Gauging *C* on the Lattice

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Based on **[** 2406.12075 **]**

Why study QFT on the lattice?

Numerical simulations

One of the few nonperturbative tools to study generic strongly-coupled QFTs

Explore strong coupling phenomena

Analytic access to strong coupling regimes

e.g. confinement and chiral symmetry breaking **[** Wilson, Banks, Kogut, Susskind **]**
Higgs vs. confinement **[** Osterwalder, Seiler, Fradkin, Shenker **]**

Study kinematic features

- Dualities (e.g. Jordan-Wigner, Kramers-Wannier, Particle-Vortex, …)
- **Global symmetries + anomalies** (focus of this talk)

Symmetry on the lattice

Continuum symmetries are often broken or modified by the lattice…

- Lorentz
- Chiral symmetries (Nielsen-Ninomiya)
- “Topological” symmetries requiring quantized topology in field space
    - e.g. winding symmetries of compact boson, magnetic symmetries in gauge theories

Symmetry on the lattice

Continuum symmetries are often broken or modified by the lattice…

… while other symmetry structures appear quite naturally

- Higher-form symmetries
- Discrete gauge theories, discrete anomalies
- Self-duality symmetries

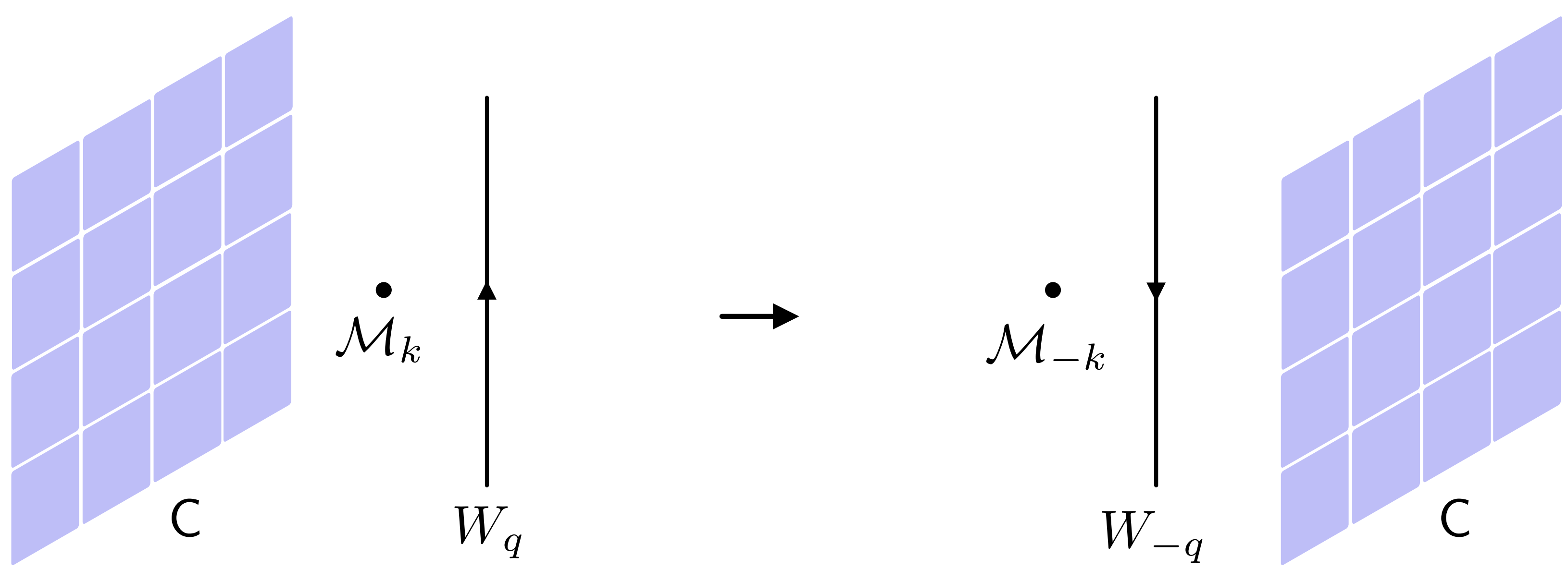
Many recent works trying to clarify and understand generalized symmetries of various kinds on (various kinds of) lattices

Why charge conjugation?

Loosely speaking:

- **0**-form symmetries act on **local** operators
- **1**-form symmetries act on **line** operators
- …

Charge conjugation acts intrinsically
on **both** local and extended operators



Why gauge charge conjugation?

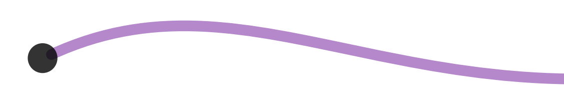
Question of how to understand what
happens when we gauge / orbifold

Local operators

$\mathcal{O} \to -\mathcal{O}$

Gauge →

Projected out
(goes to twisted sector)



Line operators

$\mathcal{L} \to -\mathcal{L}$

Gauge →

???

Why gauge charge conjugation?

Gauging charge conjugation can lead to interesting generalized symmetries

Focus: $O(2) = U(1) \rtimes \mathbb{Z}_2$ gauge theory

- **Higher-group** symmetry **[** Hsin, Turzillo '19 **]**
- **Non-invertible** symmetries **[** Nguyen, Tanizaki, Unsal '21, Heidenreich, McNamara, Montero, Reece, Rudelius, Valenzuela '21, … **]**

Goal: Construct a lattice theory that realizes these symmetries and study their implications

A sequence of gaugings

Continuum:

Higgs SO(3) to O(2) using a spin-2 Higgs field

[Kiskis '78, Schwarz '82]

$$\langle \Phi \rangle = \begin{pmatrix} v & & \\ & v & \\ & & -2v \end{pmatrix}$$

Lattice:

“Non-abelian Villain formulation”

$\mathbb{R}$ gauge theory → $U(1) = \mathbb{R}/2\pi\mathbb{Z}$ gauge theory → $O(2) = U(1) \rtimes \mathbb{Z}_2$ gauge theory

(flat) gauging of $\mathbb{Z}$ 1-form symmetry

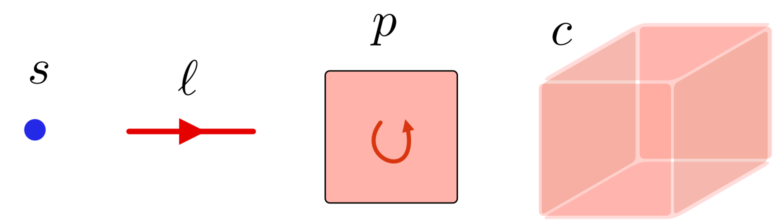
(flat) gauging of $\mathbb{Z}_2$ charge conjugation symmetry

flat gauging = no monopoles

flat gauging = no vortices

Lattice ingredients

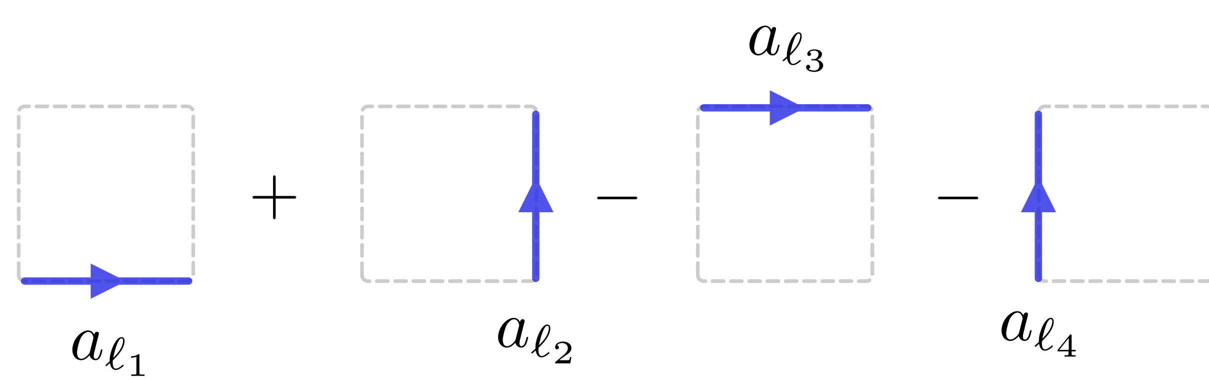
Euclidean spacetime lattices
(w/ periodic BCs)



