ππ scattering on the lattice and its applications

work with RBC & UKQCD Collaborations

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Hadrons & Hadron Interactions in QCD 2024

RIKEN BNL Research Center

UCONN
UNIVERSITY OF CONNECTICUT

Introduction

ππ scattering near threshold

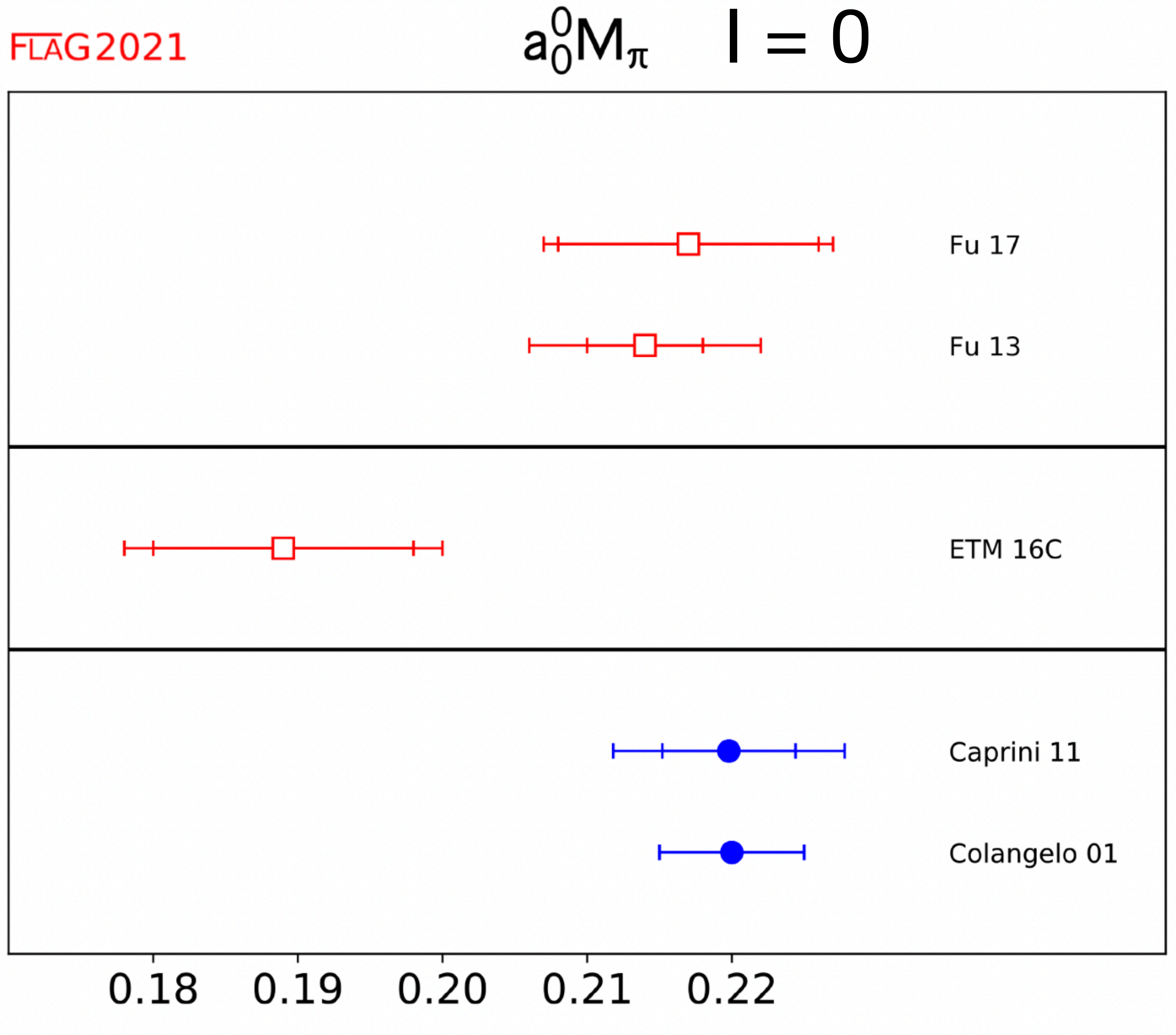
- Scattering property dominated by scattering length a^I_ℓ
- Can be determined both experimentally and on lattice
  - FLAG 2021

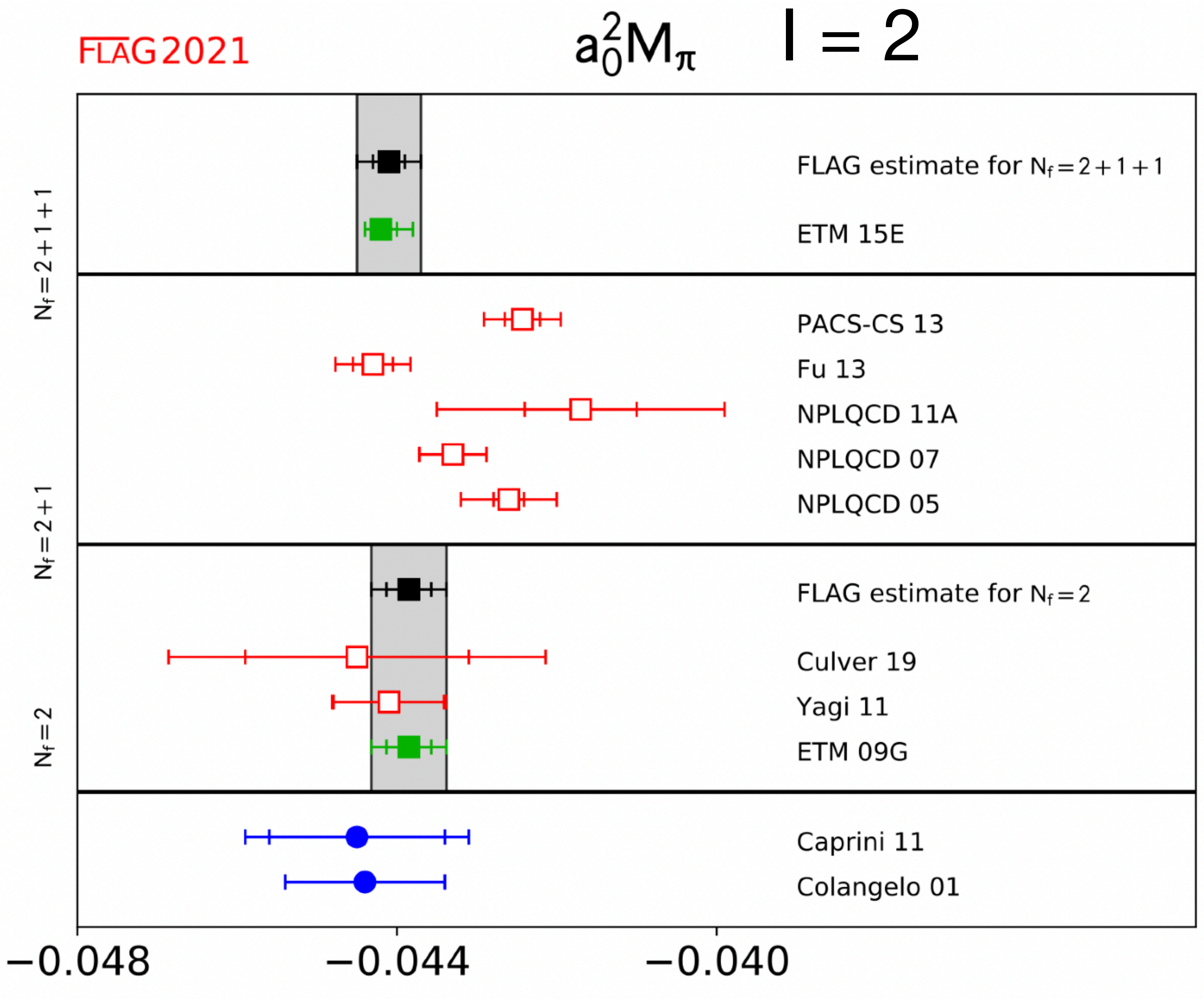
**All plotted here are from unphysical m_π simulations & chiral extrapolation
→ See backup slides for our physical m_π calculation

N_f = 2+1

N_f = 2

Pheno





N_f = 2+1+1

N_f = 2+1

N_f = 2

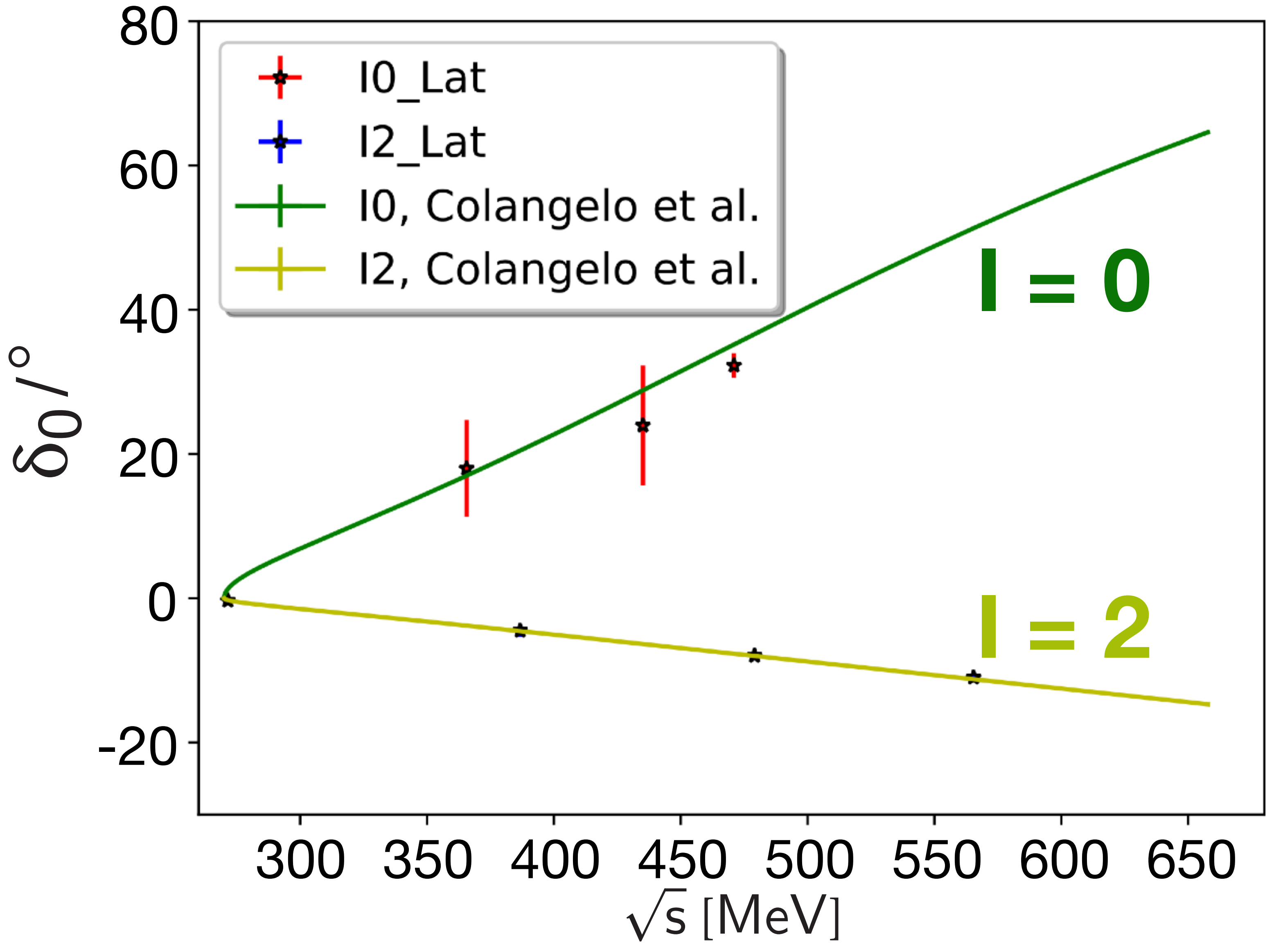
Pheno

ππ scattering above threshold

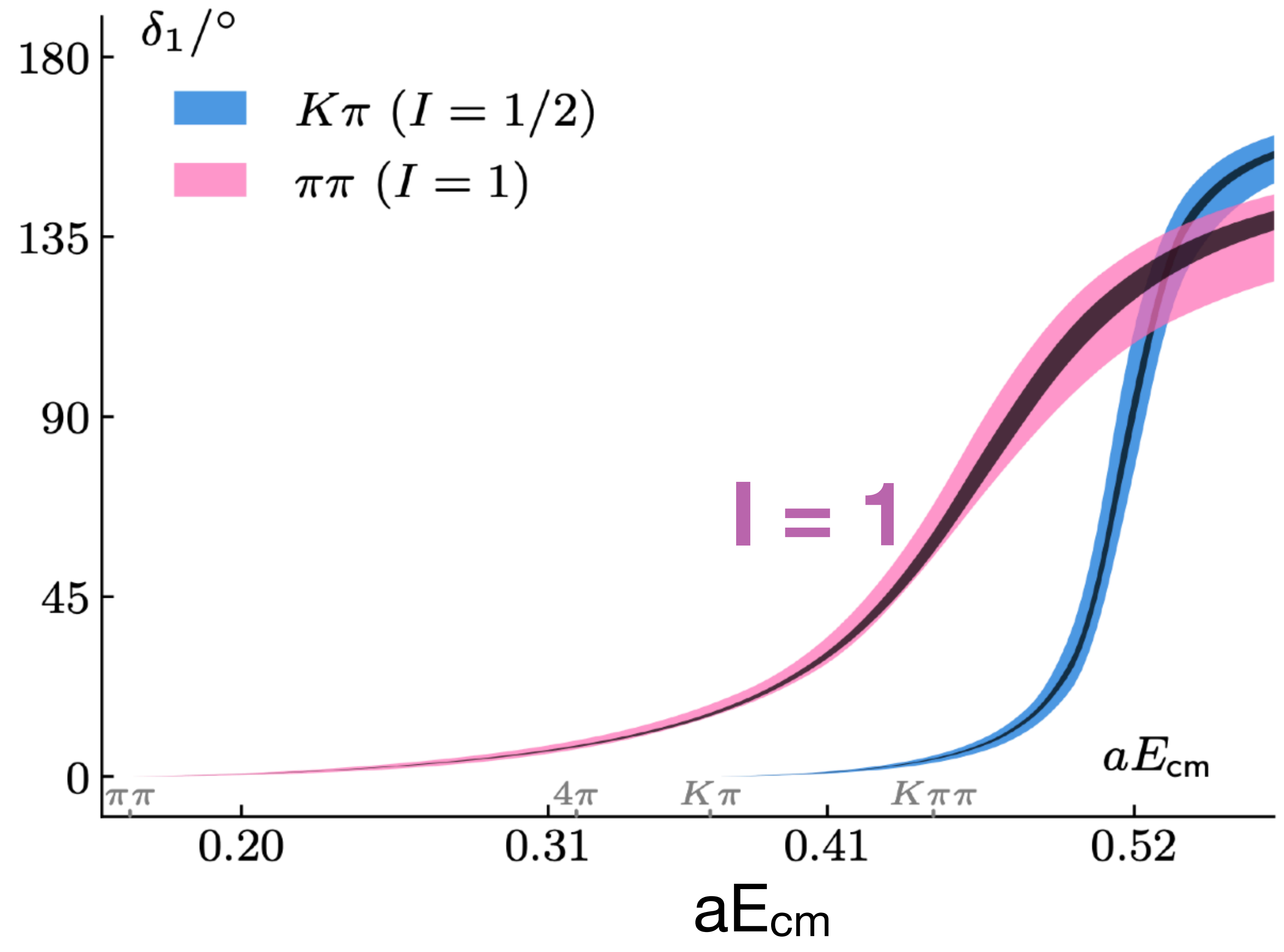
s-wave $(\ell = 0)$

p-wave $(\ell = 1)$

Phase shifts



PRD104,114506(2021)



2406.19193

Resonance

I = 0

m_σ = 400–550 MeV?

Γ_σ = 200–350 MeV?

PDG

I = 1

m_ρ = 775.26(23) MeV

Γ_ρ = 147.4(8) MeV

PDG (neutral channel)

Lüscher's FV method

CMP104,177(1986)
CMP105,153(1986)
NPB354,531(1991)

- Finite volume → discrete energy spectrum of multi-hadron states
  - E_0, E_1, …
- FV interacting momentum:

$$E_n = 2\sqrt{m_\pi^2 + k_n^2} \quad \Rightarrow k_n = \sqrt{\frac{E_n^2}{4} - m_\pi^2}$$

(for rest frame)

- Lüscher's formula (valid in elastic region)

$$k_n \cot\delta(E_n) = \frac{2}{\sqrt{\pi}L} Z_{00}(1; q_n^2) \qquad q_n = \frac{k_n L}{2\pi}$$

$$Z_{00}(s; q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\vec{n}\in\mathbf{Z}^3} \frac{1}{\left(|\vec{n}|^2 - q^2\right)^s}$$

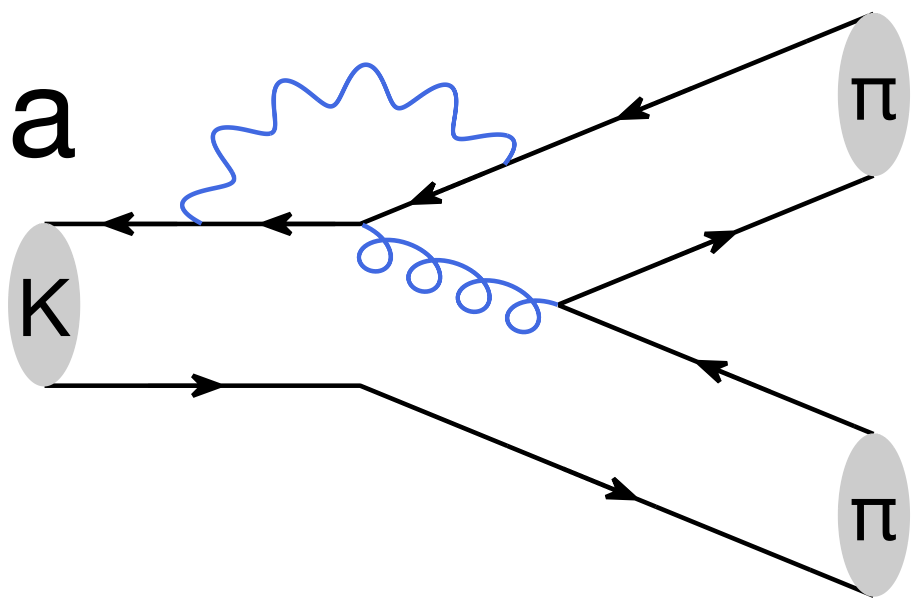
(for periodic boundaries)

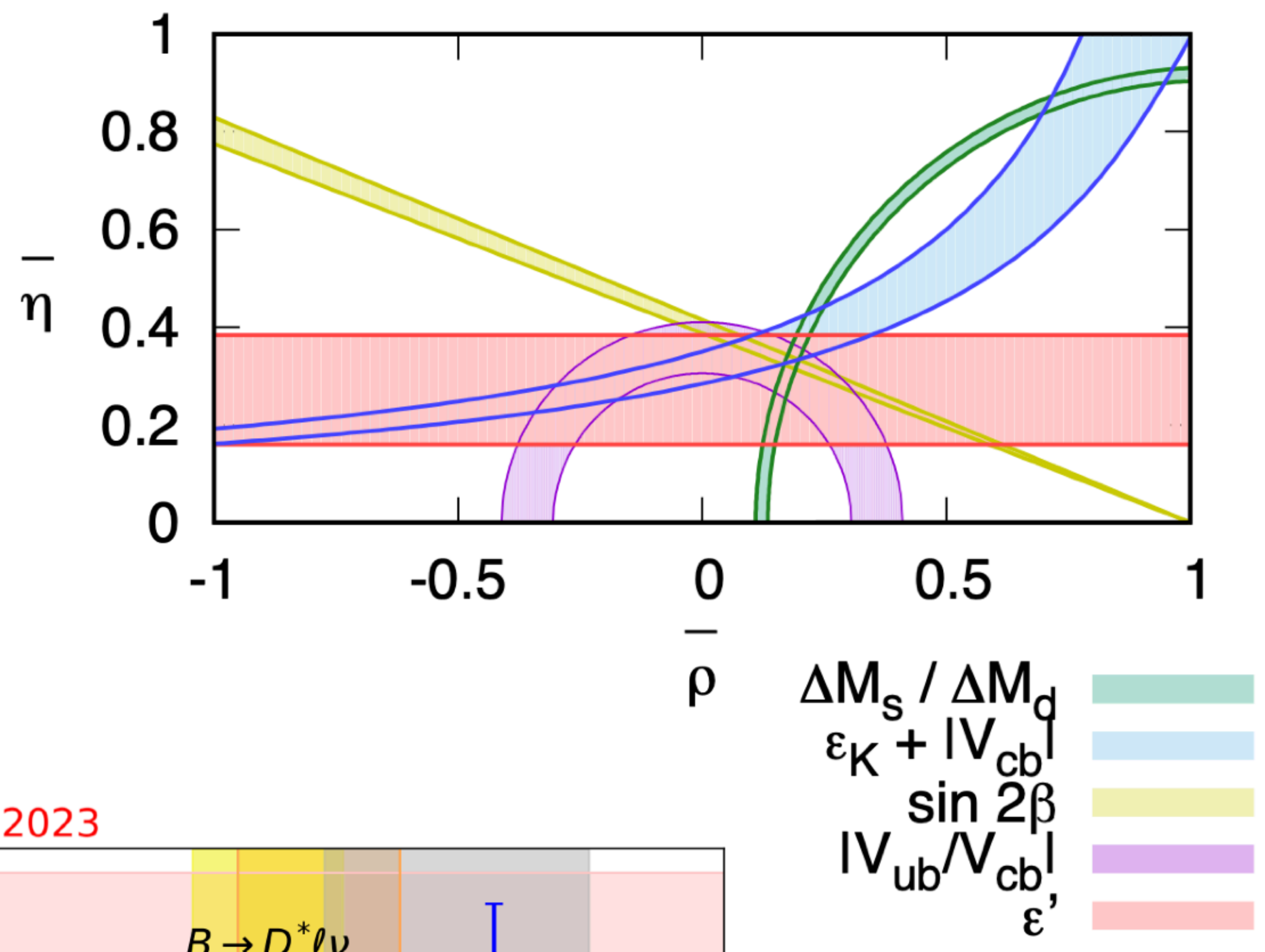
By calculating finite-volume energy levels, we can determine phase shifts and discuss scattering properties

Alternative approach – HAL QCD method [Doi's Lecture Fri Oct 18 and many other talks]

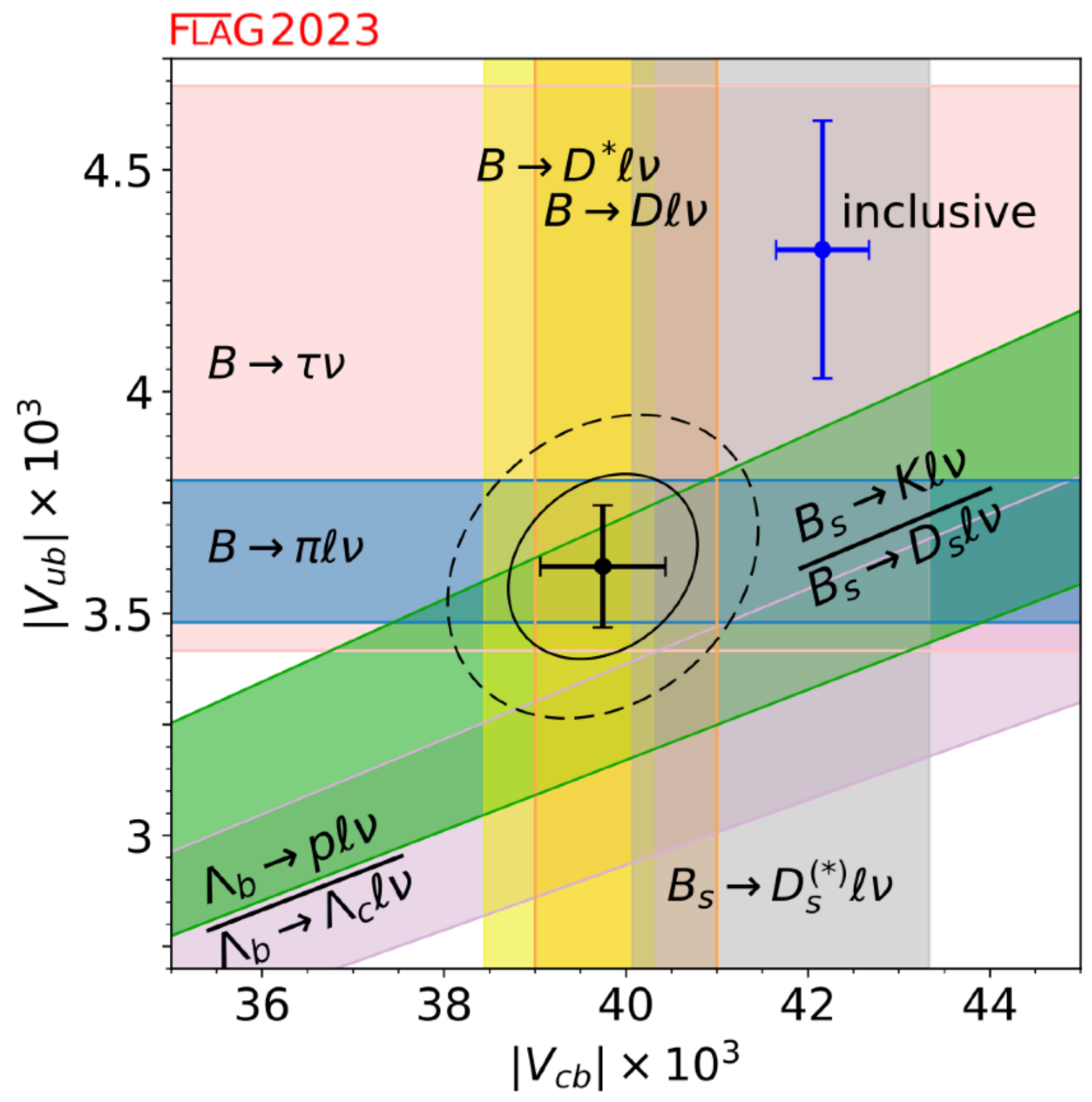
Application 1 – Weak decays

- $K \to \pi\pi$
  - Direct CP violation measure ε' provides a great test of SM & constraint on CKM parameters
  - Long-time challenge of lattice QCD





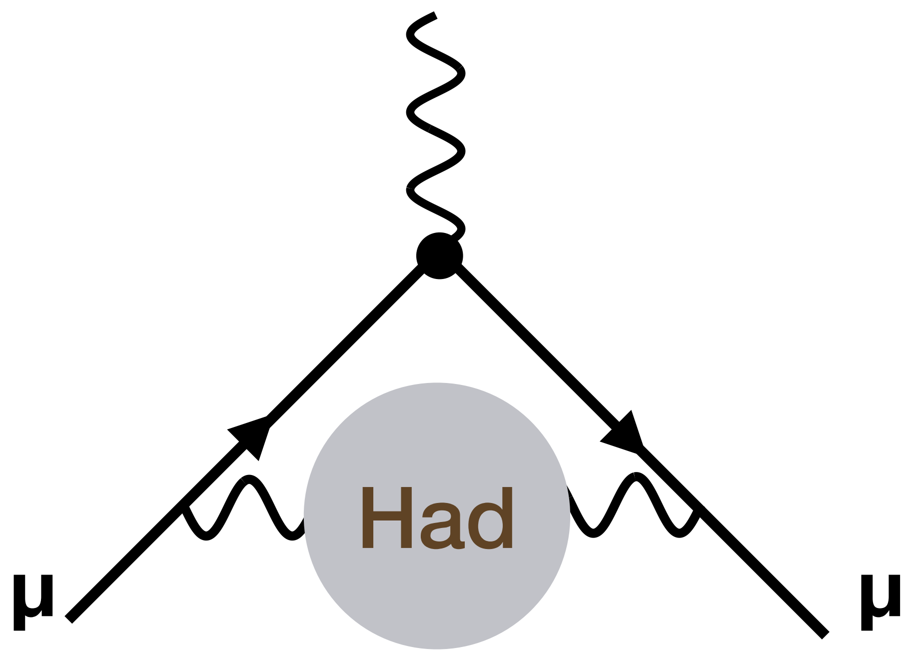
- $B \to \rho(\to\pi\pi)\ell\nu$ **2212.08833, 2401.02495 (lattice proceedings)**
  - Resonance-contained variant of $B \to \pi\ell\nu$
  - Can give us a hint about the V_{ub}-V_{cb} anomaly (tension b/w inclusive & exclusive determinations)



It is important to well control two-pion states on the lattice in order to accomplish these calculations

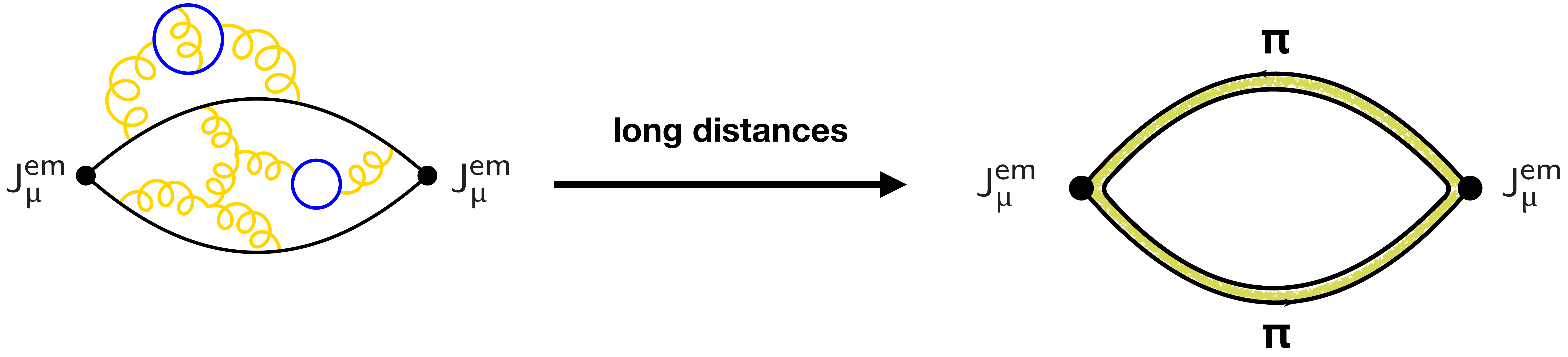
Application 2 – Muon g-2

- HVP contribution to g_μ-2



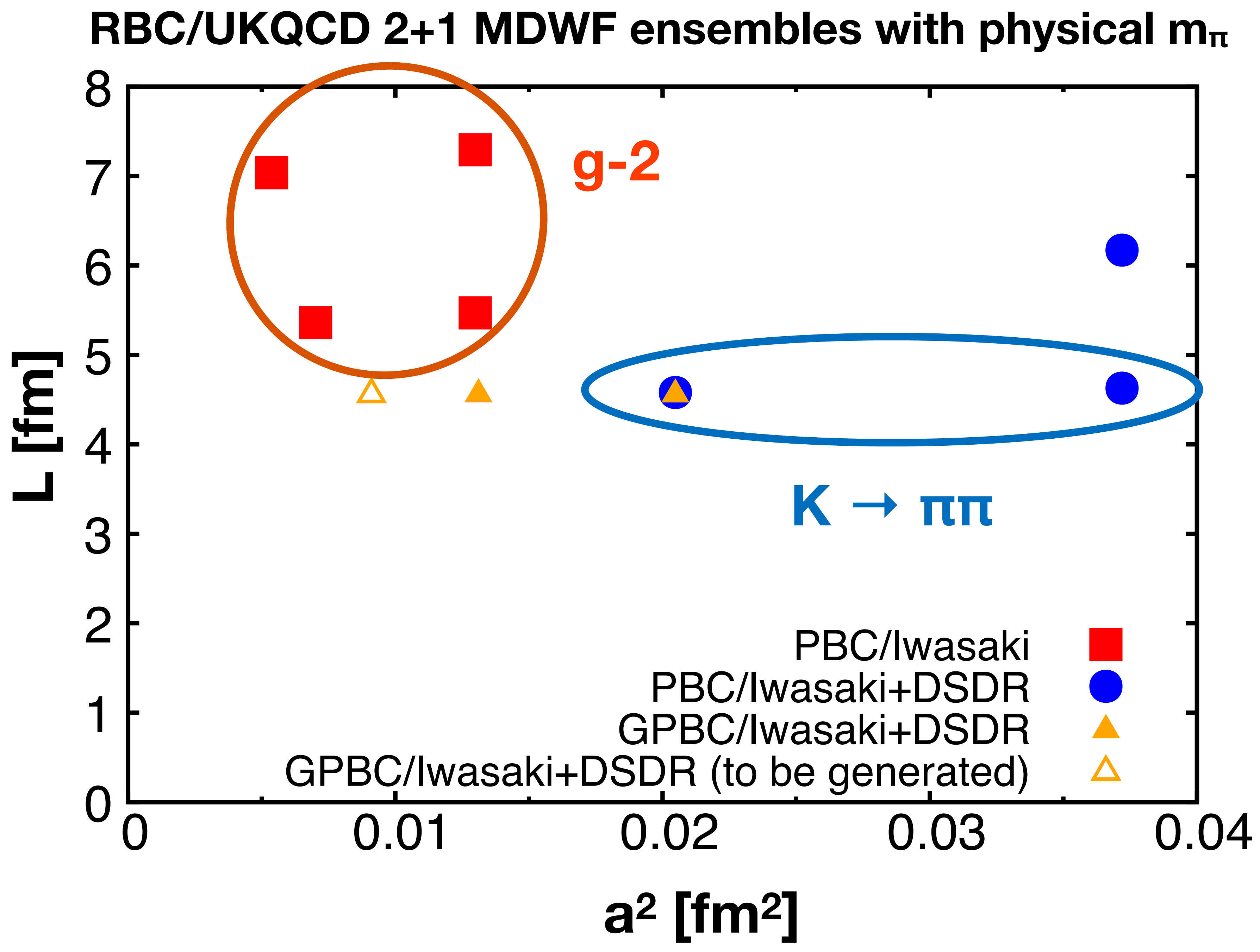
  - Dominant error source of lattice prediction of g-2
  - Key to understanding the well-known tension b/w exp & SM

- HVP at LD dominated by ππ (where g-2 calculation on lattice is the noisiest)



Contents

- Introduction (✓)
- K → ππ
- Long-distance HVP contribution to muon g-2
- Summary & Outlook



$K \to \pi\pi$

$K \to \pi\pi$ & CP violation

CP odd CP even

$$|K_L\rangle = |K_2\rangle + \varepsilon|K_1\rangle$$

ε' direct CPV

ε indirect CPV

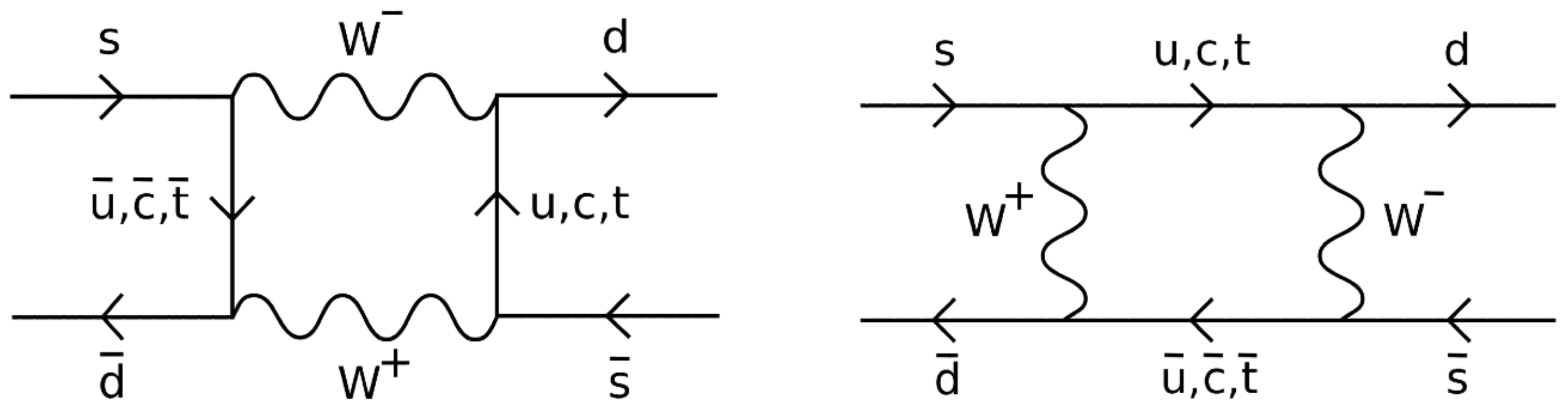
$|\pi\pi\rangle$

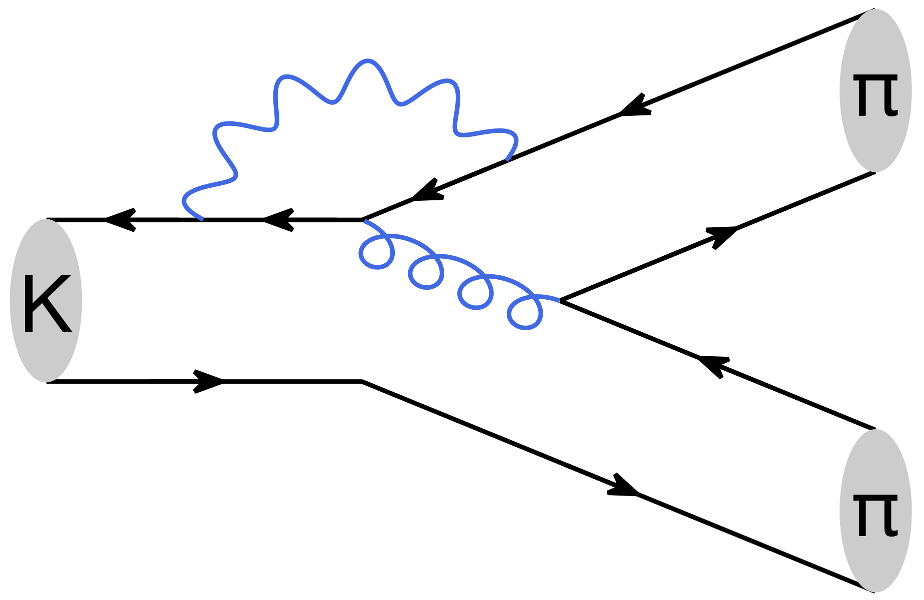
CP even

Discovered in 1964

$$\frac{\Gamma(K_L \to \pi^0\pi^0)}{\Gamma(K_S \to \pi^0\pi^0)} \Big/ \frac{\Gamma(K_L \to \pi^+\pi^-)}{\Gamma(K_S \to \pi^+\pi^-)} = 1 - 6\mathrm{Re}(\varepsilon'/\varepsilon)$$

- $|\varepsilon| = 2.228(11) \times 10^{-3}$ from "odd" mixing b/w K^0 & $\overline{K}^0$
- ε' only found in decays **Discovered in 1999**
  - $\mathrm{Re}(\varepsilon'/\varepsilon)_{exp} = 1.66(23) \times 10^{-3}$ (KTeV & NA48)
  - Consistent with SM?





