LATTICE QCD AT NON-ZERO BARYON DENSITY

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Outline

1) QCD in the grand canonical ensemble and the sign problem

2) Baryon number fluctuations

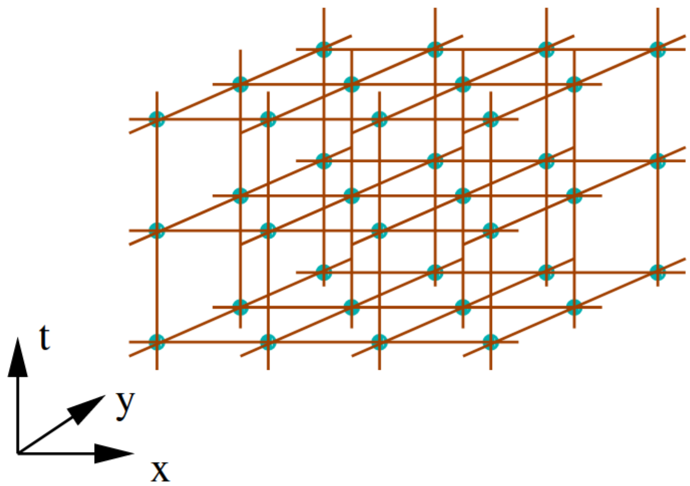
3) Chiral and deconfinement aspects of QCD at μ_B>0

4) Summary and outlook

The lattice formulation of QCD

Finite space-time lattice: $N_s^3 N_t$

Equilibrium physics: $T = \frac{1}{N_t a}$



1. Continuum limit:

For fixed temperature $a \to 0 \Leftrightarrow N_t \to \infty$

(Fixed N_t: Lower $T \Rightarrow$ Larger a (coarser))

2. Thermodynamic limit:

Size is often measured in units of 1/T

Aspect ratio: $LT = N_s/N_t$

Infinite volume limit: $LT \to \infty$

QCD in a small box is physics, a coarse lattice in a large box is not!

QCD in the grand canonical ensemble

$$\hat{p} := \frac{p}{T^4} = \frac{1}{(LT)^3} \log \mathrm{Tr}\left(e^{-(H-\mu_B B-\mu_S S-\mu_Q Q)/T}\right) \quad \text{(pressure)}$$

We set $\mu_Q = 0$ in what follows.

$$\chi_{ij}^{BS} = \frac{\partial^{i+j}\hat{p}}{\partial\hat{\mu}_B^i \partial\hat{\mu}_S^j} \qquad \left(\hat{\mu}_{B/S} := \frac{\mu_{B/S}}{T}\right) \quad \text{(susceptibilities)}$$

DERIVATIVES ⇔ FLUCTUATIONS/CORRELATIONS:

$$\chi_1^B \propto \langle B\rangle \propto n_B; \quad \chi_2^B \propto \langle B^2\rangle - \langle B\rangle^2; \quad \chi_{11}^{BS} \propto \langle BS\rangle - \langle B\rangle\langle S\rangle$$

Observables for confinement

Center symmetry:

- $\mathrm{m_q} \to \infty$ (pure gauge theory)
- discrete Z_3 symmetry
- spontaneously broken at high T (weak 1st order transition)

Polyakov-loop: $\mathrm{P} = \frac{1}{\mathrm{V}} \frac{1}{\mathrm{N_c}} \sum_{\vec{x}} \mathrm{Tr} \prod_{t=0}^{N_t-1} U_4(\vec{x},\ t)$

An order parameter for confinement in pure gauge theory

$$|\langle P \rangle| = e^{-F_Q/T}$$

Static quark free energy $F_Q = -T \log |\langle P \rangle|$

needs additive renormalization, e.g. $F_Q^R = F_Q(T) - F_Q(T_0)$

Static quark entropy $S_Q = -\frac{\partial F_Q}{\partial T}$ (See [Bazavov et al, PRD93 (2016)])

Obervables for chiral symmetry

Chiral symmetry:

- $\mathrm{m_q} \to 0$ (chiral limit)
- $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$
- spontaneously broken at low T (likely second order for N_f=2)