Fields are p-forms or “p-cochains” e.g. $a_\ell \in \mathbb{R} \quad (a^{(1)} \in \mathbb{R})$

- Lattice exterior derivative $d : p\text{-form} \rightarrow (p+1)\text{-form} \qquad (d^2 = 0)$

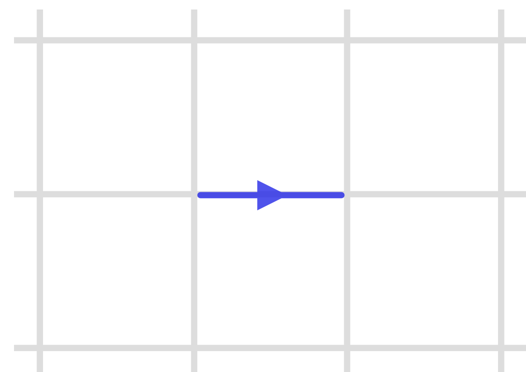
$(da)_p =$



- Cup product $\cup : (p\text{-form}, q\text{-form}) \rightarrow (p+q)\text{-form}$ obeys Leibniz rule

Lattice building blocks: Villain gauge field

Real (non-compact) gauge field



$$a_\ell \in \mathbb{R}$$

$$a_\ell \to a_\ell + (d\lambda)_\ell$$

$$S_{\text{cont}} = \frac{1}{2e^2} \int (da)^2 \longrightarrow S_{\text{lat}} = \frac{\beta}{2} \sum_p (da)_p^2 \qquad \beta = \frac{1}{e^2}$$

- Action is gauge-invariant $(d^2 = 0)$
- Action is invariant under $\mathbb{R}$ 1-form symmetry $a_\ell \to a_\ell + \epsilon_\ell \,, \; (d\epsilon)_p = 0$

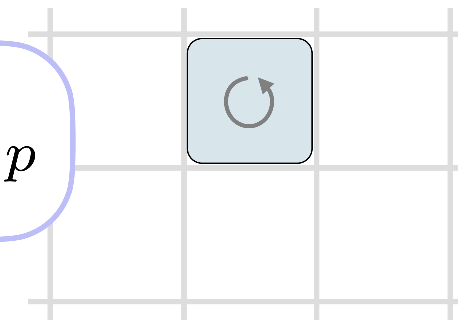
Gauge-invariant operators: Wilson lines $W_q(\gamma) = e^{iq \sum_{\ell \in \gamma} a_\ell}$ $\quad q \in \mathbb{R}$

Lattice building blocks: Villain gauge field

U(1) (compact) gauge field

Gauge shifts by 2π: $a_\ell \to a_\ell + 2\pi m_\ell \,, \; m_\ell \in \mathbb{Z}$

Integer gauge field: $n_p \to n_p + (dm)_p$



Integer-quantized magnetic flux: $\frac{1}{2\pi}\int_S da \in \mathbb{Z} \longrightarrow \sum_{p\in S} n_p \in \mathbb{Z}$

Action: $$S_{\text{Villain}} = \frac{\beta}{2}\sum_p ((da)_p - 2\pi n_p)^2$$

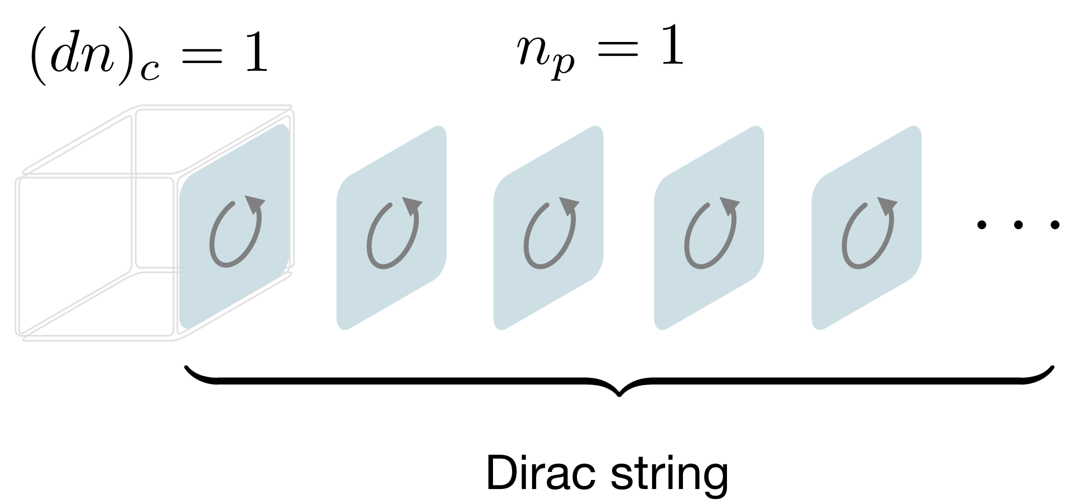
- Invariant under small $\lambda \in \mathbb{R}$ and large $m \in \mathbb{Z}$ gauge transformations
- Invariant under $U(1)_e^{(1)}$ 1-form symmetry $a_\ell \to a_\ell + \epsilon_\ell \,, \; (d\epsilon)_p = 0$

Monopole proliferation

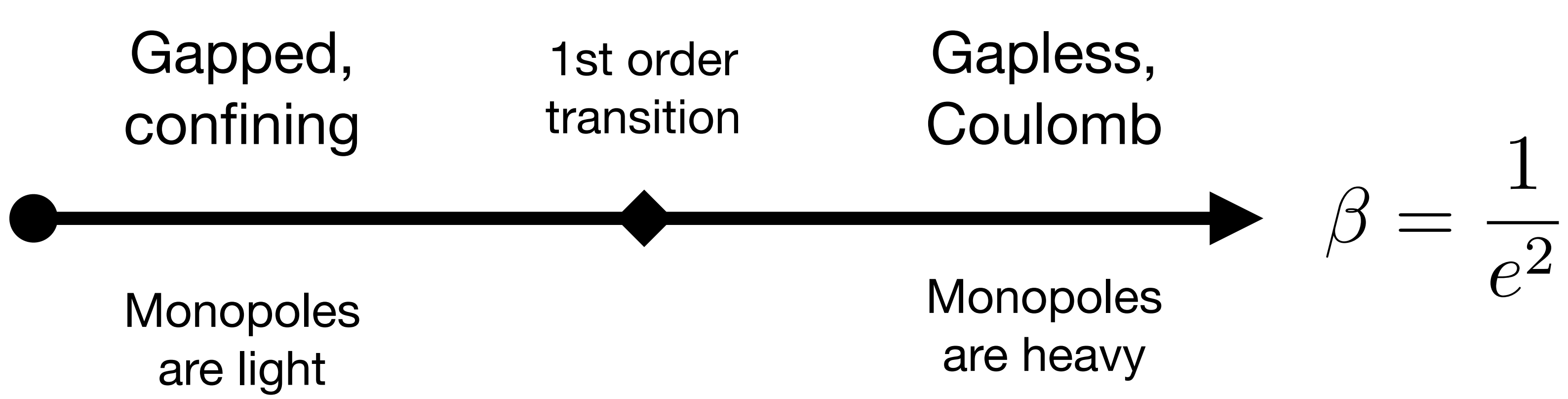
Common lore:

All UV completions of U(1) gauge theory come with finite-action monopoles

Indeed, in the compact theory we sum over monopole configurations



d = 4



Reason: the standard Villain discretization fails to preserve the magnetic symmetry of Maxwell theory ($dF = 0$)

Modified Villain: removing monopoles

Introduce Lagrange multiplier to remove monopole configurations $(dn)_c = 0$ [Gattringer, Sulejmanpasic '19]

$$S_{\text{Modified}} = S_{\text{Villain}} + i \sum_{d\text{-cells}} \tilde{a}^{(d-3)} \cup dn$$

