Chiral gauge theory on the lattice *(and questions for the continuum)*

DB Kaplan: Phys. Rev. Lett. 132 (2024) 141603, arXiv:2312.01494 DB Kaplan, S. Sen: Phys. Rev. Lett. 132 (2024) 141604, arXiv:2312.04012 DB Kaplan, S. Sen: in progress

- Chiral gauge theory and the Nielsen-Ninomiya theorem
- Edge states and topological phases
- A single connected phase boundary: a disc
- How to see free Weyl fermions on the lattice
- Gauging the theory
- A puzzle, and implications for the continuum?

Outline of this talk:

Chiral gauge theory, Nielsen-Ninomiya theorem

- A chiral gauge theory is one where a fermion mass term necessarily violates the
	-

gauge symmetry (i.e. the Standard Model)

A nonperturbative regulator does not exist for such theories $\circled{3}$

Fundamental tension between the need for a UV mass scale to tame divergences,

and a chiral gauge symmetry that forbids masses.

- •What does it mean to have a theory one cannot compute? Is it actually well-defined?
- How can we calculate nonperturbative physics without one? (E.g. EW baryon violation in the early universe)
- •Might a definition on the computer imply the need for new physics we do not expect in our continuum definition?

"SLAC derivative" violates #1

\n
$$
\widetilde{D}(p) = \sum_{\mu} i \gamma_{\mu} p_{\mu}
$$
\nNaive lattice fermions violate #3

\n
$$
\widetilde{D}(p) = \sum_{\mu} i \gamma_{\mu} \sin p_{\mu}
$$
\nWilson fermions violate #4

\n
$$
\widetilde{D}(p) = \sum_{\mu} i \gamma_{\mu} \sin p_{\mu} + M + \frac{r}{2} \sum_{\mu} (1 - \cos p_{\mu})
$$

Examples (a=1):

"SLAC derivative" violates #1 *D*

Wilson fermions violate #4

D. B. Kaplan ~ Chiral gauge theory on the lattice ~ YITP 11-11-24 . Kaplan \sim Chiral gauge theory on the lattice \sim YITP 11-11-24 symmetries of the quantum theory. The only way a symmetries of the only way a symmetry current can have a symmetry c

Nielsen-Ninomiya:

1. $\tilde{D}(\mathbf{p})$ is a periodic, analytic function of p_{μ} ; 2. $D(\mathbf{p}) \propto \gamma_{\mu} p_{\mu}$ for $a|p_{\mu}| \ll 1$; 3. $\tilde{D}(\mathbf{p})$ invertible everywhere except $p_{\mu} = 0$; 4. $\{\Gamma, \tilde{D}(\mathbf{p})\} = 0.$

Heuristic reasons behind NN theorem:

* If a chiral Dirac fermion existed, one could consider a lattice Weyl fermion using $P_{\pm}=(1 \pm \Gamma)/2$ projectors...

- * If the lattice had exact chiral symmetry and its continuum limit gave a massless Dirac
	-
	-

fermion, how could **anomalies** ever arise in the continuum?

…but how can a continuous periodic function P-D(p)P+ cross p=0 only once?

NN theorem tells us that there should be mirror fermions: incompatible with chiral gauge theory

Attempts to get rid of mirror fermions on the lattice:

1. Decouple them by breaking gauge symmetry and giving them a mass; restore gauge symmetry in continuum limit Golterman, Shamir

3. Eliminate mirror fermions by sacrificing locality (this work)

2. Gap the system and give masses to the mirrors without breaking gauge symmetry (many-body effects) Eichten, Preskill X.G. Wen

Chiral edge states appear naturally in the Integer Quantum Hall Effect: Analog for Dirac fermions with domain wall mass [Jackiw & Rebbi]:

With this domain wall mass profile, ϕ_+ is normalizable \rightarrow massless chiral edge state

$$
\left[\partial\!\!\! /+\gamma_5\partial_5+m(x_5)\right]\Psi=0
$$

Has solutions: $\Psi = \phi_{\pm}(x_5)\chi_{\pm}$

can be in two different topological phases depending on the sign of the

Why does a Dirac equation have a massless chiral edge state? Answer from condensed matter physics:

- A QFT with a free massive Dirac fermion in odd spacetime dimension mass…
- …so a domain wall is a boundary between two topological phases…
- gapless at the interface

•…the only way to connect two topological phases is for the theory to go

What is a topological phase?