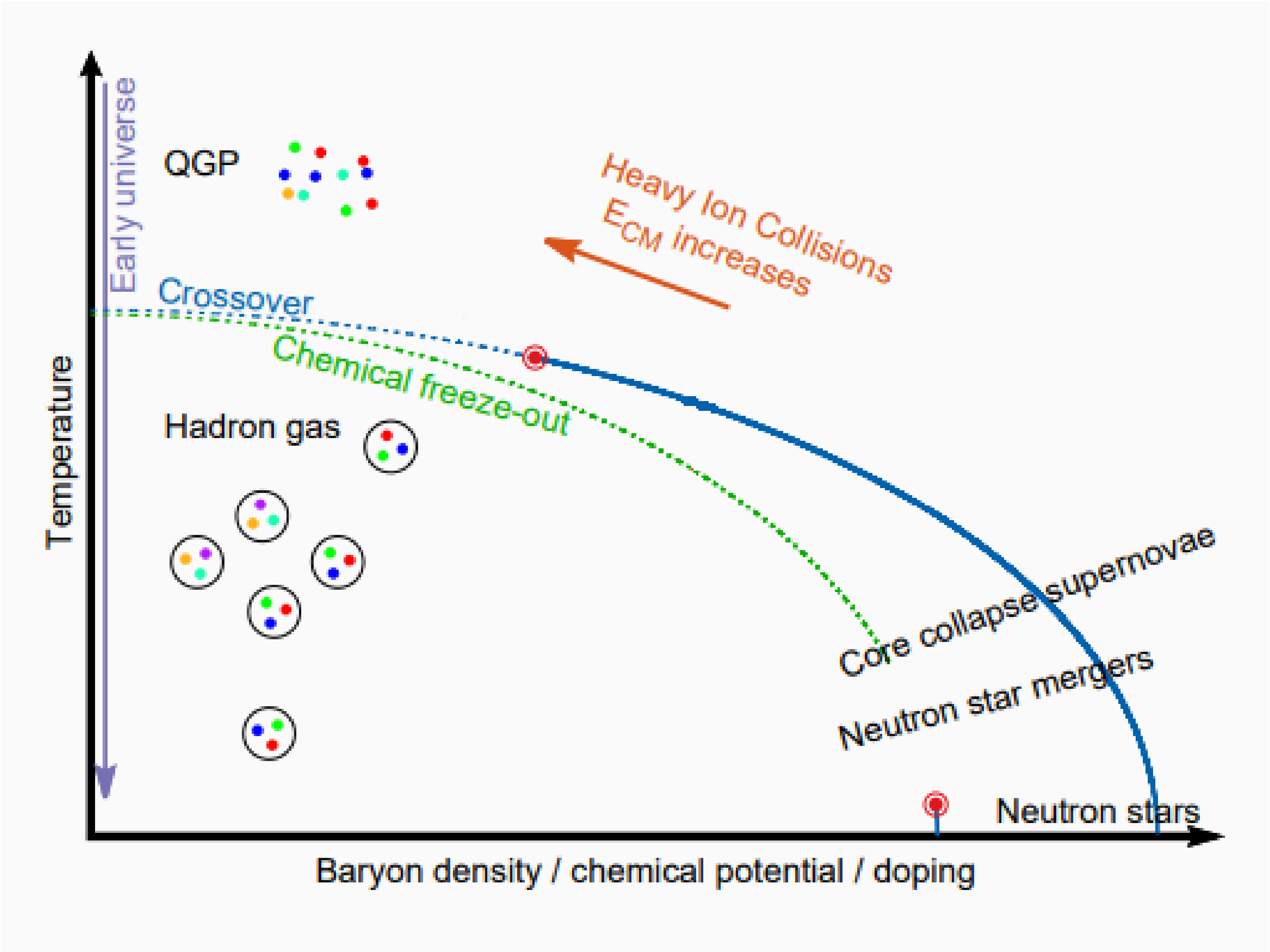
Chiral condensate: order parameter for chiral symmetry breaking in the two-flavour chiral limit

$$\langle\bar{\psi}\psi\rangle = \frac{T}{V}\frac{\partial \ln Z}{\partial m_{ud}} \qquad \langle\bar{\psi}\psi\rangle_R = -\left[\langle\bar{\psi}\psi\rangle_T - \langle\bar{\psi}\psi\rangle_0\right]\frac{m_{ud}}{f_\pi^4}$$

Chiral susceptibility:

$$\chi = \frac{T}{V}\frac{\partial^2 \ln Z}{\partial m_{ud}^2} \qquad \chi_R = \left[\chi_T - \chi_0\right]\frac{m_{ud}^2}{f_\pi^4}$$

The conjectured phase diagram



What to do with strangeness?

In the phase diagram sketch on the prev. slide, there were only two axes.

1. Zero strangeness chemical potential (simpler):

$$\mu_S = 0$$

2. Zero strangeness density (more realistic):

Tune $\mu_S(T, \mu_B)$ such that $\chi_1^S(T) = 0$

Later, I will compare the phase diagram with these two conditions.

The QCD path integral

$$Z = \int DA_\mu D\bar{\psi}D\psi e^{-S_{YM}-\bar{\psi}M(A_\mu,m,\mu)\psi} = \int DA_\mu \det M(A_\mu,m,\mu)\, e^{-S_{YM}}$$