- 3d $\tilde{a}^{(0)} \sim \tilde{a}^{(0)} + 2\pi$ $\mathcal{M}_k(s) = e^{ik\tilde{a}_s}$

  Dual photon — Monopole operator

- 4d $\tilde{a}^{(1)} \sim \tilde{a}^{(1)} + d\tilde{\lambda} + 2\pi$ $H_k(\gamma) = e^{ik\sum_{\ell\in\gamma}\tilde{a}_\ell}$

  Magnetic gauge field — 't Hooft line

Modified Villain: removing monopoles

Introduce Lagrange multiplier to remove monopole configurations $(dn)_c = 0$ **[** Gattringer, Sulejmanpasic '19 **]**

$$S_{\text{Modified}} = S_{\text{Villain}} + i \sum_{d\text{-cells}} \tilde{a}^{(d-3)} \cup dn$$

- Without monopoles, theory is in the Coulomb phase for any coupling

  ⟶ Consequence of mixed anomaly between electric and magnetic symmetries

- Exact electric-magnetic duality on the lattice

$$\frac{\beta}{2}\sum(da - 2\pi n)^2 + i\sum \tilde{a}\cdot dn \quad \longleftrightarrow \quad \frac{1}{2(2\pi)^2\beta}\sum(d\tilde{a} - 2\pi\tilde{n})^2 + i\sum a \cdot d\tilde{n}$$

- Analogous construction exists for 2d compact boson (momentum + winding symmetries, mixed anomaly, T-duality)

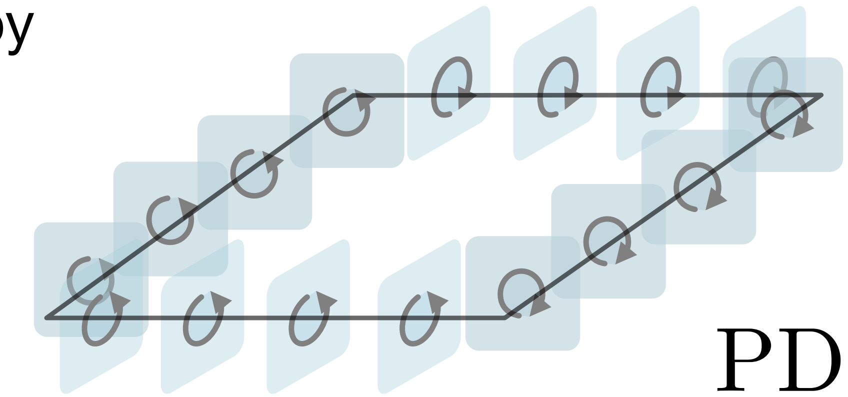
[Gorantla, Lam, Seiberg, Shao '21 **]**

Symmetries of the modified Villain theory

$$S_{\text{Modified}} = \frac{\beta}{2}\sum_p ((da)_p - 2\pi n_p)^2 + i \sum_{d\text{-cells}} \tilde{a} \cup dn$$

$U(1)_e^{(1)}$ $\quad a \to a + \epsilon\,,\ \ d\epsilon = 0$ $\quad$ (Continuum: Gauss law $d \star F = 0$)

- Lattice: generated by



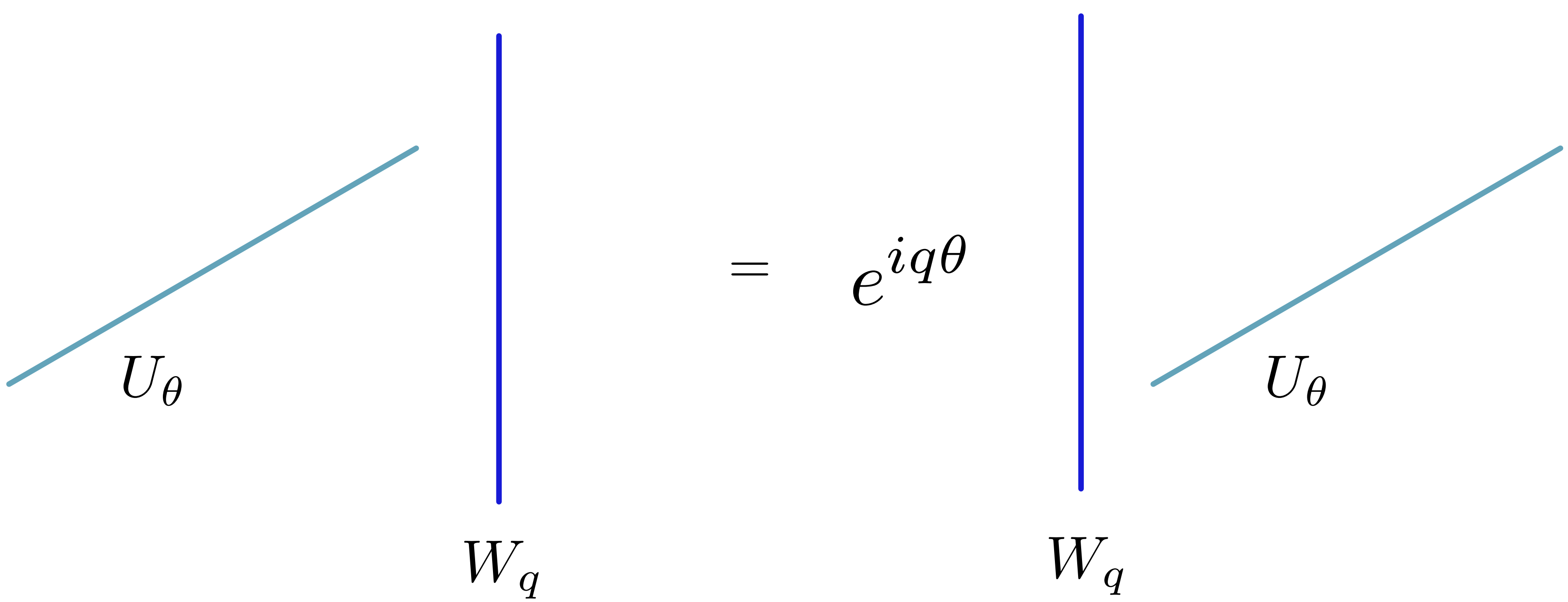
Replace $\quad \frac{\beta}{2}((da)_p - 2\pi n_p)^2 \longrightarrow \frac{\beta}{2}((da)_p - 2\pi n_p \pm \theta)^2$

Symmetries of the modified Villain theory

$$S_{\text{Modified}} = \frac{\beta}{2}\sum_p ((da)_p - 2\pi n_p)^2 + i \sum_{d\text{-cells}} \tilde{a} \cup dn$$

$U(1)_e^{(1)}$ $\qquad a \to a + \epsilon\,, \; d\epsilon = 0 \qquad$ (Continuum: Gauss law $d \star F = 0$)