Toy example: topological insulator in 0+1 dimensions — quantum mechanics with a gap

$$
H(s)\psi = E(s)\psi \ , \quad |E(s)| > \Delta
$$

Define topological quantum number: *ν = # of negative energy states.*

Theories with different parameter *s* are then topologically equivalent.

For the topology to change, e.g. # negative energy states, theory has to go gapless.

where S(p) is the fermion propagator. When the theory is regulated, this is a winding

$$
\int \frac{d^d p}{(2\pi)^d} \operatorname{Tr} S(p) \frac{\partial S^{-1}(p)}{\partial p_{\mu_1}} \cdots S(p) \frac{\partial S^{-1}(p)}{\partial p_{\mu_d}}
$$

What is topologically quantized in a QFT of massive Dirac fermions? for the anomalous divergences of the chlral fermton currents on the domain wall, then evidently its coefficient must also depend discontinuously on *mo/r* in a very particular way. We show in this letter that that does indeed

- In the Integer Quantum Hall Effect it is the Hall conductivity
- MONT The QFT analog is the coefficient of the Chern-Simons term GOOT obtained by integrating out the massive fermion in a background

number for the map $S^{-1}(p)$ from S^d (momentum space) to $S^d = SO(d+1)/SO(d)$ \blacksquare

FUCH A in the integer quantum Ha

coefficient c, will be dimensionless and the operator will not decouple for large for α $\alpha \geqslant \alpha$ in anti-shall position is the relevant portman in α " d2n+ Ip] $\frac{2000}{2000}$ gauge field.

regulator cannot change the divergence of the current, however. We thank M. Lfischer for this comment.

Using Ward identity, Chern-Simons coefficient in d= 2n+1 is proportional to rd Identity

$$
\kappa\epsilon_{abc...}\mathrm{Tr} A_a\partial_bA_c...
$$

E.g. Wilson fermions (DBK 1992; K. Jansen, M. Schmaltz 1993; M. Golterman, K. Jansen, DBK, 1993): and *R* = 34. The resulting lattice is shown in Fig. **??**. dilseli, ivi. Schiffaltz 1995, ivi. Golteffildif, N. Jalist
.

$$
\epsilon_{\mu_1...\mu_d}\int \frac{d^dp}{(2\pi)^d} \mathop{\rm Tr}\nolimits S(p) \frac{\partial S^{-1}(p)}{\partial p_{\mu_1}}\cdots S(p) \frac{\partial S^{-1}(p)}{\partial p_{\mu_d}}
$$

 $\epsilon_{\mu_1...\mu_d}$ $\int d^dp$ ϵ induced chern-Simons acuon for abelands and ϵ $\mathcal{V} \mu_1 ... \mu_d$ / Ω values \mathcal{V} late do not consider $\mathcal{L}\mathcal{L}\mathcal{L}$ \boldsymbol{v} index and the analysis by the angle \boldsymbol{v}

Remarkable fact:

Since the topology is in **momentum/spin space**, topological phases and massless edge states appear at domain wall boundaries on an infinite spacetime lattice $\frac{1}{2}$ **:ntum/spin space**, topological phases and massiess retaining those with nonzero eigenvalue to span our Hilbert space. For Figs. 2,3 we used *L* = 70

$$
\mathcal{D} = \gamma_{\mu}\partial + M + \frac{r}{2}\Delta
$$
\n
$$
\tilde{\mathcal{D}}(p) = M + \sum_{\mu} \left[i \sin p_{\mu}\gamma_{\mu} + \frac{r}{2}(1 - \cos p_{\mu}) \right]
$$
\n
$$
\phi_{\mu}\psi(x) = \frac{\psi(x + a\hat{\mu}) - \psi(x - a\hat{\mu})}{2a},
$$
\n
$$
\Delta\psi(x) = \frac{\psi(x + a\hat{\mu}) - 2\psi(x) + \psi(x - a\hat{\mu})}{a^2}.
$$

Nontrivial topological phases for 0 *< M r <* 2*d* with phase boundaries at

$$
\frac{M}{r}=0,2,\ldots,
$$

Phase diagram for QCD with Wilson fermions in 5d Euclidian spacetime

Obtain *almost* massless RH & LH Weyl fermions... mass $\propto e^{-2ML}$

periodic BC

Won't there be doubled copies of fermions on each wall?

