Twisted-sector-shuffle = "primary Witten effect".
  - Charge fractionalization = "secondary Witten effect".
- The primary Witten effect involves fewer assumptions. It is the right starting point for generalizations.

Monopoles and magnetic holonomy

- Primary Witten effect has a physical implication: it leads to generalized AB effects. Suppose we insert a static monopole into U(1) gauge theory:

$$da \to (da)_{\rm monopole} + da'$$

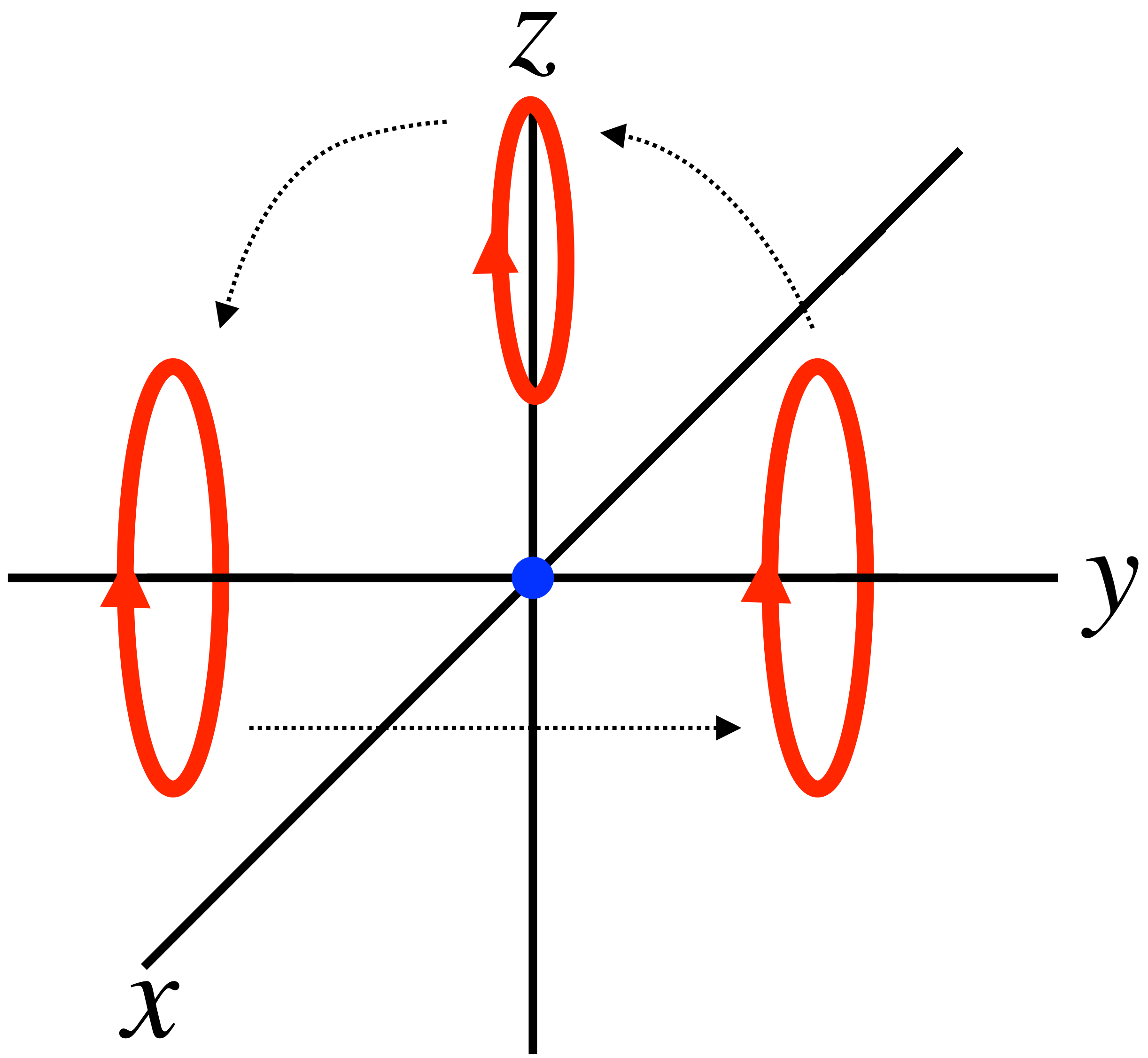
$$\Rightarrow \frac{i\theta}{8\pi^2}\int da \wedge da \to i\int \frac{\theta}{2\pi}(da)_{\rm monopole} \wedge \frac{da}{2\pi} + \frac{i\theta}{8\pi^2}\int da' \wedge da'$$