- Acts on Wilson lines

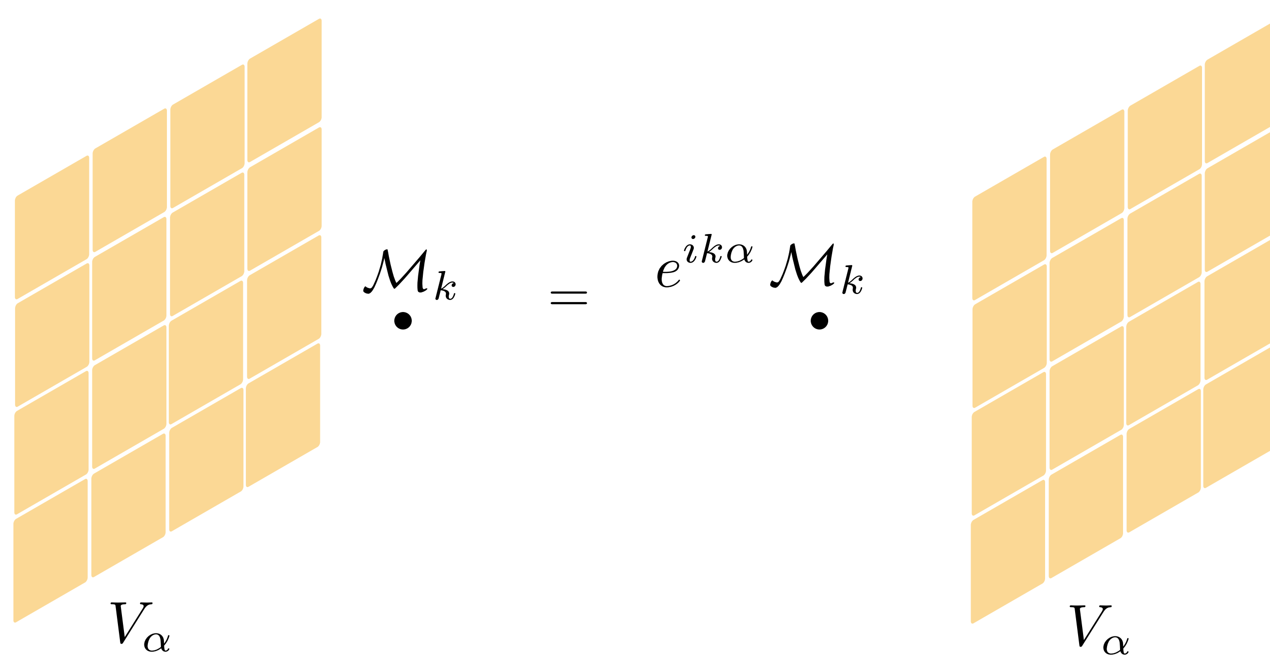


Symmetries of the modified Villain theory

$$S_{\text{Modified}} = \frac{\beta}{2}\sum_p ((da)_p - 2\pi n_p)^2 + i \sum_{d\text{-cells}} \tilde{a} \cup dn$$

$U(1)_m^{(d-3)}$ $\tilde{a} \to \tilde{a} + \tilde{\epsilon}\,,\ \ d\tilde{\epsilon} = 0$ (Continuum: Bianchi identity $dF = 0$)

- Lattice: generated by $V_\alpha(S) = e^{i\alpha \sum_{p\in S} n_p}$
- Acts on monopole/'t Hooft operators

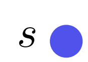


Charge conjugation symmetry

- Global action: $a \to -a\,,\ n \to -n\,,\ \tilde{a} \to -\tilde{a}$
- Turn on background gauge field $C_\ell \in \mathbb{Z}_2 = \{0, 1\}$ $\quad C_\ell \to C_\ell + (dG)_\ell$