 $-M_{\text{eff}}x_5$ of zeromode

At critical |pcrit| < π, Mef changes sign, state **delocalizes**

$$
M_{\text{eff}} \simeq M + r \sum_{i=1}^{d} (\cos p_i - 1)
$$

- 1. $D(p)$ is a periodic, analytic function of p_{μ} ;
- 2. $D(\mathbf{p}) \propto \gamma_{\mu} p_{\mu}$ for $a|p_{\mu}| \ll 1$;
- 3. $\tilde{D}(\mathbf{p})$ invertible everywhere except $p_{\mu} = 0$; $4. \{ (\sqrt{7}) \} = 0.$ The first condition is required for locality of the Fourier transform of *D*˜(p) in

can be the gained: *Cananda* satisfy all four of the four of the four of the following conditions simulated: $\frac{1}{2}$ What has been gained?? Wanted:

• reproduces the correct chiral anomalies **For continuum for continuum** each lattice field; Creating field; Creations are the least redundant, giving rise to the least rise to the le for employees in any produce in assuming the discussion in any contribution is an any produced in a space of the space •but still enforces multiplicative mass renormalization

coordinate space. The next two state that we want a single flavor of conventional Dirac With exponentially light Dirac fermion, #4 is violated. can try keeping that and eliminating one or more or more or α the other conditions; for example, for example, for example, α dvantage of domain wall fermions over Wilson fermions? Any advantage of domain wall fermions over Wilson fermions?

Yes...
$$
\left\{\tilde{D}, \Gamma\right\}
$$
 = $\tilde{D}\Gamma\tilde{D}$ Obeys "Ginsparg-Wilson" equation

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☜ locality

☜ correct continuum limit

☜ no doublers

 ∞ exact chiral symmetry ($\Gamma = \gamma_5$)

*Irg***-Wilson"** equation

Domain fermions have the attractive feature of being topological and "knowing" about anomalies

Proposals to use them for evading Nielsen-Ninomiya theorem and constructing a lattice chiral gauge theory:

•Ginsparg-Wilson approach (Lüscher): use GW fermions (Abelian chiral gauge theories constructed this way, but not non-Abelian). Sacrifices NN #4 (D anti-

- commuting with $y_5...$ involves $O(a)$ corrections)
- cancel
- between topological phases to regulate chiral gauge theory.

•Symmetric mass generation (Eichten, Preskill, Wen, Cenke, You, Wang…): invoke many-body physics to gap unwanted mirror fermions when anomalies

•Proposal here: use domain wall fermion with **single** connected boundary

Edge states on manifold with a single boundary:

Consider Dirac fermion in d+1 *continuum* dimensions: M^{d-1} x R² with coordinates $\{x_{\perp}$ \overline{a}

- 2*r r* 2*r r* Shouldn't this have a single Weyl fermion edge state?
	-
- Which can be realized with Wilson fermions on a lattice?
	- **D. B. Kaplan ~ Chiral gauge theory on the lattice ~ YITP 11-11-24** and \overline{a} is the angular momentum operators and \overline{a}

Which Which must be exactly massless?

and $\frac{1}{2}$ is the angular momentum operator of the angular momentum operators

d = 4). In polar coordinates we have *d* = 4). In polar coordinates we have

Why there can't be a chiral edge state: reason #1

…looks like wall/anti-wall system with finite size

…expect RH + LH modes with exponentially small chiral symmetry violating mass

- If there is an exact chiral edge state, then there must be a solution that is dimension Dirac operator on the disc
- Zeromode solutions are easy to solve for!
- And it is easy to show that there isn't a zeromode for the Dirac operator on disc!

Why there can't be a chiral edge state: reason #2

independent of angle (zero momentum) which is an exact zero-mode of the higher

NO ZERO momentum edge STATE

- Since topological phases exist with Wilson fermions on a lattice, we should be able to easily construct disc edge states on a lattice if they exist in the continuum!
- …but the Nielsen Ninomiya theorem says that we on a lattice we must have an analytic, periodic dispersion relation which cannot cross zero an odd number of times (eg, once)

$$
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

Why there can't be a chiral edge state: reason #3

Think less, calculate more

Solve the Dirac equation with this mass profile (DB Kaplan: Phys. Rev. Lett. 132 (2024) 141603, arXiv:2312.01494)