Sensitivity of ε′ to BSM

- s → d: most suppressed within SM

$$\mathrm{Re}(\varepsilon'/\varepsilon) \propto \mathrm{Im}\left(V_{td}V_{ts}^*\right)$$

$$V_{CKM} \sim \begin{array}{cc} \begin{array}{ccc} d & s & b \end{array} & \\ \begin{pmatrix} 1 & \lambda & \lambda^3 \\ -\lambda & 1 & \lambda^2 \\ \boxed{\lambda^3} & \boxed{-\lambda^2} & 1 \end{pmatrix} & \begin{array}{c} u \\ c \\ t \end{array} \end{array}$$

$$\lambda \approx 0.23$$

$$\underbrace{|V_{td}V_{ts}^*| \sim 5 \times 10^{-4}}_{s \to d} \ll \underbrace{|V_{td}V_{tb}^*| \sim 1 \times 10^{-2}}_{b \to d}, \ \underbrace{|V_{ts}V_{tb}^*| \sim 4 \times 10^{-2}}_{b \to s}$$

- ε′ highly sensitive to BSM & highly demanded by pheno

Isospin decay modes & ΔI = 1/2 rule

$$\langle(\pi\pi)_{\mathrm{I}=0}| = \sqrt{1/3}\langle\pi^0\pi^0| + \sqrt{2/3}\langle\pi^+\pi^-|, \quad \langle(\pi\pi)^{\mathrm{I}_3=0}_{\mathrm{I}=2}| = -\sqrt{2/3}\langle\pi^0\pi^0| + \sqrt{1/3}\langle\pi^+\pi^-|$$

- Isospin-definite amplitudes

$$\mathrm{A}_\mathrm{I} = \langle(\pi\pi)_\mathrm{I}|\mathrm{H_W}|\mathrm{K}\rangle \quad \begin{cases} \mathrm{I} = 0 \rightarrow \Delta\mathrm{I} = 1/2 \\ \mathrm{I} = 2 \rightarrow \Delta\mathrm{I} = 3/2 \end{cases}$$

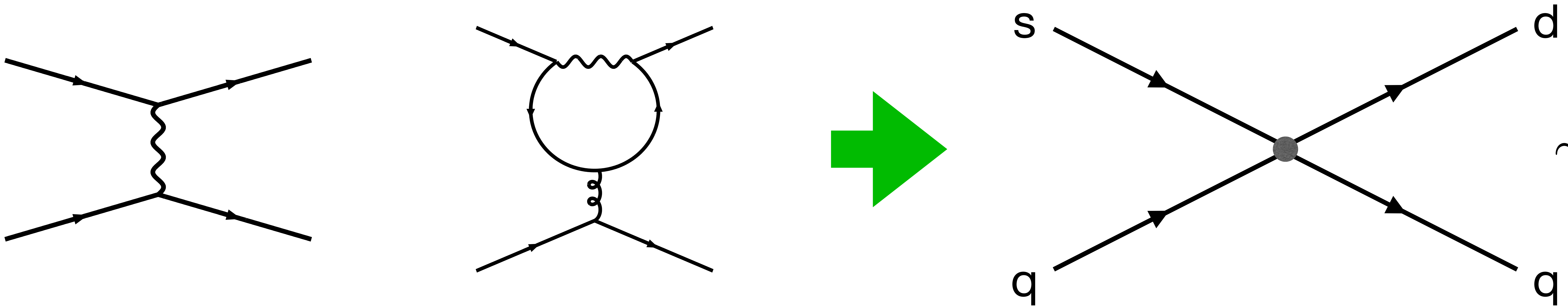
- ΔI = 1/2 rule (experimental fact)

$$\frac{\mathrm{Re\,A_0}}{\mathrm{Re\,A_2}} = 22.45(6)$$: large suppression of ΔI = 3/2 (A_2) mode

  - Factor 2 can be responsible for Wilson coefs from pQCD [Gaillard & Lee, PRL 33,108 (1974)]
  - Remaining factor 10 comes from QCD or BSM?
  - A lot of discussions happening already in 1970s
  - Firm understanding not established until lattice calculation of matrix elements wad done

Approach to weak decays

- Two typical scales
  - Electroweak scale: m_W = 80 GeV, m_Z = 91 GeV
  - QCD scale: Λ_{QCD} ~ 300 MeV
- Low-energy effective interactions @ QCD scale



$$\sim G_F \propto \frac{g^2}{m_W^2}$$

  - $H_W = \Sigma_i \, \underbrace{c_i(\mu)}_{\text{Wilson coefficients}} \, \underbrace{O_i(\mu)}_{\text{Effective operators}}$

ΔS = 1 effective operators

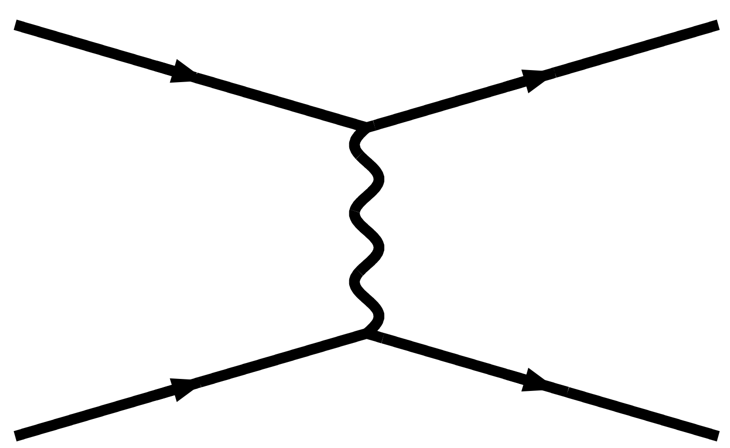
- $(\bar{s}q)_{V-A}(\bar{q}'q'')_{V\pm A} = \bar{s}\gamma_\mu(1-\gamma_5)q' \cdot \bar{q}'\gamma_\mu(1\pm\gamma_5)q''$
- α, β: color indices

$$
\begin{aligned}
Q_1 &= (\bar{s}_\alpha u_\beta)_{\mathrm{V-A}}\,(\bar{u}_\beta d_\alpha)_{\mathrm{V-A}}\,,\\
Q_2 &= (\bar{s}u)_{\mathrm{V-A}}\,(\bar{u}d)_{\mathrm{V-A}}\,,\\
Q_3 &= (\bar{s}d)_{\mathrm{V-A}}\sum_q(\bar{q}q)_{\mathrm{V-A}}\,,\\
Q_4 &= (\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q(\bar{q}_\beta q_\alpha)_{\mathrm{V-A}}\,,\\
Q_5 &= (\bar{s}d)_{\mathrm{V-A}}\sum_q(\bar{q}q)_{\mathrm{V+A}}\,,\\
Q_6 &= (\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q(\bar{q}_\beta q_\alpha)_{\mathrm{V+A}}\,,\\
Q_7 &= \frac{3}{2}\,(\bar{s}d)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}q)_{\mathrm{V+A}}\,,\\
Q_8 &= \frac{3}{2}\,(\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}_\beta q_\alpha)_{\mathrm{V+A}}\,,\\
Q_9 &= \frac{3}{2}\,(\bar{s}d)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}q)_{\mathrm{V-A}}\,,\\
Q_{10} &= \frac{3}{2}\,(\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}_\beta q_\alpha)_{\mathrm{V-A}}\,,
\end{aligned}
$$

Current-current operators

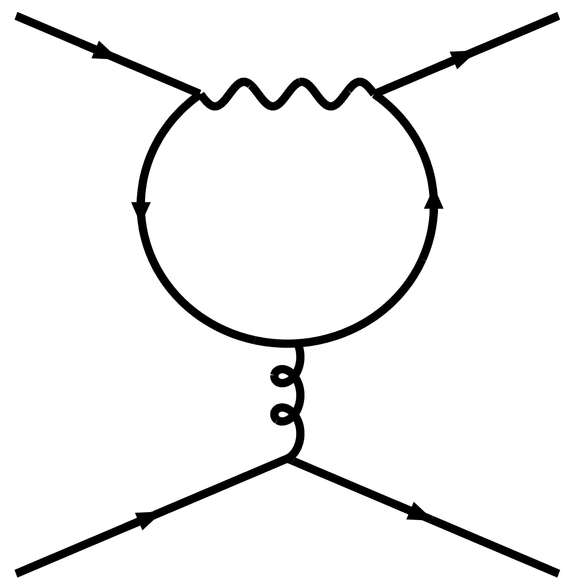
- $Q_1^c = (\bar{s}_\alpha c_\beta)_{\mathrm{V-A}}(\bar{c}_\beta d_\alpha)_{\mathrm{V-A}}$ & $Q_2^c = (\bar{s}c)_{\mathrm{V-A}}(\bar{c}d)_{\mathrm{V-A}}$

  enter when $n_f \geq 4$



QCD penguin operators

- sum over q runs for all active quarks



EW penguin operators

K → ππ Amplitude and ε′

ππ phase shifts at m_K

$$\varepsilon' = \frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}}\left[\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\right] \qquad (\omega = \mathrm{Re}A_2/\mathrm{Re}A_0)$$

Renormalization matrix

$$A_I = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_{i,j} \underbrace{[z_i(\mu)+\tau y_i(\mu)]}_{\text{Wilson coefs. pQCD}} \underbrace{Z_{ij}(\mu)}_{\text{LQCD (+pQCD)}} \underbrace{\langle (\pi\pi)_I | Q_j^{lat} | K \rangle}_{\text{LQCD}}$$

- A_2 already reached sufficient precision RBC/UKQCD PRD91 (2015) 074502
  - Re A_2 = 1.50(4)$_{stat}$(14)$_{sys}$ x 10^{-8} GeV, Im A_2 = -6.99(20)$_{stat}$(84)$_{sys}$ x 10^{-13} GeV

    cf: (Re A_2)$_{exp}$ = 1.479(4) x 10^{-8} GeV
- A_0 still challenging because of many difficulties

Challenges confronted for past few decades

- Computational cost/Statistics
  - disconnected diagrams
  - challenge enhanced due to the other difficulties

- Charm-loop effects
  - expected significant
  - directly on lattice? → am_c not small on current lattices
    ↕ window problem
  - absorb into WCs? → NLO pQCD at $\mu = m_c$ not ideal

- Chiral symmetry
  - 10 four-quark operators
  - strongly desired to prevent mixing with other operators
  - domain wall fermions preferable and used by RBC/UKQCD

Today's focus

- Two-pion final state on the euclidean lattice

  e.g. in the rest frame
  - only $E \approx 2m_\pi \approx 280$ MeV state extracted in a straightforward manner
  - $E = m_K \approx 500$ MeV state needed

Realizing on-shell kinematics

- The lightest ππ state with “2 stationary pions” in Euclidean rest frame
  - $E_{\pi\pi,0} \approx 280$ MeV → off-shell
  - need $| E_{\pi\pi} = m_K \approx 500 \text{ MeV} \rangle$ state
- Possible approaches

💡 Finite volume → two-pion spectrum not continuous

  - Moving frame (Ishizuka et al [PRD98,114512(2018)])

    e.g. $\sqrt{m_K^2 + p_{tot}^2} = m_\pi + \sqrt{m_\pi^2 + p_{tot}^2}$

  - Analyze correlation functions taking multiple states into account (GEVP, led by MT)
  - Manipulate boundary conditions → pions anti-periodic → must move → 500 MeV ground state possible [G-parity BC (GPBC) led by C. Kelly]
    - ✳ For A_2 imposing anti-periodic BC on d quark was enough to make relevant pion moving [PRL108,141601(2012), PRD91,074502(2015)]

First physical m_π result w GPBC

PRL 115,212001

$$\mathrm{Re}(\varepsilon'/\varepsilon)_{2015} = 1.38(5.15)(4.59) \times 10^{-4}$$

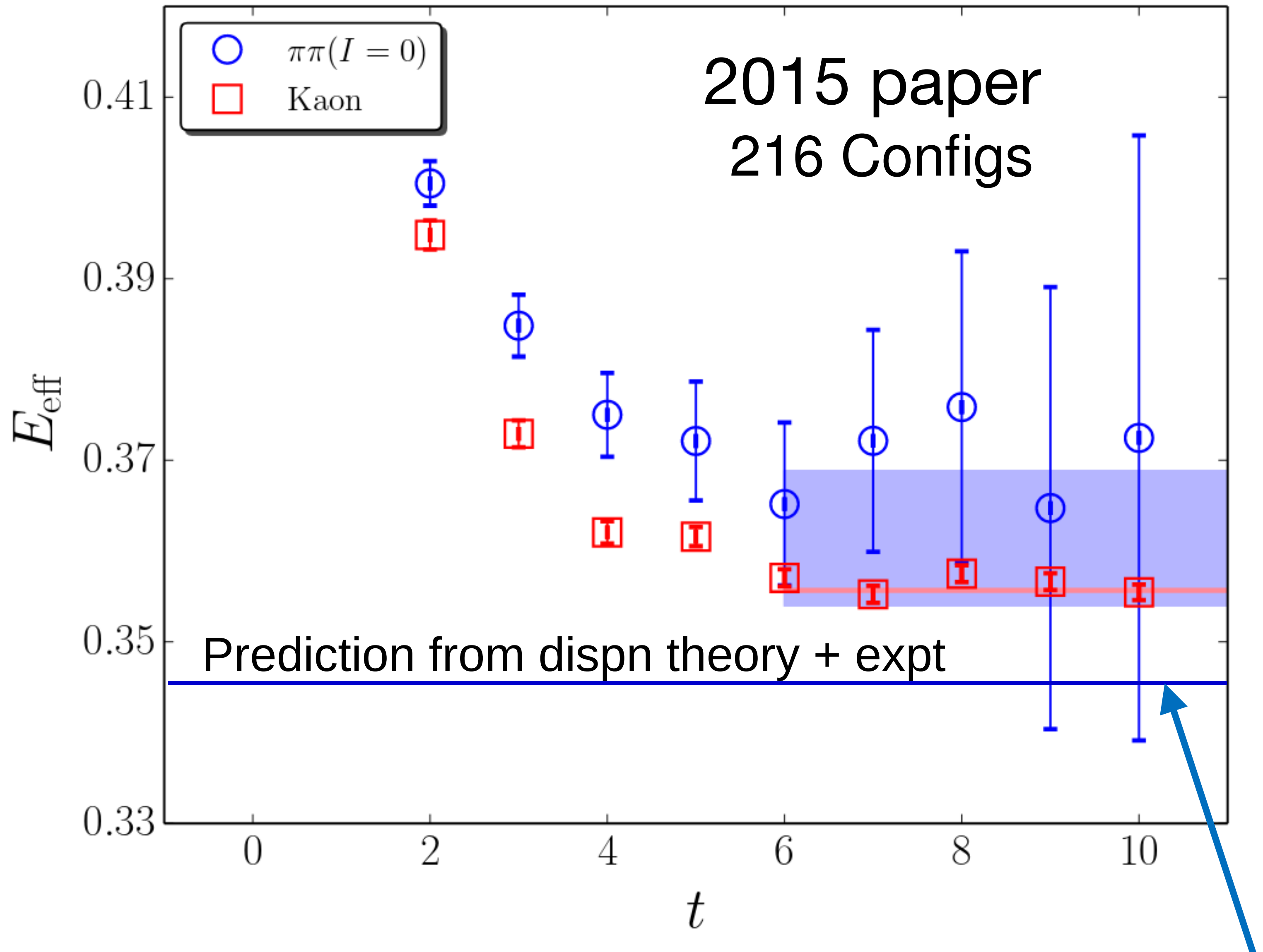
$$\updownarrow \quad \textbf{2.1}\boldsymbol{\sigma}\ \textbf{tension}$$

$$\mathrm{Re}(\varepsilon'/\varepsilon)_{\mathrm{exp}} = 16.6(2.3) \times 10^{-4}$$

- Physics or just stat/sys error?
- Needed to make lattice calculation more accurate

The “ππ puzzle”

- Large discrepancy b/w lattice & exp



exp data $\xrightarrow{\text{disp. theory}}$ $\delta(E)$ $\xrightarrow{\text{Lüscher}^{-1}}$ $E_{\pi\pi}(L)$

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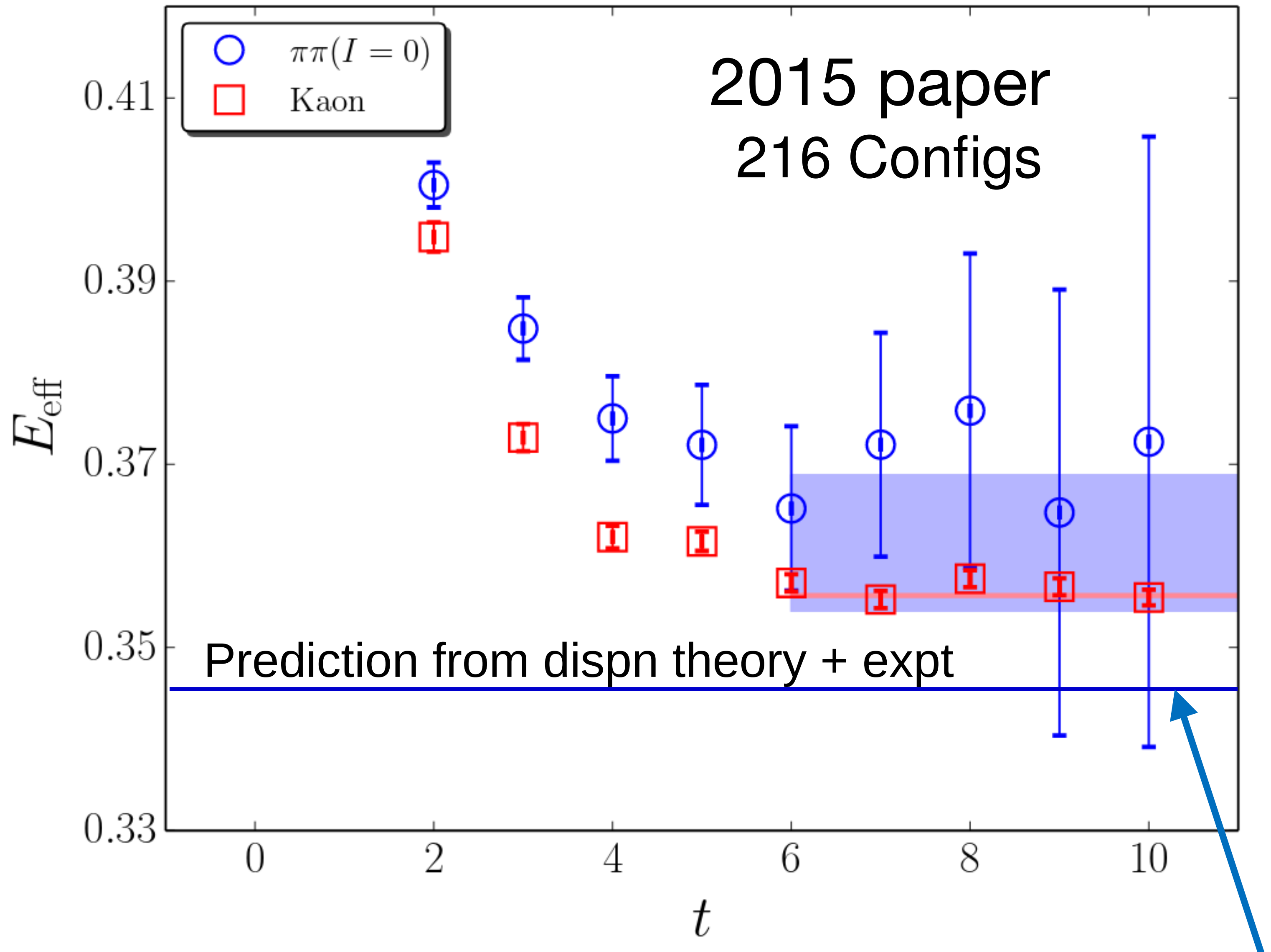
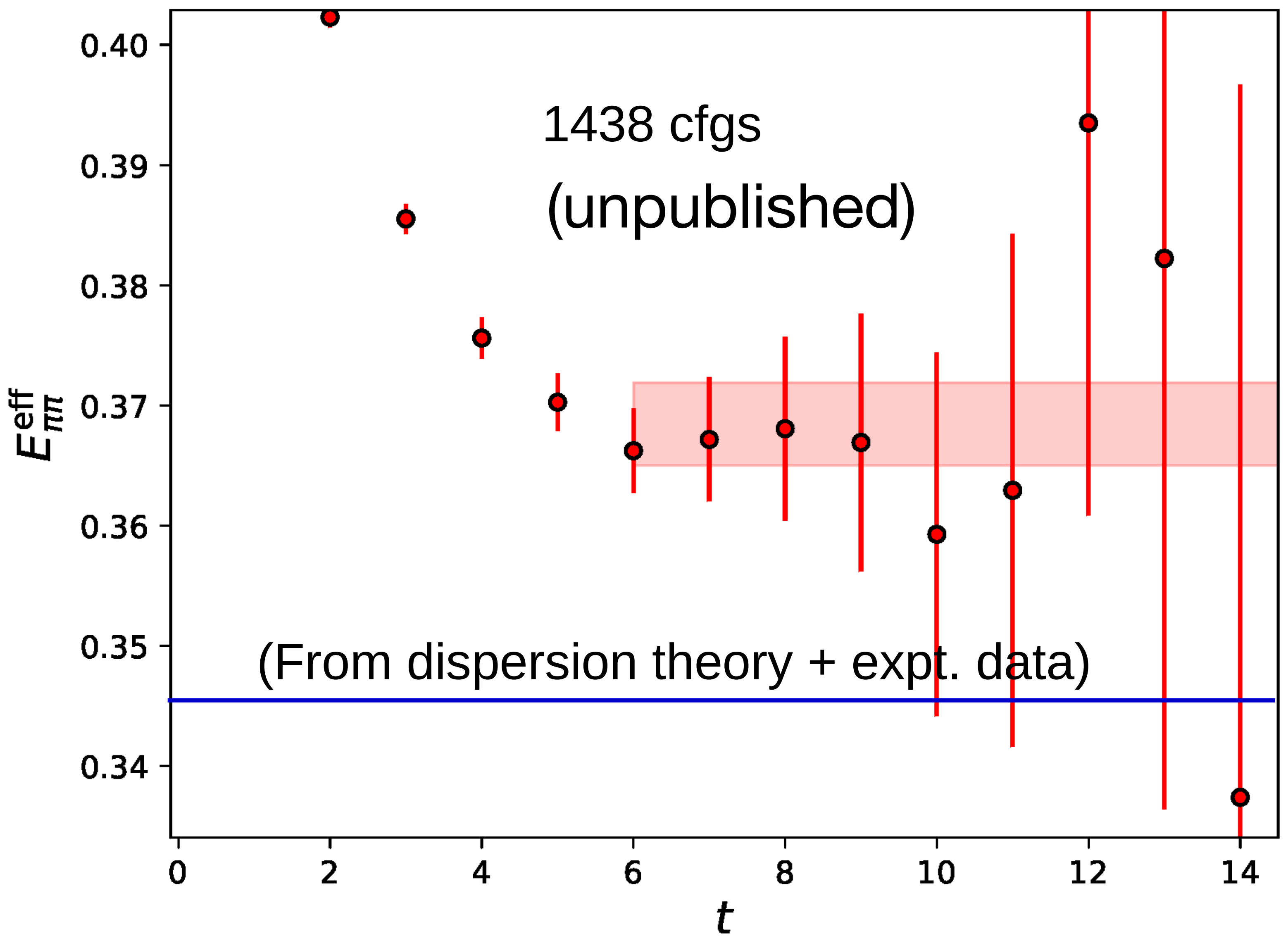


7x statistics



Discrepancy more resolved

$$\text{exp data} \xrightarrow{\text{disp. theory}} \delta(E) \xrightarrow{\text{Lüscher}^{-1}} E_{\pi\pi}(L)$$

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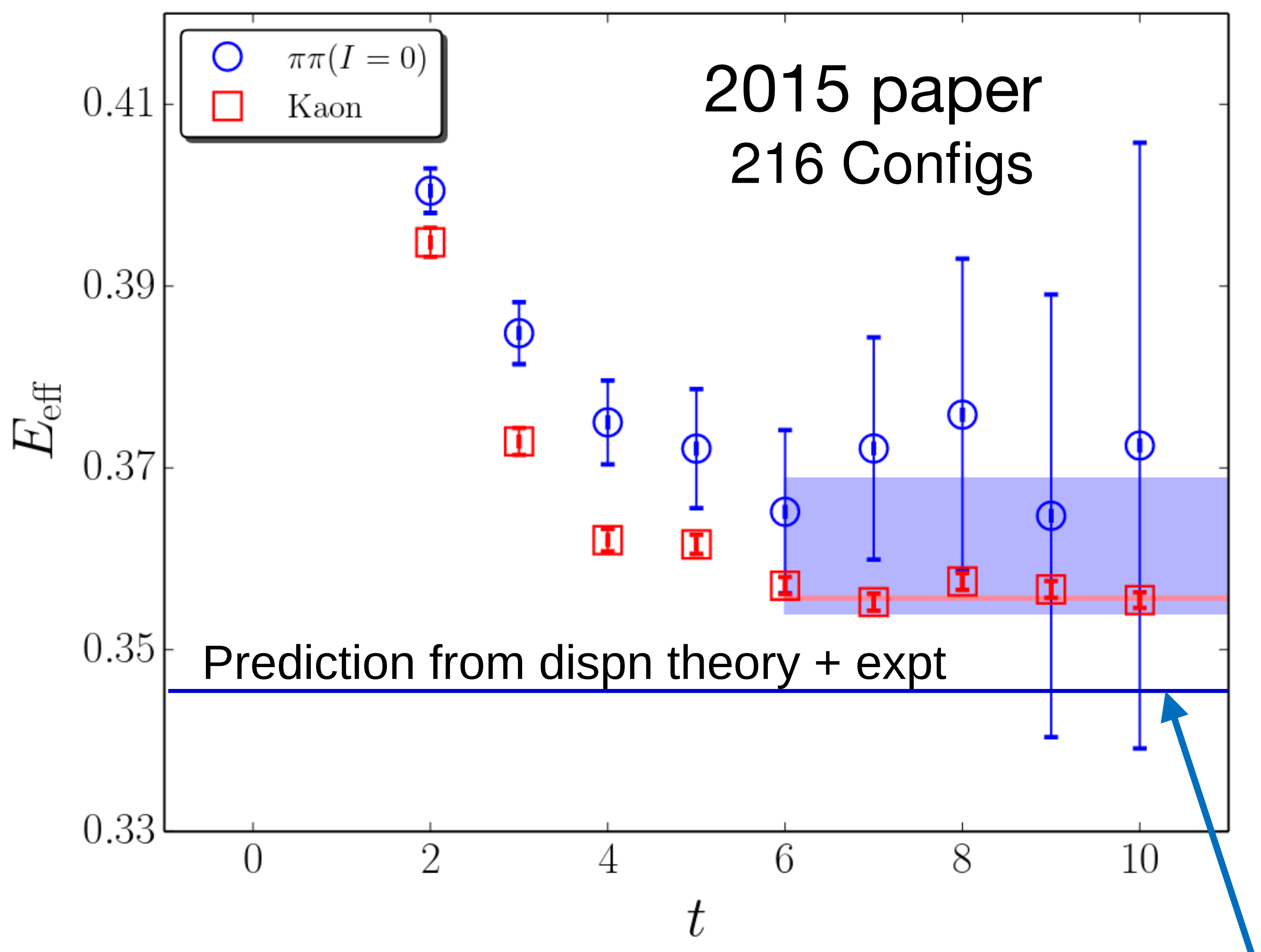

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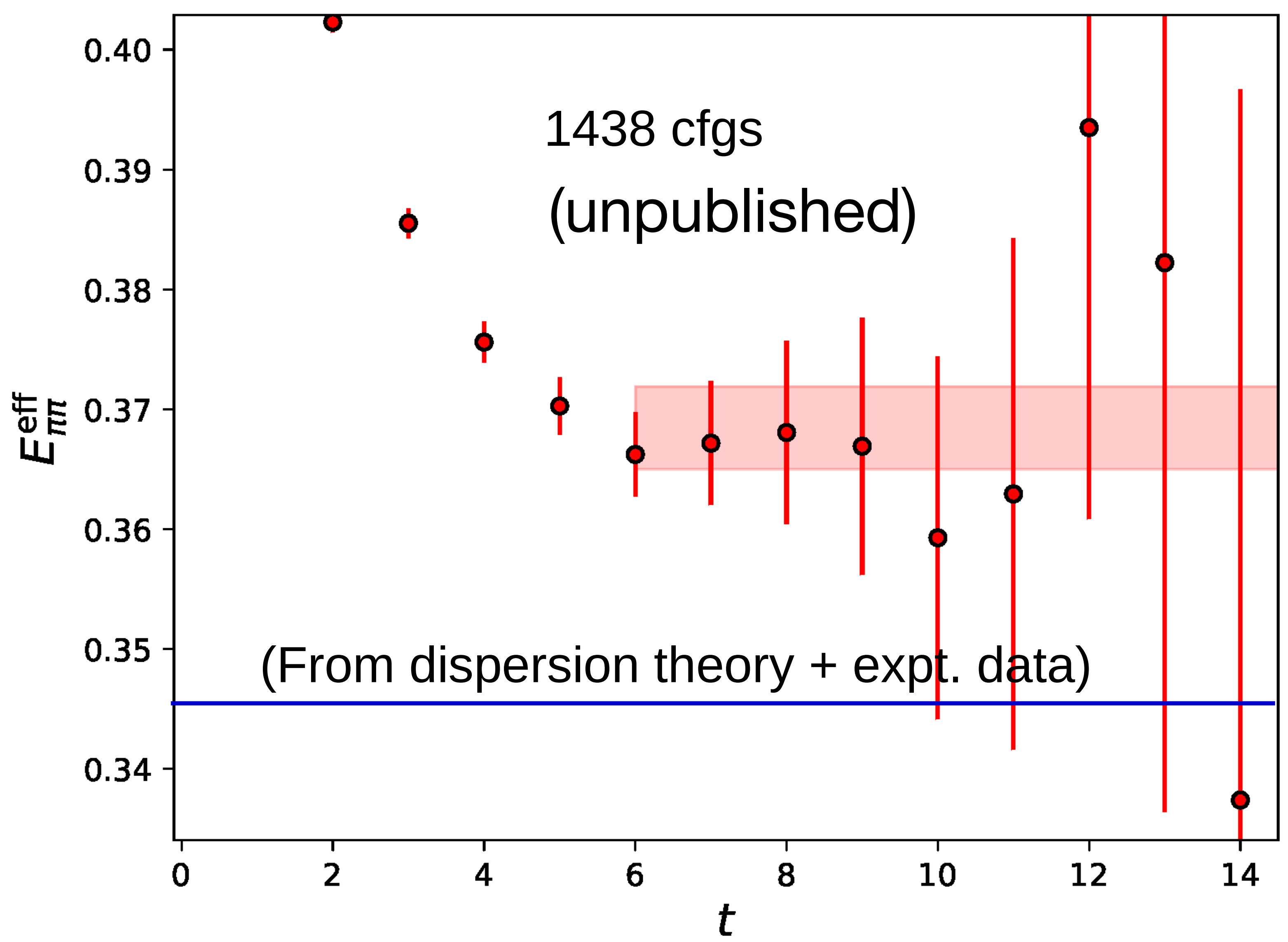


Discrepancy more resolved

exp data $\xrightarrow{\text{disp. theory}}$ δ(E) $\xrightarrow{\text{Lüscher}^{-1}}$ $E_{\pi\pi}(L)$

Let's see what $E^{\rm eff}_{\pi\pi}$ is

S/N problem of Euclidean 2pt functions

- How to extract the lowest energy from Euclidean 2pt functions

$$G_{\pi\pi}(t) = \underbrace{\int d^3x}_{\text{zero-momentum projection } (e^{i\vec{p}\cdot\vec{x}}=1)} \langle O_{\pi\pi}(t,\vec{x}) O_{\pi\pi}(0,\vec{y})^\dagger \rangle = \underbrace{\sum_n}_{} \langle 0|O_{\pi\pi}|\pi\pi, n\rangle \langle \pi\pi, n|O_{\pi\pi}^\dagger|0\rangle e^{-E_{\pi\pi,n}t}$$

all possible zero-(total)momentum states that have the same quantum numbers as $O_{\pi\pi}$

$$\xrightarrow{\text{large } t} \langle 0|O_{\pi\pi}|\pi\pi, 0\rangle \langle \pi\pi, 0|O_{\pi\pi}^\dagger|0\rangle e^{-\underbrace{E_{\pi\pi,0}}_{}t}$$

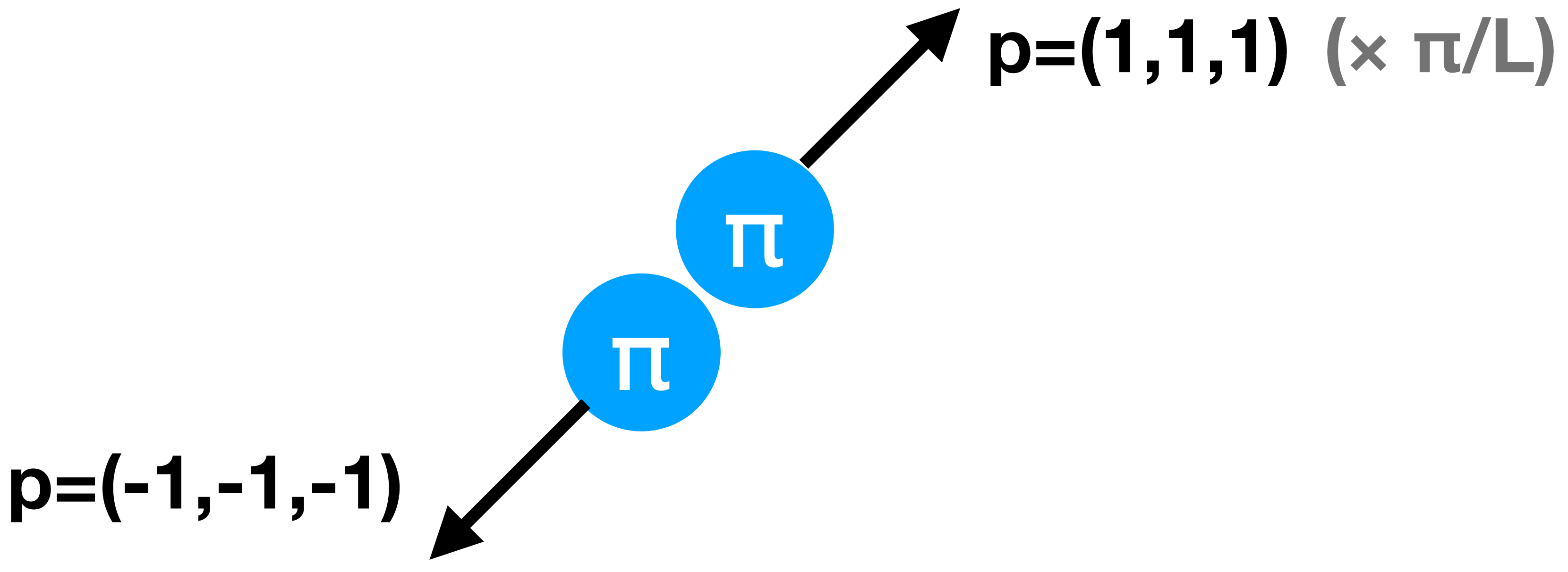
$\approx m_K$ (GPBC w tuned volume)

- S/N problem
  - Signal $\sim e^{-m_K t}$ at large t
  - Noise: $\sqrt{\langle \mathcal{O}\mathcal{O}^\dagger \rangle - \langle \mathcal{O} \rangle \langle \mathcal{O}^\dagger \rangle} \sim e^{-2m_\pi t}$ even with GPBC [PRD101,014506(2020)]
  - S/N declines by $\sim e^{-(m_K - 2m_\pi)t}$

- $$E_{\pi\pi}^{\text{eff}}(t) = \ln \frac{G_{\pi\pi}(t)}{G_{\pi\pi}(t+1)} \xrightarrow{\text{large } t} E_{\pi\pi,0}$$
  - Noisy at large t
  - NOT always a reliable indicator of ground-state saturation

Resolving the ππ puzzle

- Introduce multiple ππ operators
  - In 2015
    $O_{\pi\pi} = \pi\pi(1,1,1)$ ≡ O_a
  - Additions in 2020
    $\pi\pi(3,1,1)$ ≡ O_b $\sigma = \frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d)$ ≡ O_c



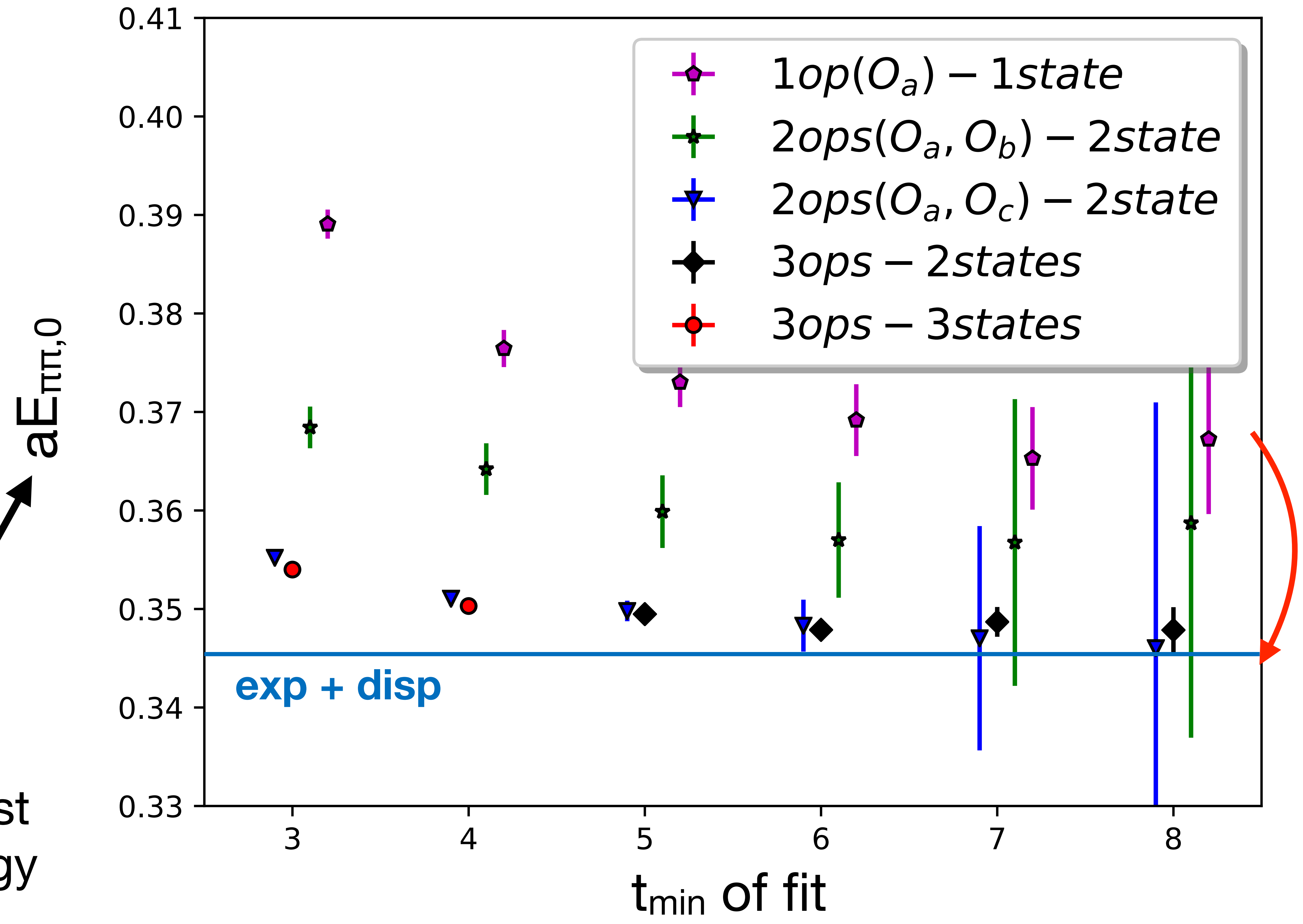
- 2pt functions

$$G_{ij}(t) = \langle O_i(t) O_j(0)^\dagger \rangle = \sum_n A_{i,n} A^\dagger_{j,n} e^{-E_n t}$$

  - better way to isolate excited-state contamination

Effect of multi operators on ππ

RBC/UKQCD PRD 104,114506 (2021)

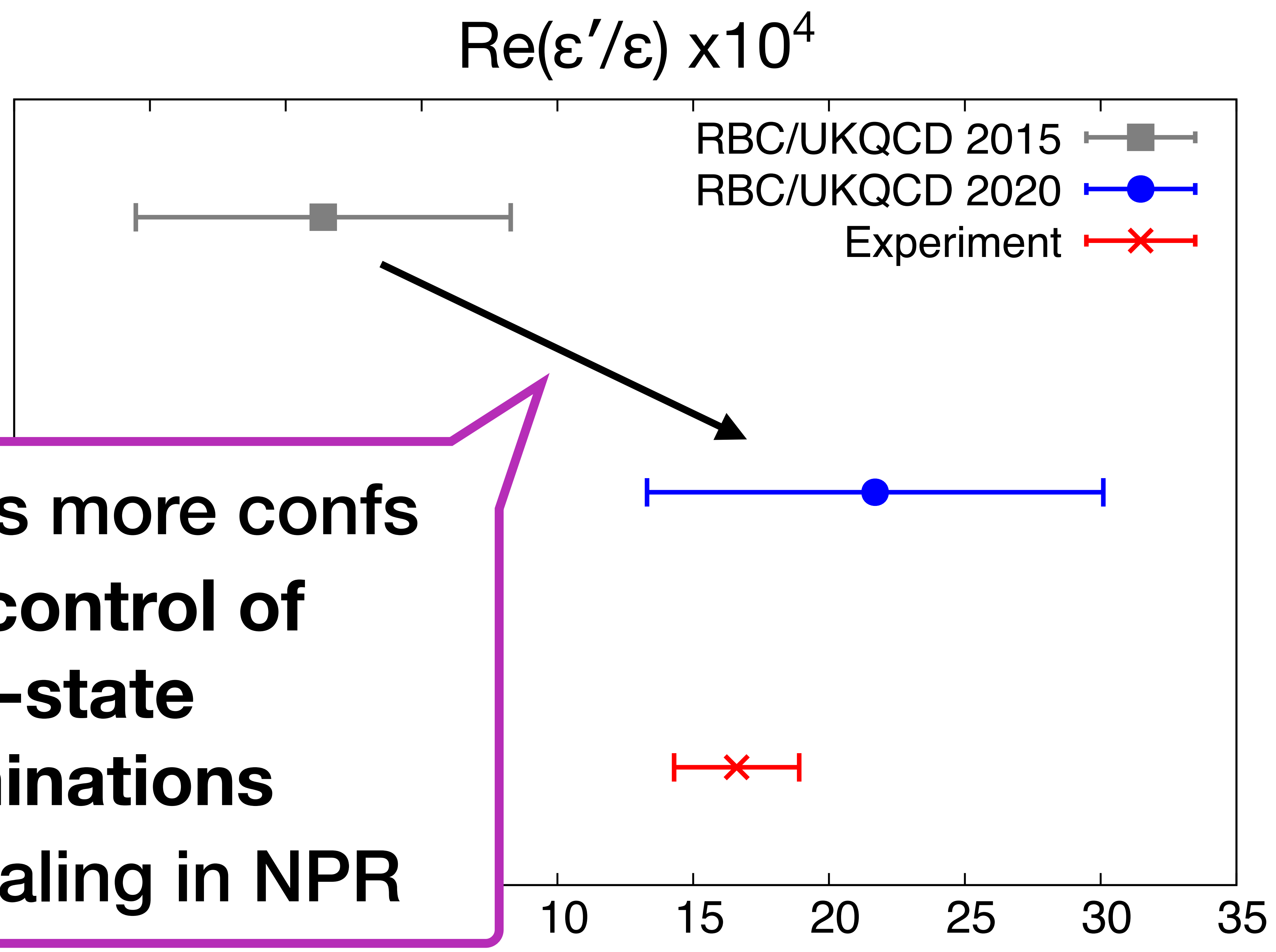


fit result for lightest (on-shell) ππ energy in lattice units

Inclusion of σ operator plays a crucial role in controling excited states

- This ππ state realizing near on-shell kinematics of K → ππ overlaps with the σ resonance
- We learned that states near a resonance energy should be isolated by introducing the corresponding composite operator

ε′ with GPBC



- 3+ times more confs
- Better control of excited-state contaminations
- Step scaling in NPR

$$\mathrm{Re}(\varepsilon'/\varepsilon)_{2020} = 21.7(2.6)_{\mathrm{stat}}(6.2)_{\mathrm{sys}}(5.0)_{\mathrm{EM/IB}} \times 10^{-4}$$

- PRD 102,054509 (2020)

$$\mathrm{Re}(\varepsilon'/\varepsilon)_{\mathrm{exp}} = 16.6(2.3) \times 10^{-4}$$

- Desire
  - Independent calculations
  - Smaller error of lattice prediction

Systematic errors in 2020

- Systematic errors on Im A_0

Finite lattice spacing	12%
Wilson coefficients/charm-loop effects	12%
Lelloch-Lüscher FV correction	1.5%
Residual FV correction	7%
Parametric error	6%
Off-shellness	5%
Renormalization	4%
Missing G_1 operator	3%
TOTAL	21%