- A_m couples as $i\int A_m \wedge \frac{da}{2\pi}$, so a monopole produces an effective $A_m = \frac{\theta}{2\pi}(da)_{\rm monopole}$, so $\exp i\int_{S^2} A_m = e^{i\theta m}$.
- So excitations created by 't Hooft lines should experience an AB effect. Subtlety: $U(1)_m^{[1]}$ is spontaneously broken in Maxwell theory, makes this hard to see.

The U(1) Higgs phase

- Let's add a charge-1 Higgs scalar field Φ. This breaks $U(1)_e^{[1]}$.
- But 1-form symmetries are a bit like zombies: hard to kill. When Φ has a large positive m^2, $U(1)_e^{[1]}$ re-emerges in IR.
- Once Φ ``condenses'', several things happen. $U(1)_m^{[1]}$ is no longer spontaneously broken, and hence the photon gets a mass. Also, $U(1)_e^{[1]}$ **is completely destroyed**, and so is the standard Witten effect.
- But the primary Witten effect remains!
- In the Higgs phase there are ANO local vortex strings with a finite tension and magnetic flux $2\pi m$. They are created by hitting the vacuum with an 't Hooft line operator.
- The θ term affects the Higgs phase physics via an AB effect.

String AB effect



Minimal vortex-string excitations moving around a unit magnetic monopole pick up a phase shift $e^{i\theta}$

- Monopoles are confined, so there are complicated string-string interactions in this process. But the **AB phase** is universal.

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Neuzil,
coming soon

Part III: The general Witten effect

Generalizing the Witten Effect

- We want to describe the most general consistent "shuffle of **fixed-charge** twisted sectors".
    - Random shuffles would violate locality, so here consistent = locality-preserving.
- To see how to do this, let's arrange the charge and twist sectors associated to a symmetry G in a table:

$\mathcal{Z}$	G-charges a			
G-twists α	$\mathcal{H}^{\alpha_0}_{a_0}$	$\mathcal{H}^{\alpha_0}_{a_1}$	$\mathcal{H}^{\alpha_0}_{a_2}$	$\cdots$
	$\mathcal{H}^{\alpha_1}_{a_0}$	$\mathcal{H}^{\alpha_1}_{a_1}$	$\mathcal{H}^{\alpha_1}_{a_2}$	$\cdots$
	$\mathcal{H}^{\alpha_2}_{a_0}$	$\mathcal{H}^{\alpha_2}_{a_1}$	$\mathcal{H}^{\alpha_2}_{a_2}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$

- Witten effect = consistent shuffles of entries within each column.

$S^\dagger TS$ and the Witten effect

- We assemble three well-known results:
  1. Gauging G with flat gauge fields, "S", transposes the table and produces a new model with a dual symmetry $\hat{G} = \text{Rep}(G)$.
  2. Gauging $\hat{G}$ with flat gauge fields, "$S^\dagger$", transposes table again, and takes us back to the original theory.
  3. Consistent reshuffles of **rows** of table, "T" for a $\hat{G}$-symmetric QFT = "stacking" a $\hat{G}$-symmetric QFT with a $\hat{G}$-symmetric invertible topological QFT.
- Consistent column reshuffles of G-symmetric QFT are generated by $S^\dagger TS$!