D†r = -*"^r*

 $D. B.$ Kaplan ~ Chiral gauge theory on the lattice ~ YITP 11-11-24 *^Â–ⁱ* ⁼ ^X \overline{D} . *ˆ^r* + \overline{a} n \sim $Chíri$ *"◊J* + *m*(*r*) \sim YITP 11-11-24 block in our direct product notation for the Dirac matrix \mathcal{L} and \mathcal{L} matrix \mathcal{L} matrix \mathcal{L} matrix \mathcal{L} **THE AND AND AND AND CONTROL CONCLUSE CONCLUSE** 2
2. B. Kaplan ~ C $Chiral$ gauge theory on the lat while \mathcal{A} is unchanged. The \mathcal{A} is unchanged. The \mathcal{A} is unchanged. The \mathcal{A}

A convenient basis: A convenient basis: **~** diagonal, such as a conv A convenient basis:

$$
\vec{\gamma}_\perp = \sigma_3 \otimes \vec{\Gamma} \ , \quad \gamma_x = \sigma_1 \otimes 1 \ , \quad \gamma_y = \sigma_2 \otimes 1
$$
\n
$$
\gamma_r = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} \ , \quad \gamma_\theta = \begin{pmatrix} 0 & -ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}
$$

$$
S = \int d\mathbf{x}_{\perp} \int r dr d\theta \, \overline{\psi} \left(\partial_{\perp} + \mathcal{D} \right) \psi
$$

\ndim= $d - 2$
\n
$$
\mathcal{D} = \gamma_x \partial_x + \gamma_y \partial_y + m(r)
$$

\n
$$
= \gamma_r \left(\partial_r + \frac{1}{2r} \right) + \frac{i}{r} \gamma_\theta \mathcal{J} + m(r)
$$

$$
\mathcal{J} = -i\partial_{\theta} + \frac{1}{2}\Sigma \; , \qquad \Sigma = -\frac{i}{2}\left[\gamma_x,\gamma_y\right]
$$

Find:

-
- higher dimension theory
- The total angular momentum coordinate (*-j/R*) plays the role of linear momentum around the disc edge two results to result and the disc edge

-j/R ~ momentum in boundary world itly in an expansion in inverse powers of *mR*, with the $-j/R \cong {\sf momentum}$ in bounda μ_j = $-\frac{j}{L}$ *R* $\sqrt{}$ $1 +$ 1 $\frac{1}{2mR}$ + 1 $\frac{1}{2m^2R^2} +$ 3 $\frac{6}{4m^3R^3}$ + 3 $2m^4R^4$ $+$ 15 $4m^5R^5$ $\overline{1}$ $+\frac{j^3}{D}$ *R* 5 1 $\frac{1}{4m^4R^4}$ + 3 $2m^5R^5$ $\overline{1}$ $+ O ((mR))$ -6 t / D χ_n *.* (16) *ˆ/* = 0 *≈ ≈* $\frac{1}{\sqrt{2}}$ *· ˆ* $\ddot{\bullet}$ $\frac{3}{3\,D^3} + \frac{3}{2\,2\,4\,D^4}$ $+\frac{1}{4m^5}$ $\frac{1}{T}$ $\sqrt{25}$ $\left[\frac{1}{2} + \frac{3}{2} \right] + O((mR)^{-6})$ (21) $\sqrt{4m^4R^4}$ + $\sqrt{2m^5}$

 $\mathcal{L}(R)$ plays the role of linear momentum *j*.

, (21)

D. B. Kaplan ~ Chiral gauge theory on the lattice ~ YITP 11-11-24 consider the Dirac operator in *d*-dimensions in a chiral ry on the lattice \sim γ ITP 1 $I_{\text{max}} \sim \text{Delta}$ b *,* Finally, compactifying the *dth* dimension to a circle of

• There is an exact Weyl edge mode circulating the disc in only one direction • Its chiral symmetry is exact: part of the exact U(1) fermion number symmetry of the ² *^b^j* (*R*) = ⁰ *,* (20) with α playing the role of α playing the role of α in addition to the surface α in addition to the surface α

Precisely: Fuclidian action Precisely: Euclidian action of edge mode is

To interpret the boundary mode action we found in eq. (1113) superiority the possible: a
Martin *eⁱ*(*j*-1*/*2)*◊ ^I*|*j*-1*/*2|(*Ÿ^j ^r*) R *ˆ/* = *≈* ~ **What happened to all those arguments that this shouldn't be possible?**