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also see backup slide

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also see backup slide

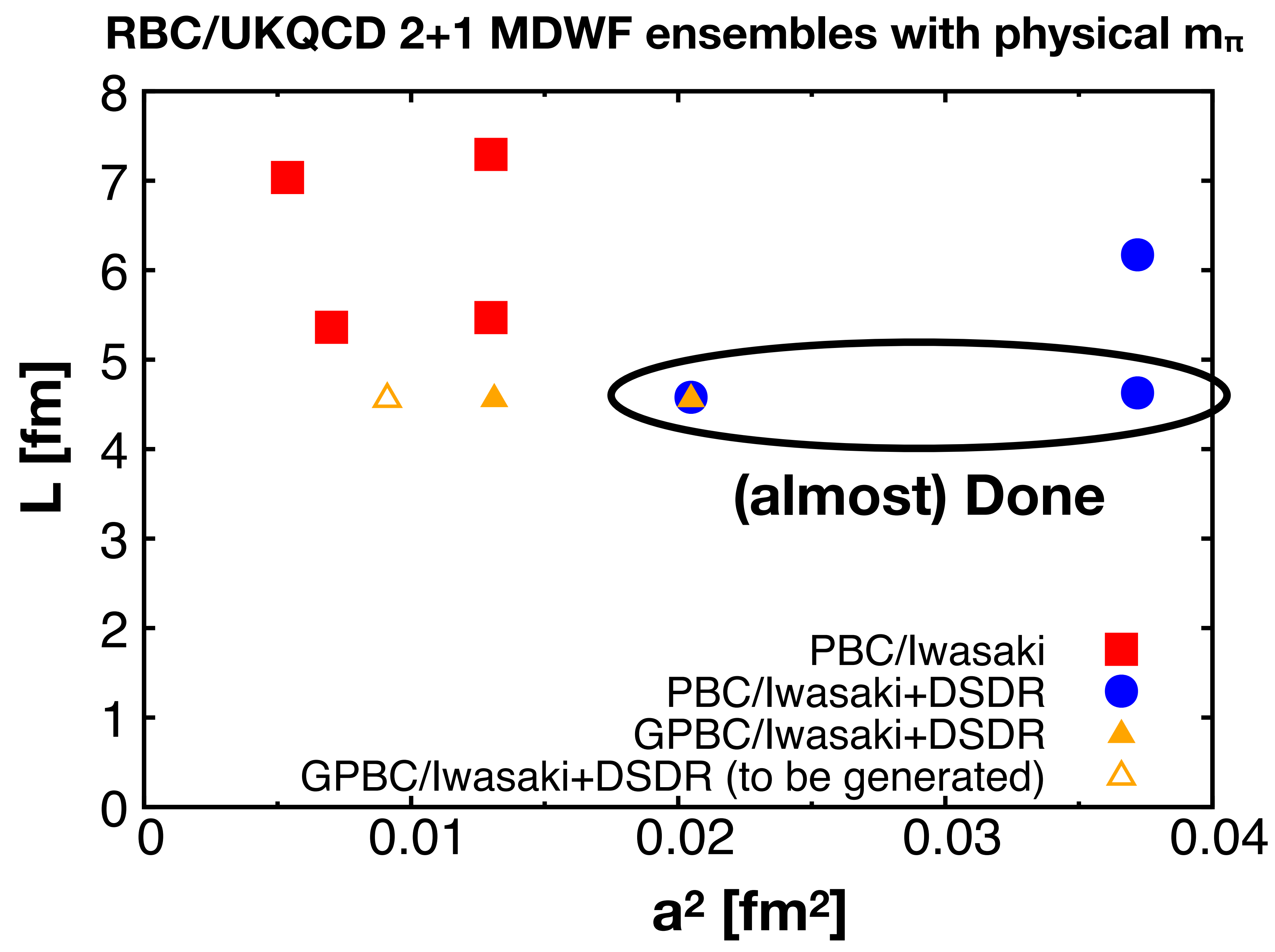
- In addition

Hope to compute near future

  - ε′ could be significantly affected by EM/IB effects (ΔI = 1/2 rule → 25%)

Finite lattice spacing error

- Can be resolved by taking **continuum limit**
  - Results with different lattice spacings needed
- G-parity BC
  - 32^3 x 64, $a^{-1} \approx 1.4$ GeV: Done (2020)
  - GPBC ensemble generation speed-up algorithm [Lat23, C. Kelly]
  - 40^3 x 64, $a^{-1} \approx 1.7$ GeV: Calculation on-going
  - 48^3 x ??, $a^{-1} \approx 2.1$ GeV: in the future as needed
- **Fine ensembles already generated for PBC**



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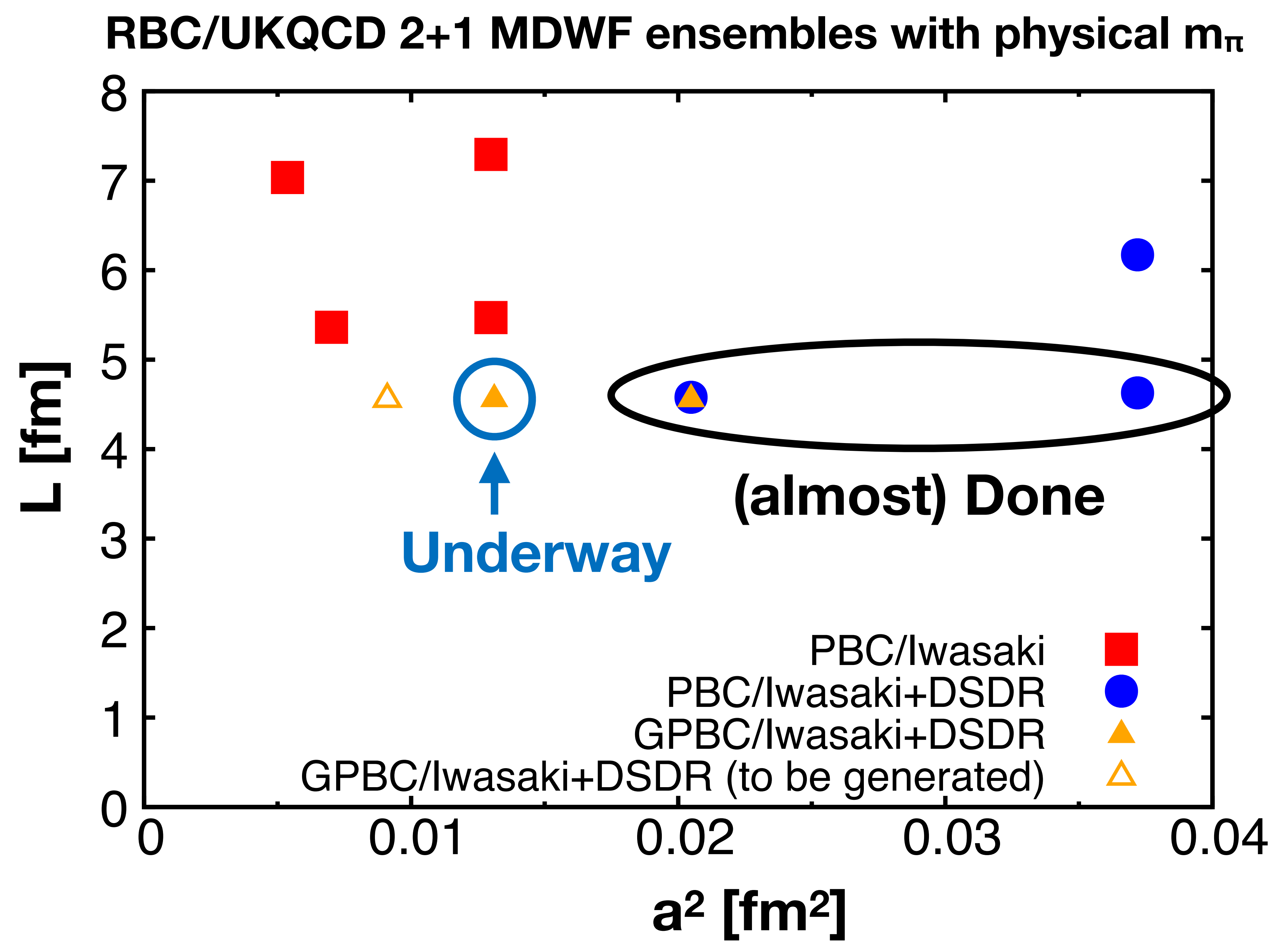

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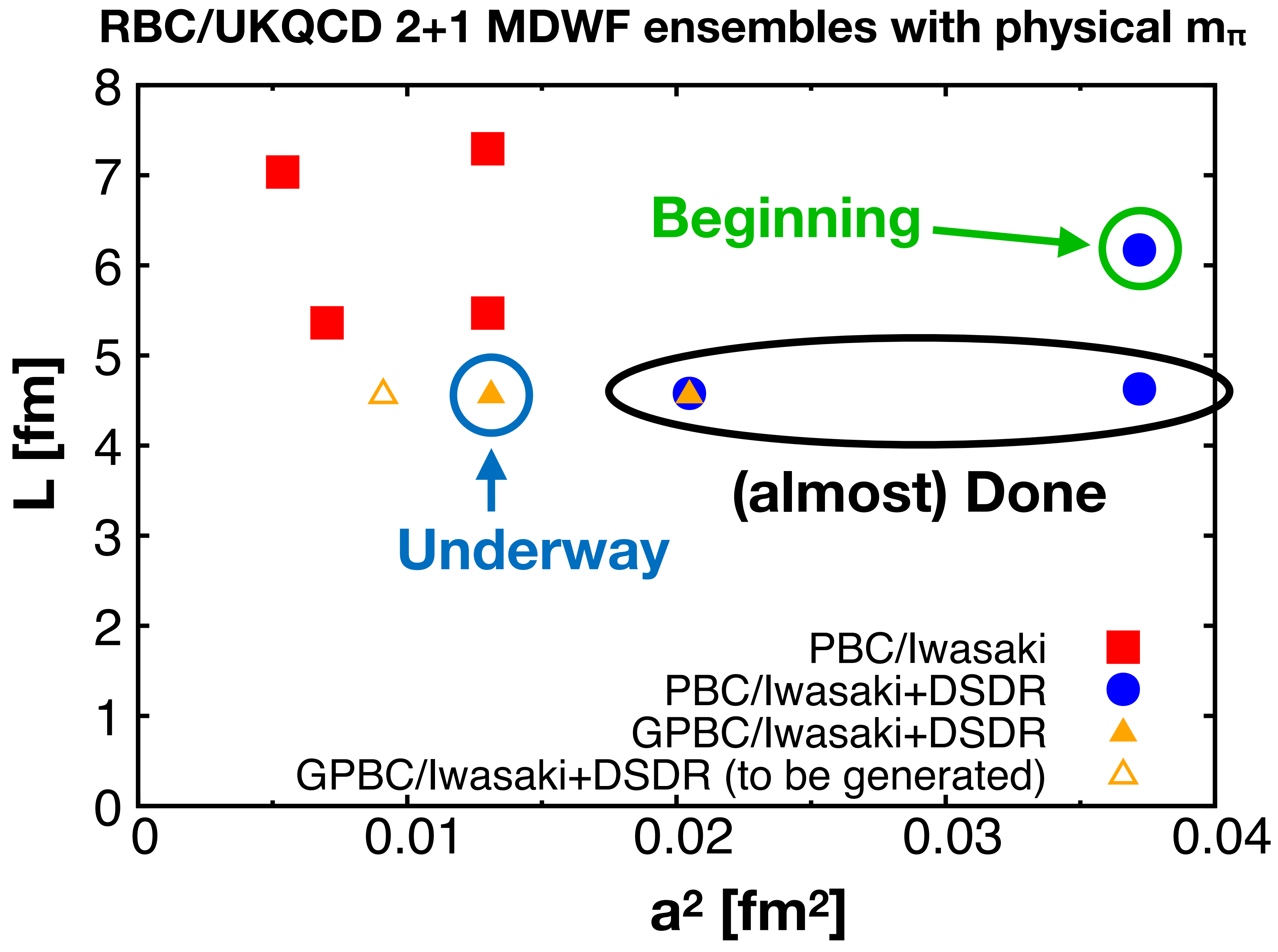
  - $48^3 \times$??, $a^{-1} \approx 2.1$ GeV: in the future as needed
- **Fine ensembles already generated for PBC**



Finite lattice spacing error

- Can be resolved by taking **continuum limit**
  - Results with different lattice spacings needed
- G-parity BC
  - $32^3 \times 64$, $a^{-1} \approx 1.4$ GeV: Done (2020)
  - GPBC ensemble generation speed-up algorithm [Lat23, C. Kelly]
  - $40^3 \times 64$, $a^{-1} \approx 1.7$ GeV: Calculation on-going
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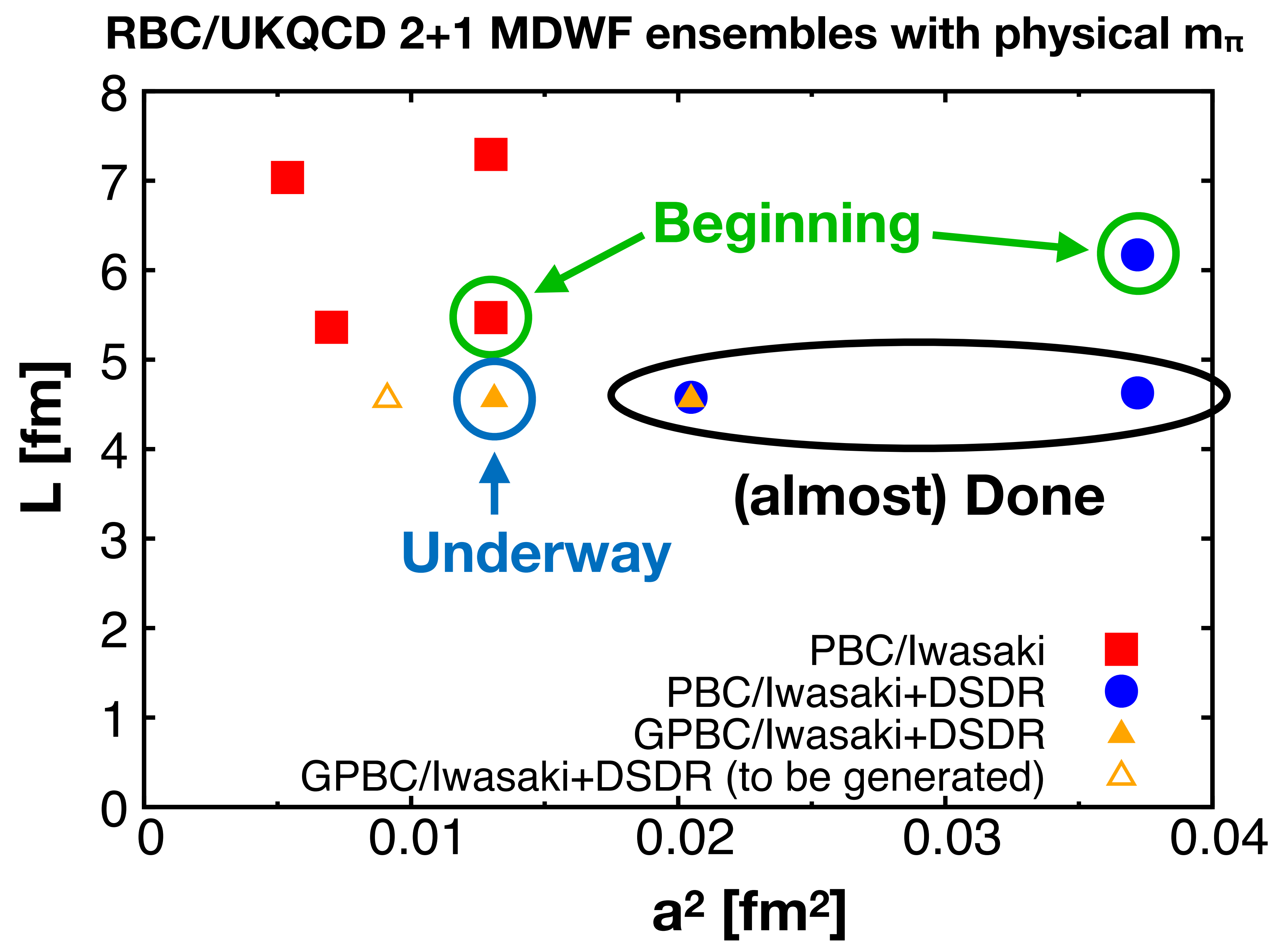

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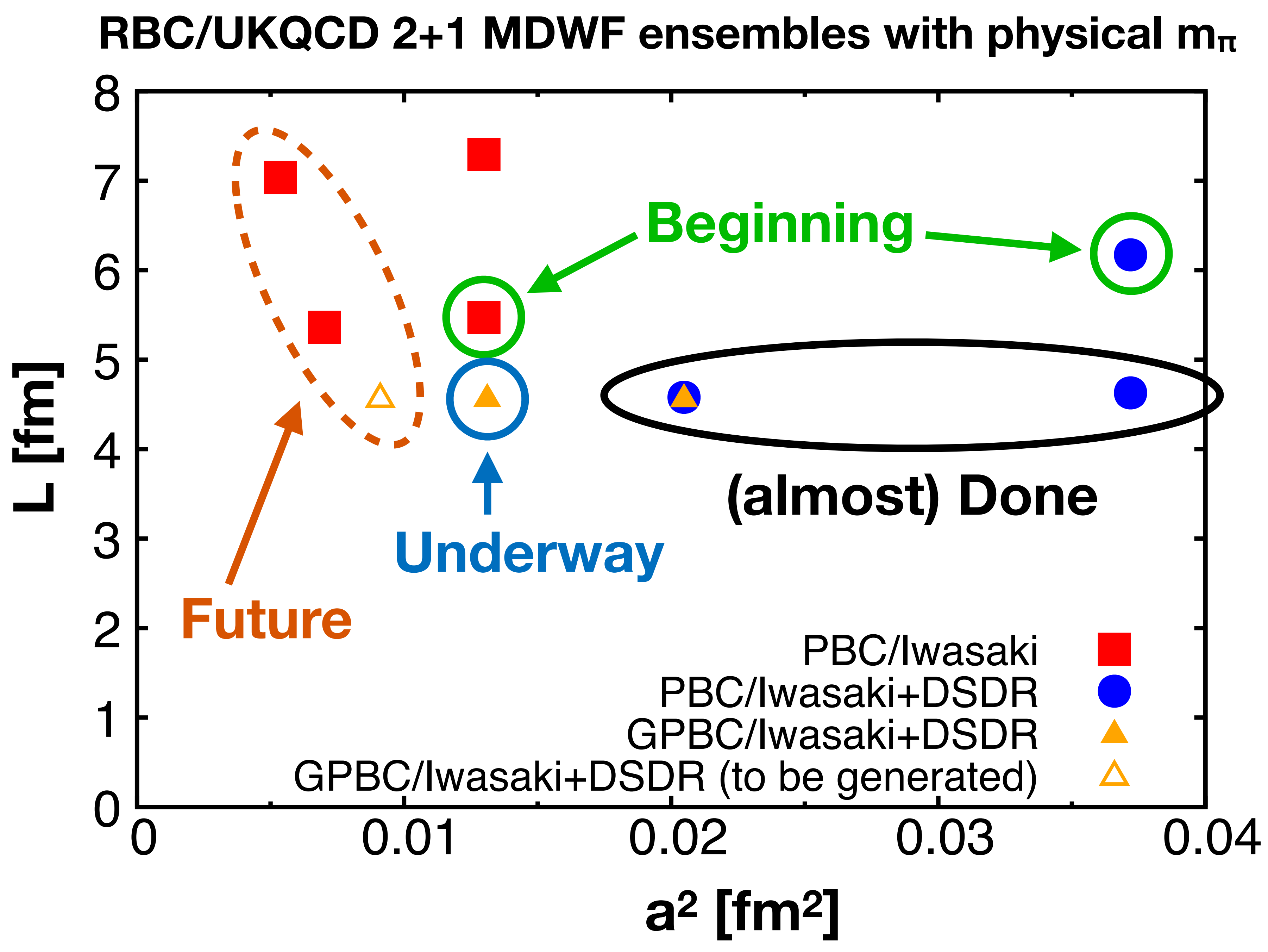


EM/IB effects

- Usually O(1%) but …

$$\frac{\varepsilon'}{\varepsilon} = \frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}\varepsilon}\left[\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\right] = -\frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}\varepsilon}\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\left[1 - \frac{1}{\omega}\frac{\mathrm{Im}A_2}{\mathrm{Im}A_0}\right] \qquad (\omega = \mathrm{Re}A_2/\mathrm{Re}A_0)$$

Even small correction to this can amplified for ε' ($1/\omega \approx 22.5$: $\Delta I = 1/2$ rule)

Ciligriano et al, JHEP 02, 032 (2020)
NLO ChPT + large N_c
(example estimation)

$$\frac{\varepsilon'}{\varepsilon} = \frac{i\omega_+ e^{i(\delta_2-\delta_0)}}{\sqrt{2}\varepsilon}\left[\frac{\mathrm{Im}A_2^{\mathrm{emp}}}{\mathrm{Re}A_2^{(0)}} - \frac{\mathrm{Im}A_0^{(0)}}{\mathrm{Re}A_0^{(0)}}(1-\hat{\Omega}_{\mathrm{eff}})\right] \qquad \boxed{\hat{\Omega}_{\mathrm{eff}} = 0.170\binom{+91}{-90}}$$

- Developing approaches to introduce QED/IB effects on the lattice
  - Extension of Lüscher's formalism for treatment of $\pi\pi$ state in a finite box
  - Coulomb correction to $\pi^+\pi^+$ scattering [Christ et al, PRD106 (2022), 014508]
  - Contribution of transverse radiation getting understood
  - PBC appear necessary to introduce these effects

With PBC & rest frame, $\pi\pi$ excited state is necessary to realize on-shell kinematics of $K \to \pi\pi$

Variational method [NPB339,222(1990)]

- Solving GEVP (Generalized Eigenvalue Problem)

$$C(t)v_n(t,t_0) = \lambda_n(t,t_0)C(t_0)v_n(t,t_0)$$

$C(t)$: N x N correlator matrix $C_{ab}(t) = \langle O_a(t)O_b(0)^\dagger \rangle$

  - $O'_n = \sum_a v^*_{n,a} O_a$ couples mostly with n-th state
  - $\lambda_n(t,t_0) = e^{-E_n(t-t_0)}$

- ππ operators used in this work:

$$\left.\begin{array}{l} \left.\begin{array}{l} \pi_{p=(0,0,0)}\pi_{p=(0,0,0)} \\ \pi_{p=(0,0,1)}\pi_{p=(0,0,-1)} \\ \pi_{p=(0,1,1)}\pi_{p=(0,-1,-1)} \\ \pi_{p=(1,1,1)}\pi_{p=(-1,-1,-1)} \end{array}\right\} \mathbf{I = 2} \\ \sigma \sim \bar{u}u + \bar{d}d \\ KK \sim \overline{K}K + K^+K^- \text{ : turned out insignificant for } K \to \pi\pi \end{array}\right\} \mathbf{I = 0}$$

Overlap b/w GEVP signals

- Energies from GEVP unresolved with insufficient statistics (107confs, 32^3)
- Plateau not well seen for excited states
- Possible problem of traditional GEVP

$$A v_n = \lambda_n B v_n$$

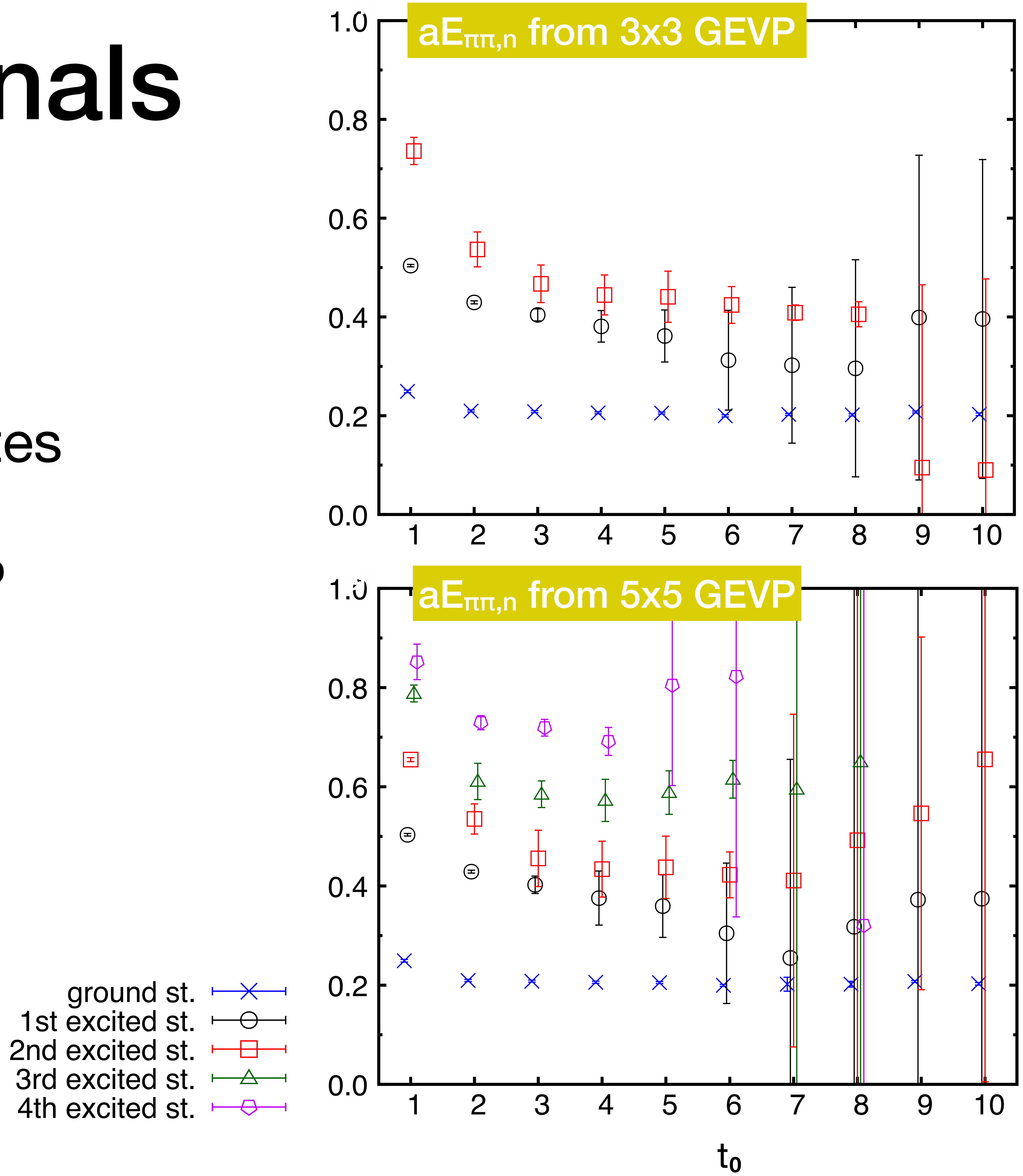
$$\Updownarrow$$

$$B^{-1/2} A B^{-1/2} (B^{1/2} v_n) = \lambda_n (B^{1/2} v_n)$$

small statistics

→ B becomes singular (zero-consistent eval(s))

→ GEVP singular



Rebased GEVP

- Re-based GEVP
  - Large size GEVP at short time separations
  - Switch to smaller-size GEVP at larger time any eigenvalue is becoming zero-consistent

- Example:

$$\begin{pmatrix} \mathrm{O}_{(000)} \\ \mathrm{O}_{(001)} \\ \mathrm{O}_{(011)} \\ \mathrm{O}_{(111)} \\ \mathrm{O}_{\sigma} \end{pmatrix} \xrightarrow{\textbf{5x5 GEVP at t = 2}} \left\{ \begin{array}{l} \mathrm{O}_0^{t=2} \\ \mathrm{O}_1^{t=2} \\ \mathrm{O}_2^{t=2} \\ \mathrm{O}_3^{t=2} \\ \mathrm{O}_4^{t=2} \end{array} \right.$$

- $\mathrm{O}_0^{t=2}$ - couples well with the ground st.
- $\mathrm{O}_1^{t=2}$ - couples well with the 1st excited st.
- $\mathrm{O}_2^{t=2}$ - …
- $\mathrm{O}_4^{t=2}$ - incorporating 5th op and hence 5th state can spoil the signal of lower states at larger time separations

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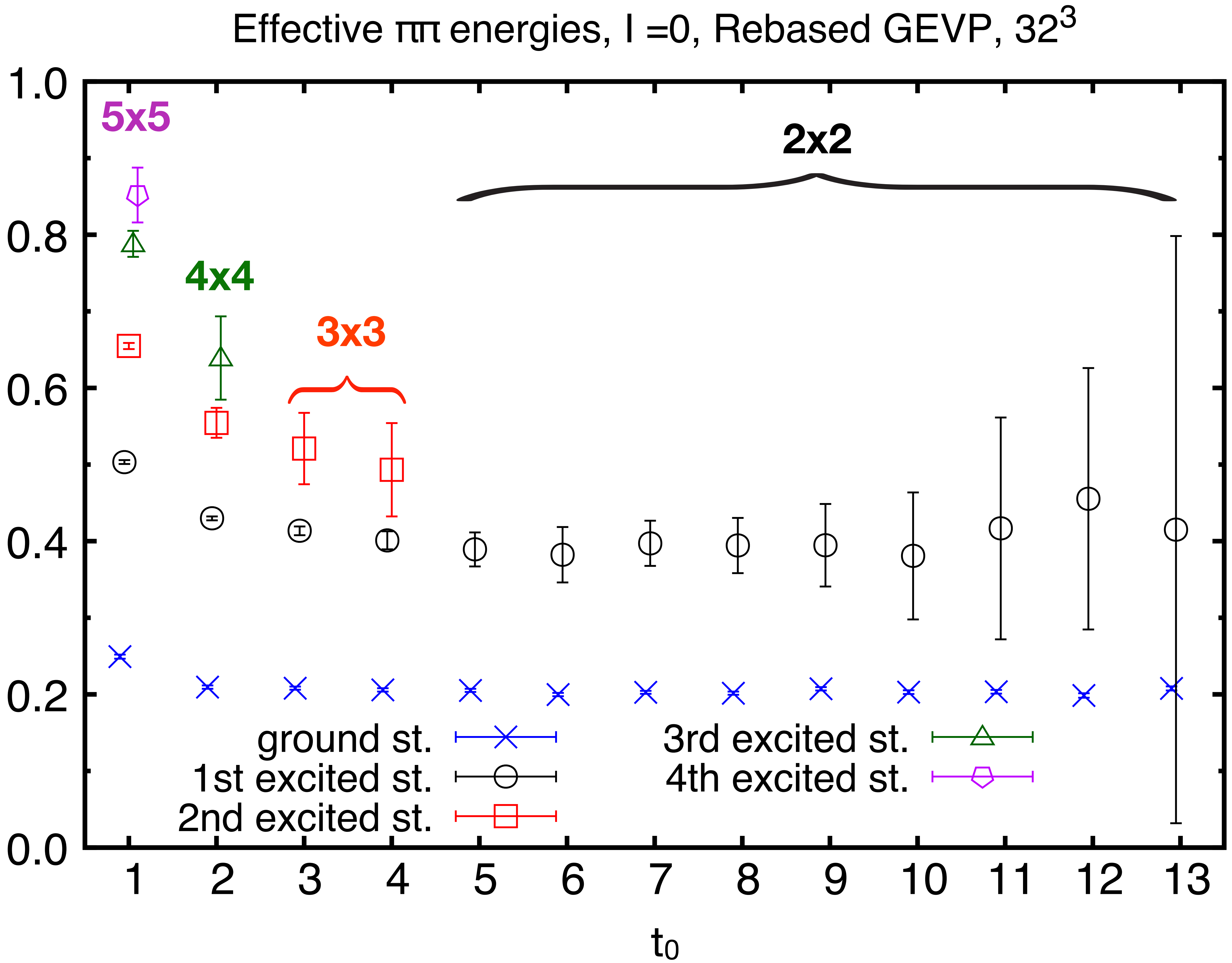
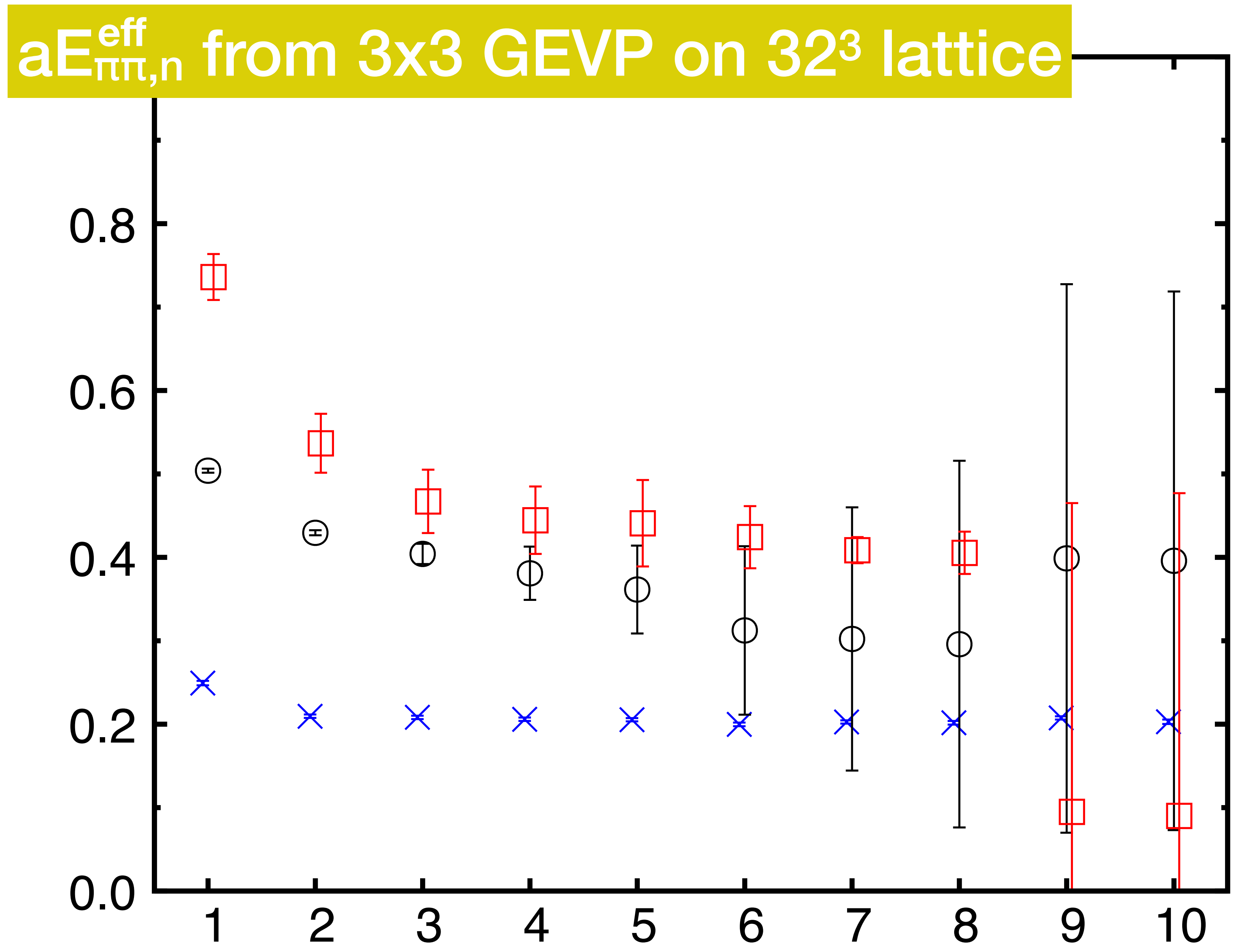
- couples well with the ground st.
- couples well with the 1st excited st.
- …
- incorporating 5th op and hence 5th state can spoil the signal of lower states at larger time separations

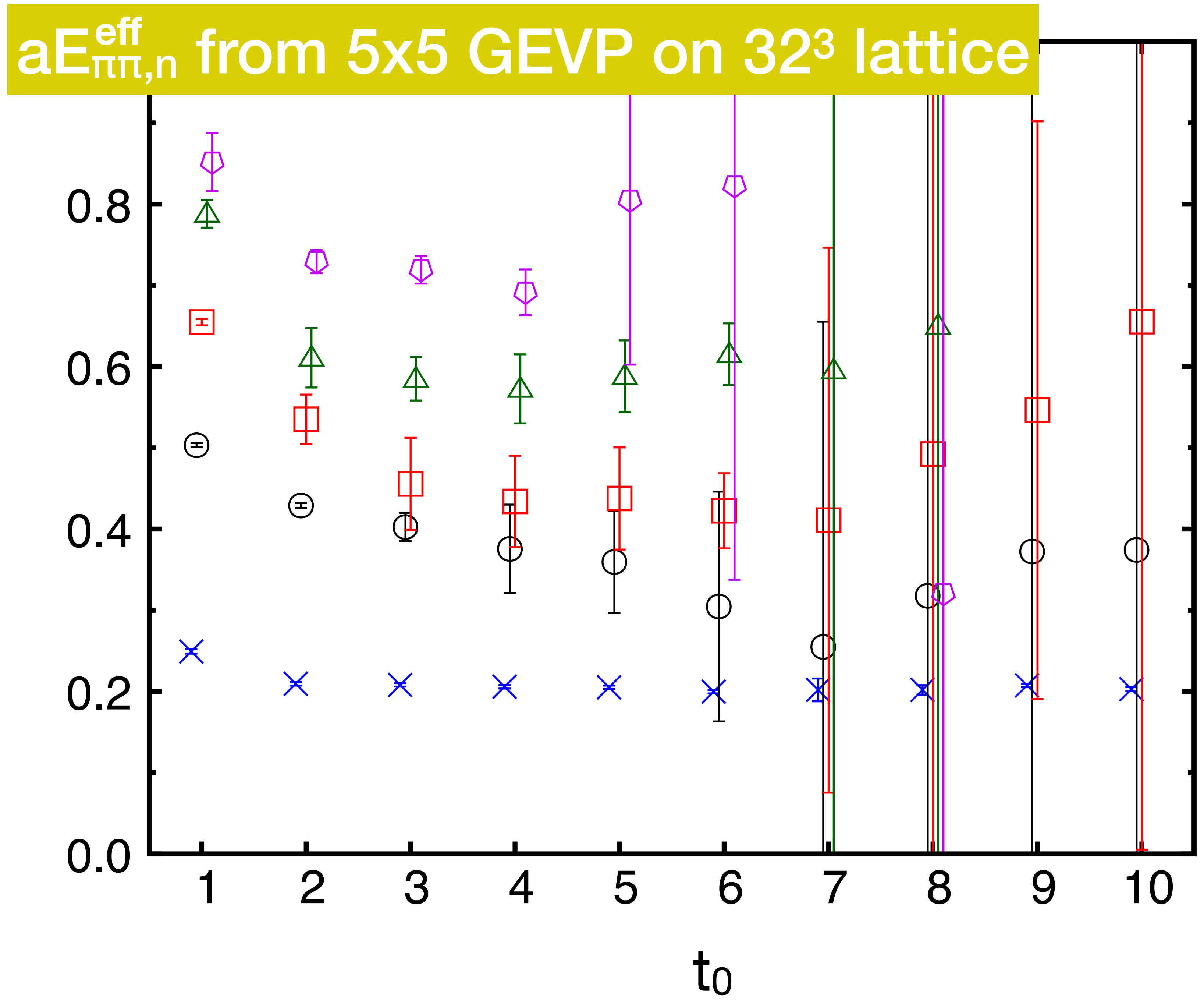
$$\begin{pmatrix} O_0^{t=2} \\ O_1^{t=2} \\ O_2^{t=2} \\ O_3^{t=2} \end{pmatrix} \xrightarrow{\text{4x4 GEVP at t = 3, ...}}$$

- 3 operators to form next basis
- 1 operator to be excluded from next basis

Rebased GEVP signals



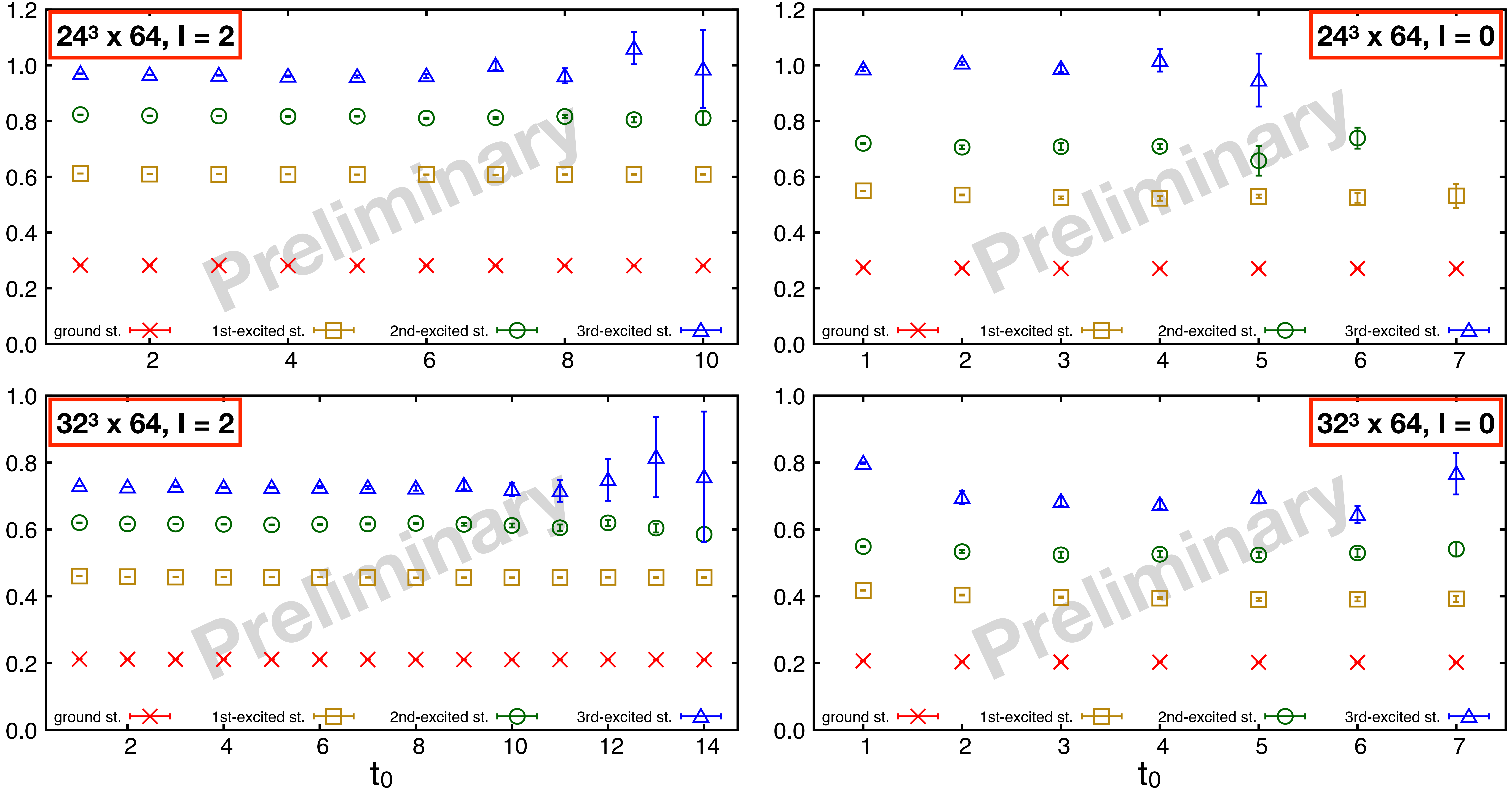




- Plateau seen for the first two states!
- Increasing statistics

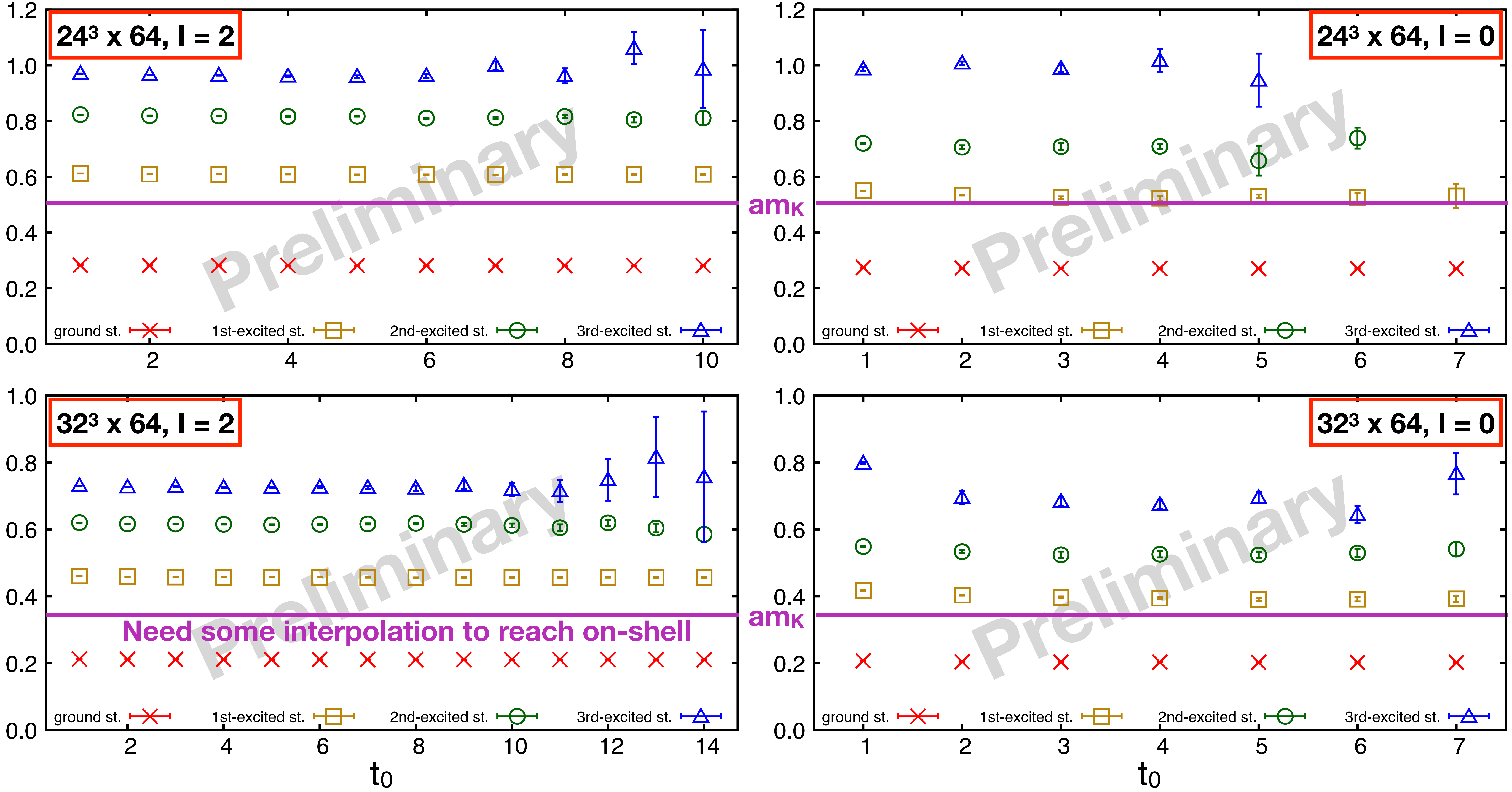
$aE_{\pi\pi}^{\text{eff}}$ with more statistics