Most complicated way to get Maxwell θ term

- This constructions is a way to think about generalized θ terms. For example, consider $U(1)$ Maxwell theory **without** a θ term.
  - Gauge $U(1)_m^{[1]}$ with flat gauge fields.
  - Stack with any one of a family $\hat{G} = \mathbb{Z}^{[1]}$-symmetric invertible theories labeled by an angle θ.
  - Gauge $\hat{G} = \mathbb{Z}^{[1]}$ with flat gauge fields.
- Result:

$$\frac{i\theta}{8\pi^2} \int da \wedge da$$

Generalized θ terms

- So θ terms of QFTs with symmetry G can be defined to arise from non-trivial $\hat{G}$-symmetric invertible TQFTs through the $S^{\dagger}TS$ map.
  - Applying the techniques to "-1-form symmetries" gives (in a rather trivial way) e.g. SU(N) YM θ term.
  - More generally, we seem to reproduce all known (to us) θ terms and Witten effects, and get lots of new ones.
- Finally, we can try to say something about confining strings in 4d YM theory.

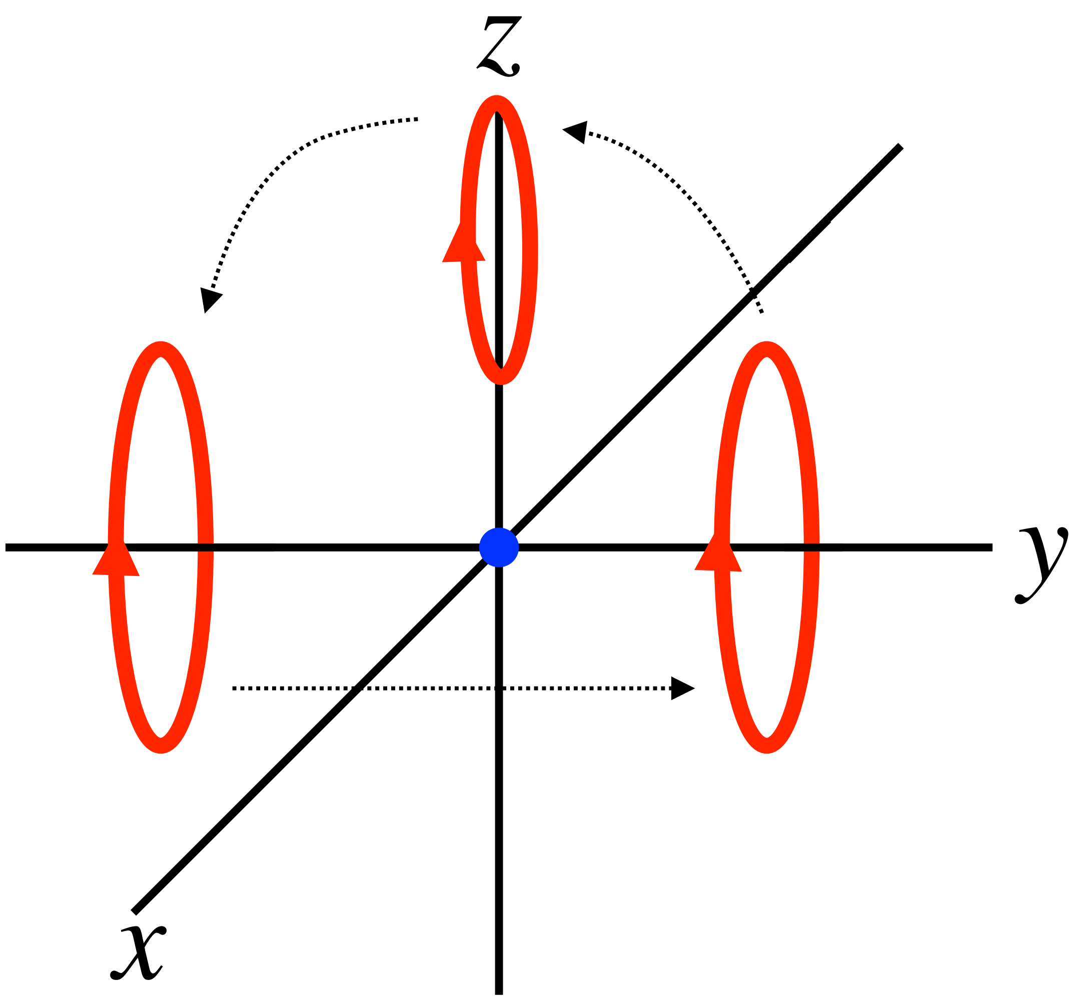
New discrete θ parameter in YM

- 4d SU(N) YM theory has an `electric' $\mathbb{Z}_N^{[1]}$ symmetry
    - Gauge $\mathbb{Z}_N^{[1]}$ with gauge field σ, getting dual symmetry $\hat{G} = \mathbb{Z}_N^{[1]}$.
    - Stack a $\hat{G}$-invariant TQFT, labeled by $\hat{\theta} \in \mathbb{Z}_N$.
    - Gauge $\hat{G}$ with a gauge field α.
- Tempting to view $\hat{\theta}$ as a new-ish discrete θ parameter in YM theory. (NB: not equivalent to $\theta = 2\pi k$ in PSU(N).)
- Unfortunately the simplest expression we found is

$$Z_{YM}(\hat{\theta};\sigma') \sim \sum_{\alpha,\sigma \in H^2(M,\mathbb{Z}_N)} Z_{YM}(\hat{\theta}=0;\sigma') \,\exp\left\{ i\frac{2\pi}{N}\int \Big[\alpha \cup (\sigma'-\sigma) + \hat{\theta} Q(\alpha)\Big]\right\},\ Q(\alpha) \sim \alpha \cup \alpha.$$

- Good news: the **physical effect** same as in Abelian Higgs!

String AB effect