· ˆ

‹ - *p*^Î 0

$$
S = \int d\mathbf{x}_{\perp} \sum_{n} \bar{\chi}_{n} (\vec{\Gamma} \cdot \vec{\partial}_{\perp} + \mu_{n}) \chi_{n} .
$$

\nIn d=1+1, $\vec{\Gamma} = 1$
\nIn d=3+1, $\vec{\Gamma} = \vec{\sigma}$

Why there can't be a chiral edge state: reason #1

…looks like wall/anti-wall system with finite size

…expect RH + LH modes with exponentially chiral symmetry violating mass

…**but** the wall/anti-wall system had constant γ5… ... but the wall/anti-wall system had const $\overline{\mathsf{S}^{\mathsf{t}}}$

on disc, analog of y_5 for edge states is on disc, analog of v_5 for edge states is

... it violates locality though! *^Â*¯*–ⁱ* ⁼ ^X

which changes sign on opposite which changes sign on opposite side of disc!

I
P in higher dimen Exponentially small interaction is still there, but preserves chirality 100% (= fermion number in higher dimension theory)

$$
\gamma_r = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}
$$

- If there is an exact chiral edge state, then there must be a field that is higher dimension Dirac operator on the disc
- Zeromode solutions are easy to solve!
- And it is easy to show that there isn't one!

Why there can't be a chiral edge state: reason #2

independent of angle (zero momentum) which is an exact zero-mode of the

- ...but we have seen that momentum about the edge is given by $-j/R$ and $j=\pm 1/2$, $\pm 3/2...$
- There is no zero momentum edge state it behaves as if anti-periodically quantized!

…and therefore no exact zeromode in Euclidian spacetime.

Why there can't be a chiral edge state: reason #3

- Since topological phases exist with Wilson fermions on a lattice, we should be able to easily construct disc edge states on a lattice if they exist in the continuum!
- …but the Nielsen Ninomiya theorem says that on a lattice we must have an analytic, periodic dispersion relation which cannot cross zero an odd number of times (eg, once)
- …but we have already argued that there must be some nonlocality in the theory, so perhaps the dispersion is not analytic & periodic and that's OK?

Let's look at what happens on a lattice DB Kaplan, S. Sen: Phys. Rev. Lett. 132 (2024) 141604 arXiv:2312.04012

Free Wilson fermions on a 2d spatial lattice: consider 3 different boundary conditions tice: consider 3 different houndary conditions we can significantly work on a square lattice. As discussed in a square lattice \mathcal{L}

• **Periodic** boundary conditions in x & y

topology = torus, no boundaries

• Mixed: **periodic** in y + **open** BC in x

topology = open cylinder, 2 disconnected boundaries

• **Open** boundary conditions in x & y

topology = disc, 1 connected boundary

D. B. Kaplan \sim Chiral gauge theory on the lattice \sim YITP 11-11-24 We will take *M* = *r* = 1 in lattice units since for those

$$
\begin{array}{lcl} H & = & \gamma_0 \mathcal{D} \; , \\ & \\ \mathcal{D} & = & \displaystyle \sum_{\mu=1}^2 \gamma_\mu \partial_\mu + M + \frac{r}{2} \varDelta \end{array}
$$

10

20

30

40

50

60

70

Work on a lattice disc with open BC

Weyl edge state? Look at 1+1 dispersion relation

$$
P_R = \begin{cases} 0 & x^2 + y^2 \ge R^2 \\ 1 & x^2 + y^2 < R^2 \end{cases}
$$

 $H_{\text{disc}} = P_R H_{L \times L} P_R$

We took $L=70$, $R = 34$.

If you want E vs p for the edge state, plot E vs J

Nielsen-Ninomiya would have you believe this is not possible for sensible system

[5] David B. Kaplan, "A method for simulating chiral fermions on the lattice," Phys. Lett. B **288**, 342–347 (1992), arXiv:heplate: chi [6] Karl Jansen and Martin Schmaltz, "Critical momenta of lattice chiral fermions," Phys. Lett. B **296**, 374–378 (1992), concent of acool Note: chiral edge states on a 2-sphere boundary were previously discussed in the [5] David B. Kaplan, "A method for simulating chiral fermions on the lattice," Phys. Lett. B **288**, 342–347 (1992), arXiv:hepcontext c [4] H. B. Nielsen and M. Ninomiya, "Absence of Neutrinos on a Lattice. 1. Proof by Homotopy Theory," Nucl. Phys. B **185**, context of describing Weyl fermions in a gravitational background:

[8] Shoto Aoki and Hidenori Fukaya, "Curved domain-wall fermions," Progress of Theoretical and Experimental Physics **2022**, 063B04 (2022). choto Acki and Hidenori Fukaya "Curved domain-wall fermions" Progress of Theoretical and Experimental Physics 2022 Phys. 1.011 and 111 and 11 2016, 21, 31, 32, 3011 and 10111
063R04 (9099) [8] Shoto Aoki and Hidenori Fukaya, "Curved domain-wall fermions," Progress of Theoretical and Experimental Physics **2022**, finato Aoki and Hidenori Fukaya, "Curved domain-wall fermions," Progress of Theoretical and Experimental Physics 2022 $063B04(2022)$ $\mathcal{N}_{\mathrm{max}}$ Maarten F. L. Golden, and David B. Kaplan, "Chern-simons currents and chiral fermions on the lattice," $\mathcal{N}_{\mathrm{max}}$

Experimental Physics **2023**, 033B05 (2023). Shoto Aoki and Hidenori Fukaya, "Curved domain-wall fermion and its anomaly filliow, "Frogress of Theoretical and of Theoretical and Experimental Physics **2024**, 043B05 (2024). Shoto Aoki and Hidenori Fukaya, "Curved domain-wall fermion and its anomaly inflow," Progress of Theoretical and Experimental Physics **2023**, 033B05 (2023).

[12] Dorota M Grabowska and David B Kaplan, "Nonperturbative regulator for chiral gauge theories?" Phys. Rev. Lett. **116**, Shoto Aoki, Hio of Theoretical and Experimental Physics 2024, 043B05 (2024). Shoto Aoki, Hidenori Fukaya, and Naoto Kan, "A lattice formulation of weyl fermions on a single curved surface," Progress of Theoretical and Experimental Physics **2024**, 043B05 (2024).

D. B. Kaplan ~ Chiral gauge theory on the lattice ~ YITP 11-11-24 [16] Maarten Golterman and Yigal Shamir, "Conserved currents in five-dimensional proposals for lattice chiral gauge theories,"

Last (important!) piece of the puzzle: how to gauge?

d+1 theory with *Nf* flavors has exact *U(Nf)* global symmetry…can easily gauge a subgroup in the continuum or the lattice. The gauge measure is well defined because its a regulated a theory of Dirac fermions $d+1$ theory with N_f flavors has exact $U(N_f)$ global symmetry...can easily gauge a sists of Dirac fermions with an exact *U*(*N*) global symsubgibup in the continuum of the lattice. phase is no consequence in the free theory, has no consequence the free theory, $\frac{1}{2}$ Badde medsare is wen defined in defections org. As in Ref. [34], the phase ambiguity is resolved by a single by a single

values on the surface, and not have independent bulk de-surface, and not have independent bulk de-surface, and
The surface, and not have independent bulk de-surface, and not have independent bulk de-surface, and the surfa

Define bulk gauge fields B_{μ} to be functionals of the boundary values A_{μ} ; integrate only over the A_μ Dofine bulk gauge fields R to be function **BUP**
P
P

$$
B_{\mu}(\mathbf{x}_{\perp}, r, \theta)\Big|_{r=R} = A_{\mu}(\mathbf{x}_{\perp}, \theta)
$$

Earovample *P* can be colution to Euclie For example, \boldsymbol{b}_{μ} can be solution to Eucht For example, B_μ can be solution to Euclidian YM eq. subject to this BC.

D. B. Kaplan ~ Chiral gauge theory on the lattice ~ YITP 11-11-24 \sum_{MSE} INSTITUTE for
 \sum_{MSE} V Measure \sum_{MSE} S*S* **The** *d* different definition and the *d*-dimensional A-dimensional A-dimension. r the <mark>Latti</mark>c

...but want a d-dimensional gauge theory, not d+1...unlike CM systems *v*, not d+1...unlike CM systems ulated sum of signs of the eigenvalues of the eigenvalues of the eigenvalues of the eigenvalues of the bulk Di
Diracted sum of the bulk Diracted sum of the bulk Diracted sum of the bulk Diracted sum of the eigenvalues of of the houndary values A_{\cdot} . in the presence of the *B^µ* bulk gauge field² The functional

is some functional of *B^µ* that needs to be uniquely de-

only depends on its boundary value, the physical gauge

$$
B_{\mu}(\mathbf{x}_{\perp}, r, \theta) \Big|_{r=R} = A_{\mu}(\mathbf{x}_{\perp}, \theta)
$$

For example. B_u can be solution to Euclid μ can be finding integrated over integrating over integrating μ For example, B_{μ} can be solution to Euclidian YM eq. subject to this BC.