All preliminary with new data set

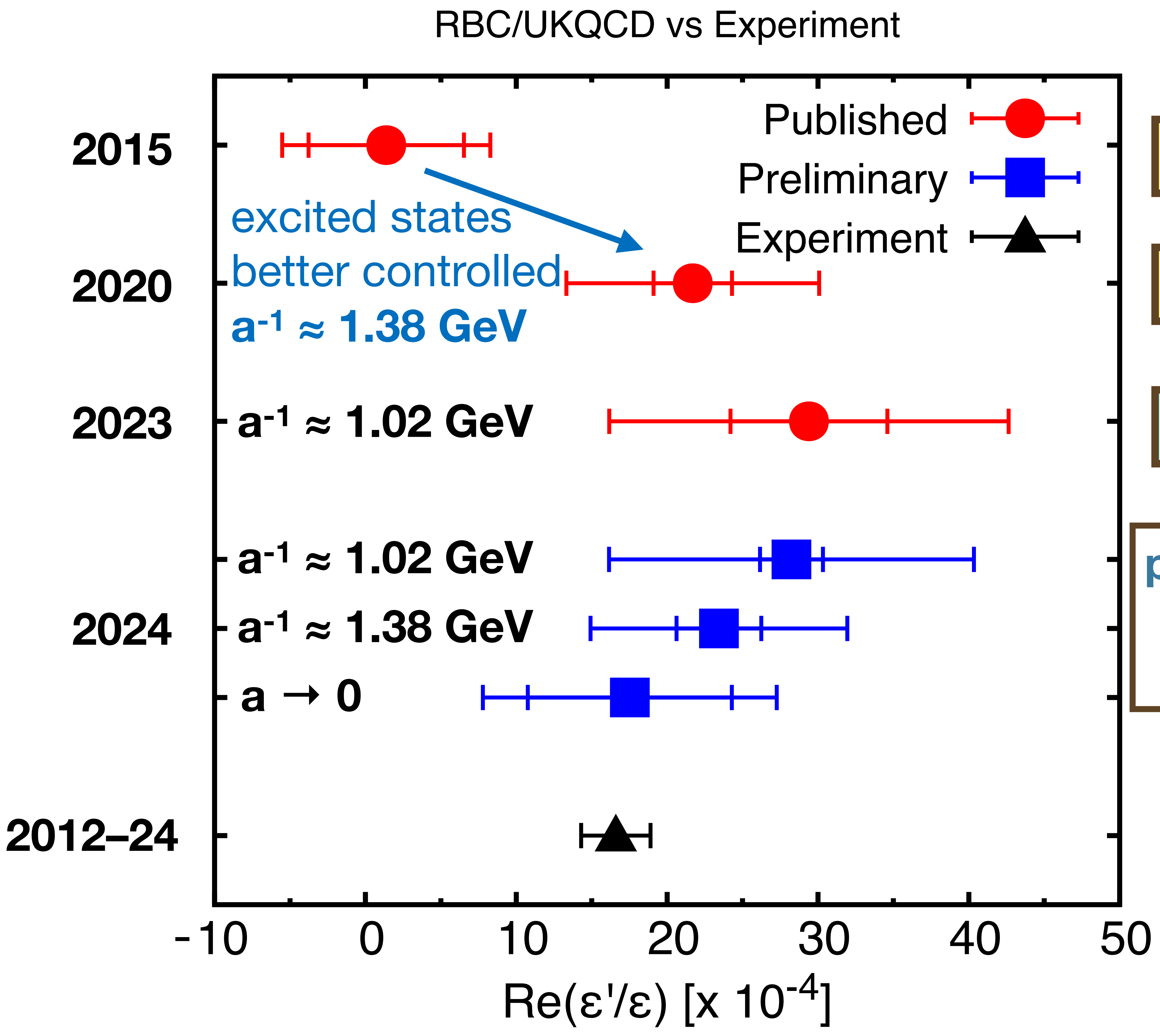


$aE^{eff}_{\pi\pi}$ with more statistics

All preliminary with new data set



Current status of ε′/ε



PRL 115,212001

PRD 102,054509

PRD 102,094517

preliminary results with multiple lattice spacings

- G-parity Boundary Conditions (GPBC)
  - $a^{-1} \approx 1.38$ GeV
  - efforts started by early 2000s
  - continuing calculation on finer lattice(s) C. Kelly's talk at Lattice24

- Periodic Boundary Conditions (PBC)
  - newer project
  - important for introducing EM/IB effects
  - Led by MT (see backup slides for more details)

- **$a^{-1} \approx 1.38$ GeV almost done, wrapping up**
- **starting calculation at $a^{-1} \approx 1.73$ GeV**

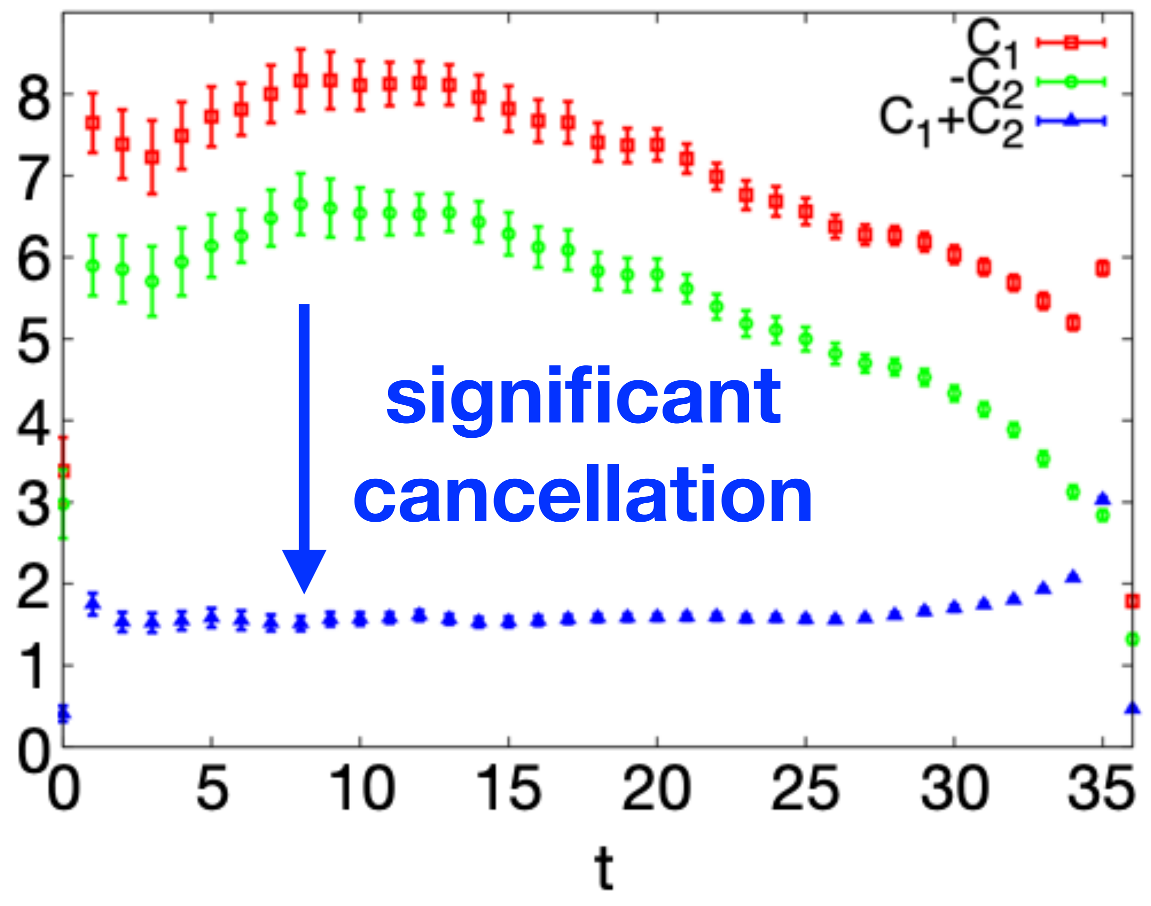
⁕ Result from another group, Ishizuka et al 2018: Re(ε′/ε) = (19 ± 57) x 10^{-4} (calculated at unphysical m_π, m_K)

The ΔI = 1/2 rule

- Experimental fact

$$\frac{\text{Re}\,A_0}{\text{Re}\,A_2} = 22.45(6)$$: large suppression of ΔI = 3/2 (A_2) mode

- Significant suppression of ReA_2 (2012/2015)
  - C_1, C_2 contributions of different color structure to $K \to \pi\pi$ correlation function most significant to ReA_2
  - Naïvely $C_1 = -3C_2$ based on color counting
  - Significant cancellation at physical m_π observed



RBC/UKQCD, PRD91,074502 (2015)

- Numerical confirmation of the ΔI = 1/2 rule with the lattice result for A_0 (2020)

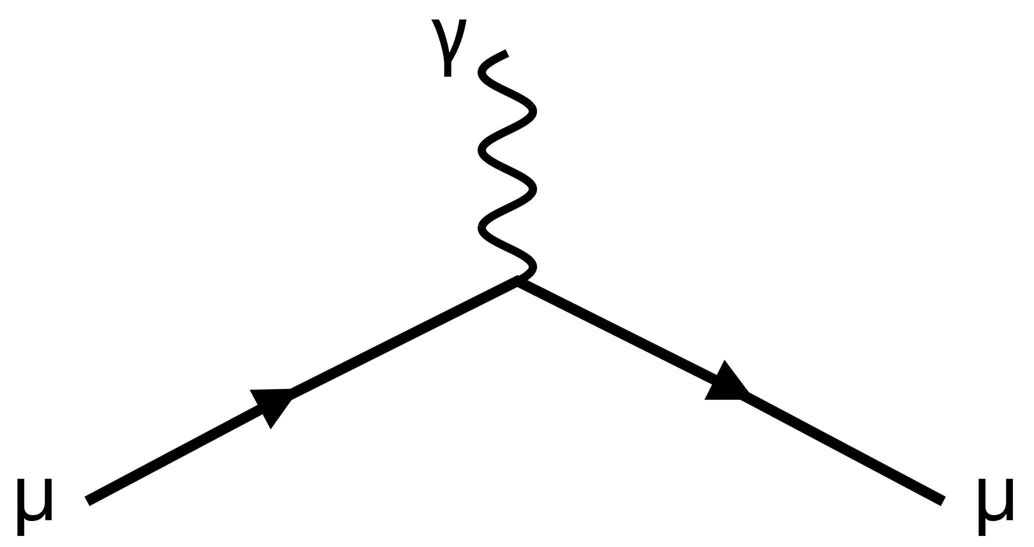
$$\frac{\text{Re}A_0}{\text{Re}A_2} = 19.9(2.3)_{\text{stat}}(4.4)_{\text{sys}}$$

Long-distance HVP contribution to muon g-2

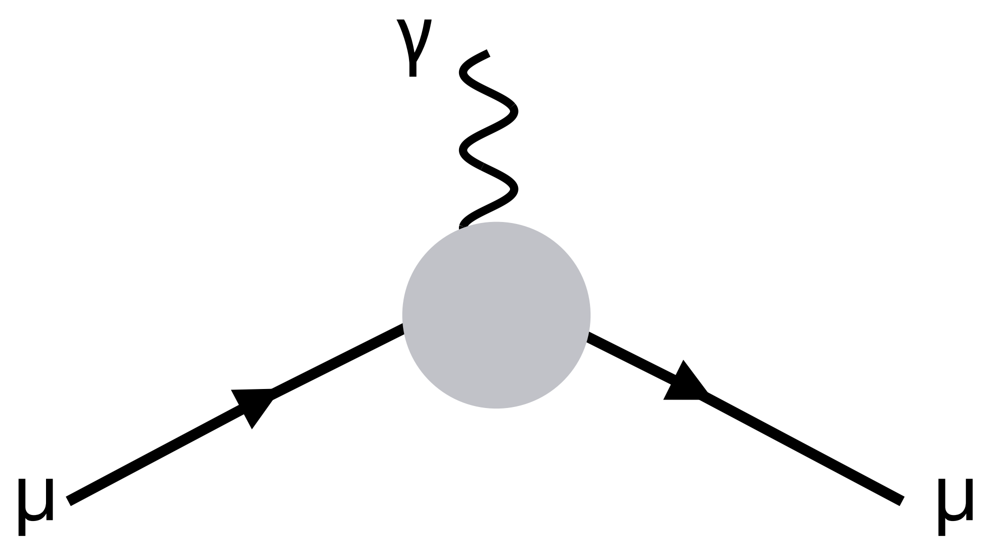
Muon anomalous magnetic moment

Magnetic moment

$$\vec{\mu}_\mu = g_\mu \left(\frac{e}{2m_\mu} \right) \vec{S}$$



Classically $g_\mu = 2$



Actually $g_\mu \neq 2$

Anomalous magnetic moment

$$a_\mu = \frac{g_\mu - 2}{2}$$

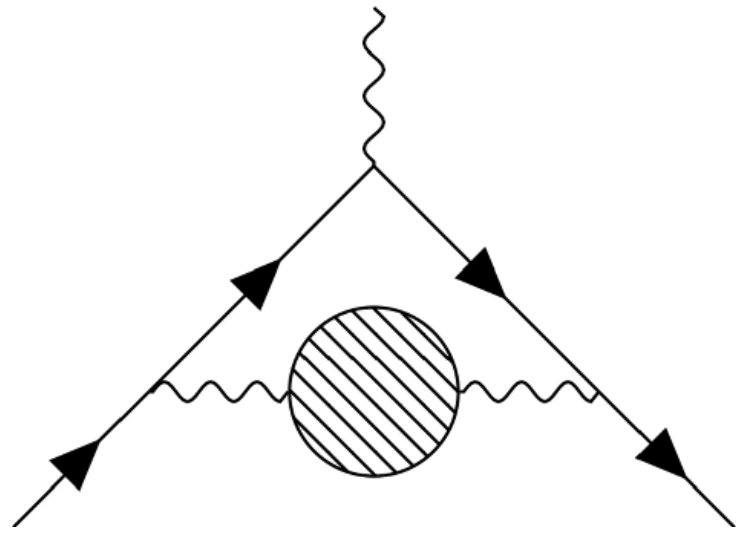
Experiment vs theory

- Experiment (2023): $10^{10}\,a_\mu = 11659205.9(2.2)$ PRL131,161802(2023)
- Theory white paper (2020): $10^{10}\,a_\mu = 11659181.0(4.3)$ 2006.04822
  - QED: 11658471.893(10)
  - EW: 15.36(10)
  - QCD
    - LO HVP: 693.1(4.0)
      - data-driven
      - dominant uncertainty
    - NLO HVP: -98.3(7)
    - NNLO HVP: 12.4(1)
    - HLbL: 9.0(1.7)

$\longleftarrow$ 2.2σ $\longrightarrow$

BMW (2020): 707.5(5.5)

Nature593,51(2021)



LO HVP contribution to a_μ

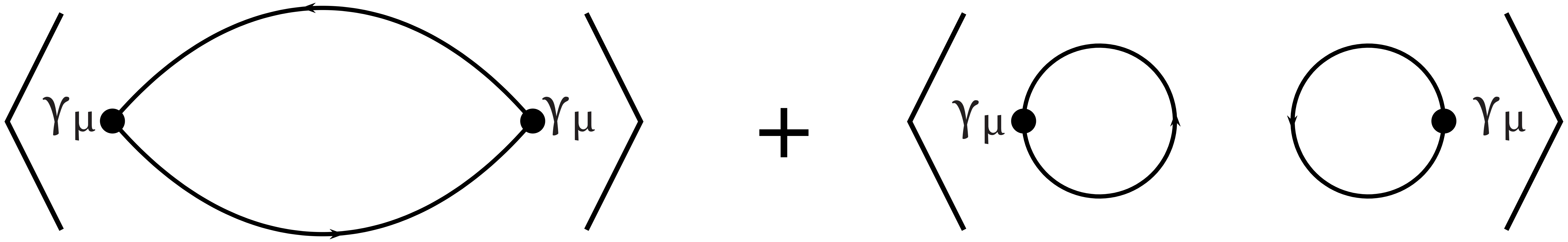
$$a_\mu^{\mathrm{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, t^3 G(t) \underline{\widetilde{K}(t)}$$

known function
~ t at $t \ll 1/m_\mu$
~ t^{-1} at $t \gg 1/m_\mu$

$$G(t) = \frac{1}{3}\sum_{\mu=1}^{3} \left\langle J_\mu^{em}(t) J_\mu^{em}(0)^\dagger \right\rangle$$

$$\xrightarrow{\text{lattice}} \sum_t G(t) w_t$$

G(t) =



Practically convenient to treat separately

What parts cause significant error?

$$a_{\mu}^{\mathrm{LO-HVP}} = \sum_{q=(ud),s,c,b} a_{\mu,\mathrm{conn}}^{\mathrm{LO-HVP}}(q) + a_{\mu,\mathrm{disc}}^{\mathrm{LO-HVP}} + \underline{a_{\mu,\mathrm{SIB}}^{\mathrm{LO-HVP}} + a_{\mu,\mathrm{EM}}^{\mathrm{LO-HVP}}}$$

from 2020 WP [10^{-10}]

$a_\mu^{\text{HVP, LO}}(ud)$	$a_\mu^{\text{HVP, LO}}(s)$	$a_\mu^{\text{HVP, LO}}(c)$	$a_{\mu,\text{disc}}^{\text{HVP, LO}}$	$\delta a_\mu^{\text{HVP}}$
650.2(11.6)	53.2(0.3)	14.6(0.1)	−13.7(2.9)	7.2(3.4)

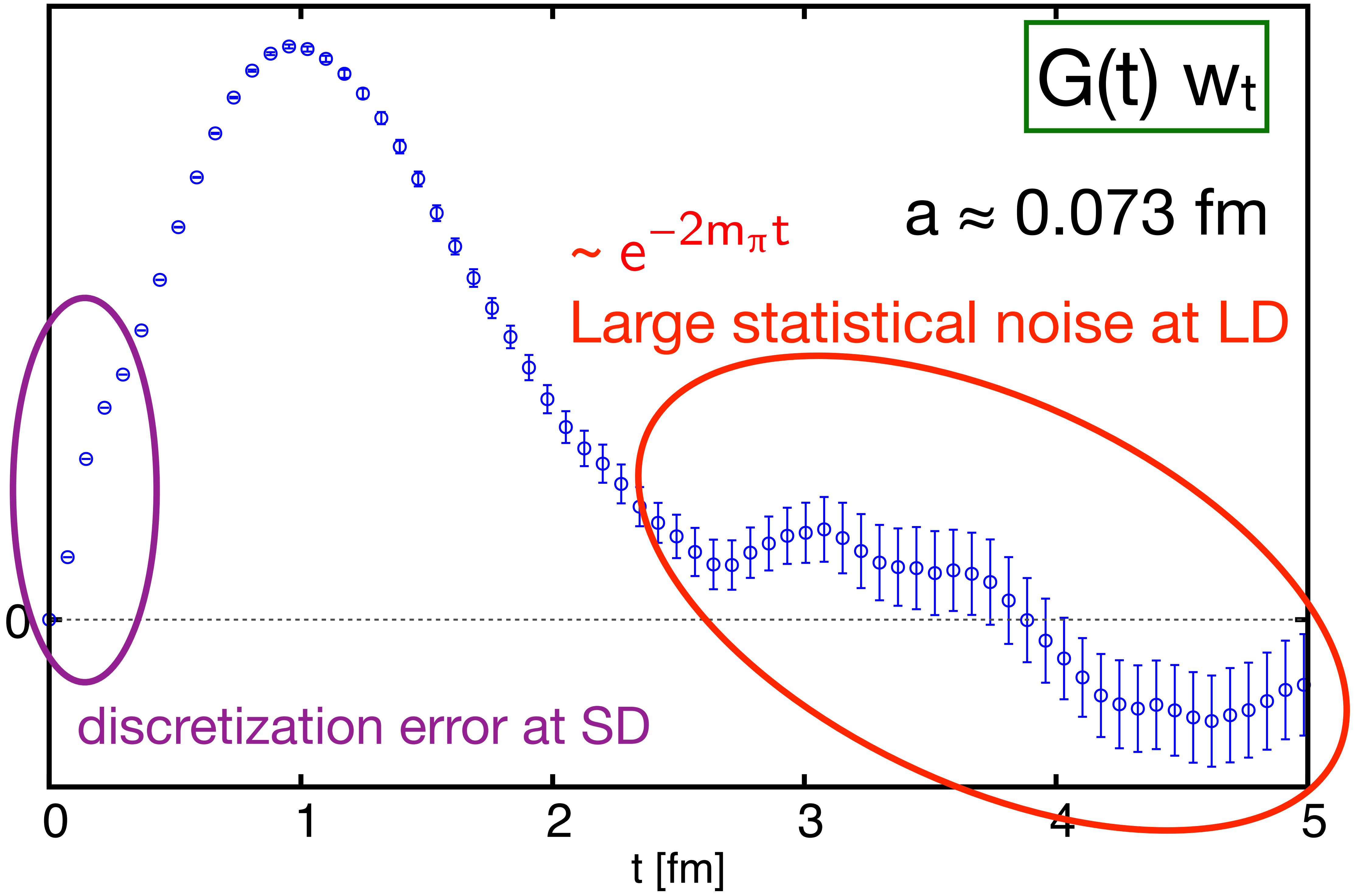
**** final precision goal: ~1.5x10^{-10} error**

** $a_{\mu,\mathrm{conn}}^{\mathrm{LO-HVP}}(ud)$: Dominant error source of lattice calculation (focus of this work)

LO HVP contribution to a_μ

$$a_\mu^{\mathrm{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, t^3 G(t) \underline{\tilde{K}(t)} \xrightarrow{\text{lattice}} \sum_t G(t) w_t$$

known function
$\sim t$ at $m_\mu t \ll 1$
$\sim t^{-1}$ at $m_\mu t \gg 1$



- Window method [PRL121,022003(2018)]

$$a_\mu^{\mathrm{LO-HVP}} = a_\mu^{\mathrm{SD}} + a_\mu^{\mathrm{W}} + a_\mu^{\mathrm{LD}}$$

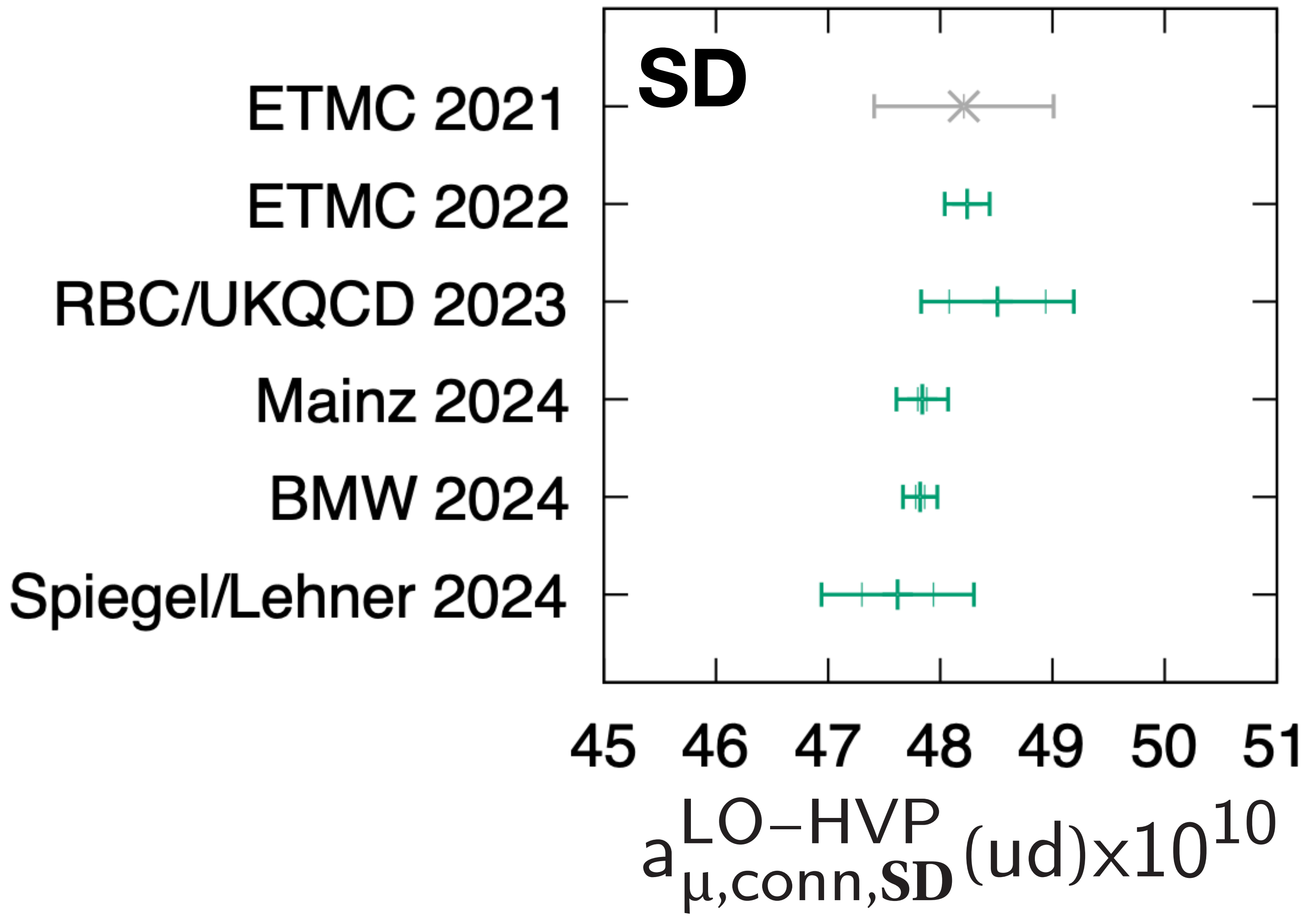
$$\begin{pmatrix} a_\mu^{\mathrm{SD}} \\ a_\mu^{\mathrm{W}} \\ a_\mu^{\mathrm{LD}} \end{pmatrix} = \sum_t G(t) w_t \begin{pmatrix} 1 - \Theta(t, t_1, \Delta t) \\ \Theta(t, t_1, \Delta t) - \Theta(t, t_2, \Delta t) \\ \underline{\Theta(t, t_2, \Delta t)} \end{pmatrix}$$

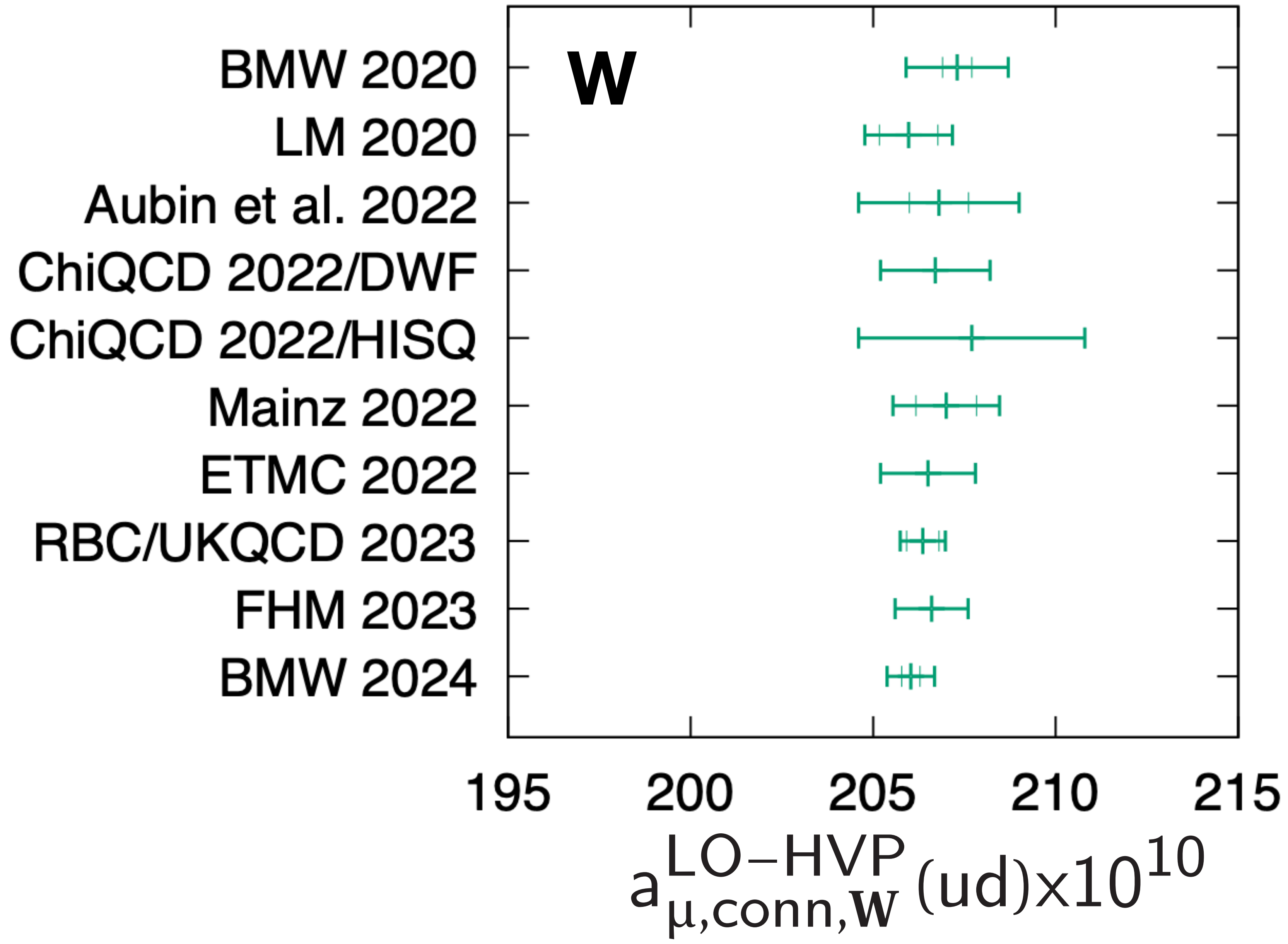
step function with width Δt

- Calculate G(t) with a way suitable for the respective region
- $(t_1, t_2, \Delta t)$ = (0.4 fm, 1.0 fm, 0.15 fm)

Status after WP

- Precision goal: ~ 1.5 x 10^{-10}
- SD & W reaching the goal
- LD desired to be as precise on lattice to achieve full first-principle prediction
- Challenges
  - Large error of long tail
  - Finite-volume effects





Status after WP

New paper by RBC/UKQCD 2410.20590

CERN-TH-2024-182

The long-distance window of the hadronic vacuum polarization for the muon $g-2$

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(RBC and UKQCD Collaborations)

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[8] *Fakultät für Physik, Universität Regensburg, Universitätsstraße 31, 93040 Regensburg, Germany*
[9] *RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973, USA*
[10] *Nuclear and Chemical Sciences Division, Lawrence Livermore National Laboratory, Livermore, CA 94550, USA*

(Dated: October 29, 2024)

We provide the first ab-initio calculation of the Euclidean long-distance window of the isospin symmetric light-quark connected contribution to the hadronic vacuum polarization for the muon $g-2$ and find $a_\mu^{\mathrm{LD,iso,conn,ud}} = 411.4(4.3)(2.4) \times 10^{-10}$. We also provide the currently most precise calculation of the total isospin symmetric light-quark connected contribution, $a_\mu^{\mathrm{iso,conn,ud}} = 666.2(4.3)(2.5) \times 10^{-10}$, which is more than 4σ larger compared to the data-driven estimates of Boito et al. 2022 and 1.7σ larger compared to the lattice QCD result of BMW20.

Reconstruction of G(t) at LD

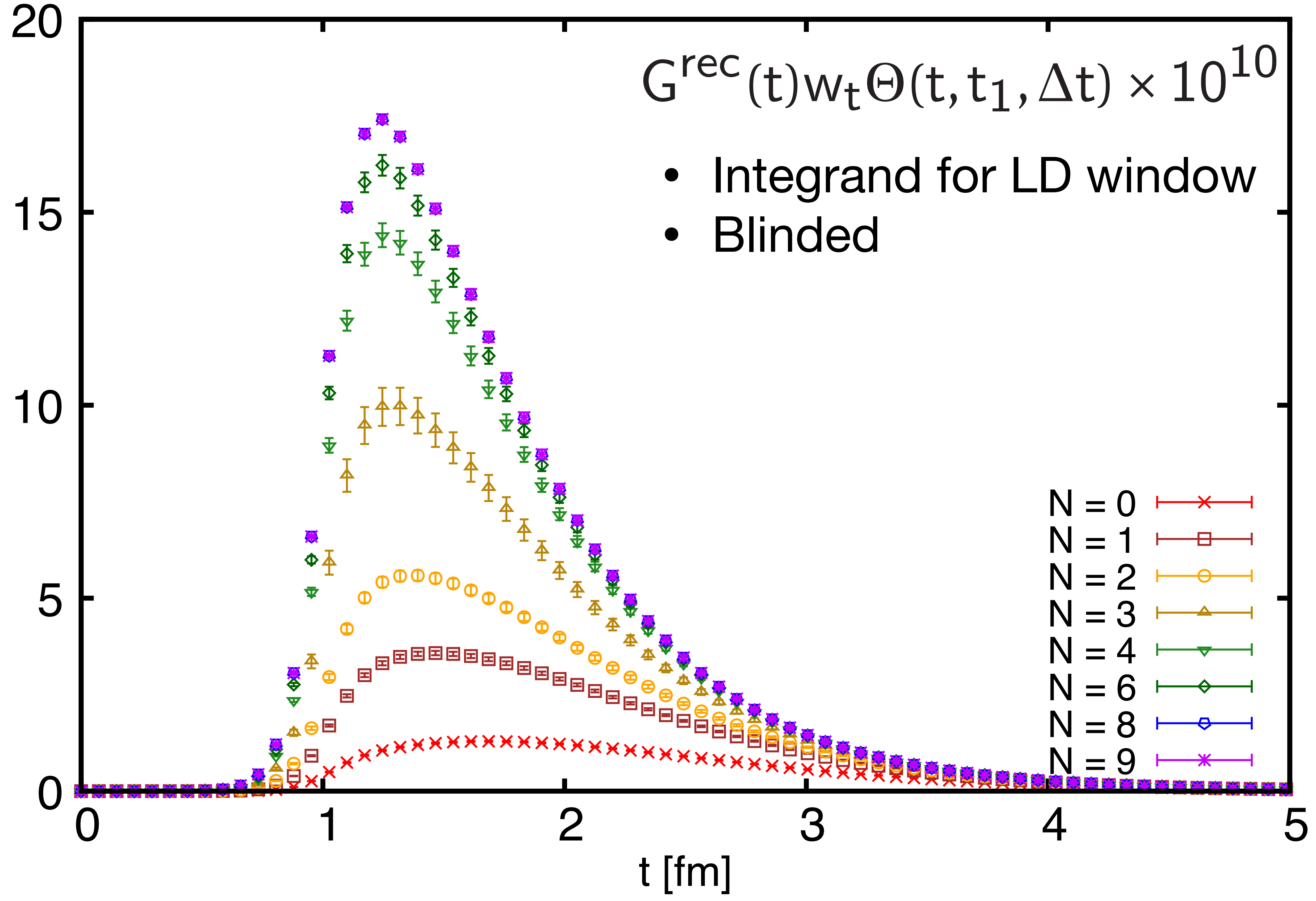
EPJWC175,06031 (2018)
PoSLat2019,239 (2019)

$$G(t) = \langle V_\mu(t) V_\mu(0)^\dagger \rangle = \sum_n \langle 0|V_\mu|n\rangle\langle n|V_\mu^\dagger|0\rangle e^{-E_n t}$$

- If we know $< 0 \,|\, V_\mu \,|\, n >$ and E_n for n = 0, 1, 2, …, N, we can approximate G(t) as a finite sum
  - Contribution from n > N suppressed exponentially at LD
  - The long-tail noise will be small enough if $< 0 \,|\, V_\mu(t) \,|\, n >$ and E_n are determined with a sufficient precision
- This work is focused on light-quark connected contribution → I = 1
- GEVP method capable of determining $< 0 \,|\, V_\mu \,|\, n >$ and E_n
  - ππ-like operators: $\pi_{p=(0,0,1)}\pi_{p=(0,0,-1)}$, $\pi_{p=(0,1,1)}\pi_{p=(0,-1,-1)}$, …
  - quark currents: V_μ (local & smeared), …