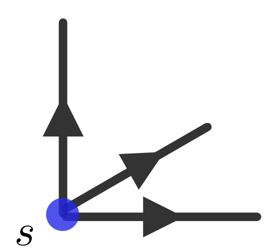
Flatness: $dC = 0 \bmod 2$

0-forms



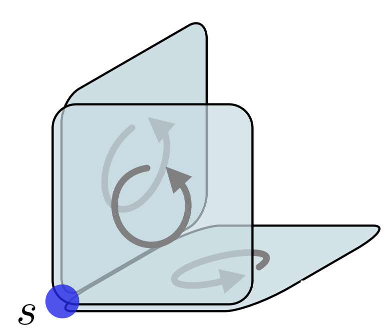
$$\varphi_s \to (-1)^{G_s}\,\varphi_s$$

1-forms



$$a_{(s,i)} \to (-1)^{G_s}\,a_{(s,i)}$$

2-forms



$$n_{(s,ij)} \to (-1)^{G_s}\,n_{(s,ij)}$$

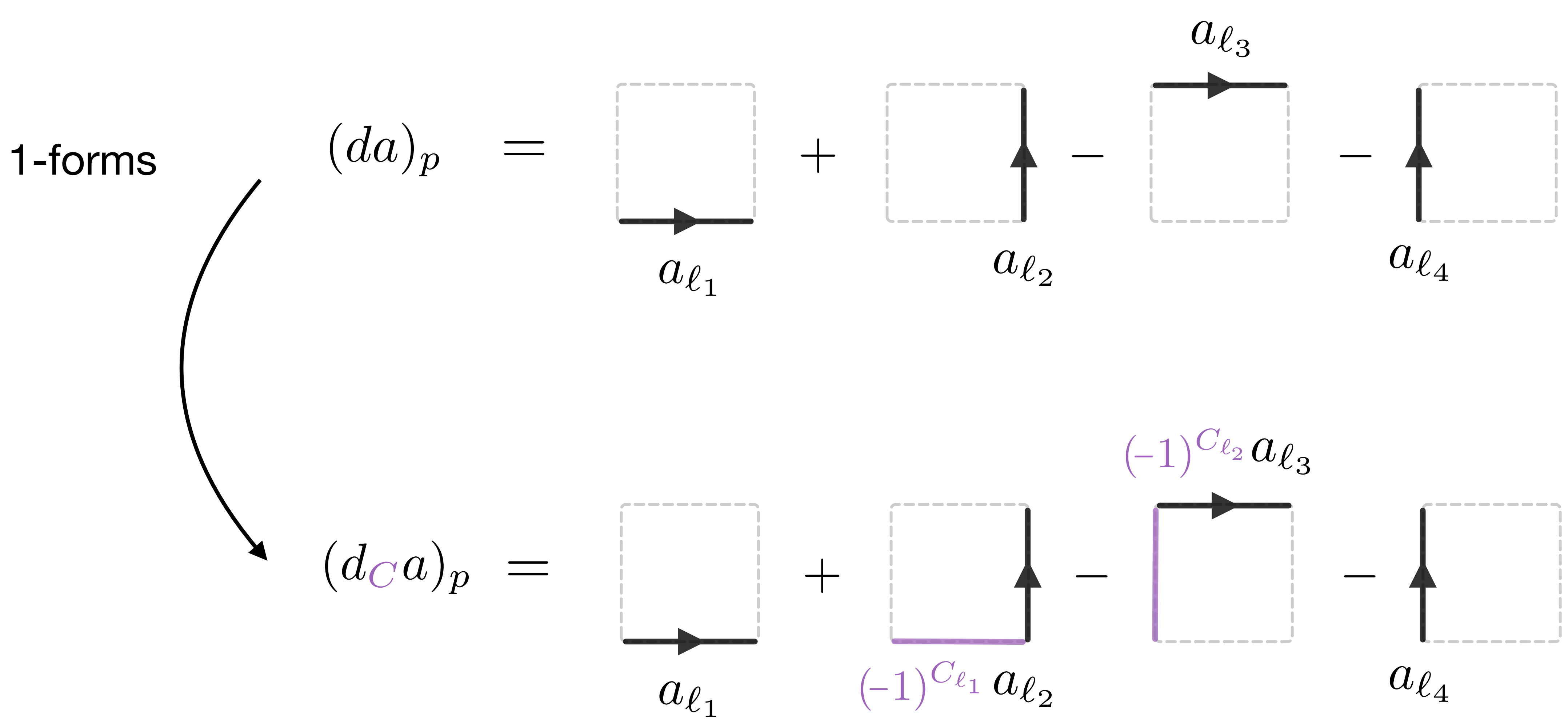
Covariant derivative

0-forms

$$(d\varphi)_\ell = \varphi_{s_2} - \varphi_{s_1}$$

$$(d_{C}\varphi)_\ell = (-1)^{C_\ell}\,\varphi_{s_2} - \varphi_{s_1}$$

Covariant derivative



note: $d_C^2 = 0$ only if $dC = 0 \bmod 2$

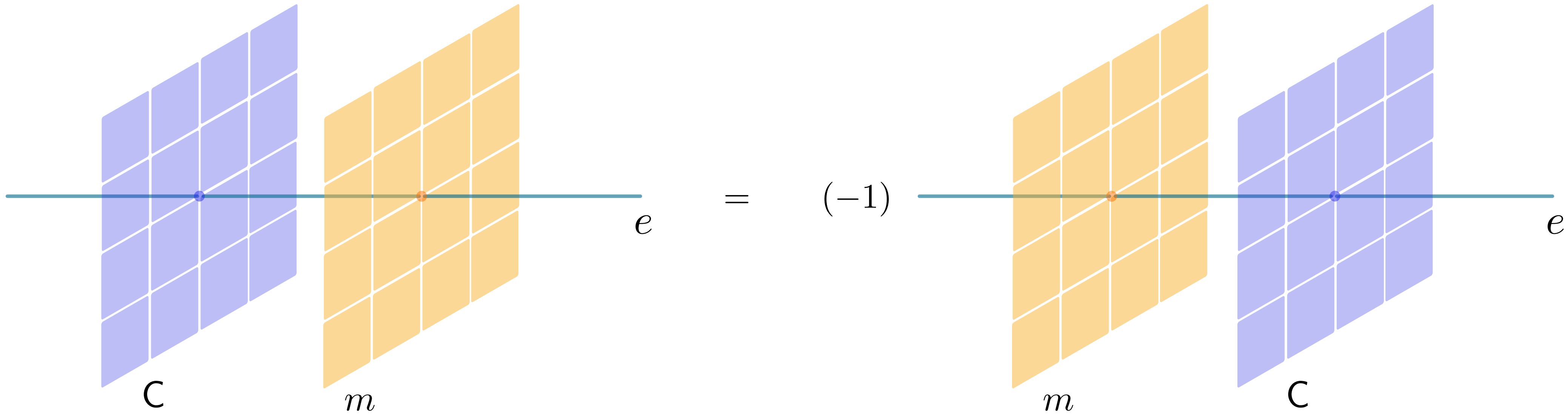
Coupling to background fields

$$S[C] = \frac{\beta}{2}\sum_p ((d_C a)_p - 2\pi n_p)^2 + i \sum_{d\text{-cells}} \tilde{a} \cup_C d_C n$$

I Lagrange multiplier only sets $d_C n = 0 = dn \pmod 2$

$\longrightarrow$ $U(1)_m^{(d-3)} \to \mathbb{Z}_{2,m}^{(d-3)}$ • Similarly $U(1)_e^{(1)} \to \mathbb{Z}_{2,e}^{(1)}$

II Mixed 't Hooft anomaly (type III) for $\mathbb{Z}_{2,\mathsf{C}}^{(0)} \times \mathbb{Z}_{2,e}^{(1)} \times \mathbb{Z}_{2,m}^{(d-3)}$



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⟶ $U(1)_m^{(d-3)} \to \mathbb{Z}_{2,m}^{(d-3)}$ • Similarly $U(1)_e^{(1)} \to \mathbb{Z}_{2,e}^{(1)}$

II Mixed 't Hooft anomaly (type III) for $\mathbb{Z}_{2,\mathrm{C}}^{(0)} \times \mathbb{Z}_{2,e}^{(1)} \times \mathbb{Z}_{2,m}^{(d-3)}$

I leads to a **non-invertible** symmetry when we dynamically gauge C

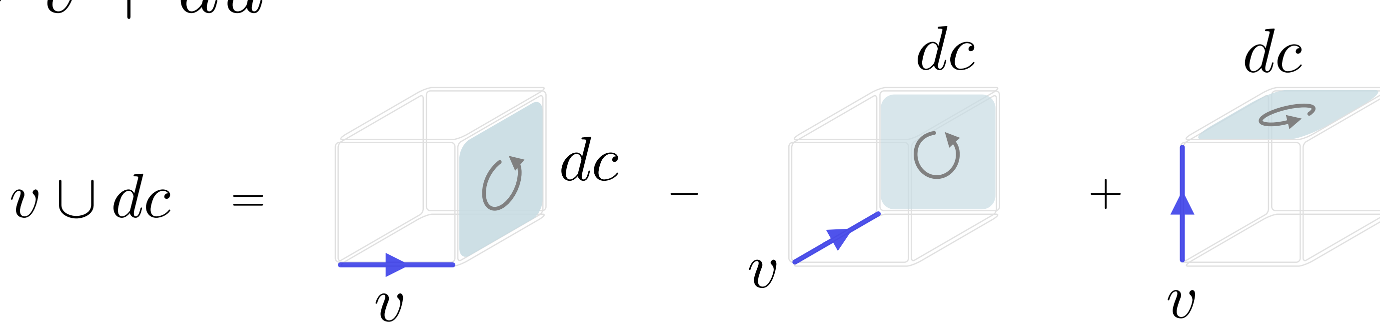
II leads to a **higher-group** symmetry when we dynamically gauge C

O(2) gauge theory

Promote background field to dynamical field, summed over in path integral $C \to c$

$$S_{O(2)} = \frac{\beta}{2}\sum_p ((d_c a)_p - 2\pi n_p)^2 + i \sum_{d\text{-cells}} \tilde{a}^{(d-3)} \cup_c (d_c n) + i\pi \sum_{d\text{-cells}} v^{(d-2)} \cup dc$$

- $v^{(d-2)} \in \mathbb{Z}_2$ is a Lagrange multiplier setting $dc = 0 \bmod 2$
- $v \to v + du$



- Gauge-invariant $a \to a + d_c\lambda,\ \tilde{a} \to \tilde{a} + d_c\tilde{\lambda}$ provided $dc = 0 \bmod 2$

O(2) gauge theory

Promote background field to dynamical field, summed over in path integral $C \to c$

$$S_{O(2)} = \frac{\beta}{2} \sum_p ((d_c a)_p - 2\pi n_p)^2 + i \sum_{d\text{-cells}} \tilde{a}^{(d-3)} \cup_c (d_c n)$$

$$+ i\pi \sum_{d\text{-cells}} v^{(d-2)} \cup dc$$

- **Sign problem?** In practice, **no**: propose updates satisfying constraints
- **Rotation invariance?** Not manifest, but **preserved** if $dc = 0 \bmod 2$

New operators

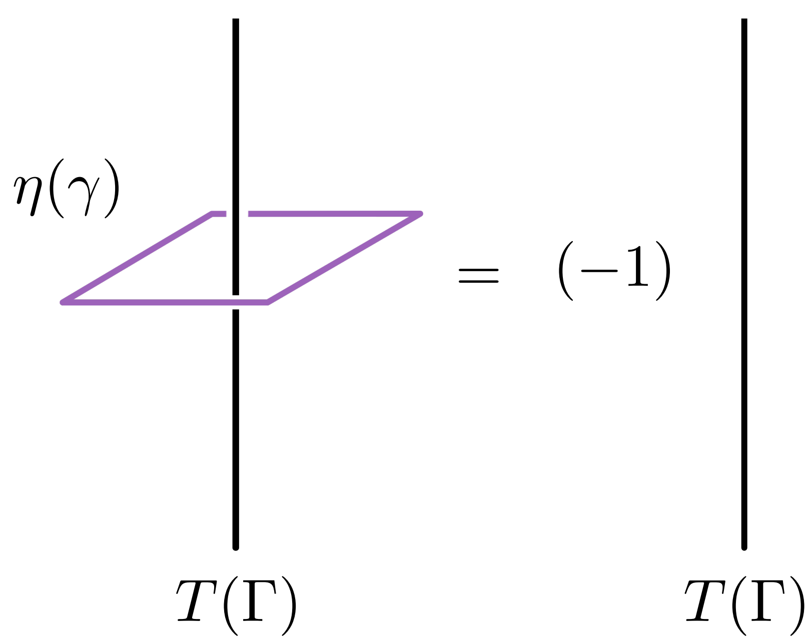
- **C Wilson line** $\eta(\gamma) = e^{i\pi \sum_{\ell \in \gamma} c_\ell}$ = Wilson line in the “det” rep of O(2)

  Flat gauge field -> topological Wilson line $\eta(\gamma + \partial\Sigma) = \eta(\gamma)$

  η generates a (d-2)-form symmetry $\mathbb{Z}_{2,v}^{(d-2)}$ which acts on:

- **Twist vortex** $T(\Gamma_{d-2}) \overset{?}{=} e^{i\pi \sum_\Gamma v^{(d-2)}}$ = Gukov-Witten operator for conjugacy class of reflections

AKA “Alice” or
“Cheshire” strings



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Note: gauging $\mathbb{Z}_{2,v}^{(d-2)}$ condenses (trivializes) η

and *ungauges* charge conjugation

Fate of old operators

Roughly, keep C-even operators and throw out C-odd operators

Local operators

e.g. keep $\mathcal{M}_k(s) + \mathcal{M}_{-k}(s)$

Extended operators

e.g. keep $W_q(\gamma) + W_{-q}(\gamma)$?

this expression is not **locally** gauge-invariant !