Minimal confining-string excitation moving around a heavy static fundamental-representation quark picks up a phase shift $e^{i\theta}$

- The quark is confined, so there are complicated string-string interactions in this process. But the **AB phase** is universal.

Conclusions

- There are peculiar `Cheshire' θ parameters of models with compact scalars.
- Witten effect is all about AB effects, and Witten and θ terms can be generalized a lot. Many new examples from our general $S^\dagger T S$ construction!
- When are these exotic θ terms induced via RG flows?
  - Which UV axion models induce Cheshire $\hat{\theta}'s$ in the axion-SM EFT?
- Can there be implications for phase structure of QFTs from these θ terms?
- One can write an ``η' - baryon'' $\hat{\theta}$ term in the large N QCD chiral EFT. Is $\hat{\theta}_{\text{large N}} = 0$ or $\hat{\theta}_{\text{large N}} = \pi$?

Thank you

Optional: derivation of θ_{Maxwell} from $S^\dagger TS$

$$Z_{\text{Maxwell}}(\sigma) = \int \mathcal{D}a\ \mathcal{Y}_{\text{Maxwell}}(a) \exp\left(i \int \sigma \cup \frac{[da]}{2\pi}\right)$$

$$\begin{aligned}
Z_{\text{Maxwell}}(\sigma' = 0, \theta) &= \sum_{\alpha \in H^2(M,\mathbb{Z})} \int \mathcal{D}\sigma Z_{\text{Maxwell}}(\sigma) \exp\left\{-i \int \alpha \cup \sigma + i\frac{\theta}{2} \int \alpha \cup \alpha\right\} \\
&= \int \mathcal{D}a \sum_{\alpha \in H^2(M,\mathbb{Z})} \int \mathcal{D}\sigma \mathcal{Y}(a) \exp\left\{i \int \sigma \cup \left(\frac{[da]}{2\pi} - \alpha\right) + i\frac{\theta}{2} \int \alpha \cup \alpha\right\} \\
&= \int \mathcal{D}a \sum_{\alpha \in H^2(M,\mathbb{Z})} \mathcal{Y}(a)\ \delta\left(\frac{[da]}{2\pi} - \alpha\right)\ \exp\left\{i\frac{\theta}{2} \int \alpha \cup \alpha\right\} \\
&= \int \mathcal{D}a\ \mathcal{Y}(a)\ \exp\left\{i\frac{\theta}{2} \int \frac{[da]}{2\pi} \cup \frac{[da]}{2\pi}\right\}\ = \int \mathcal{D}a\ \mathcal{Y}(a)\ \exp\left(i\frac{\theta}{8\pi^2} \int da \wedge da\right)
\end{aligned}$$