Reconstruction of G(t) at LD

a = 0.073 fm
L = 7.0 fm



Reconstruction of G(t) at LD

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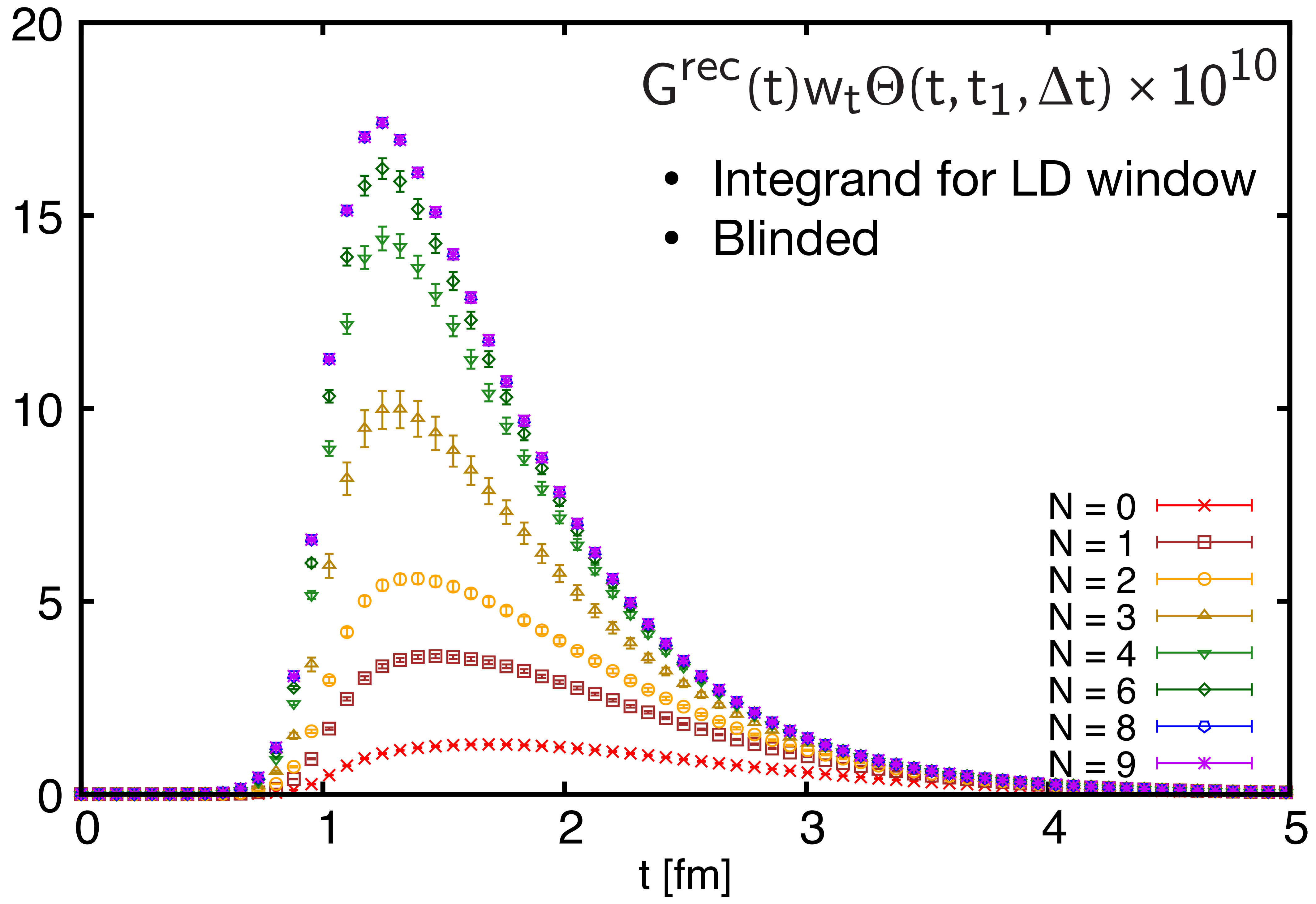
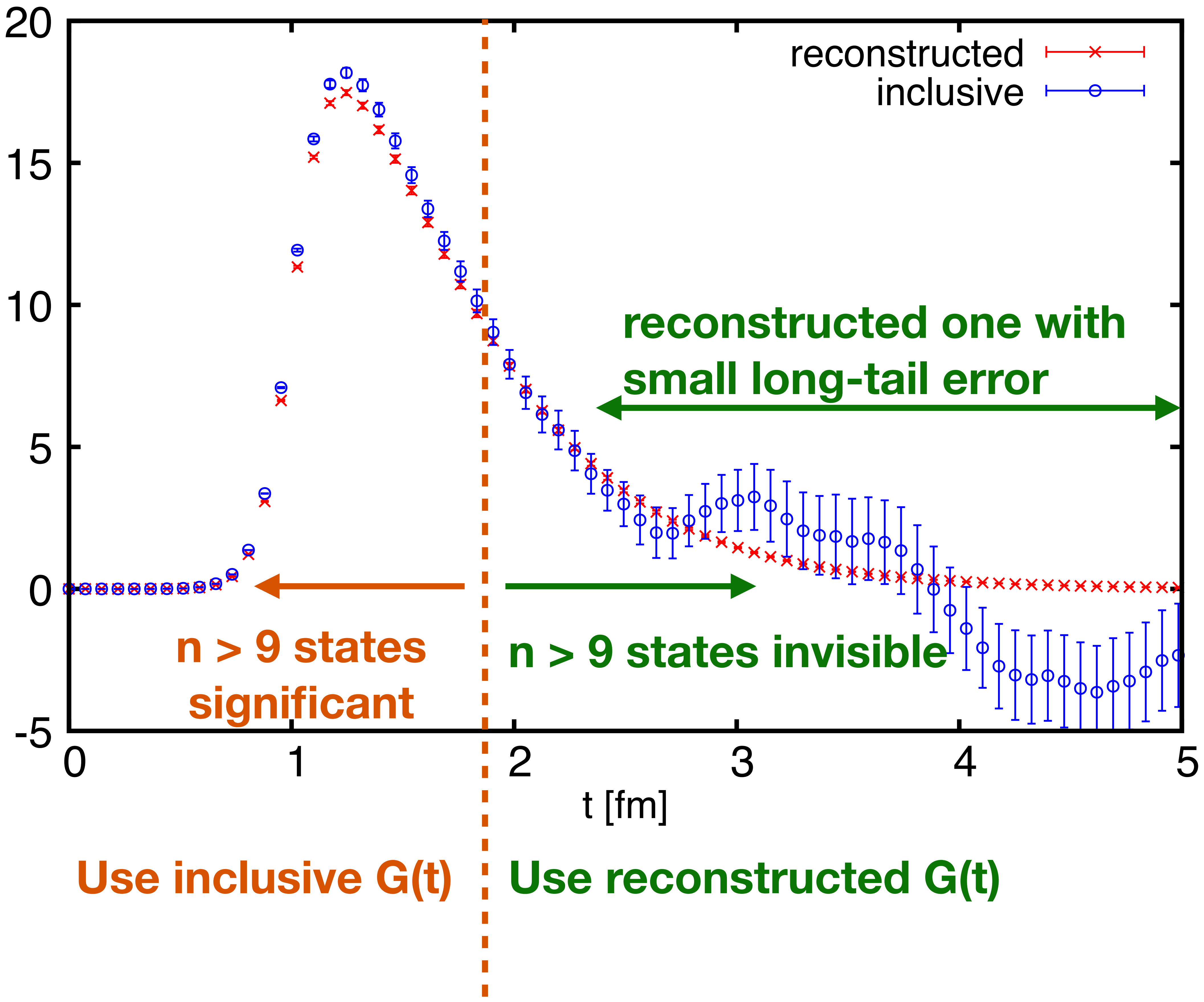
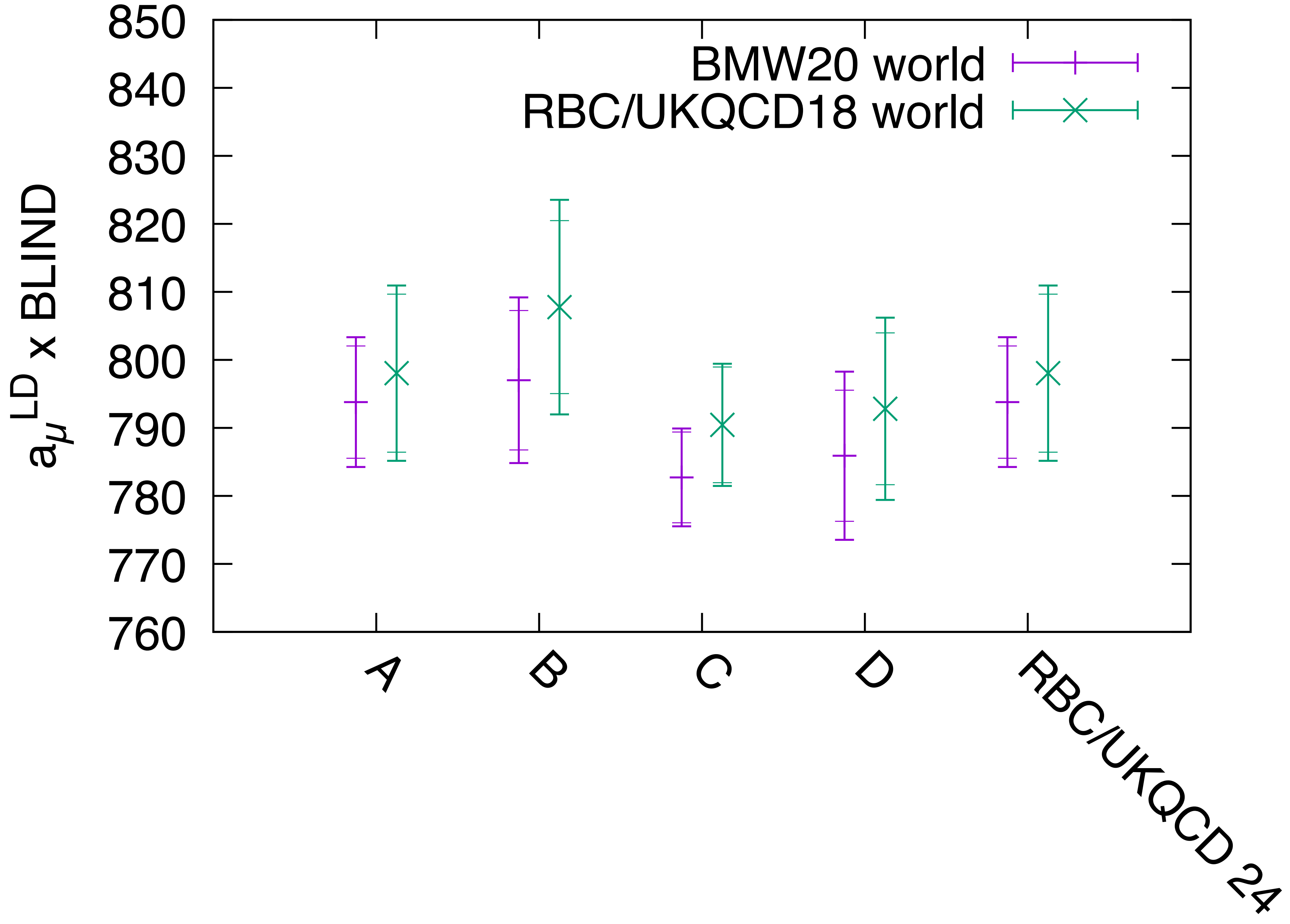


N = 9 vs Inclusive

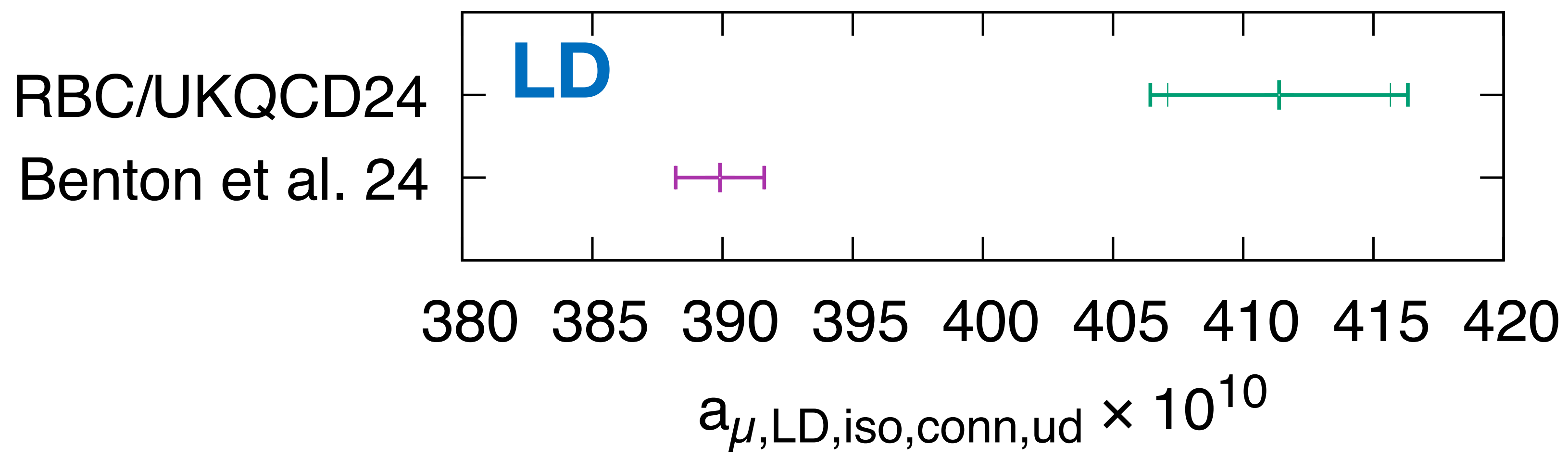


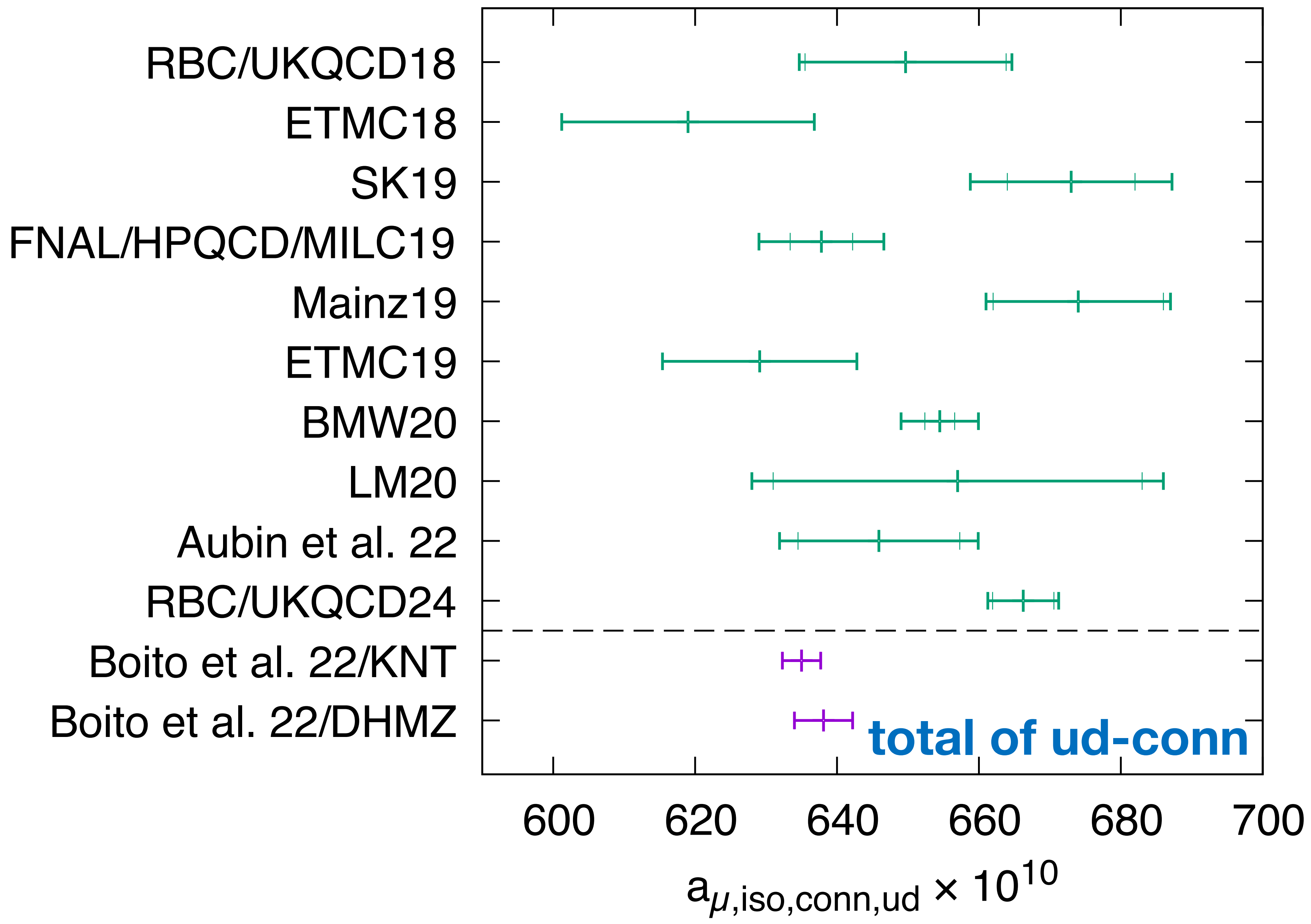
Unblind and result

- Relative unblinding



- Comparison with other results





Unblind and result

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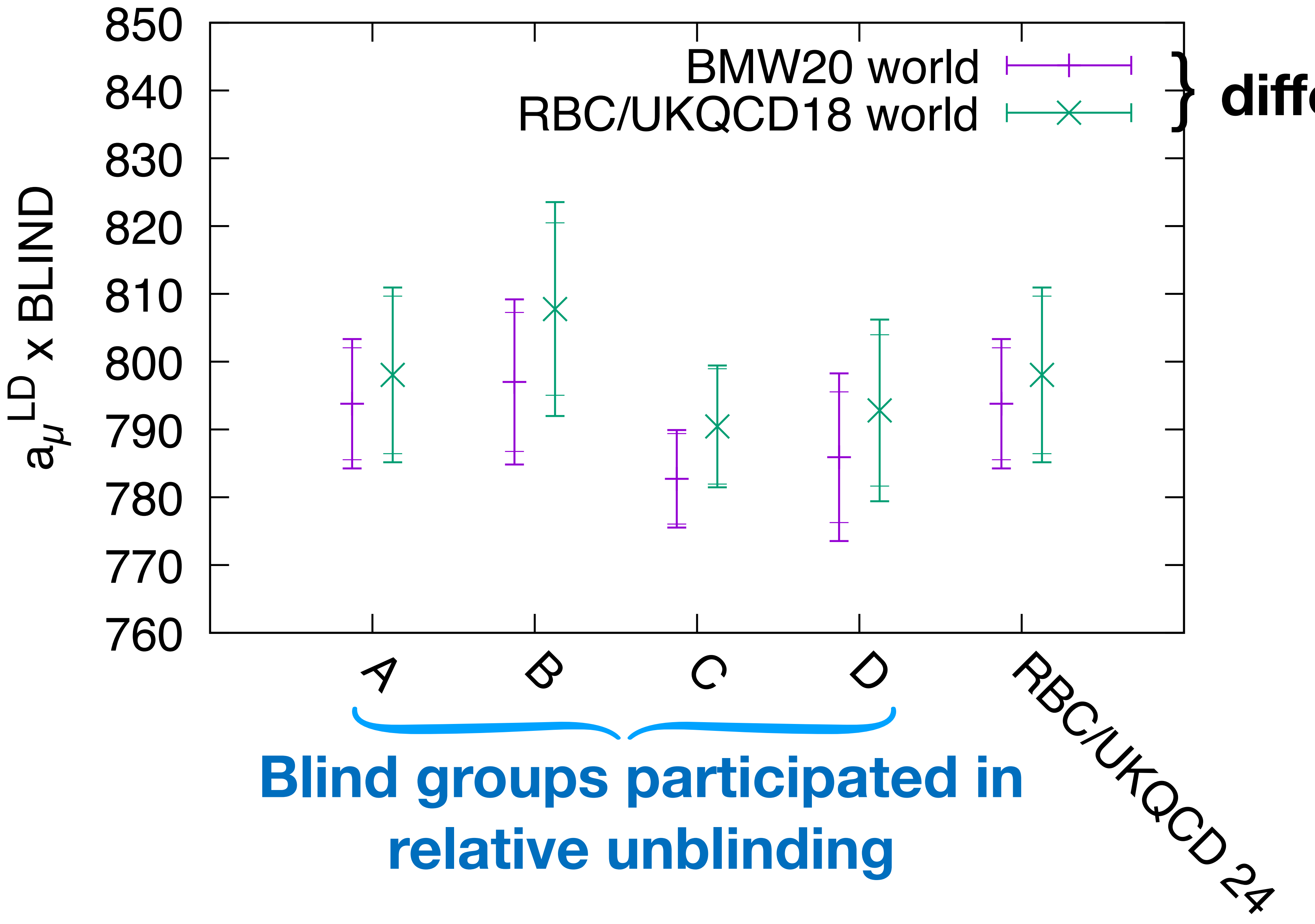

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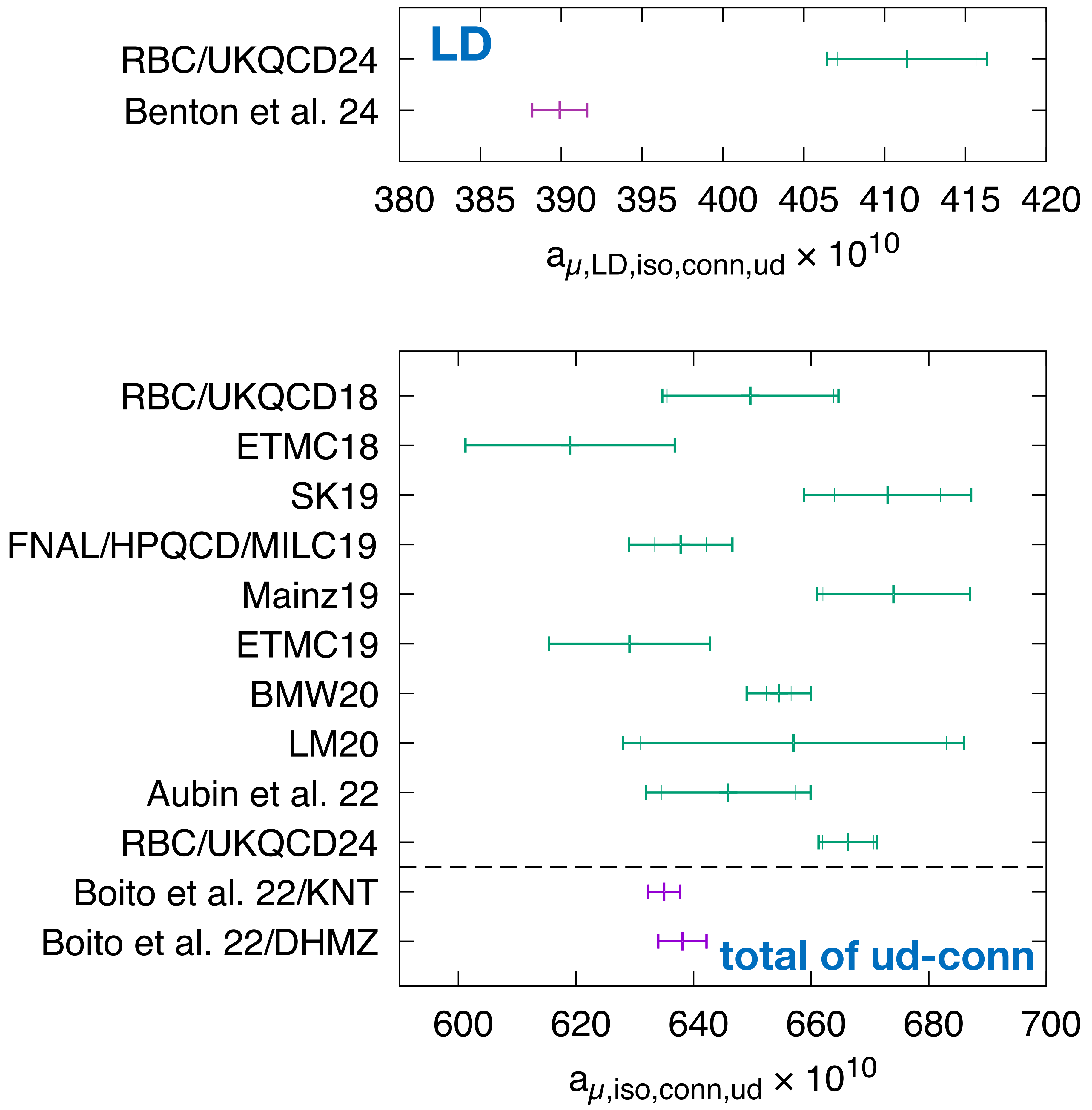
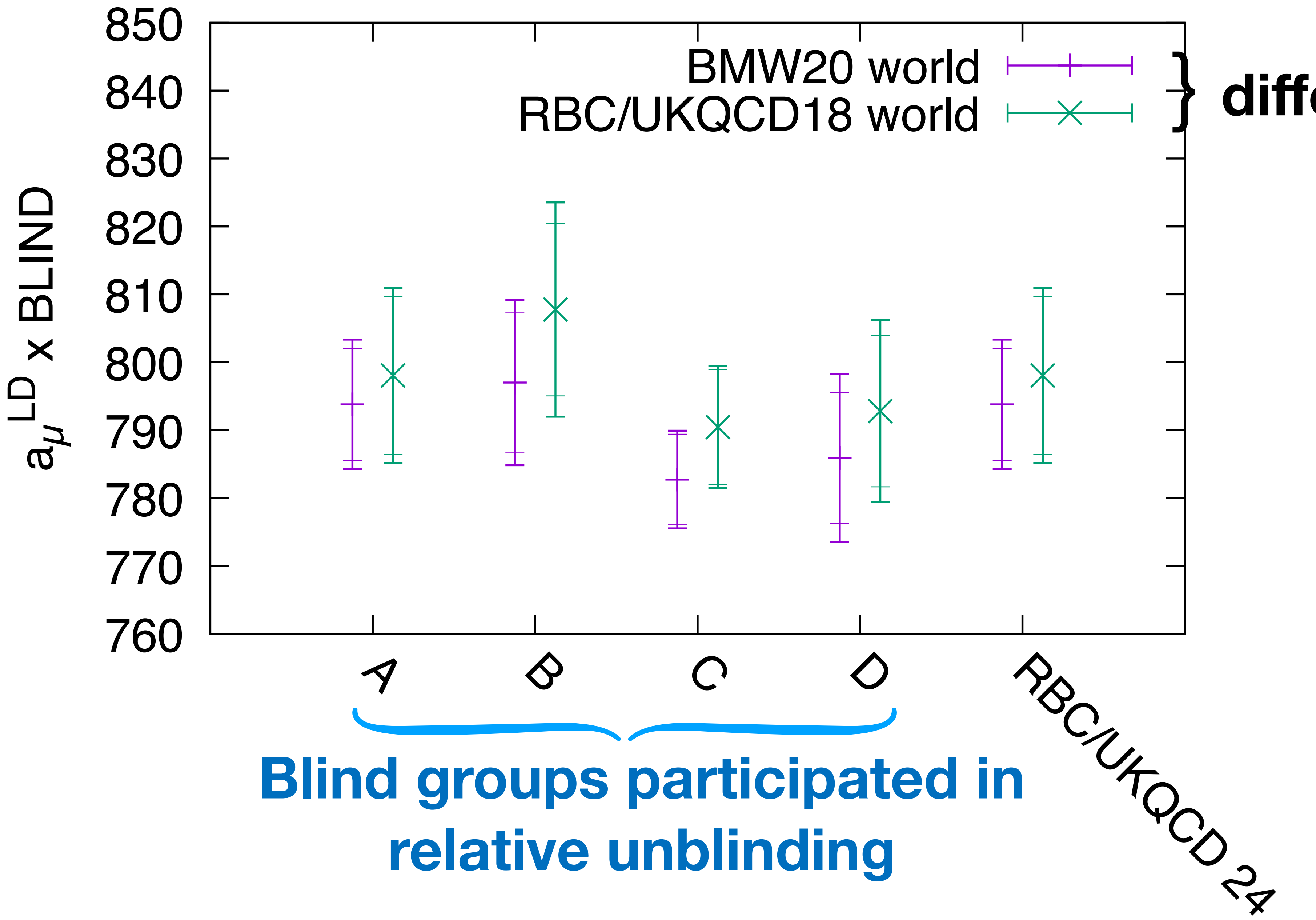


Unblind and result

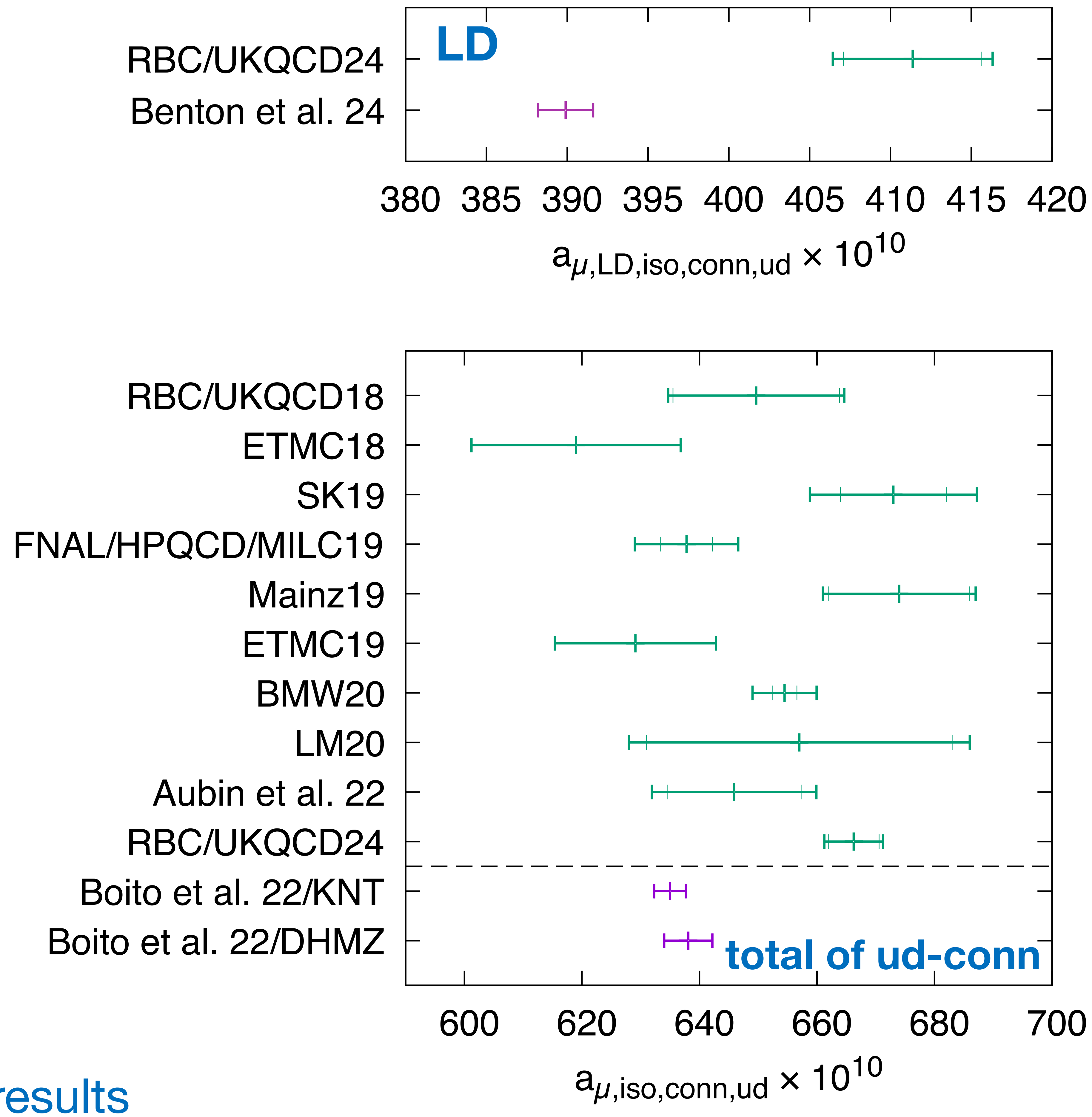
- Relative unblinding



} **different scale setting**

Blind groups participated in relative unblinding

- Comparison with other results



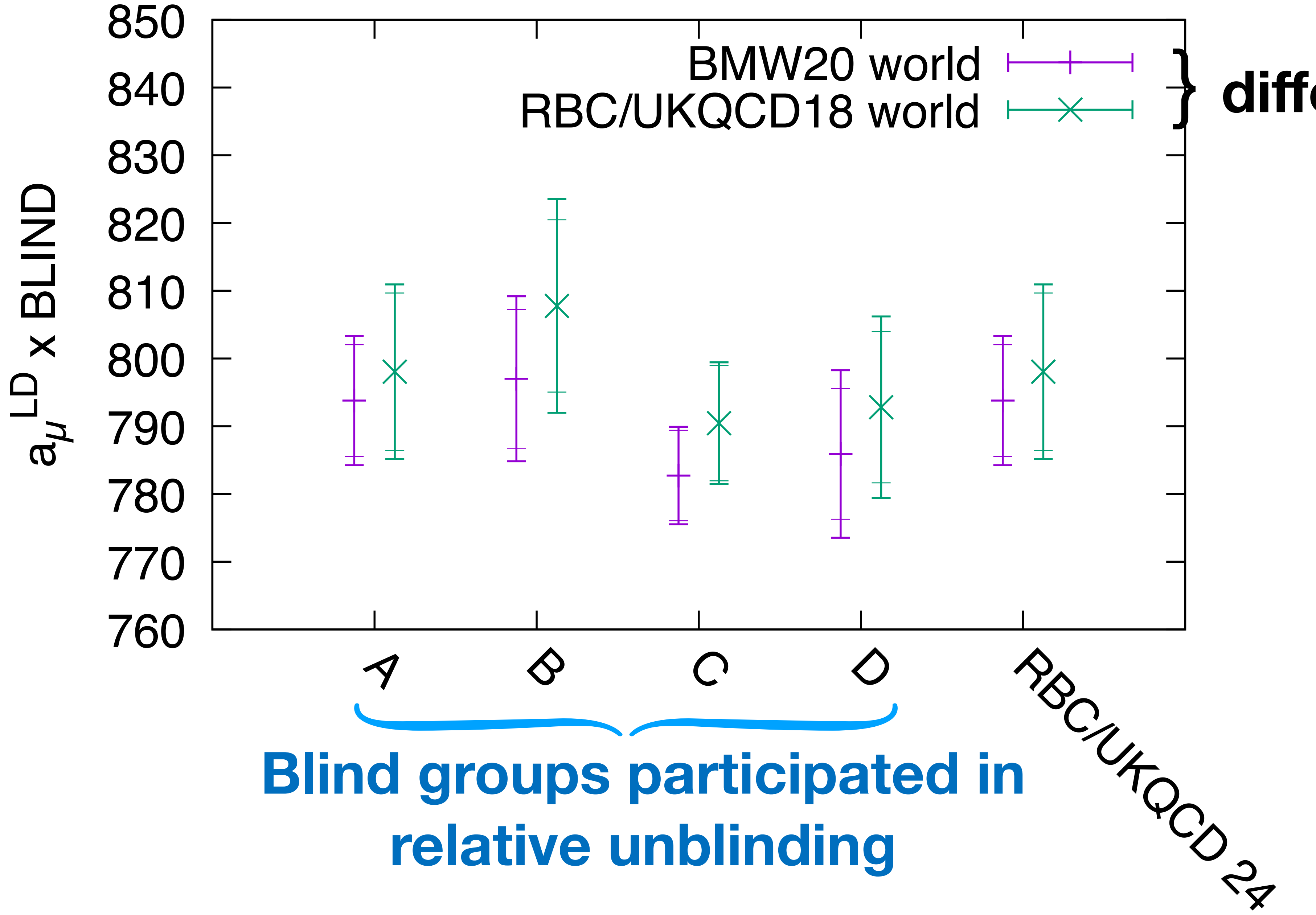
- Absolute unblinding (BMW20 World)

$$a^{LD}_{\mu,conn}(ud) = 411.4(4.3)_{stat}(2.4)_{syst}\times 10^{-10}$$

$$a^{LO-HVP}_{\mu,conn}(ud) = 666.2(4.3)_{stat}(2.5)_{syst}\times 10^{-10}$$ ←inclusion of SD & W results

Unblind and result

- Relative unblinding



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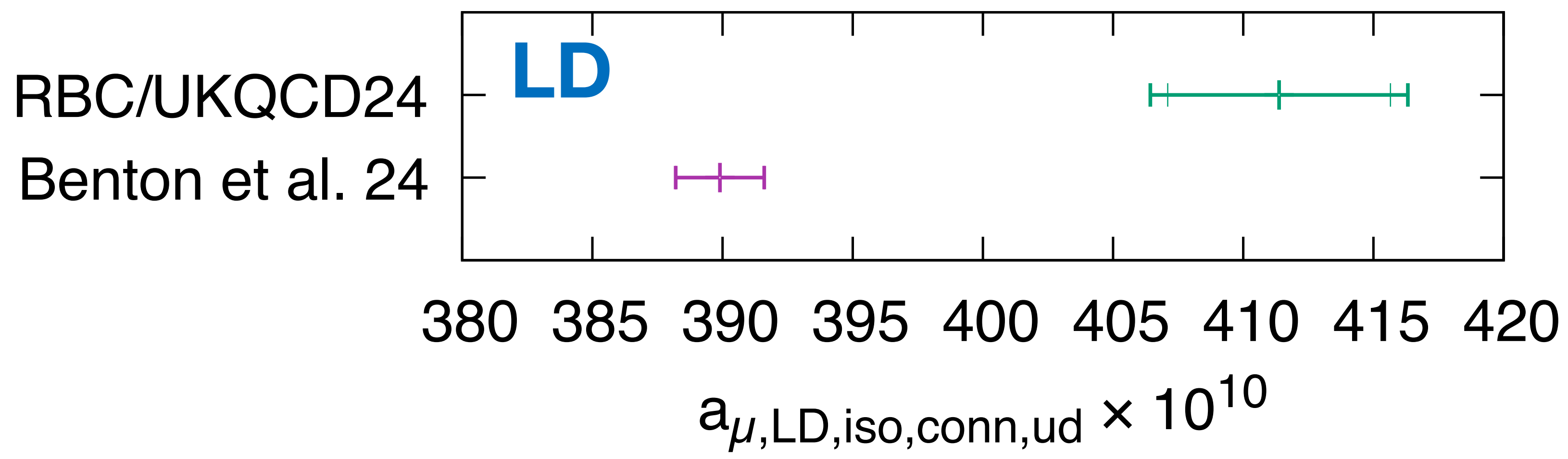

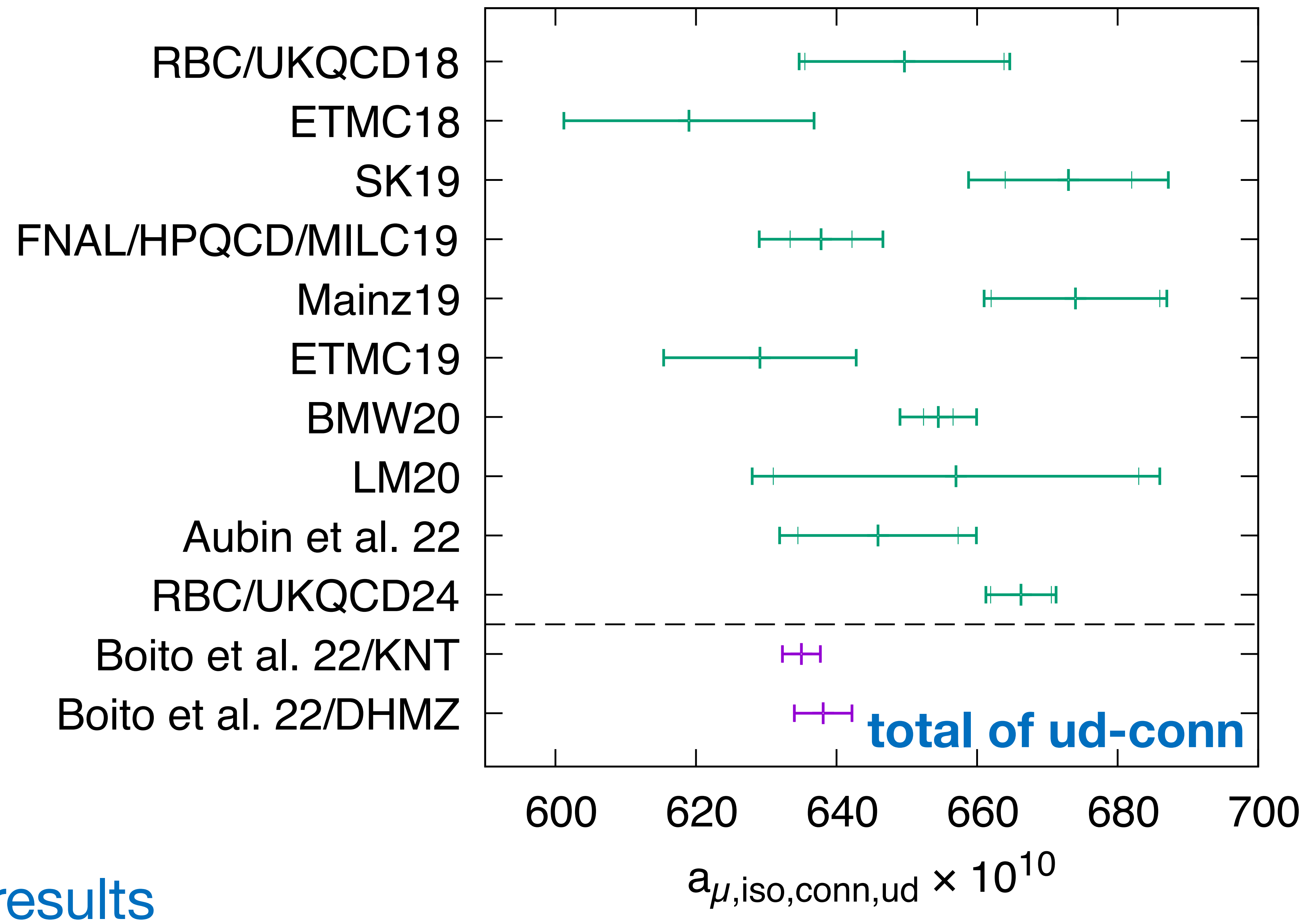

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Similar calculation done by Mainz group (see KEK workshop last month)

Summary

- ππ system related to various remarkable topics
- Scattering lengths (see backup slides for our physical m_π calculation)
- Resonance (ρ & σ)
- K → ππ
  - Long-standing challenge for LQCD
  - SM prediction for ε′: 4x larger error than experiment
  - Working on main error sources: 1. $O(a^2)$ 2. charm-loop, 3. EM/IB correction
- LD HVP contribution to g-2
  - Necessary for LQCD to improve to achieve the precision similar to Fermilab exp
  - Exclusive reconstruction method significantly improves lattice calculation
  - Next steps: 1. increase statistics, 2. disconnected contribution, 3. IB & EM corrections, 4. strange & charm contributions