How to make extended operators fully gauge-invariant?

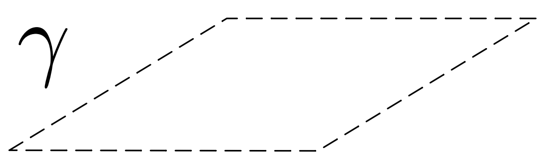
Constructing Wilson lines

Approach inspired by **[** Alford, Lee, March-Russell, Preskill ’92 **]**

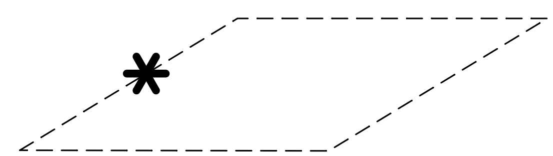
First make the Wilson line transform “like a local operator”

Ingredients:

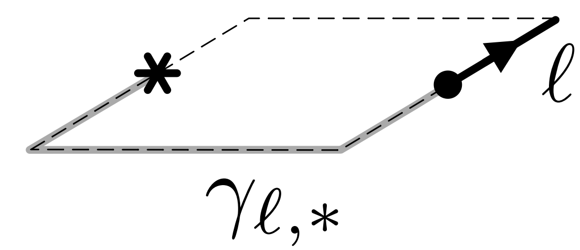
A closed curve



A basepoint on the curve



A set of paths on the curve connecting each link to the basepoint



$$\sum_{\ell\in\gamma} a_\ell \longrightarrow \sum_{\ell\in\gamma} \eta(\gamma_{\ell,*})\, a_\ell$$

all terms transform differently under C

transforms covariantly at the basepoint

Constructing Wilson lines

$$W_q(\gamma) \stackrel{?}{=} \sum_{G_*=\pm 1} \exp\left(iq\, G_* \sum_{\ell\in\gamma} \eta(\gamma_{\ell,*})\, a_\ell\right)$$

sum over gauge transformations at the basepoint

- C invariant !
- Independent of choice of basepoint

- **Not** U(1) invariant
- Depends on the choices of paths

Both solved if $\eta(\gamma) = +1$

$$W_q(\gamma) = \frac{1+\eta(\gamma)}{2} \sum_{G_*=\pm 1} \exp\left(iq\, G_* \sum_{\ell\in\gamma} \eta(\gamma_{\ell,*})\, a_\ell\right)$$

$P(\gamma)$ projects onto states with trivial C holonomy

Consistency check: fusion

Fusion of lines can be performed directly on the lattice, without ambiguity

O(2) irreps	**Wilson lines**
$\mathbb{1}$	1
$\mathbb{1}_{\rm det}$	η
$\mathfrak{2}_{q,\ q\geq 1}$	$W_q = W_{-q} \qquad (W_0 = 1+\eta)$

$\mathbb{1}_{\rm det} \otimes \mathbb{1}_{\rm det} = \mathbb{1}$	$\eta^2 = 1$
$\mathfrak{2}_q \otimes \mathbb{1}_{\rm det} = \mathfrak{2}_q$	$W_q\, \eta = W_q$ (b/c of projector)
$\mathfrak{2}_q \otimes \mathfrak{2}_{q'} = \mathfrak{2}_{q+q'} \oplus \mathfrak{2}_{\lvert q-q'\rvert}$	$W_q\, W_{q'} = W_{q+q'} + W_{q-q'}$
$\mathfrak{2}_q \otimes \mathfrak{2}_q = \mathfrak{2}_{2q} \oplus \mathbb{1} \oplus \mathbb{1}_{\rm det}$	$W_q\, W_q = W_{2q} + 1 + \eta$

Twisted sector extended operators

"$W_q - W_{-q}$"

$$\widetilde{W}_q(\gamma) = \frac{1+\eta(\gamma)}{2} \sum_{G_*=\pm 1} G_*\, \eta(\gamma_{*,\infty}) \exp\left(iq\, G_* \sum_{\ell\in\gamma} \eta(\gamma_{\ell,*})\, a_\ell \right)$$

Fate of old operators

Wilson line construction generalizes to any extended operator

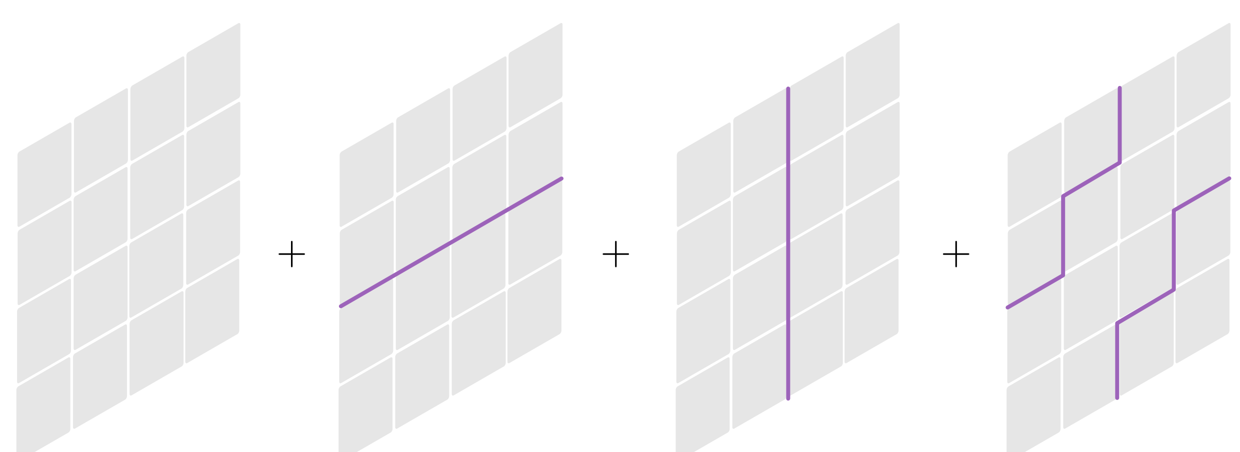
$$\mathcal{O}(M_j) = e^{i\sum_M X^{(j)}} \longrightarrow P(M_j) \sum_{G_*=\pm 1} e^{iG_* \sum_M \eta(\gamma_{j,*})\, X^{(j)}}$$

$$P(M_j) = \frac{1}{|H^1(M_j,\mathbb{Z}_2)|} \sum_{\gamma \in H^1(M_j,\mathbb{Z}_2)} \eta(\gamma)$$

"condensation defect" which **gauges** $\mathbb{Z}_{2,v}^{(d-2)}$ on M_j

[Roumpedakis, Seifnashri, Shao '22 **]**

(i.e. it **ungauges C** on M_j)

e.g. $P(T^2) = \frac{1}{4}\Big($  $\Big)$

Non-invertible symmetries

- **Electric 1-form symmetry** $\text{“}U_\theta(\tilde{\Gamma}_{d-2}) + U_{-\theta}(\tilde{\Gamma}_{d-2})\text{”}$
- **Magnetic (d-3)-form symmetry** $\text{“}V_\alpha(S) + V_{-\alpha}(S)\text{”}$