 $\mathcal{P}_{\text{NUCLEAR THE GRY}}$ **D. B. Kaplan ~ Chiral gauge theory on the lattice ~ YITP 11-11-24**

but the phase exp(-*ifi÷*[*B*]) of the fermion determinant \boldsymbol{v} independent of the gauge field in the gauge field in the \boldsymbol{v}

erator, but contains additional information. When the contains additional information. When the contains a second
The contains addition in the contains and the contains and the contains and the contains and the contains an edge states are in a representation free of gauge anomalies, not only does the Chern-Simons contribution cancel, only depends on its boundary value, the physical gauge

\mathcal{L}_{max} 4d boundary fields are quantum; 5d bulk fi *D^µ* = *ˆ^µ* + [*Bµ, ·*] *, ICAI SUDJECL LO QUANTUIN DU* 4d boundary fields are quantum; 5d bulk fields are classical subject to quantum BC

of gauge anomalies in defining a *d*-dimensional chiral generate chern simons operator in the buik functional of the d-dimensional gauge fields A_μ

$$
B_{\mu}(\mathbf{x}_{\perp}, r, \theta)\Big|_{r=R} = A_{\mu}(\mathbf{x}_{\perp}, \theta)
$$

For example. B_u can be solution to Euclid μ can be finding integrated over integrating over integrating μ but the phase exp(-*ifi÷*[*B*]) of the fermion determinant \boldsymbol{v} independent of the gauge field in the gauge field in the \boldsymbol{v} For example, B_{μ} can be solution to Euclidian YM eq. subject to this BC.

In general this will give a ter It the possible then become possible the role of the r In general this will give a terribly nonlocal theory:

its coefficient vanishes if edge chiral aguments. What its coefficient vanishes if edge chiral aguments. the *B^µ* fields being nonlocal functionals of the *A^µ* gauge th above boundary condition, f_{max} *…but its coefficient vanishes if edge chiral gauge theory is anomaly-free**

Coniecture: this theory will be a local d-dimensional theory in of the edge states as being a local *d*-dimensional gauge theory. The child gauge theory is infrared *iff* the chiral gauge theory is anomaly-free Conjecture: this theory will be a local d-dimensional theory in the

 $*$ Mare precisely: α term α ainniel includes no represented the gauge term of the gauge theory in *<u>I Anomalies (see Witten, Yonekura)*</u> **More precisely: CS term -> eiπη[A] , includes nonperturbative anomalies (see Witten, Yonekura)*

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Generate Chern Simons operator in the bulk which is a function of B_μ and therefore a nonlocal

Criticisms by Aoki, Fukaya, Kan and by Golterman & Shamir:

Aoki, Fukaya, Kan:

When there are nontrivial gauge field configurations on the boundary, there fermion zeromodes

on the surface are paired with zeromodes in the bulk interior

Shoto Aoki and Hidenori Fukaya, "Curved domain-wall fermion and its anomaly inflow," Progress of Theoretical and Experimental Physics 2023, 033B05 (2023). Shoto Aoki, Hidenori Fukaya, and Naoto Kan, "A lattice formulation of Weyl fermions on a single curved surface," (2024), $arXiv:2402.09774$ [hep-lat]. Shoto Aoki, "Study of curved domain-wall fermions on a lattice," (2023), arXiv:2404.01002 [hep-lat].

There is an exactly conserved, gauge invariant current for every Weyl fermion on the boundary,

Golterman and Shamir: unlike in target 4d theory

> Maarten Golterman and Yigal Shamir, "Conserved currents in five-dimensional proposals for lattice chiral gauge theories," Phys. Rev. D 109, 114519 (2024).