Backup slides

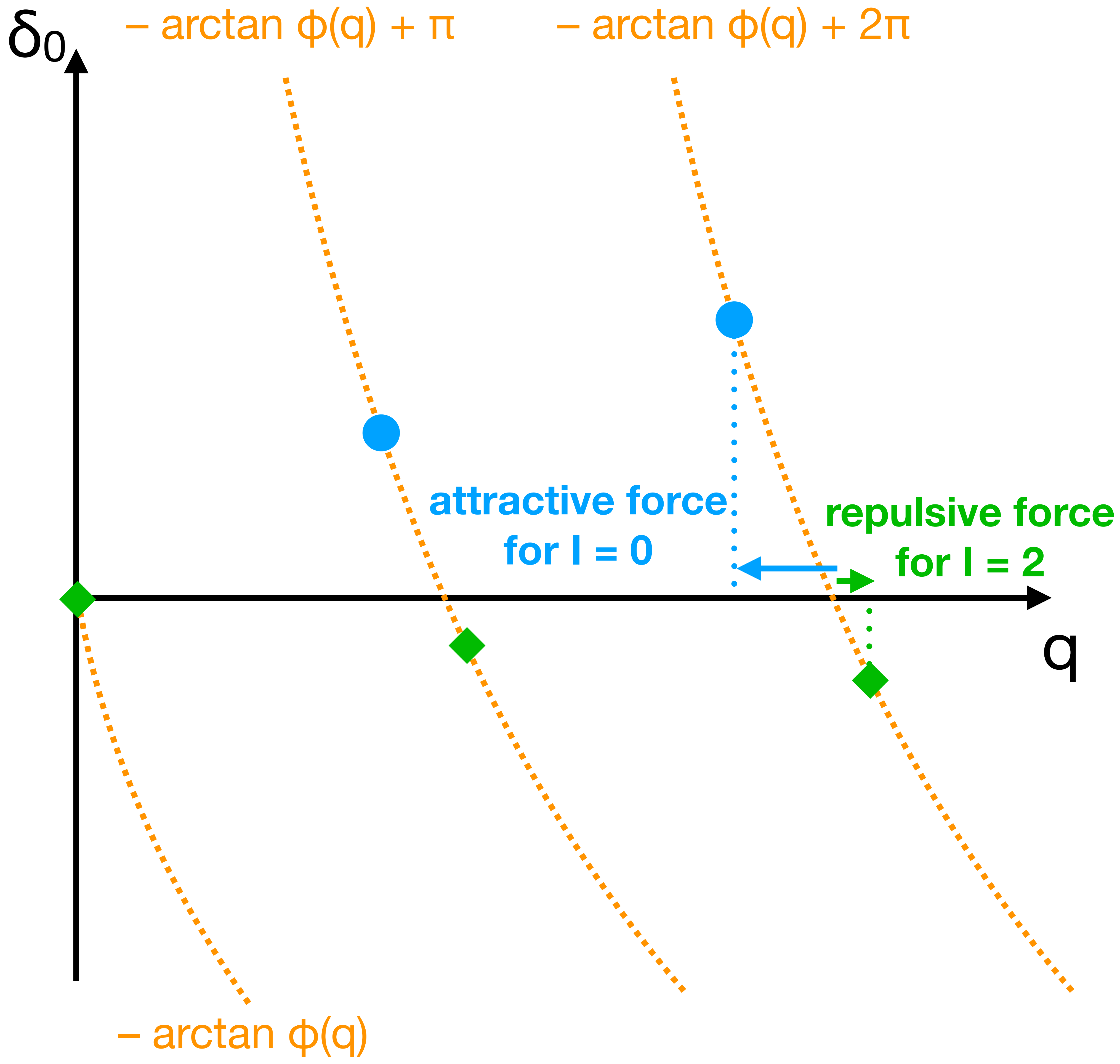
Phase shifts δ_l

- Lüscher 1991 (valid in $2m_\pi < E_{\pi\pi} < 4m_\pi$)

$$\tan\delta_l = -\frac{\pi^{3/2}q}{Z_{00}(1;q^2)} \equiv -\phi(q)$$

$$q = \frac{L}{2\pi}\sqrt{\frac{E_{\pi\pi}^2}{4} - m_\pi^2}$$

$$Z_{00}(s;q^2) = \frac{1}{\sqrt{4\pi}}\sum_{\vec{n}}(|\vec{n}|^2 - q^2)^{-s}$$



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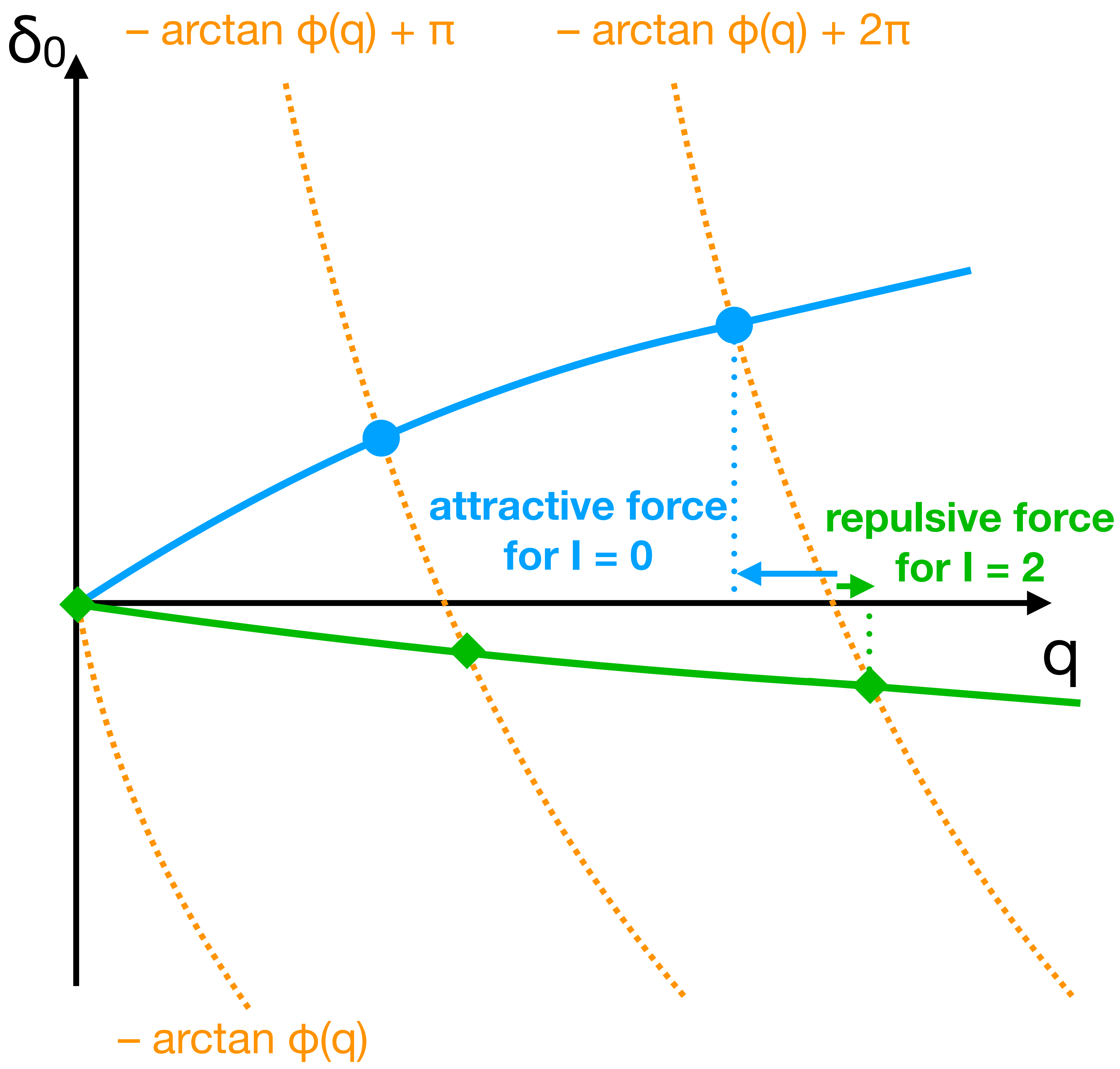

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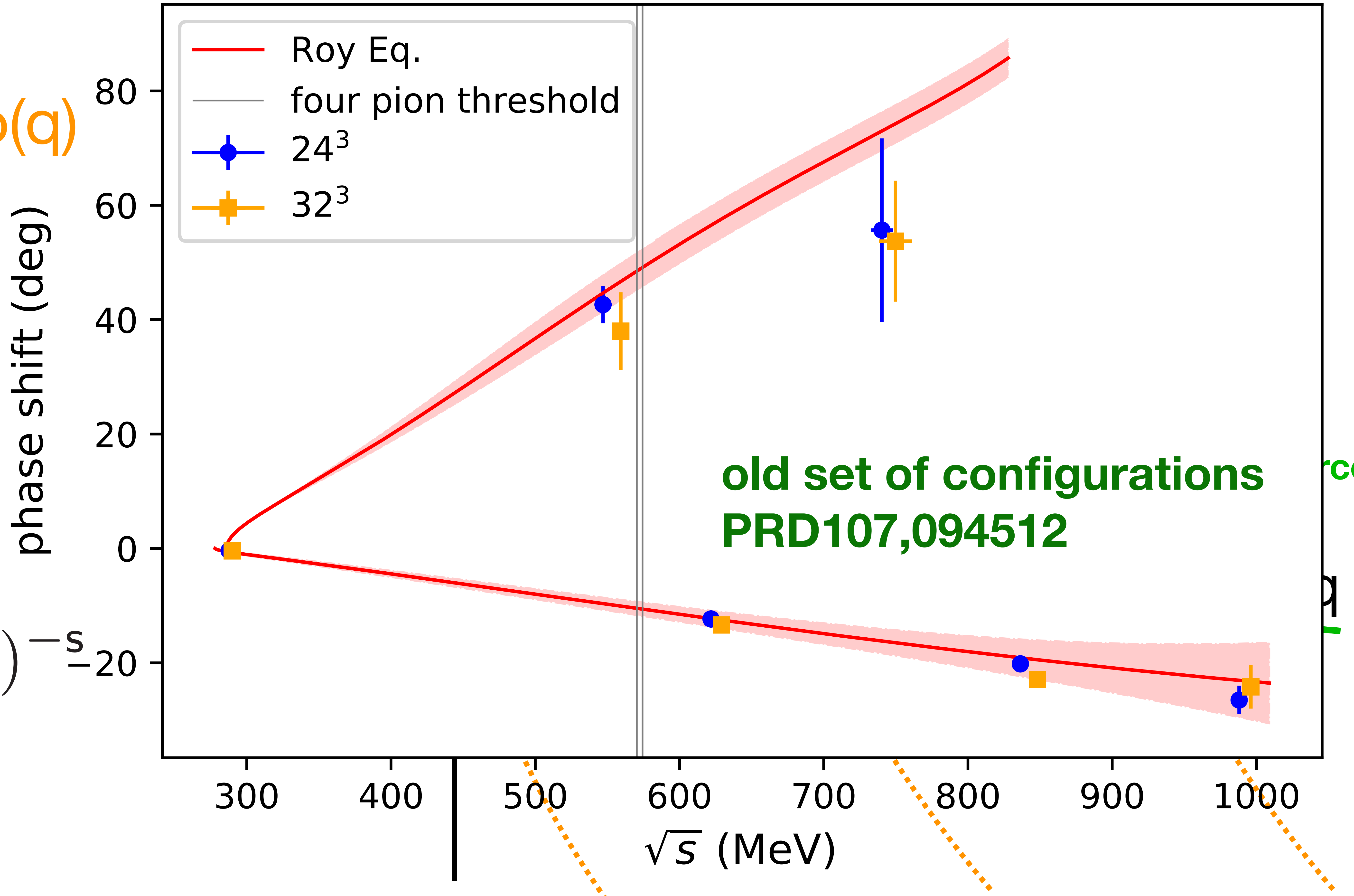


– arctan $\phi(q)$

Scattering lengths

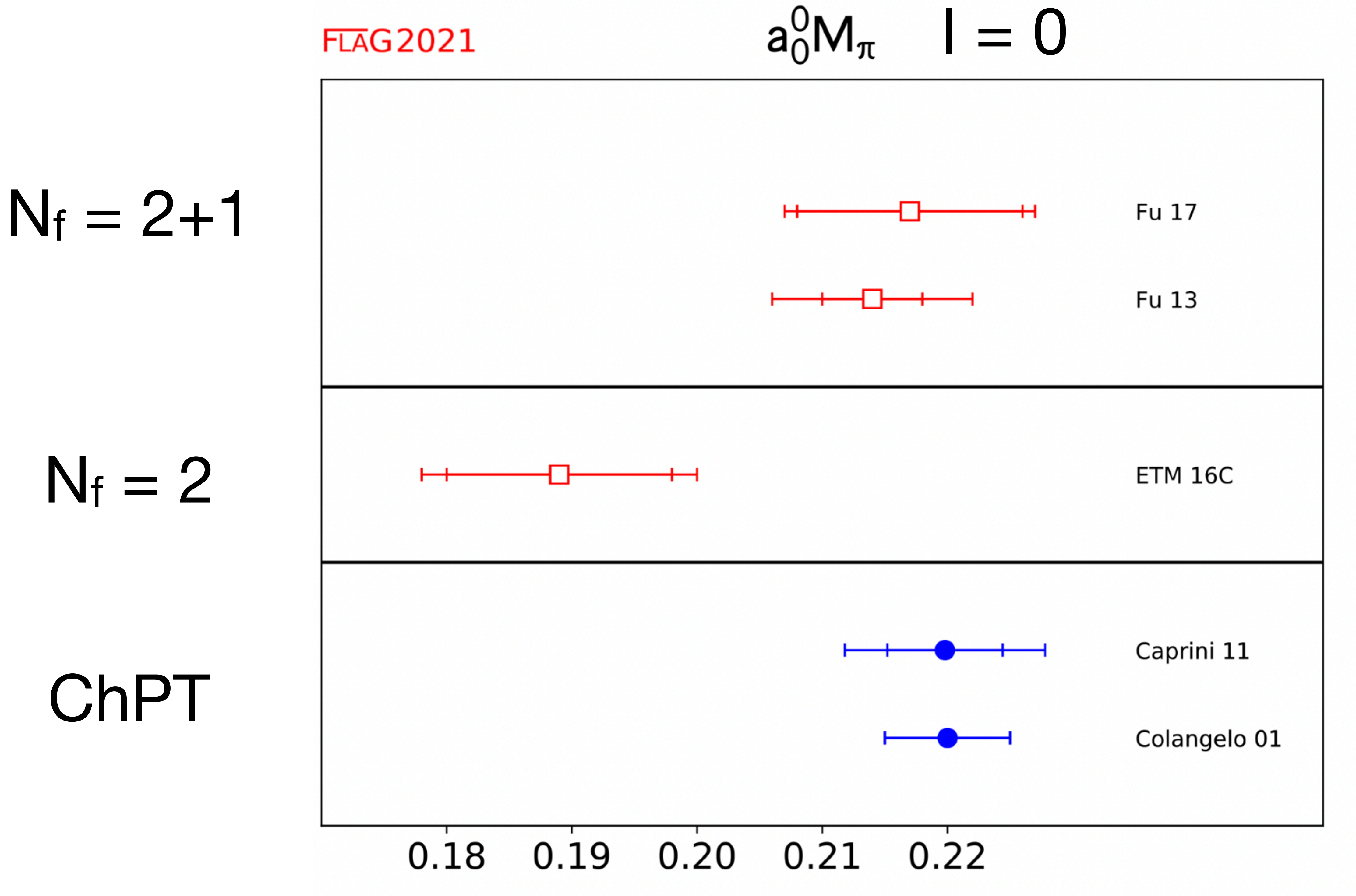
$$k \cot \delta_0^I(k) = \frac{1}{a_0^I} + \frac{1}{2} r_0^I k^2 + O(k^4) \quad \Rightarrow \quad a_0^I \simeq \frac{\tan \delta_0^I(k)}{k}$$

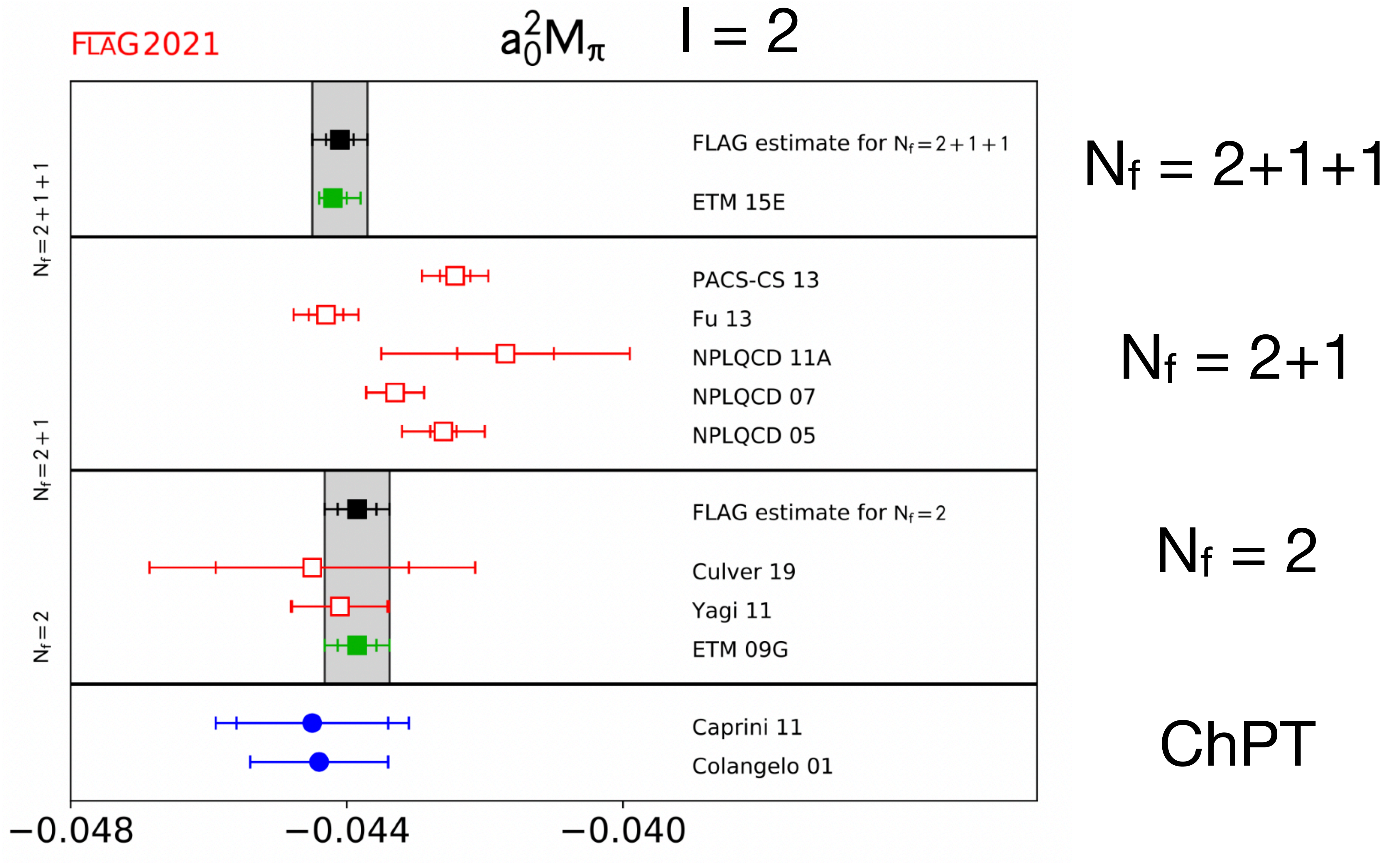
for k of the ground state

scattering length

$$\left(k = \sqrt{\frac{E_{\pi\pi}^2}{4} - m_\pi^2} \right)$$

- FLAG 2021





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scattering length

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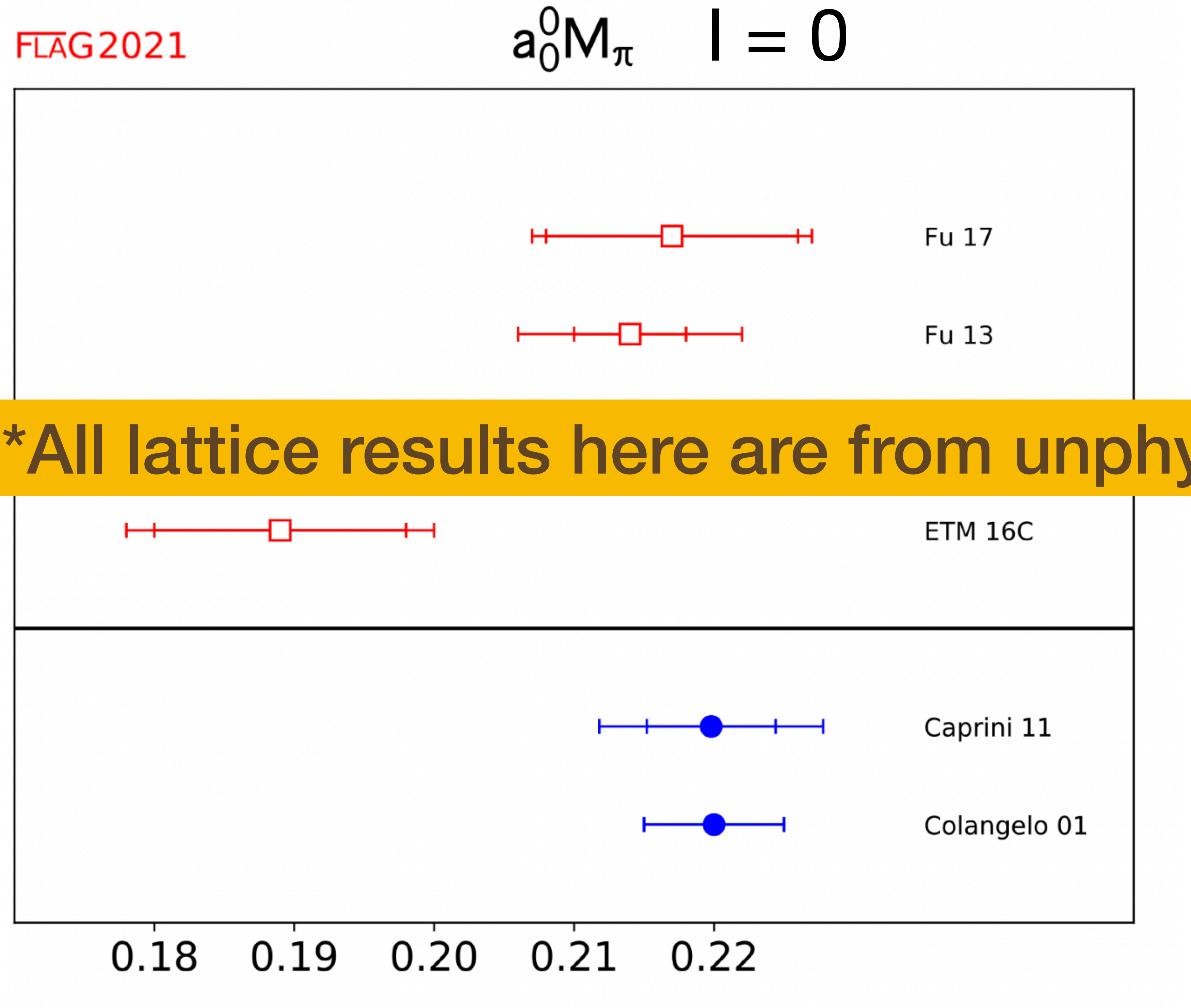
- FLAG 2021

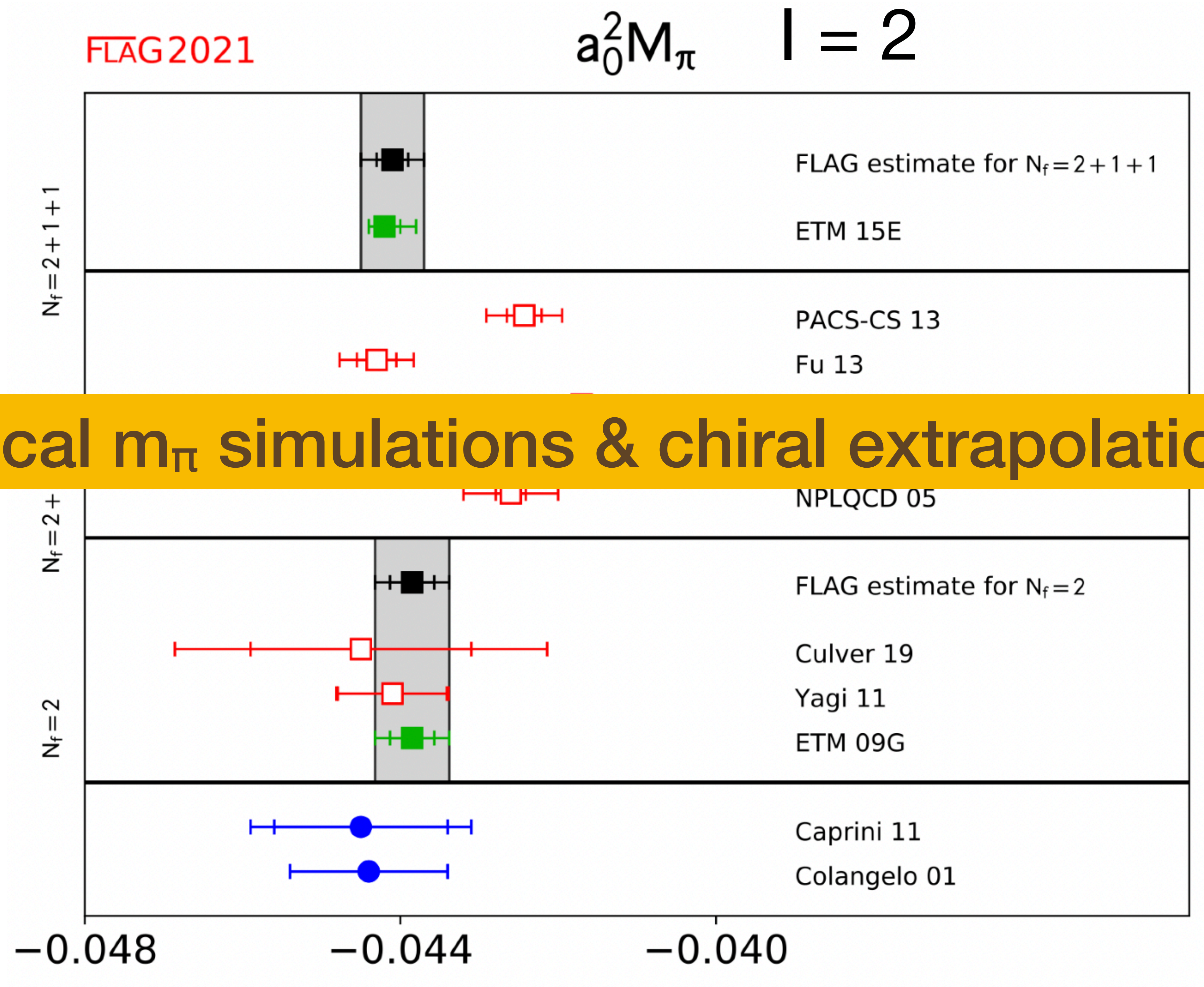
$N_f = 2+1$

$N_f = 2$

ChPT



**All lattice results here are from unphysical m_π simulations & chiral extrapolation



$N_f = 2+1+1$

$N_f = 2+1$

$N_f = 2$

ChPT

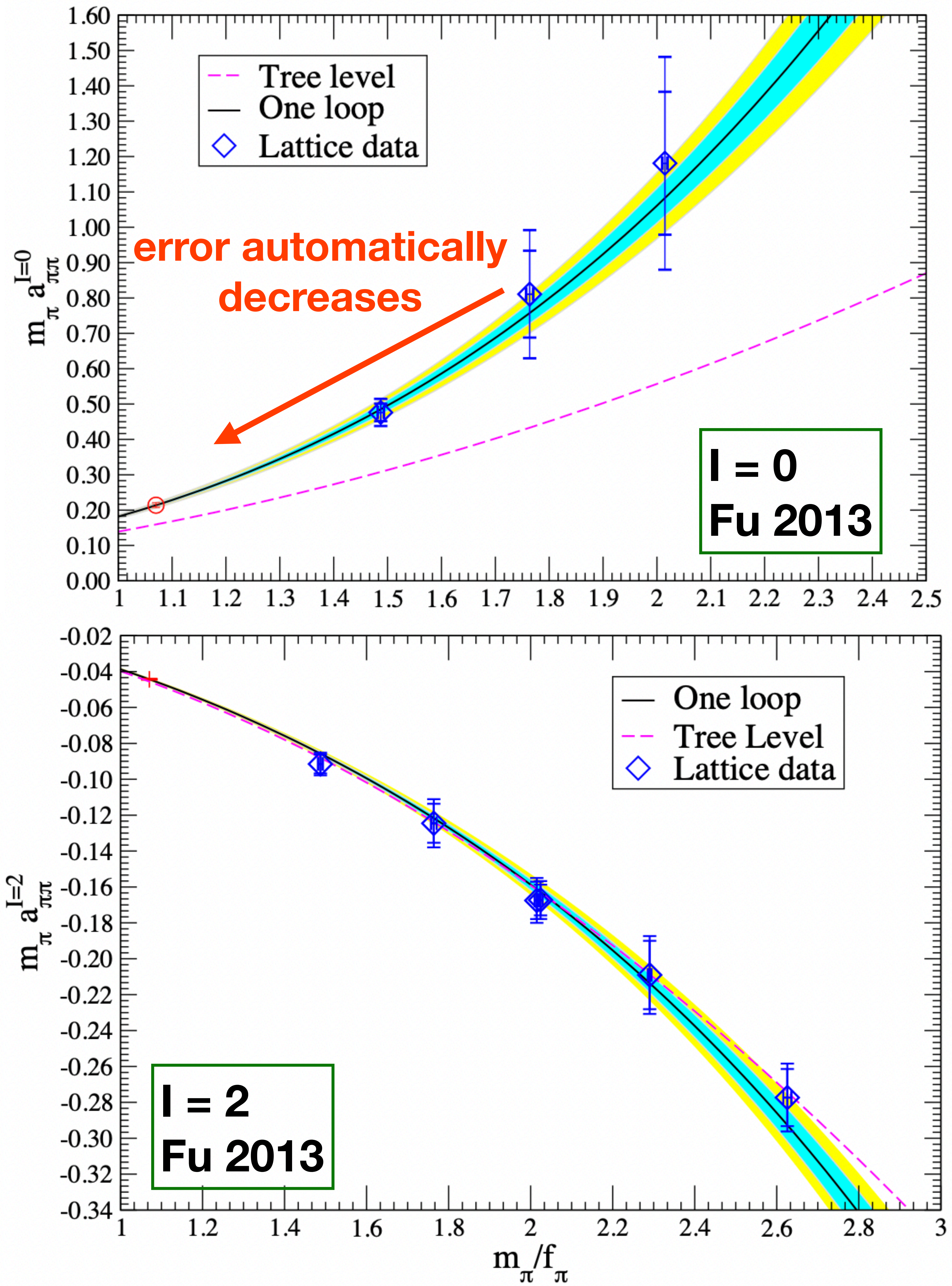
Chiral extrapolation of $a_0^I m_\pi$

- Fit functions (in earlier works using ChPT)

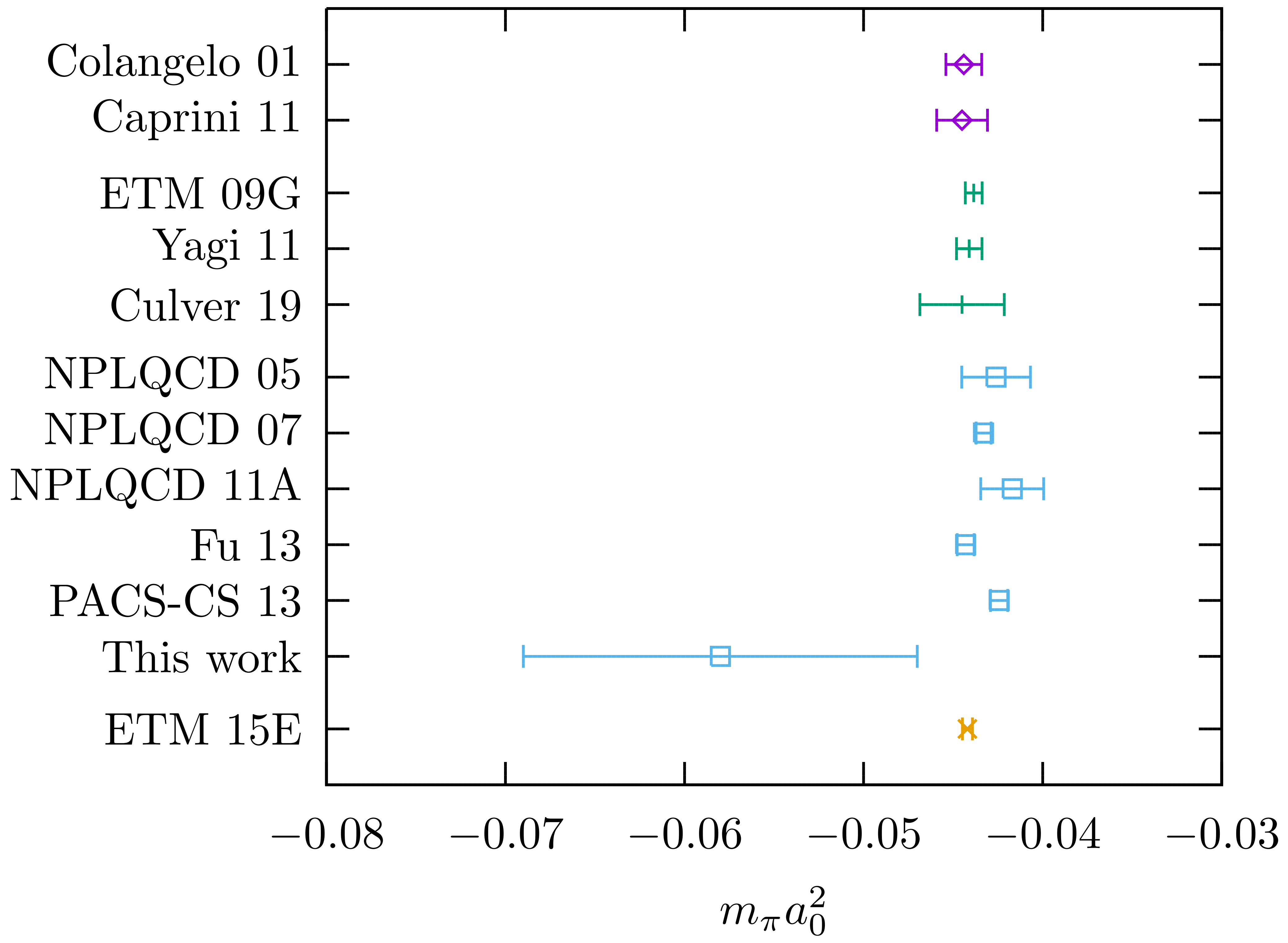
$$m_\pi a_0^0 = \frac{7m_\pi^2}{16\pi f_\pi^2}\left\{1 - \frac{m_\pi^2}{16\pi^2 f_\pi^2}\left[9\ln\frac{m_\pi^2}{f_\pi^2} - 5 - l_{\pi\pi}^0\right]\right\}$$

$$m_\pi a_0^2 = -\frac{m_\pi^2}{8\pi f_\pi^2}\left\{1 + \frac{m_\pi^2}{16\pi^2 f_\pi^2}\left[3\ln\frac{m_\pi^2}{f_\pi^2} - 1 - l_{\pi\pi}^2\right]\right\}$$

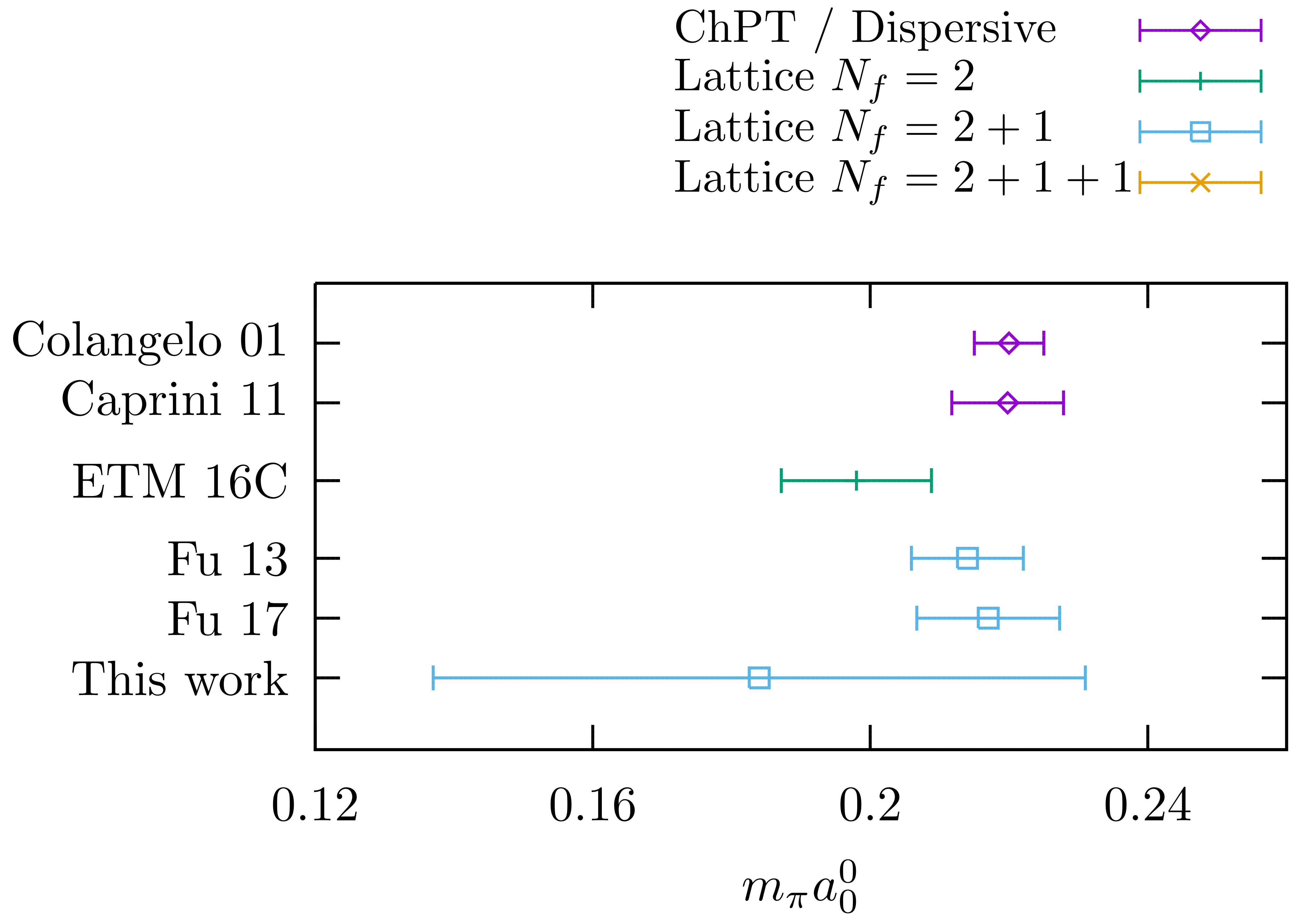
  - with only $l_{\pi\pi}^I$ as the free parameter
  - input m_π/f_π gives precise LO
  - lattice data only contribute to NLO
- Result from physical m_π simulations meaningful
- Ambitious for physical m_π simulations to try to surpass the precision



$a_0^I m_\pi$ with physical m_π from 2023

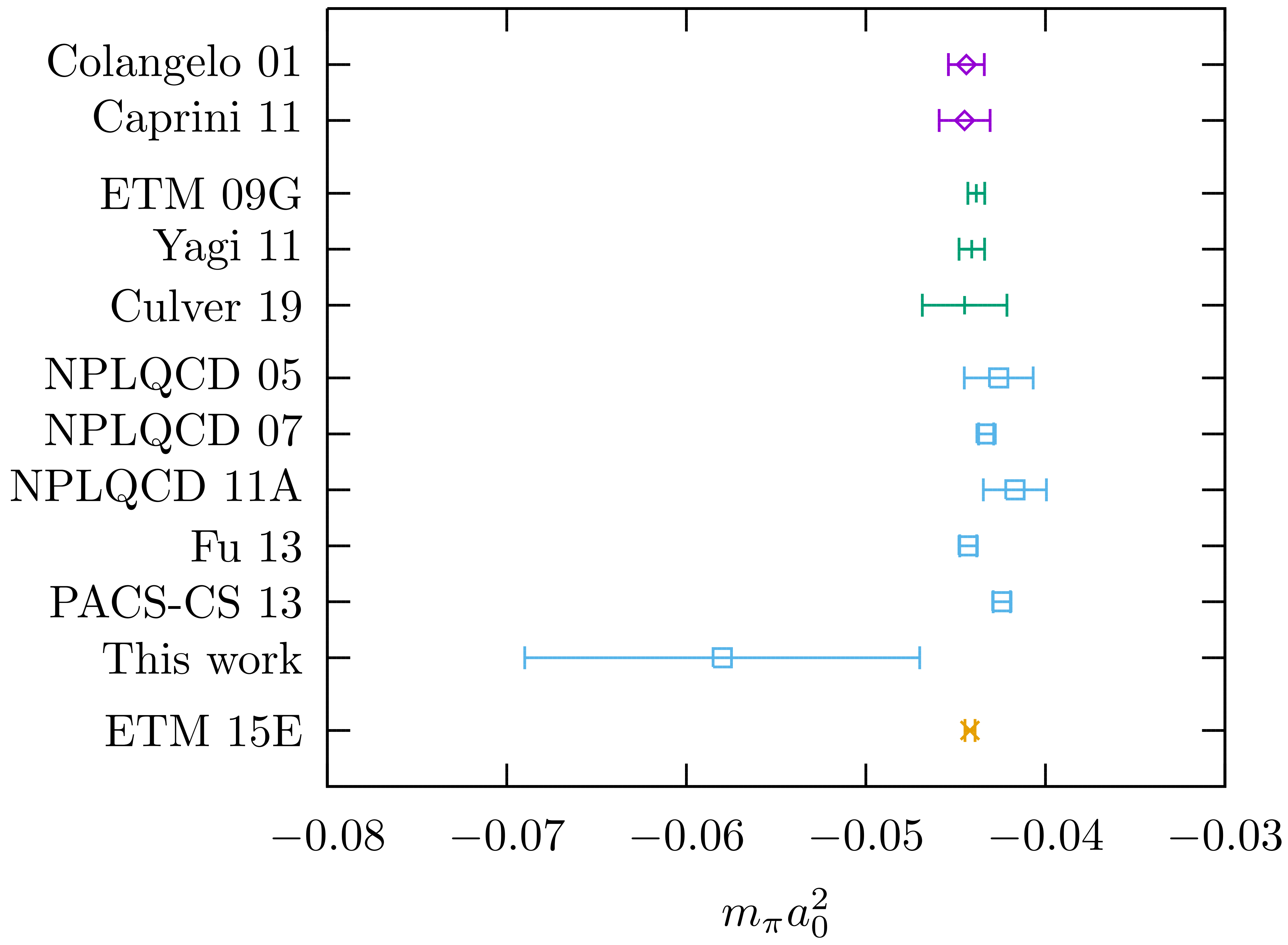


I = 2

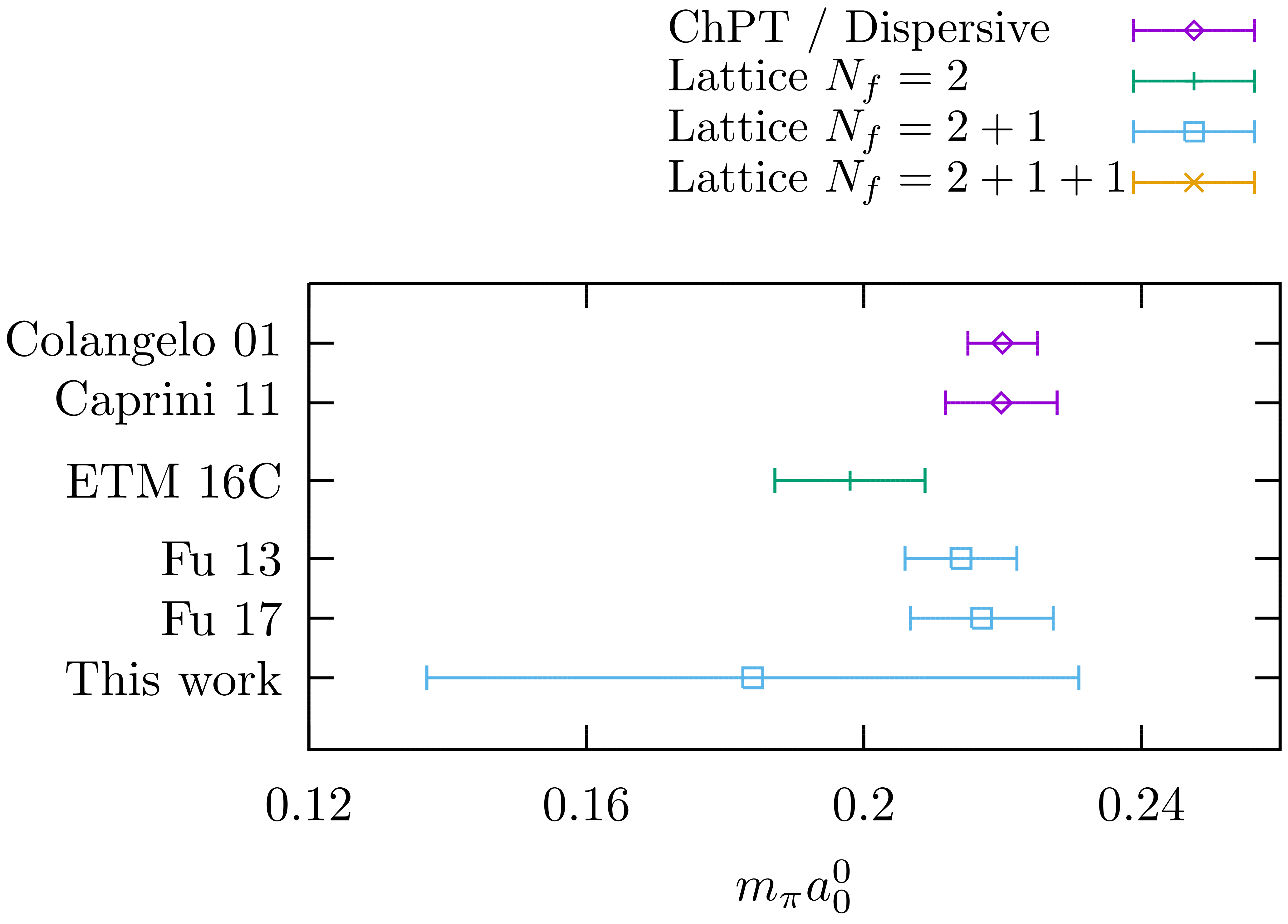


I = 0

$a_0^I m_\pi$ with physical m_π from 2023



I = 2



I = 0

Improving the signal of the ground state crucial

Extra contributions to ππ 2pt functions

- More precise evaluation of 2pt functions on the lattice

$$C_{ab}(t) = \sum_n A_{n,a} A^*_{n,b} e^{-E_n t}$$

– vacuum effect $+\langle O_a \rangle \langle O_b \rangle$ – needs to be subtracted for I = 0