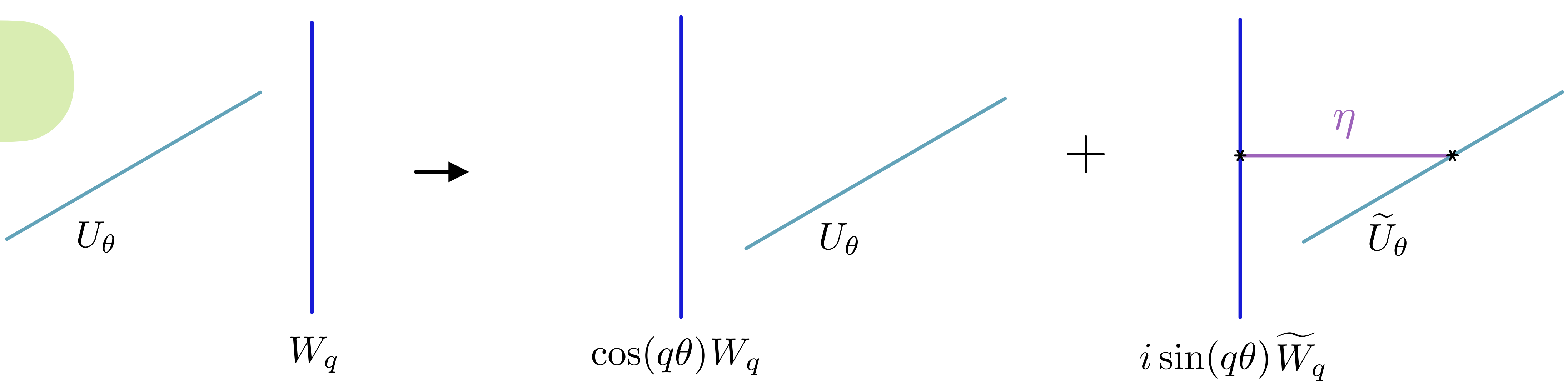
Both become non-invertible symmetries (non-invertible b/c of projector)

e.g. electric 1-form symmetry $U_\theta = U_{-\theta} = U_{\theta+2\pi}$

labelled by conjugacy classes of rotations in O(2)

Fusion $U_\theta U_{\theta'} = U_{\theta+\theta'} + U_{\theta-\theta'}$

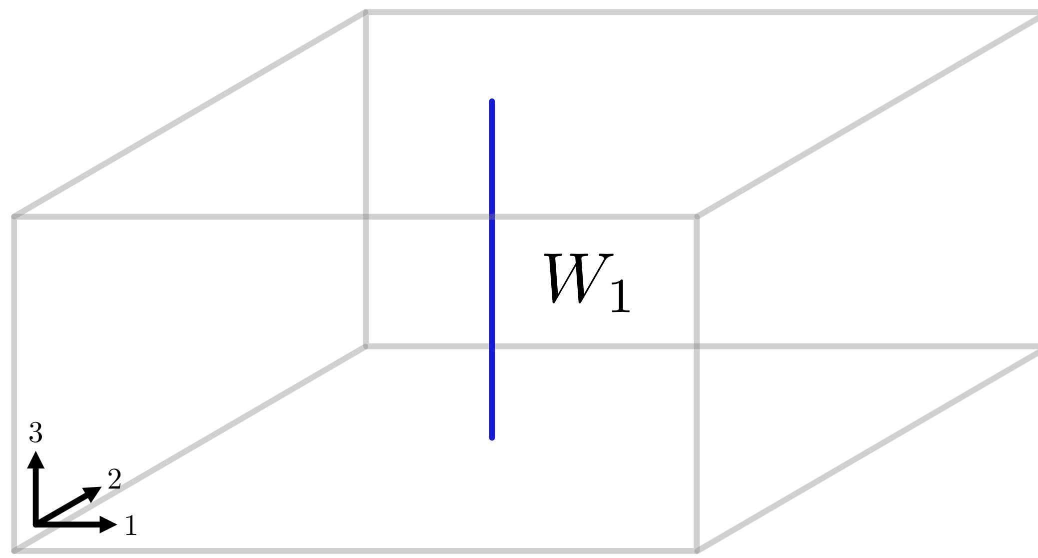
Action



Selection rules from non-invertible symmetries

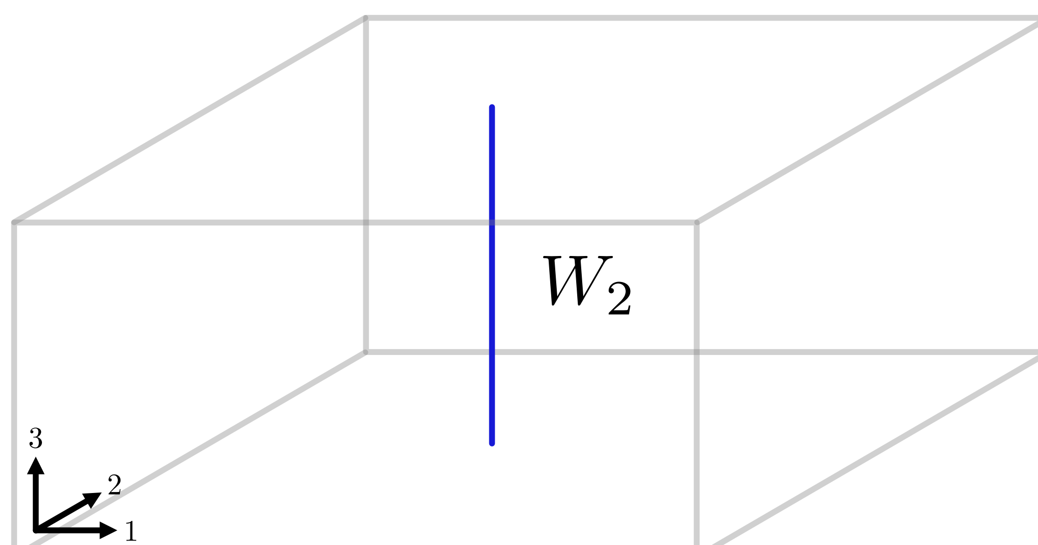
Peculiar action implies a difference in the nature of selection rules

Charged under invertible 1-form symmetry



$$= 0$$

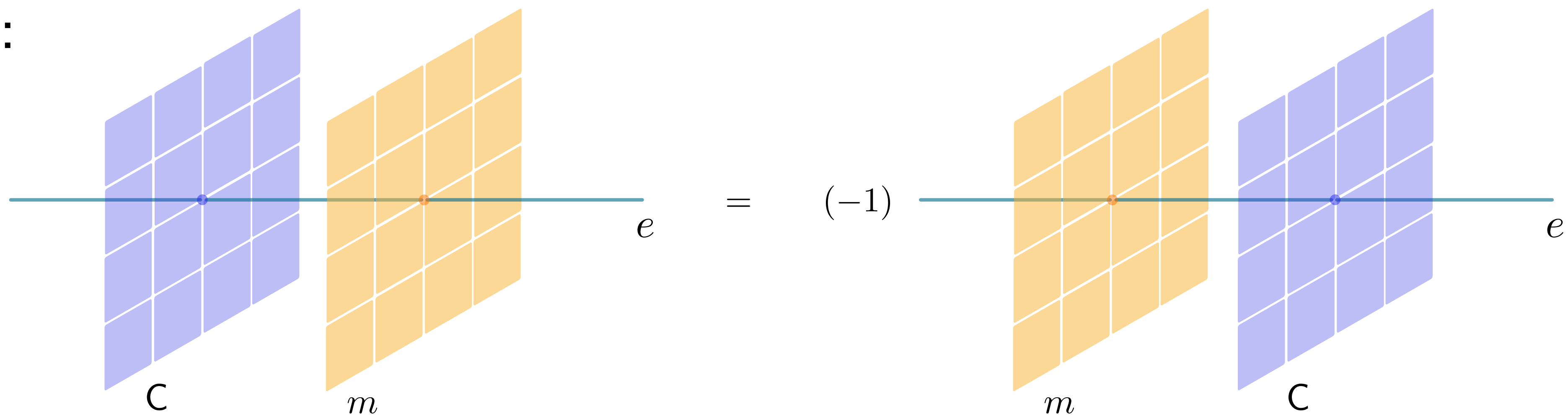
Charged under non-invertible 1-form symmetry