These criticisms are apparently related: bulk zeromodes appear because of conserved U(1)

Golterman and Shamir:

- For every boundary Weyl fermion, have one bulk massive Dirac fermion
- Exact global $U(1)$ symmetry for each bulk fermion with 5d conserved current
- Can construct exactly conserved 4d currents by integrating 5d currents over r
- Leads to too much symmetry for boundary theory…eg, $N_f=1$ QCD on boundary has exact U(1) x U(1) symmetry

Bug or feature ?? Integrate out massive bulk modes, find for 5d conserved current: Problem! E.g., 4d QCD with N_f=1 would have exact $U(1)_V \times U(1)_A$ symmetry

GS currents:
$$
j_{\mu}(x) = \int r dr J_{\mu}
$$

Can show: $\partial_{\mu} j_{\mu}(x) = 0$

$$
\epsilon_{\mu b c d e} F_{b c} F_{d e} \qquad \qquad \mu = 1, \ldots, 4
$$

 $J_5(x,r) = \kappa \theta(R-r) \epsilon_{5bcde} F_{bc} F_{de}$ $J_{\mu}(x,r) = \delta(R-r)\bar{\chi}\sigma_{\mu}D_{\mu}\chi + \kappa\,\theta(R-r)\epsilon$ chiral edge state contribution

bulk gauge field contribution

 $J(f) = \mu b c d e F b c F d e$ $\mu = 1, \ldots, 4$

is found to be equivalent to the conventional anomalous Ward identity on the boundary

$$
\kappa\epsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}(x)
$$

This is a **feature**, not a bug! Current conservation in the 5d theory = 4d "anomaly inflow"

However… integrating out bulk modes assumed no light states in interior… What about Aoki-Fukaya-Kan-Golterman-Shamir zeromodes??

$$
J_{\mu}(x,r) = \delta(R-r)\bar{\chi}\sigma_{\mu}D_{\mu}\chi + \kappa \theta(R-r)
$$

chiral edge state contribution

$$
J_{5}(x,r) = \kappa \theta(R-r)\epsilon_{5bcde}F_{bc}F_{de}
$$

Golterman-Shamir equation $\partial_{\mu} j_{\mu}(x)=0$ $\partial_{\mu}(\bar{\chi}\sigma_{\mu}\chi(x))=-$

bulk gauge field contribution

Integrating out bulk modes is not justified when boundary gauge field has nontrivial topology… Aoki-Fukaya-Kan-Golterman-Shamir criticism is a **bug** then, not a feature

It seems that expected theory cannot be achieved for nontrivial topology in boundary gauge field (e.g. instantons)

Very weird: whenever there are instantons, the 4d world becomes aware of mirror zeror lurking in the 5th dimensions?! …

For regulating the SM though, how about if we restrict to **trivial** topology? (Eg, constrain number of instantons = number of anti-instantons)

on boundary and the nonsingular gauge fermion zeromode

Instanton in boundary theory

Aoki-Fukaya-Kan-Golterman-Shamir problems seem to go away if the topology of boundary theory is trivial, Q=0. (# instantons $=$ # anti-instantons, imposed on boundary theory)

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field in interior, no bulk zeromodes

For regulating the SM though, how about if we restrict to trivial topology? (Eg, constrain number of instantons = number of anti-instantons)

Has been shown that $Q=0$ QCD is equivalent to integrating over θ ... \rightarrow yields θ =0 QCD + 1/volume corrections

- Solves strong CP problem
- No axion particle
- Saturates Goldstone theorem for spontaneously broken exact U(1)
- Work in progress (DBK & S Sen)

Does this mean one can only regulate SM with θ =0??

Integrating over θ is equivalent to having an axion field…and then throwing away all of it except the p=0 mode

Finite volume QCD at fixed topological charge Sinya Aoki, Hidenori Fukaya, Shoji Hashimoto, Tetsuya Onogi, PHYS. REV. D76, 054508 (2007)

An excitingly simple picture is emerging: Chiral gauge theory as a boundary theory, without requiring new dynamics

Does it work? Too early to tell…

….but the Nielsen-Ninomiya theorem is no longer the obstacle.

theory can be regulated, giving θ =0 theory in large volume

Conclusions

Construction "understands" anomalies: local 4D theory emerges only if gauge anomalies cancel (discrete and perturbative)

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- **It appears that the theory is not purely 4d unless boundary gauge field topology** is trivial (what does theory with nontrivial topology look like?? The η' portal \mathbb{C})
- **If gauge topology is trivial and anomalies cancel, it appears that chiral gauge**
	- Could it be that 4d chiral gauge theory can only be regulated if anomalies

P cancel *and* θ=0? Is there a BSM scenario that realizes this physics?