– thermal effect $+\langle \pi | O_a | \pi \rangle \langle \pi | O_b | \pi \rangle e^{-E_\pi T} + \ldots$ – single pion propagating backward

– thermal effect 2 $+\sum_n A_{n,a} A^*_{n,b} e^{-E_n (T-t)}$ – two pions propagating backward – taken into account after GEVP

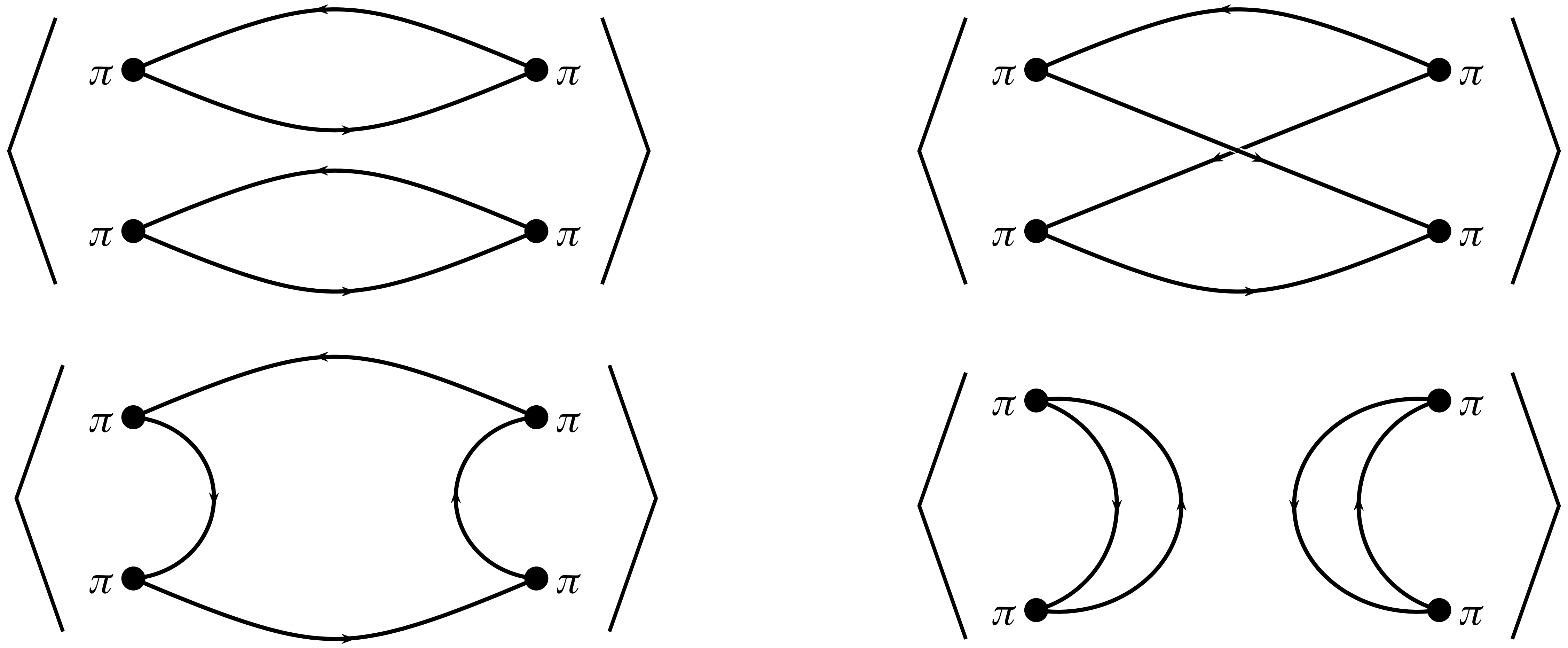
$$+\ldots$$

- Subtraction of vacuum & 1st thermal effects

$$C_{ab}(t) \to C_{ab}(t) - C_{ab}(t+\delta t) = \sum_n A_{n,a} A^*_{n,b} (1 - e^{-E_n \delta t}) e^{-E_n t}$$

Non-interacting ππ 2pt func

- Interacting ππ correlators



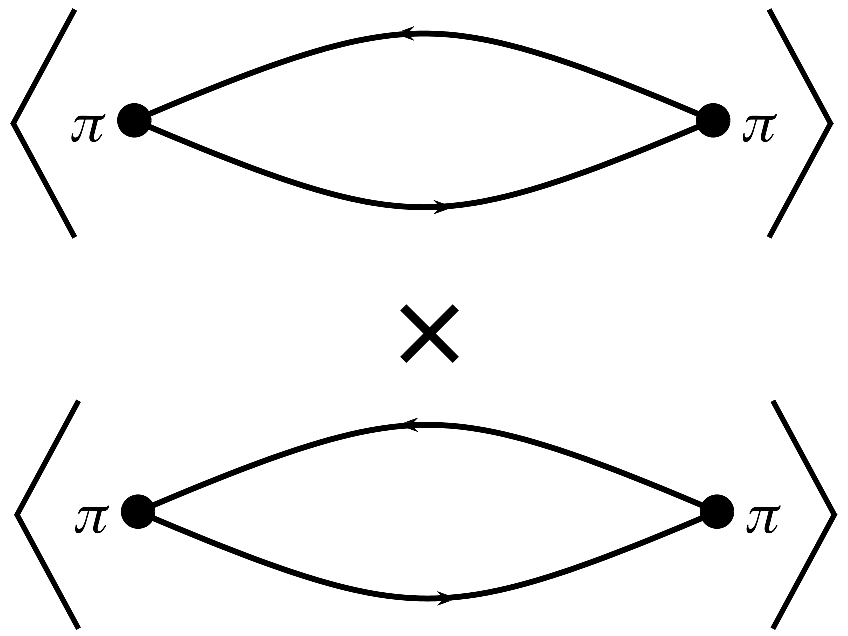
I = 2

I = 0

GEVP

$\sim e^{-E_{\pi\pi,n}t}$

- Non-interacting ones



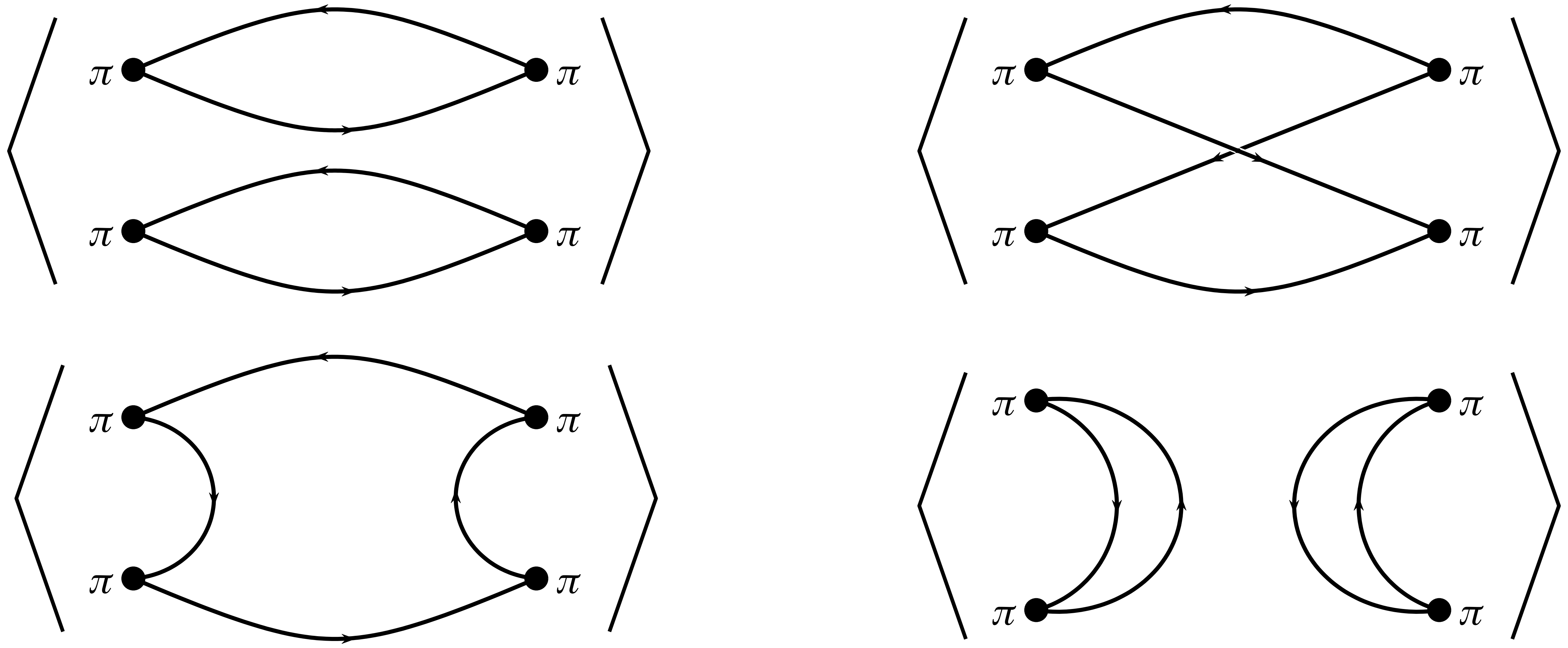
$$\sim e^{-E^{(0)}_{\pi\pi,n}t}$$

$$E^{(0)}_{\pi\pi,n} = 2E_{\pi,n} = 2\sqrt{m_\pi^2 + (2\pi/L)^2 n}$$

same value, but statistical correlation not maximized

Non-interacting ππ 2pt func

- Interacting ππ correlators

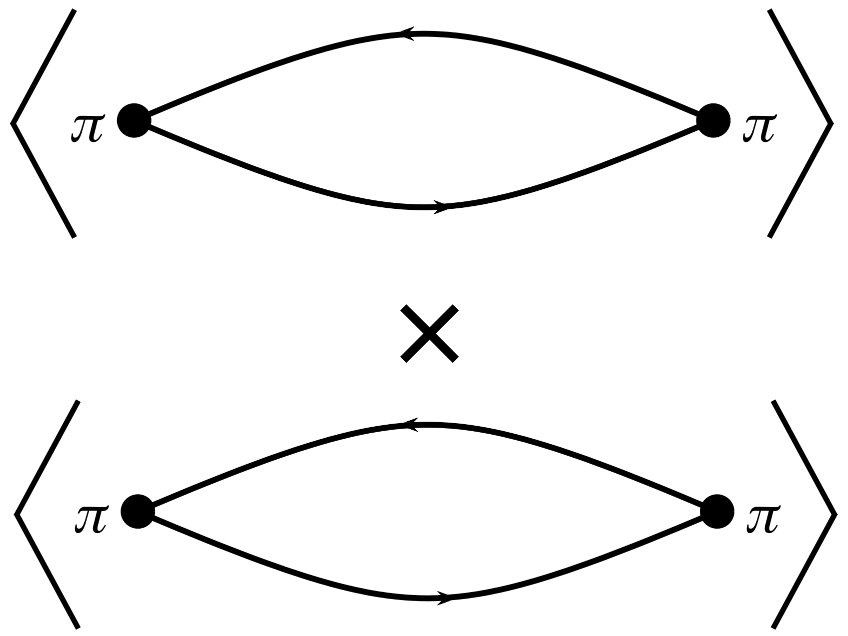


I = 2

I = 0 GEVP $\sim e^{-E_{\pi\pi,n} t}$

similar values
significant correlation

- Non-interacting ones



$$\sim e^{-E^{(0)}_{\pi\pi,n} t}$$

$$2E_{\pi,n} = 2\sqrt{m_\pi^2 + (2\pi/L)^2 n}$$

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Non-interacting ππ 2pt func

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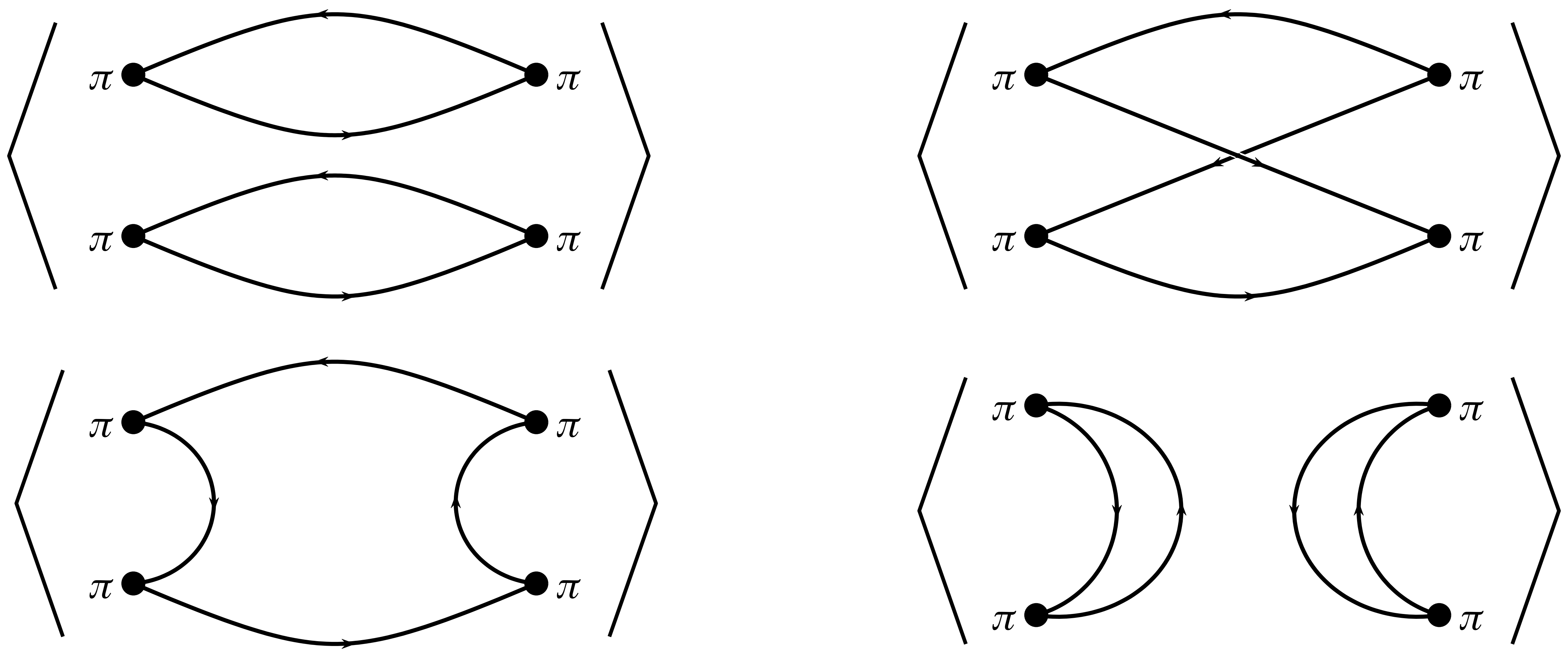

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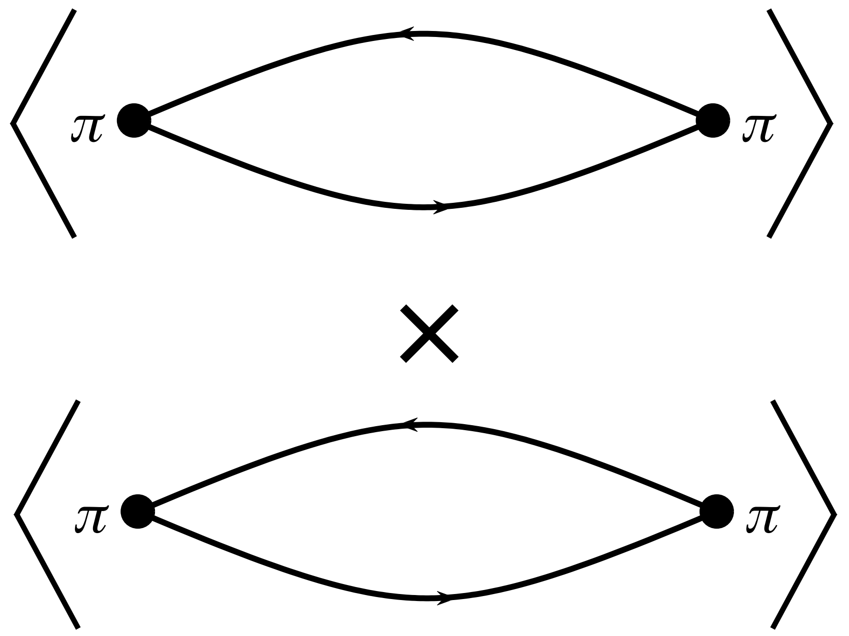


$\sim e^{-E^{(0)}_{\pi\pi,n}t}$

$$E^{(0)}_{\pi\pi,n} = 2E_{\pi,n} = 2\sqrt{m_\pi^2 + (2\pi/L)^2 n}$$

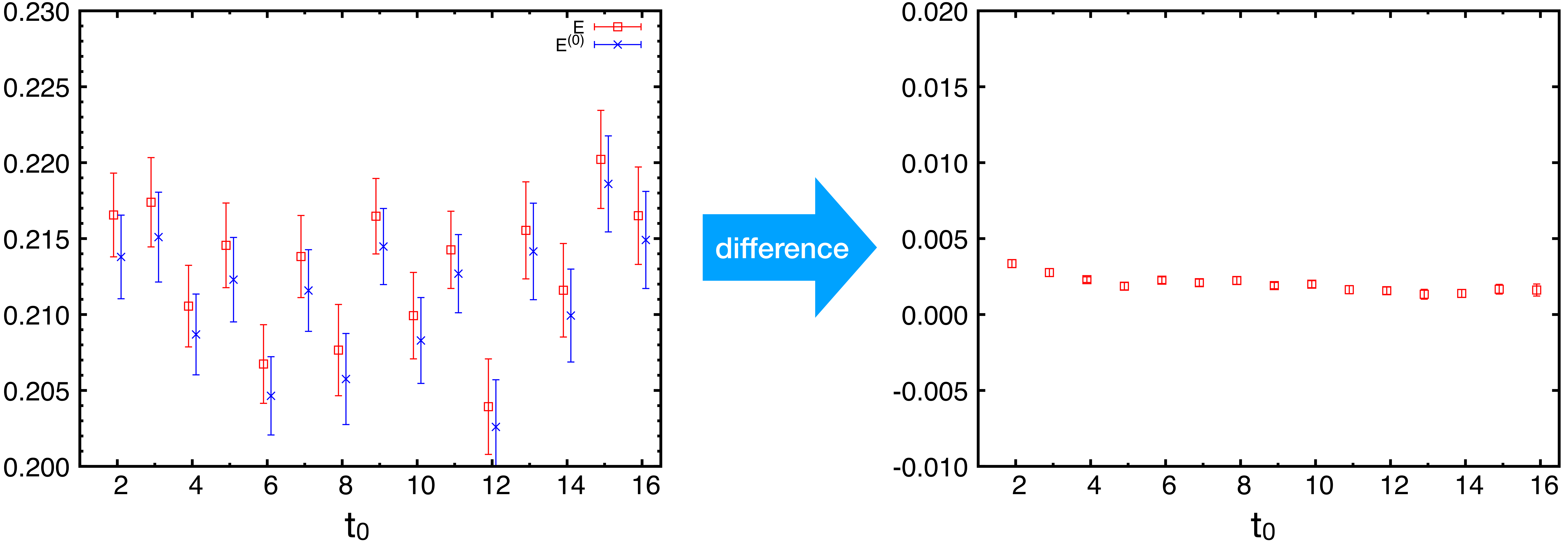
same value, but statistical correlation not maximized

similar values
significant correlation

→ **ratio $\sim e^{-\Delta E_{\pi\pi,n}t}$**

more precise

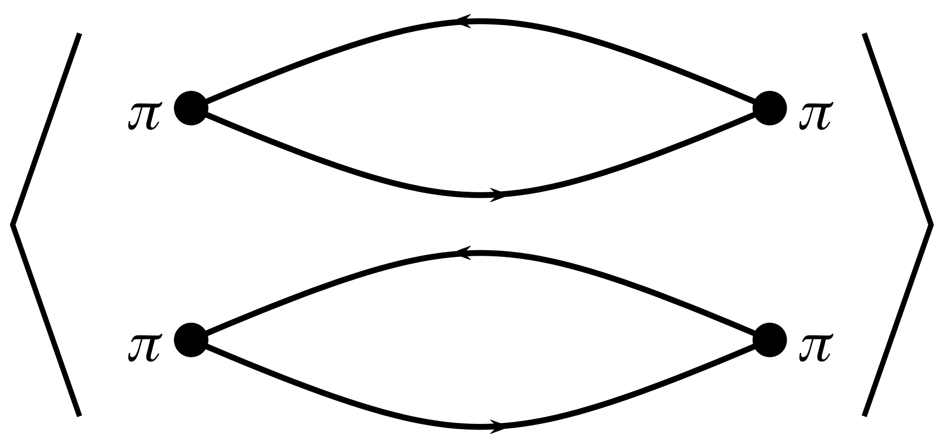
$E_{\pi\pi,n=0}$ vs $\Delta E_{\pi\pi,n=0}$



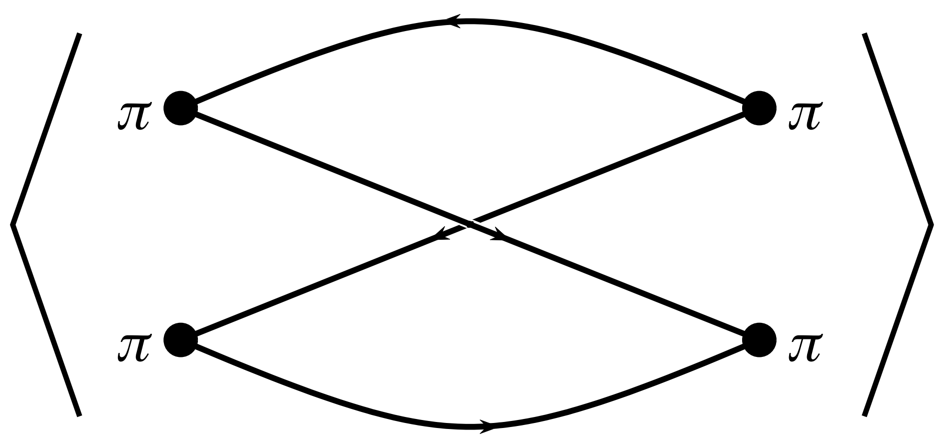
- I = 2, 32^3x64
- Error drastically decreased
- 107 configurations (data in 2023)

Translation average

$$C(t) = \frac{1}{N_{t_{src}}} \sum_{t_{src}} \langle O_{\pi\pi}(t + t_{src}) O_{\pi\pi}(t_{src})^\dagger \rangle$$



D

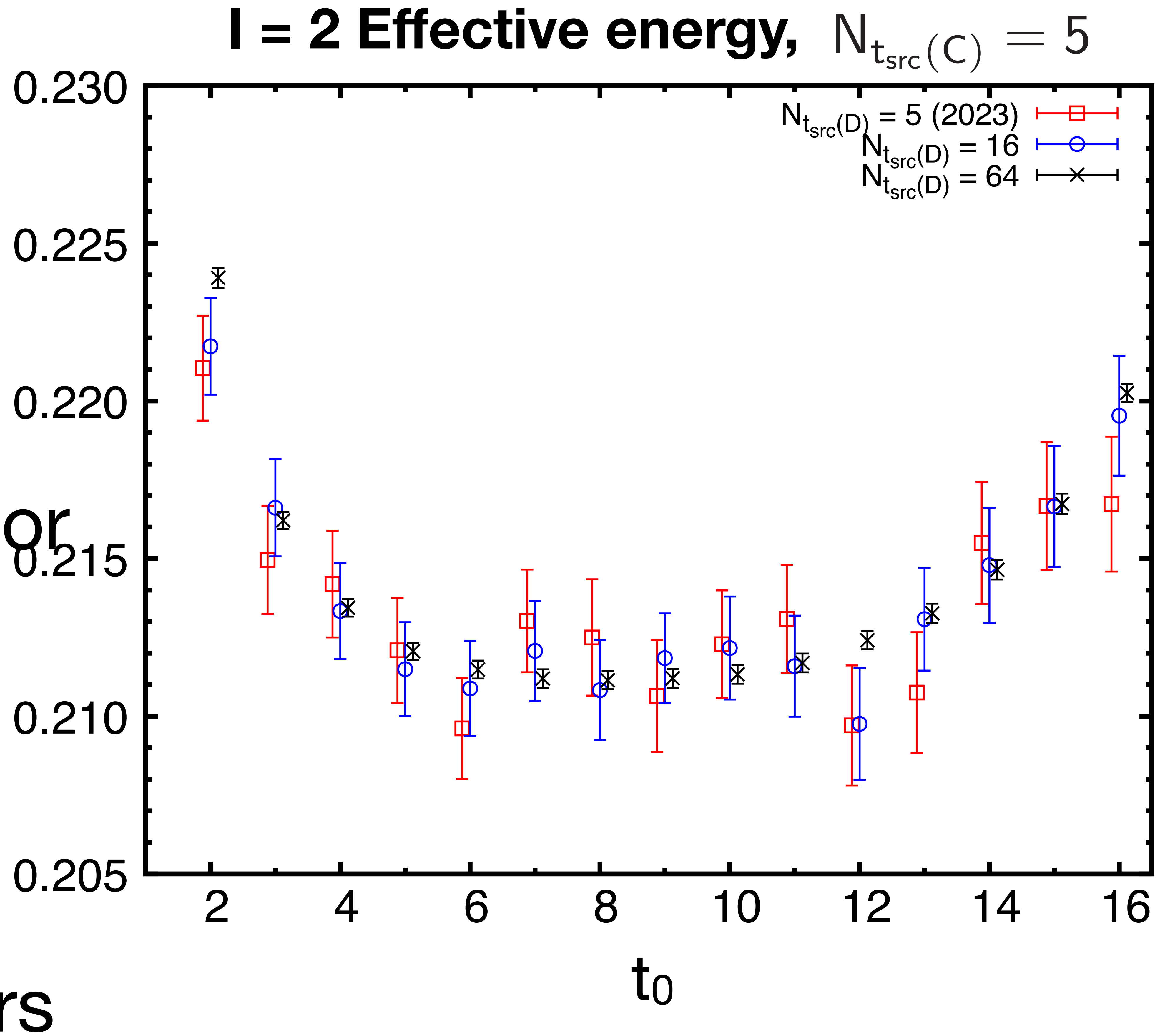


C

- Average can be taken diagram by diagram with different Ntsrc
- No cost to increase N_{tsrc} for D but C is pretty expensive
- D is dominant for I = 2 signal & noise
- Increasing N_{tsrc} for D is interesting

N_{tsrc}-dependence

- 236 confs
- Very small error for $N_{tsrc}(D) = 64$
  - Effective energy $\sim \ln \frac{C(t)}{C(t+1)}$
  - Correlation b/w numerator & denominator enhanced when average is taken every time translation
- Now error from C diagram significant
  - Ntsrc(C) → 64 for ππ(000) & ππ(001) oprs (from next slides)



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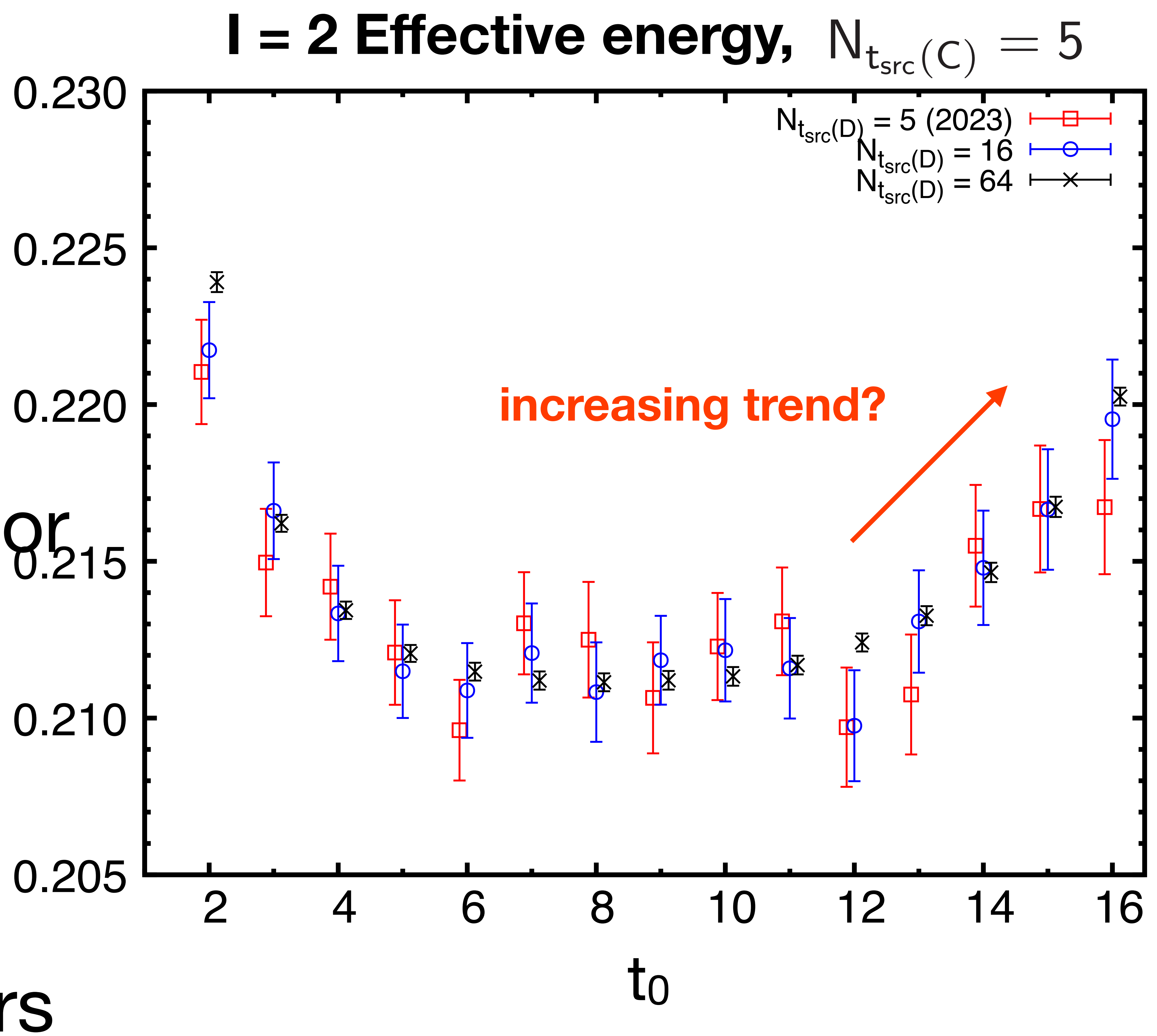


Two-pion backward propagating effect

- Subtraction of constant artifacts (vacuum & thermal effects)

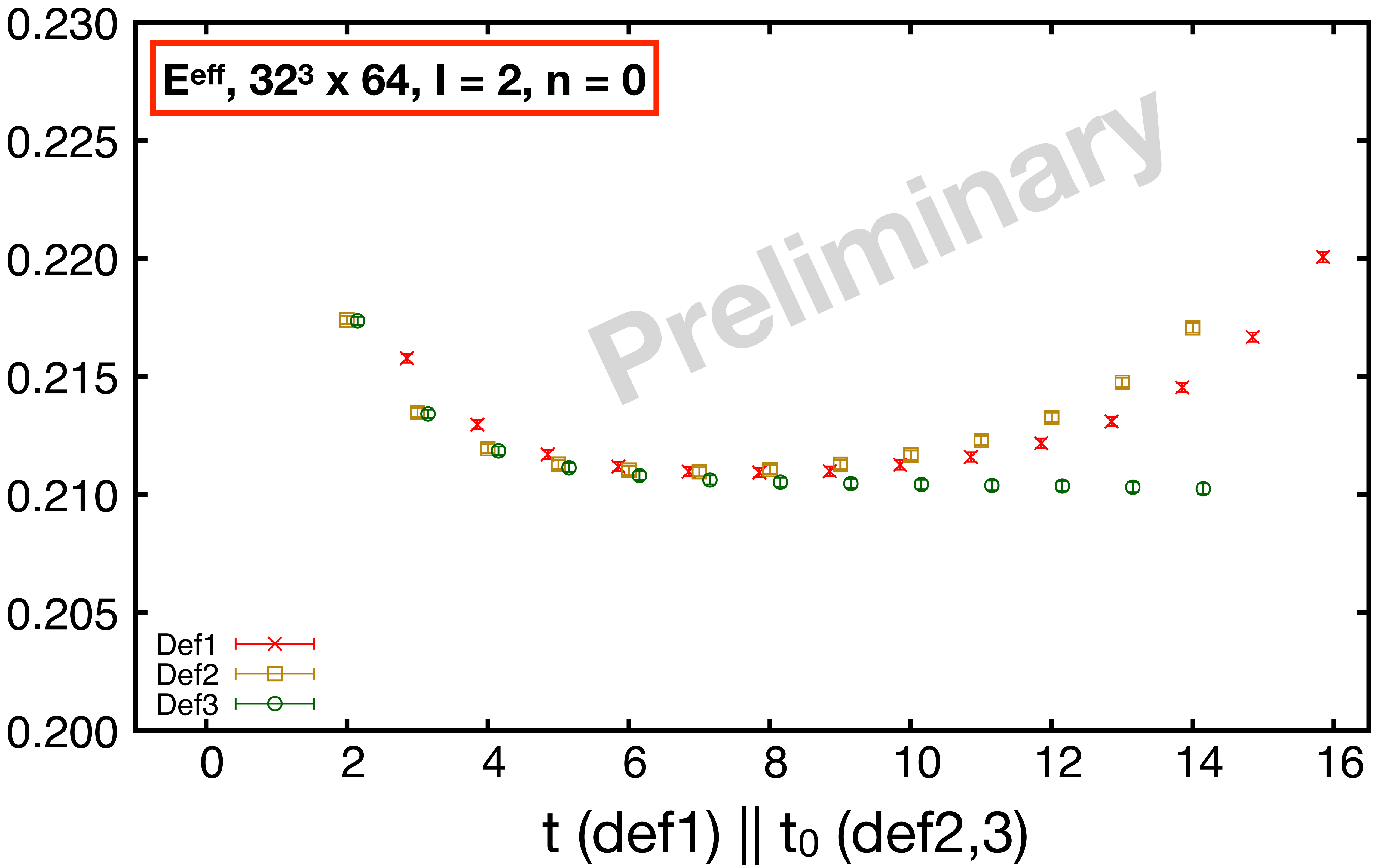
$$C_{ab}(t) \to C_{ab}(t) - C_{ab}(t+\delta_t) = \sum_n A_{n,a}A^*_{n,b}(1-e^{-E_n\delta_t})e^{-E_n t}$$

- GEVP eigenvalue

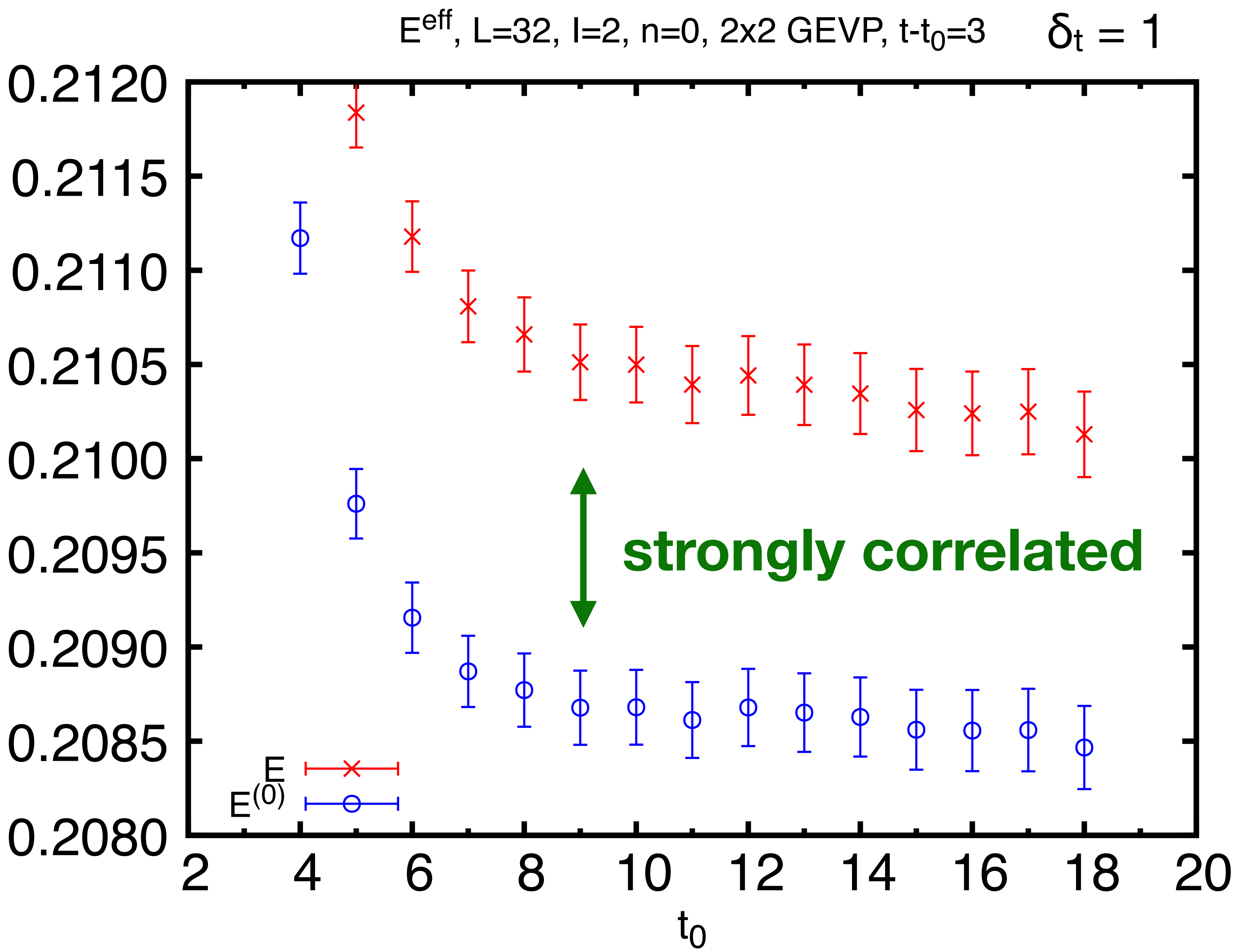
$$\lambda_n(t,t_0) \to \frac{e^{-E_n t} - e^{-E_n(T'-t)}}{e^{-E_n t_0} - e^{-E_n(T'-t_0)}} \qquad \textbf{(*)}$$

$$(T' = T - 2\Delta - \delta_t)$$

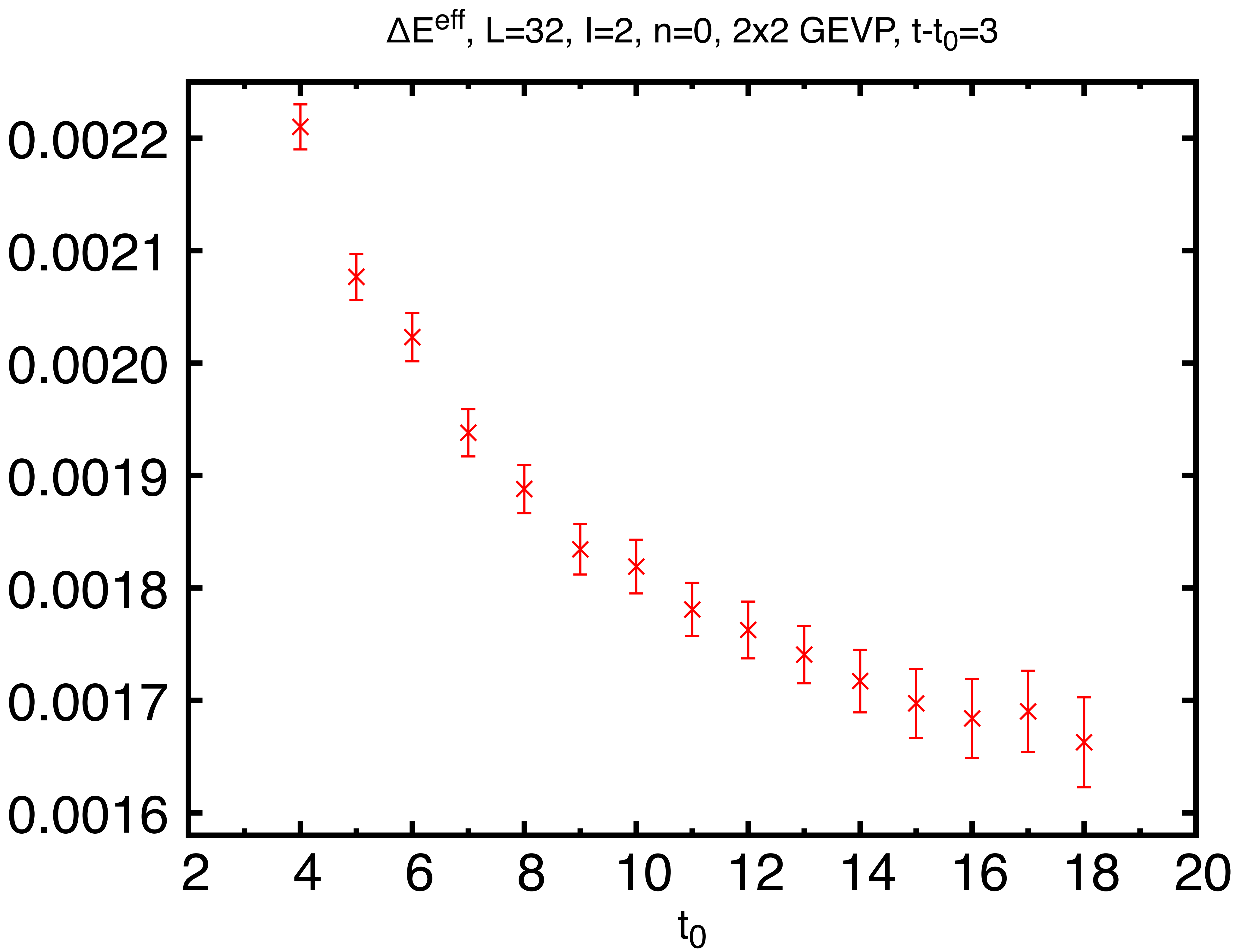
- Effective energy definitions
  - def1: $\ln(\lambda_n(t,t_0)/\lambda_n(t+1,t_0))$
  - def2: $-\ln(\lambda_n(t,t_0))/(t-t_0)$
  - def3: solution for **(*)**