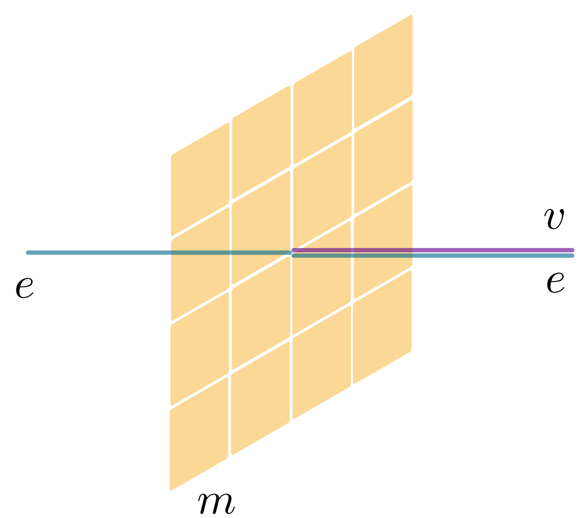
$$\sim e^{-\frac{1}{2\beta} L_3 \min(L_1, L_2)}$$

Higher-group symmetry

Before gauging:



After gauging:



Higher group: cannot activate $\mathbb{Z}_{2,e}^{(1)} \times \mathbb{Z}_{2,m}^{(d-3)}$ without also activating $\mathbb{Z}_{2,v}^{(d-2)}$

cannot **break** $\mathbb{Z}_{2,v}^{(d-2)}$ (i.e. introduce dynamical twist vortices)

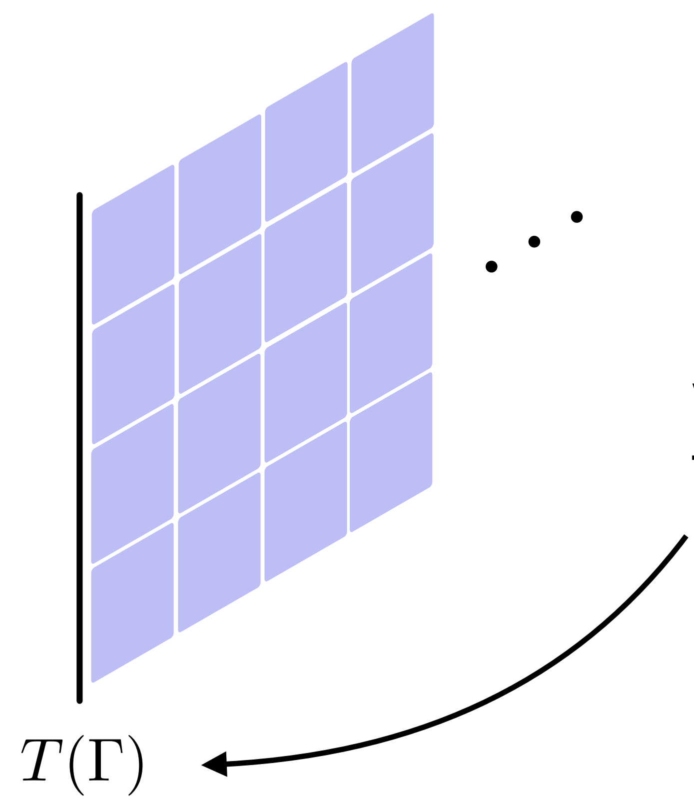
without also breaking $\mathbb{Z}_{2,e}^{(1)} \times \mathbb{Z}_{2,m}^{(d-3)}$

Revisiting the twist vortex

- **Twist vortex** $T(\Gamma_{d-2}) \stackrel{?}{=} e^{i\pi \sum_\Gamma v^{(d-2)}}$ = induces non-flat C gauge field

Invariance of the action under $\begin{array}{l} a \to a + d_c\lambda \\ \tilde{a} \to \tilde{a} + d_c\tilde{\lambda} \end{array}$ requires $dc = 0 \bmod 2$

There is **anomaly inflow** onto the twist vortex



Requires non-trivial worldvolume degrees of freedom to cancel anomaly

(Also follows from consistency w/ higher-group symmetry operators)

Anomaly matching on the twist vortex

- **d = 4** Twist-vortex = surface operator

Anomaly inflow can be cancelled with a **compact boson** on the worldsheet:

(for which there is a symmetry-preserving lattice discretization)

mixed anomaly

$$\big(U(1)_m \times U(1)_w\big) \rtimes \mathbb{Z}_{2,\mathrm{C}}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$q_m\, a_\ell \qquad q_w\, \tilde{a}_\ell \qquad c_\ell$$

$$q_m q_w = 2$$

"T_e" $q_m = 1,\ q_w = 2$

breaks $\mathbb{Z}_{2,e}^{(1)}$ preserves $\mathbb{Z}_{2,m}^{(1)}$

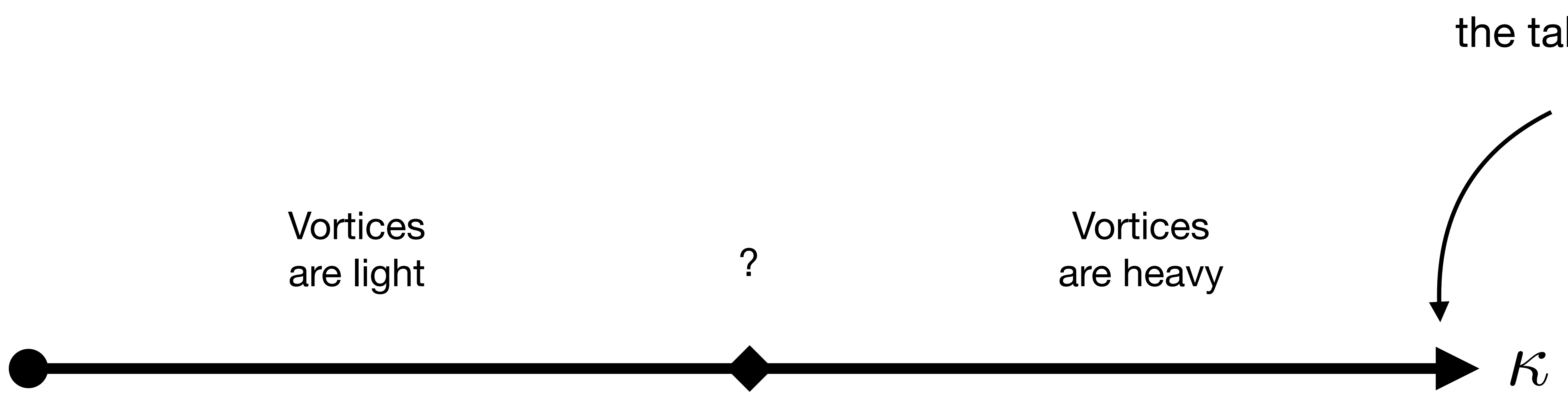
"T_m" $q_m = 2,\ q_w = 1$

breaks $\mathbb{Z}_{2,m}^{(1)}$ preserves $\mathbb{Z}_{2,e}^{(1)}$

Exchanged by **EM duality** in the bulk / **T-duality** on the worldsheet!

Adding dynamical twist vortices

$$i\pi \sum_{d\text{-cells}} v \cup dc \longrightarrow \frac{\kappa}{2} \sum_p ((dc)_p - 2h_p)^2$$



Gapped, Higgs

(worldvolume degrees of freedom condense in the bulk)

Gapless, Coulomb

(*emergent* non-invertible symmetries spontaneously broken)

Upshot: worldvolume degrees of freedom can affect **bulk** phases

Summary + conclusions

- Constructed O(2) gauge theory on the lattice
  - preserving continuum symmetries
  - tracked all operators through the gauging process
  - explored implications of various generalized symmetries

Future directions

- Hamiltonian formulation
- More exploration of non-invertible symmetries: anomalies, or higher-group type structure?
- Topological theories (non-abelian TQFT)
- General story about higher-group charged objects
- Exploring phase diagram (numerically? sign-problem free)
- General non-abelian Villain formulation (next talk by Jing-Yuan)