$\Delta E_{\pi\pi,n=0}$ vs $E_{\pi\pi,n=0}$



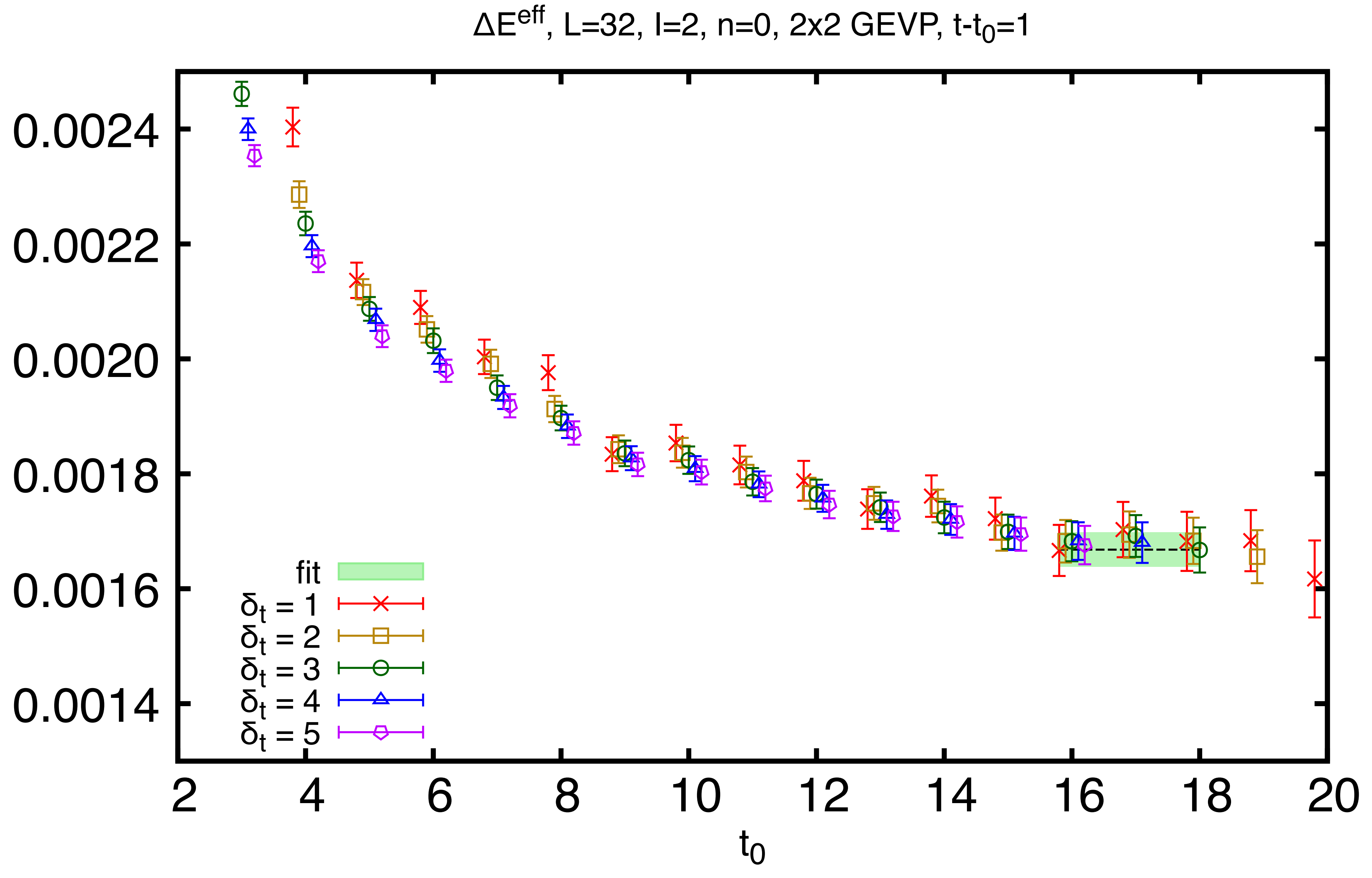
difference



- Error drastically decreased
- Plateau from t_0 = 16 (see also next slide)

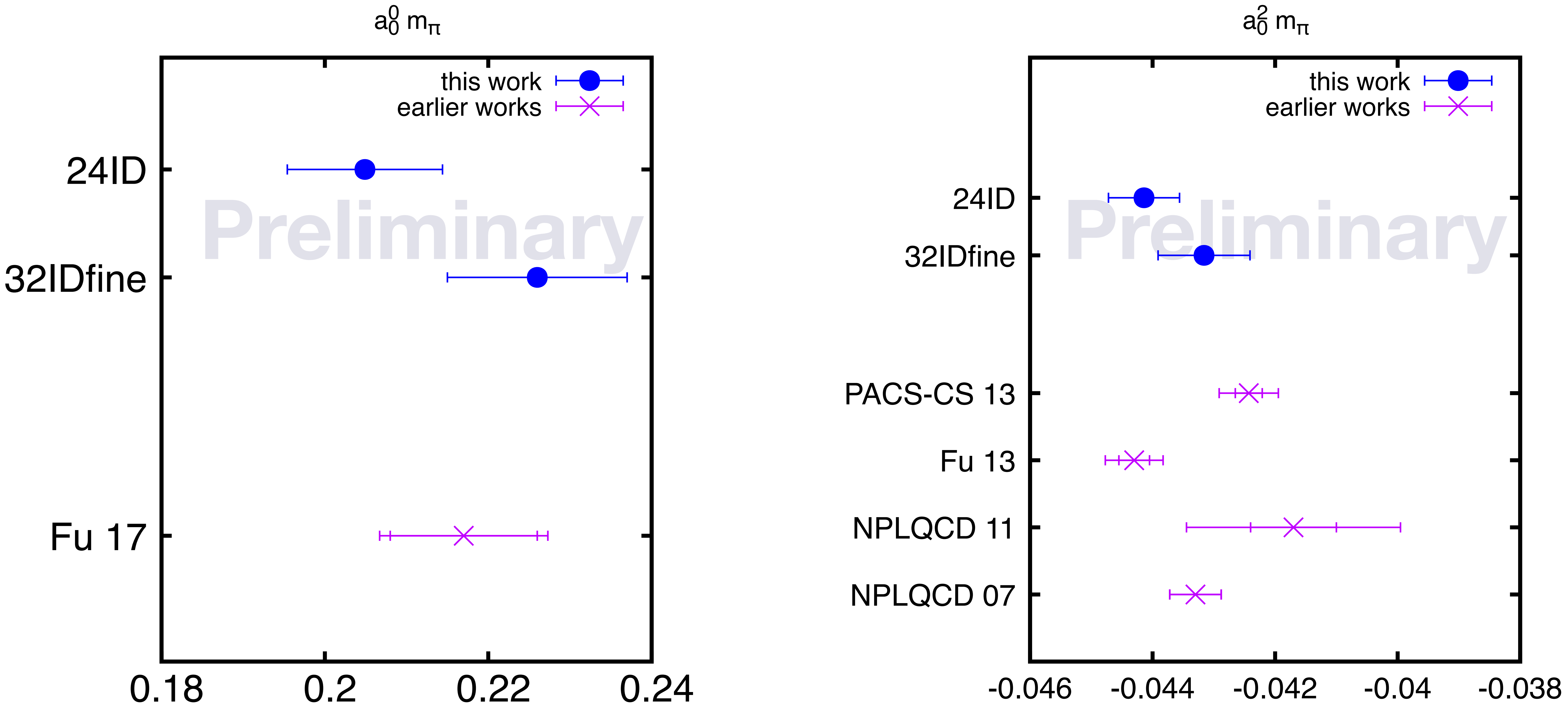
Preliminary result for $a_0^I m_\pi$

ΔE_0	fit	0.001668(30)
$E_0 = 2m_\pi + \Delta E_0$		0.21025(19)
phase shift & scattering length	Lüscher formalism	$\delta_0 = -0.3315(89)^\circ$ $a_0^2 m_\pi = -0.04566(81)$



- I = 0 needs more investigation (signal loses before $t_0 = 16$)
- I = 2 reaching the FLAG precision of 2%
- need investigation of systematic error
- may need scaling correction wrt $(m_\pi / f_\pi)^2$

Preliminary result



- Reaching the FLAG precision (5% for I = 0, 2% for I = 2)
- again, all other works are done at unphysical m_π and input LO as $(m_\pi/f_\pi)^2$
- need investigation of systematic error (expecting not significant though)

MEs from correlation functions

- Euclidean correlation function (0-momentum case)

$$\underbrace{\int d^3x_{\pi\pi} d^3x_K}_{\text{zero-momentum projection } (e^{i\vec{p}\cdot\vec{x}}=1)} \langle O_{\pi\pi}(t_{\pi\pi},\vec{x}_{\pi\pi}) Q_i(t,\vec{0}) O_K(t_K,\vec{x}_K)^\dagger \rangle$$

$$= \underbrace{\sum_{m,n}}_{} \langle 0|O_{\pi\pi}|\pi\pi, m\rangle\langle\pi\pi, m|Q_i|K, n\rangle\langle K, n|O_K^\dagger|0\rangle e^{-E_{\pi\pi,m}(t_{\pi\pi}-t)} e^{-m_{K,n}(t-t_K)}$$

all possible zero-(total)momentum states that have the same quantum numbers as $O_{\pi\pi}/O_K$

- If we were interested in the lightest (lowest-energy) states …

  look at large $t_{\pi\pi}$ - t & t - t_K:

$$\to \langle 0|O_{\pi\pi}|\pi\pi, 0\rangle\langle\pi\pi, 0|Q_i|K, 0\rangle\langle K, 0|O_K^\dagger|0\rangle e^{-E_{\pi\pi,0}(t_{\pi\pi}-t)} e^{-m_{K,0}(t-t_K)}$$

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$$= \sum_{\underline{m,n}} \langle 0|O_{\pi\pi}|\pi\pi, m\rangle \langle \pi\pi, m|Q_i|K, n\rangle \langle K, n|O_K^\dagger|0\rangle e^{-E_{\pi\pi,m}(t_{\pi\pi}-t)} e^{-m_{K,n}(t-t_K)}$$

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  look at large $t_{\pi\pi}$ - t & t - t_K:

$$\to \langle 0|O_{\pi\pi}|\pi\pi, 0\rangle \underbrace{\langle \pi\pi, 0|Q_i|K, 0\rangle}_{\text{ME of ground states}} \langle K, 0|O_K^\dagger|0\rangle e^{-E_{\pi\pi,0}(t_{\pi\pi}-t)} e^{-m_{K,0}(t-t_K)}$$

Matrix elements

- For extraction of ground-state ME

$$M^{\mathrm{eff}}(t_2, t_1) = C^{(3)}(t_2, t_1)\left[\frac{e^{E^{\pi\pi}t_2}e^{E^K t_1}}{C^{\pi\pi}(t_2)C^K(t_1)}\right]^{1/2} \xrightarrow{\text{large } t_1 \ \& \ t_2} M$$

- Excited (n-th) ππ state needed for on-shell kinematics with PBC

$$\boxed{M_n^{\mathrm{eff}}(t_2, t_1) = C_n^{(3)}(t_2, t_1)\left[\frac{e^{E_n^{\pi\pi}t_2}e^{E^K t_1}}{C_n^{\pi\pi}(t_2)C^K(t_1)}\right]^{1/2}} \xrightarrow{\text{large } t_1 \ \& \ t_2} M_n$$

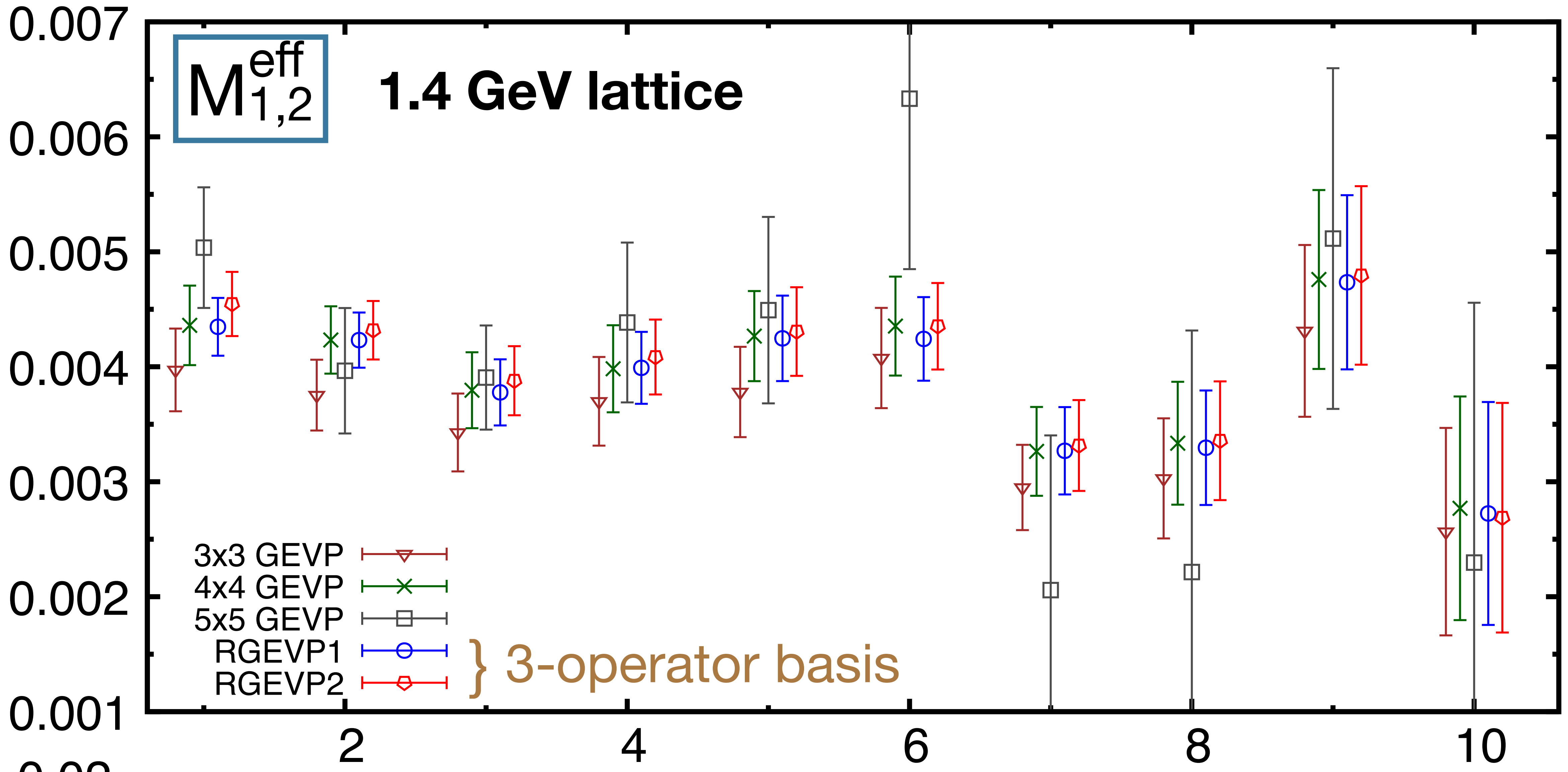
$C_n^{\pi\pi}$: 2-pt function of ππ operators diagonalized by GEVP

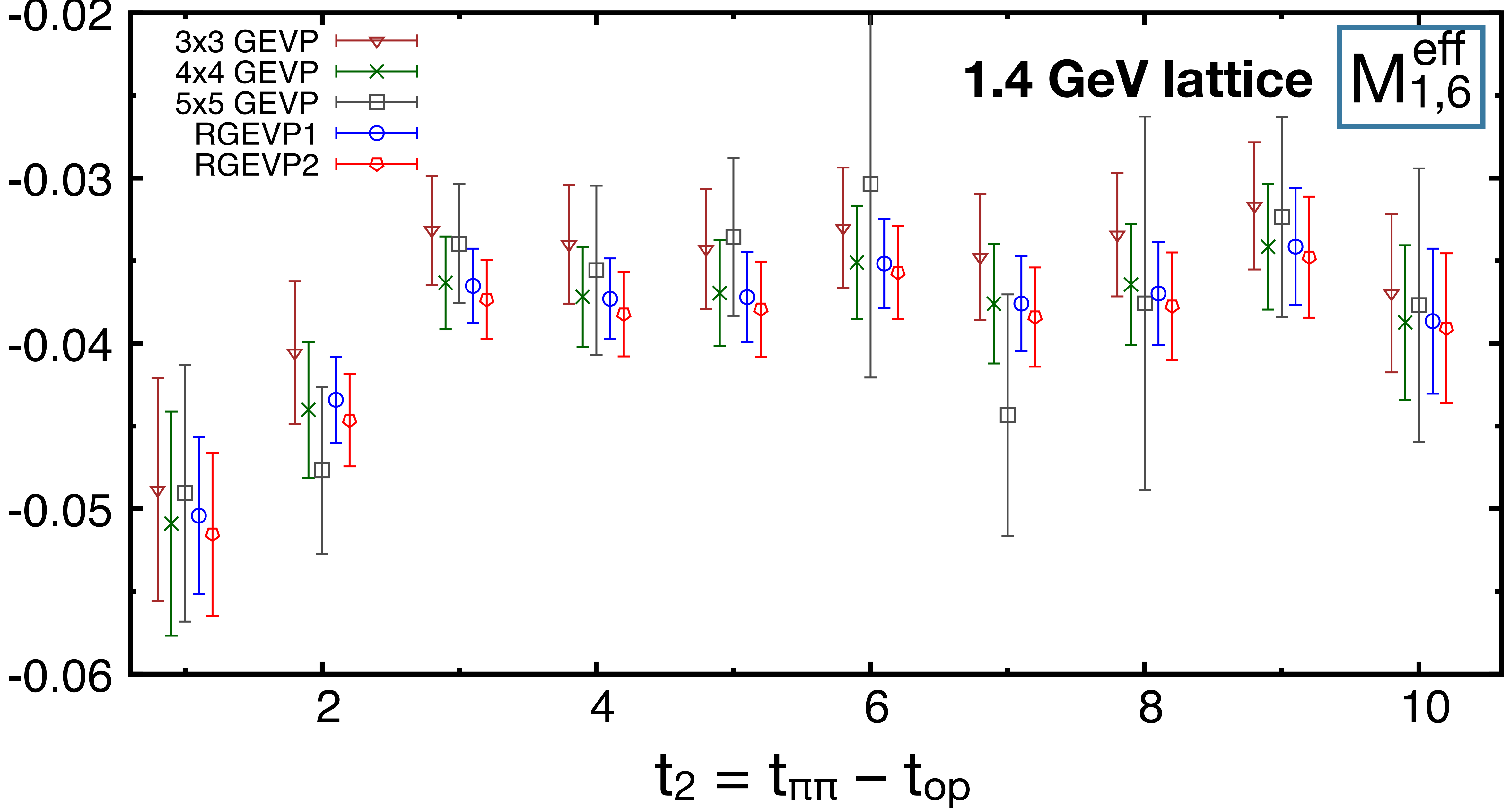
$C_n^{(3)}$: K → ππ 3-pt function with ππ operator used in $C_n^{\pi\pi}$

Effective matrix elements

$$M^{\rm eff}_{n,i}(t_2,t_1) = C^{(3)}_{n,i}(t_2,t_1)\left[\frac{e^{E^{\pi\pi}_n t_2}e^{E^K t_1}}{C^{\pi\pi}_n(t_2)C^K(t_1)}\right]^{1/2} \xrightarrow{\text{large } t_1\ \&\ t_2} M_{n,i}$$

n: state index
i: operator index

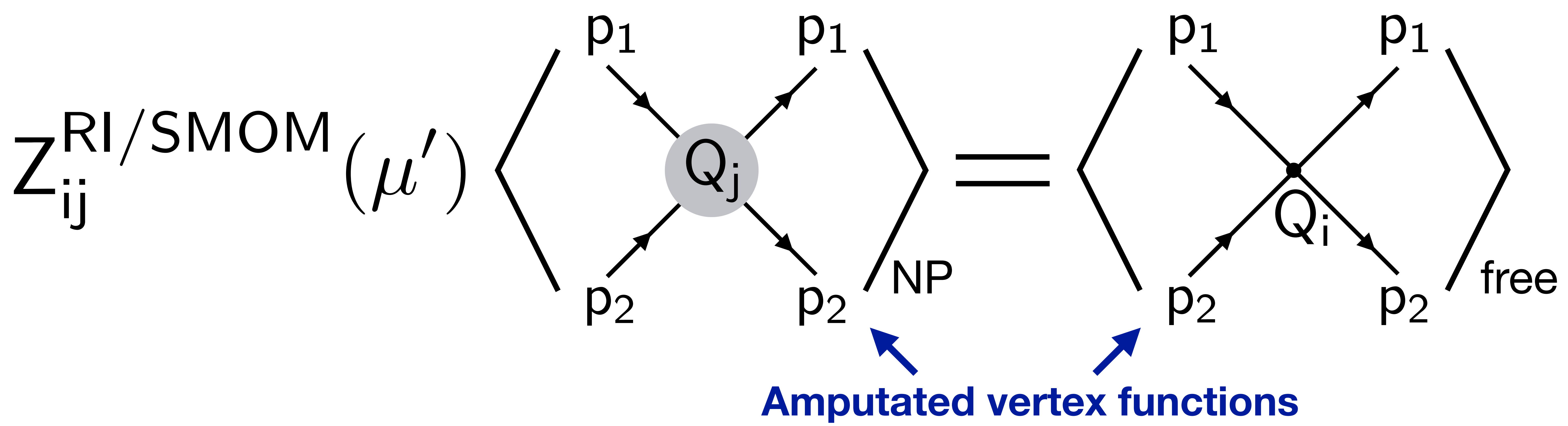




- Weighted average over $t_1 = t_{op} - t_K$ taken
- RGEVP (5→4→3 operator basis) plateauing from t_2 = 3 or 4
  - smaller error than 4x4
  - potential excited-state contamination in 3x3
  - GEVP statistically near singular for 5x5

Translating to more physical ME

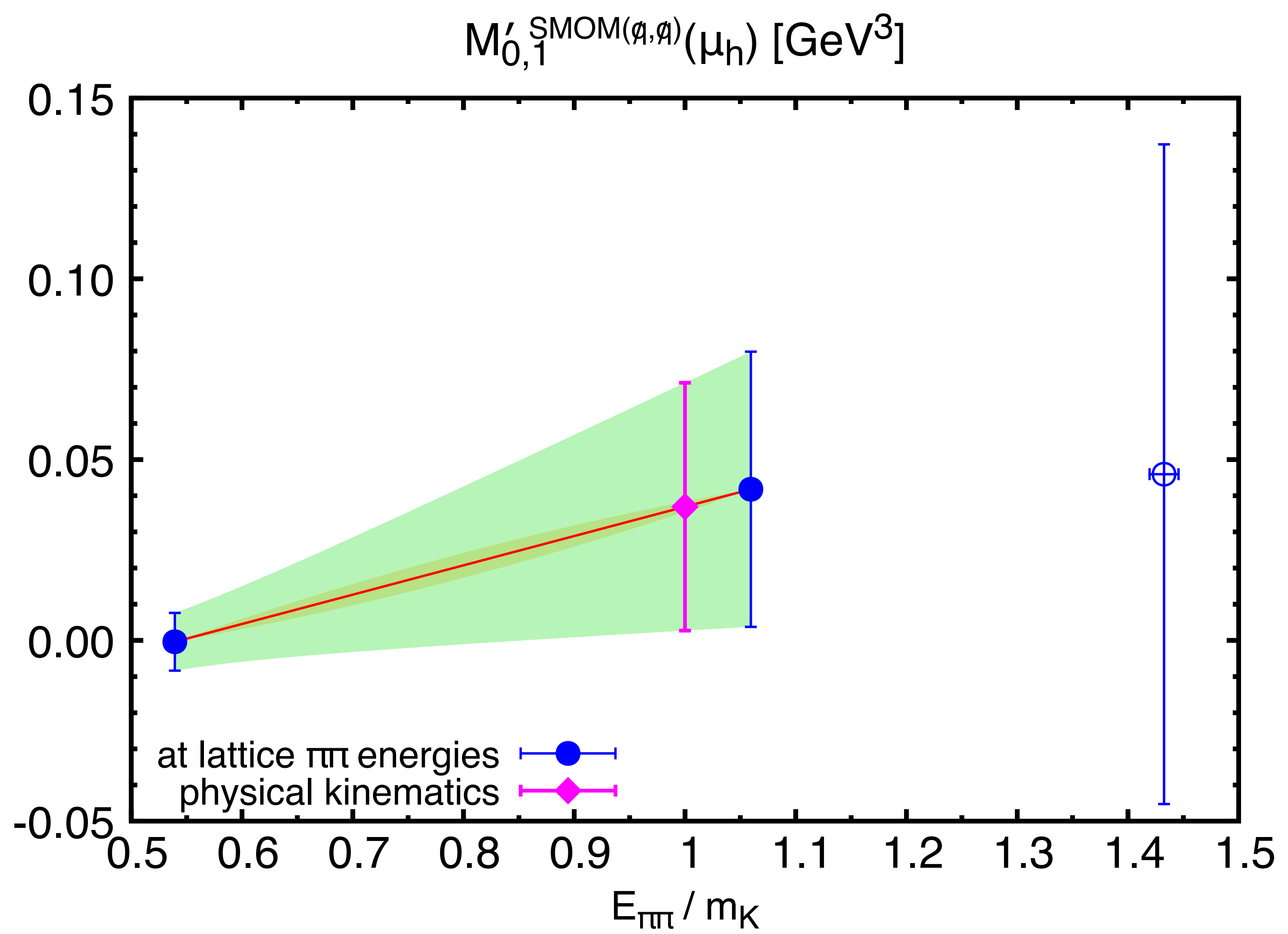
- Renormalization (RI/SMOM scheme)

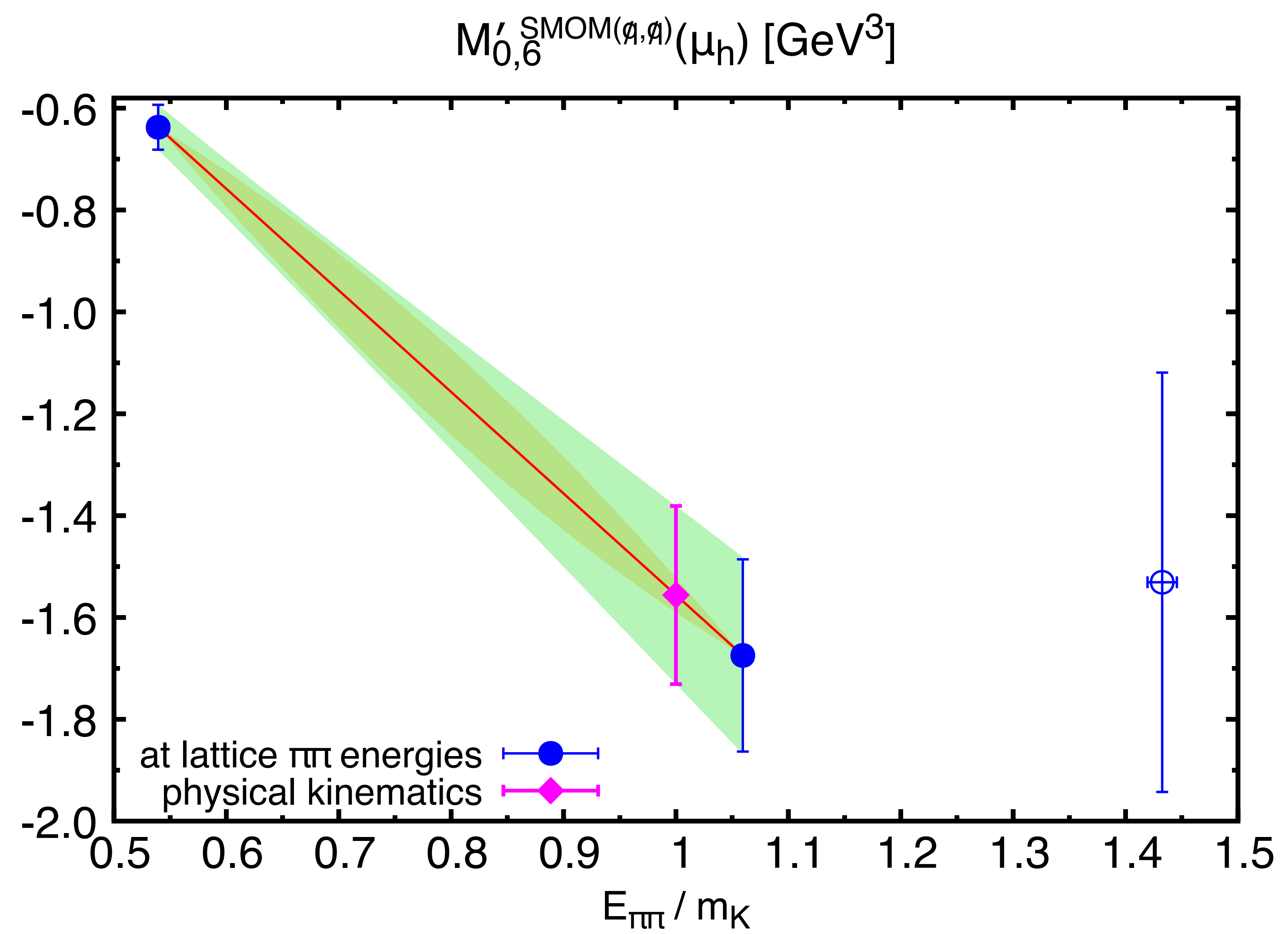


$$\mu'^2 = p_1^2 = p_2^2 = (p_1 - p_2)^2$$

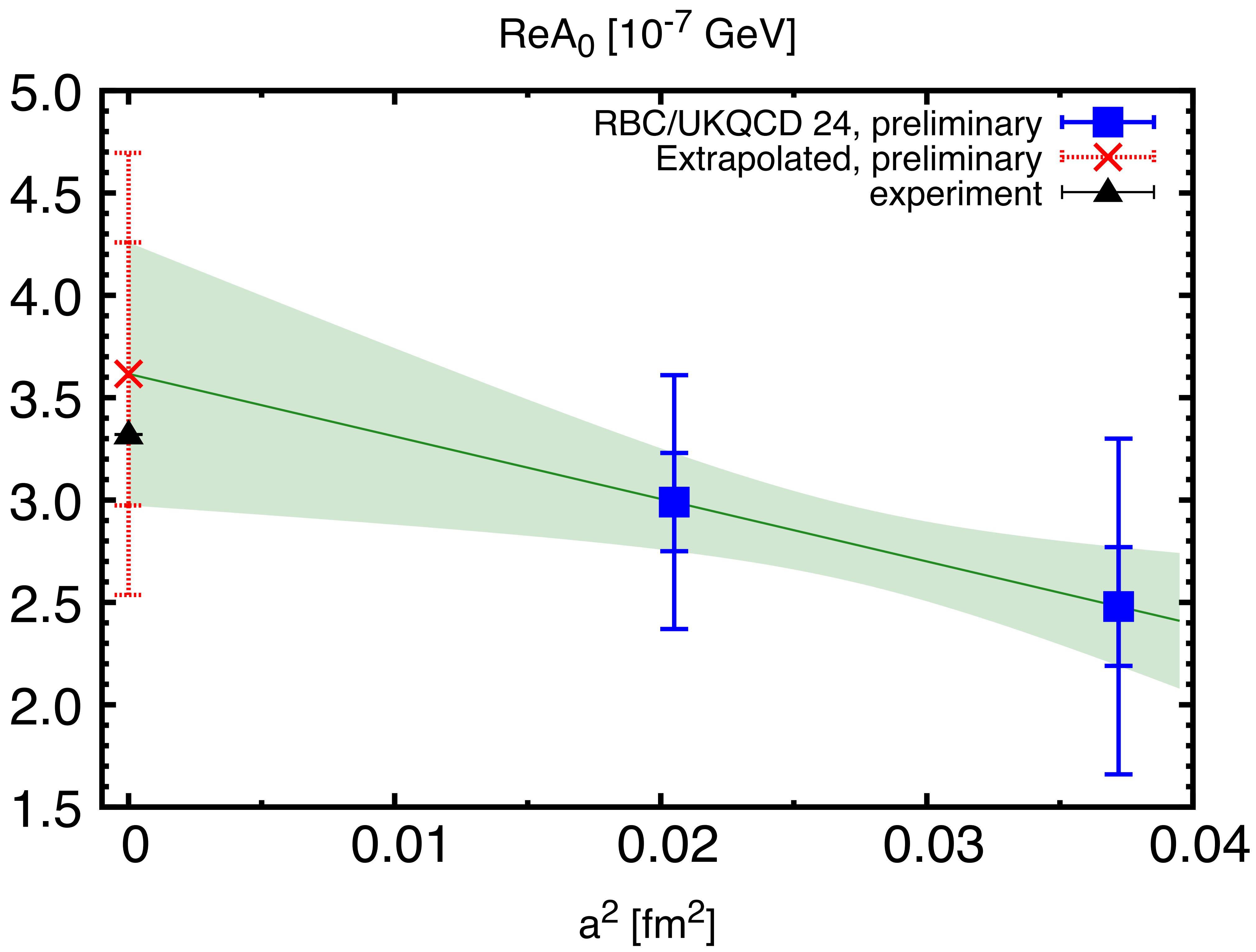
- Interpolation

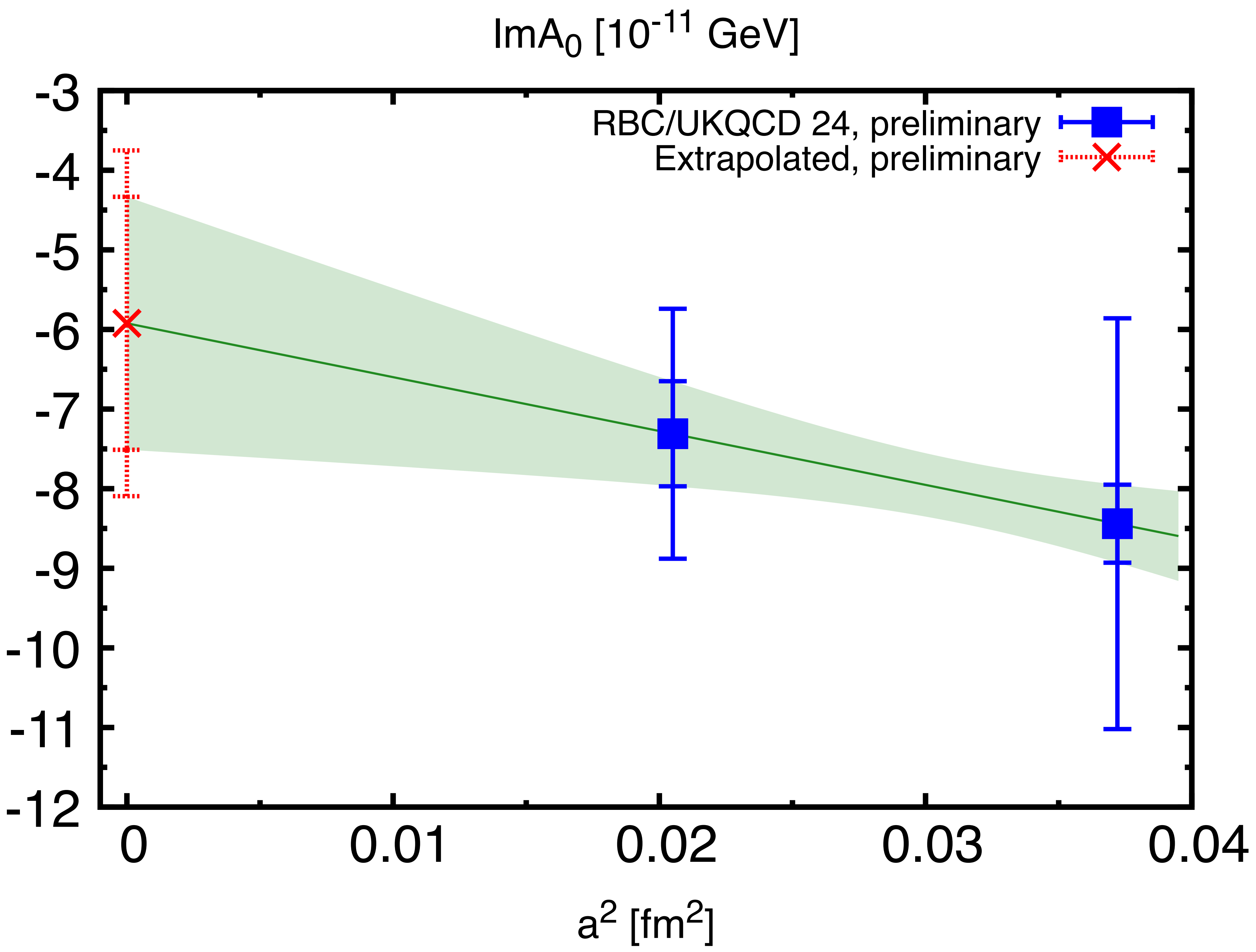
Examples of interpolation of renormalized ME
- Linear & quadratic in $E_{\pi\pi}/m_K$
- Systematic error estimated as lin vs quad is small as 1st excited st. close to on-shell





Result for A_0

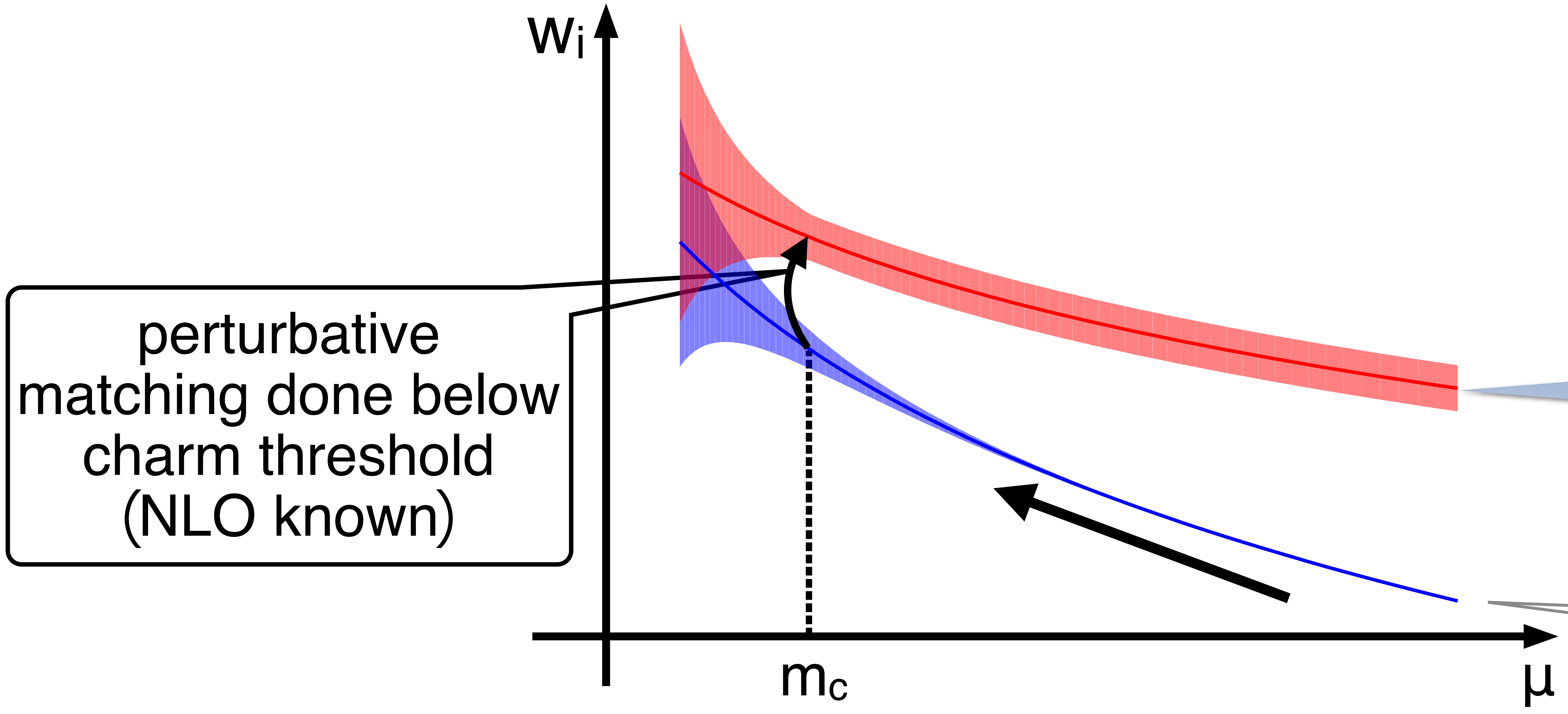




- $O(a^2)$ scaling violation potentially significant
  - Extrapolation with $c_0 + c_2\, a^2 + c_4\, a^4$ with a constraint $|c_2\, a^2| = 2\, |c_4\, a^4|$ at $a^{-1} = 1.0$ GeV corresponding to the coarser lattice did not change the result beyond statistical error

Wilson coefs

$$\langle f|H_W|i\rangle = \Sigma_i \underbrace{w_i^{3f}(\mu)}_{\text{pQCD}} \underbrace{\langle f|O_i^{3f}(\mu)|i\rangle}_{\text{LQCD}}$$



$w_i^{3f}(\mu)$: large uncertainty

12% uncertainty on A_0

- Relevant for $(\bar{s}d)_L(\bar{c}c)_{L/R}$: $\Delta I = 1/2$
- Does not happen for $\Delta I = 3/2$

$w_i^{4f}(\mu)$: precise at HEs

- Possible resolutions
  - NNLO matching only partially done [Cerda-Sevilla et al. Acta Phys.Polon.B 4 (2018) 1087-1096]
  - Nonperturbative matching underway [MT, LATTICE2019]

NP matching of WCs

- Basic idea

$$H_W = \sum_i w_i^{4f} O_i^{4f} = \sum_{i,j} \underbrace{w_i^{4f} M_{ij}}_{w_j^{3f}} O_j^{3f}$$

$$O_i^{4f} \to \sum_j M_{ij} O_j^{3f}$$

i.e. $\langle E_{out} | O_i^{4f} | E_{in} \rangle = \sum_j M_{ij} \langle E_{out} | O_j^{3f} | E_{in} \rangle$ for small E_{out} & E_{in} compared to m_c

- Strategy
  - Consider many 3pt functions on fine lattice (w unphysical m_π)

$$C_{i,ab}^{3f/4f}(t_{out}, t, t_{in}) = \langle \mathcal{O}_a(t_{out}) O_i^{3f/4f}(t) \mathcal{O}_b(t_{in}) \rangle$$

  - Perform fit with many pairs of O_a & O_b at large t_{out} – t & t - t_{in} $\boxed{C_{i,ab}^{4f} = M_{ij} C_{j,ab}^{3f}}$
  - Trying with ~ 200 relevant pairs of O_a & O_b
  - Automatic Wick contractor in use

Energy spectrum (for HVP)

Good signal observed for $E_n < 1$ GeV

$$E_n^{\rm eff} = -\frac{1}{t-t_0}\ln\lambda_n(t,t_0)$$