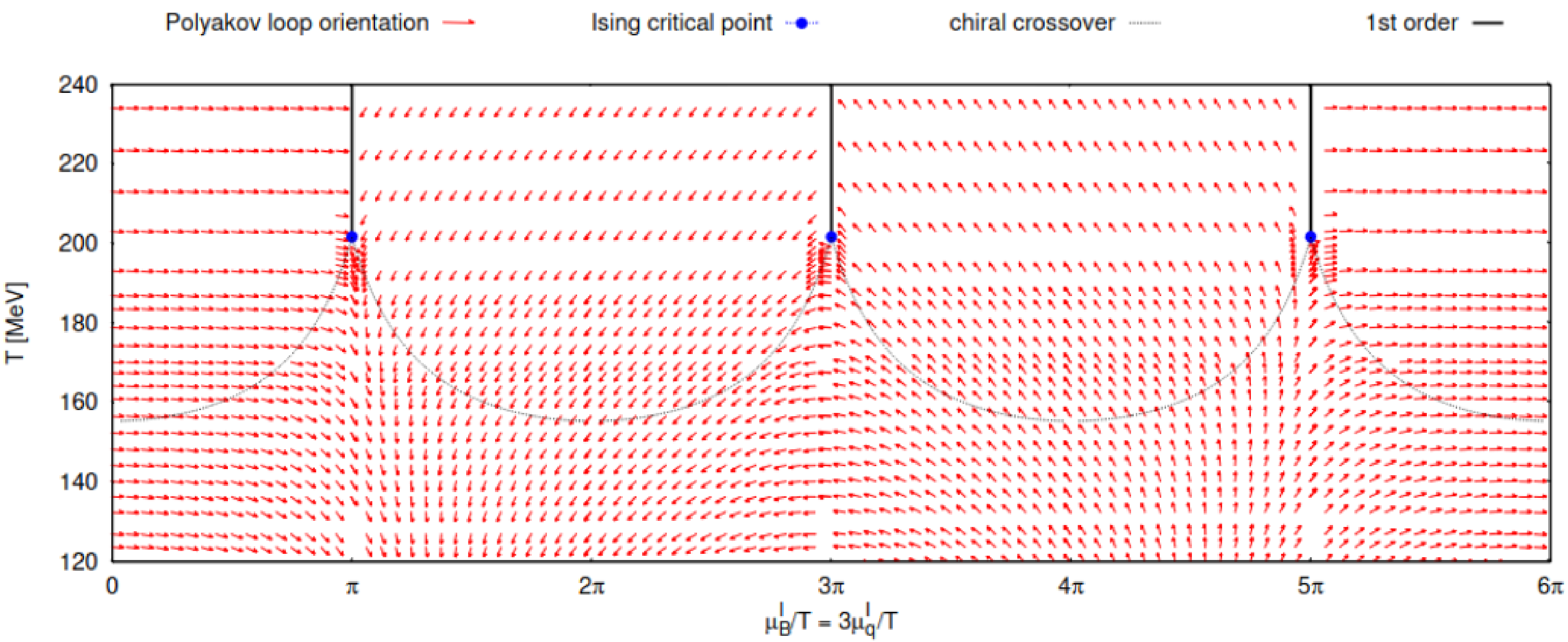
Can be simulated with Monte Carlo if $\det M\, e^{-S_{YM}}$ is real and positive:

- zero chemical potential $\mu = 0$
- purely imaginary chemical potential $\mathrm{Re}(\mu) = 0$
- isospin chemical potential $\mu_u = -\mu_d$

Otherwise: complex action/sign problem

$\Rightarrow$ desperate times, desperate measures

Physics at imaginary chemical potential



Roberge-Weiss periodicity: $Z\left(\frac{\mu_q}{T}\right) = Z(\frac{\mu_q}{T} + i\,\frac{2\pi}{3})$

From $Z = \mathrm{Tr}\left(e^{-(H-\mu_q(\mathrm{N}_u+N_d+N_s))/T}\right)$ one would naively expect a 2π periodicity, not $\frac{2\pi}{3}$

Related to confinement:

1) Remnant of center symmetry for dynamical quarks
2) Only integer baryon numbers appear in the spectrum

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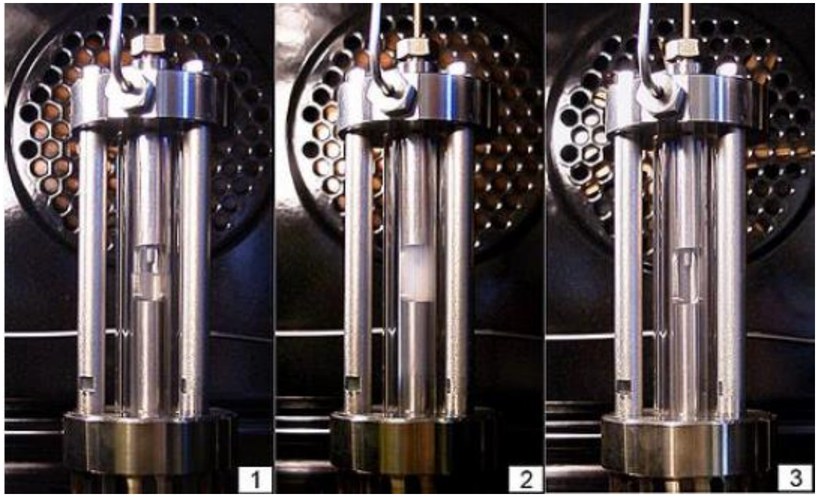
Critical fluctuations

Experiment: tune to criticality



$T < T_c$ $\quad$ $T \approx T_c$ $\quad$ $T > T_c$

HEAT THE SYSTEM

Picture from [Wikipedia]

Lattice/Taylor: try to see it from far away

$$\chi_n^B = \left(\frac{\partial^n \hat{p}}{\partial \hat{\mu}_B^n} \right)_{\mu_B=0}$$

To as large n as possible…

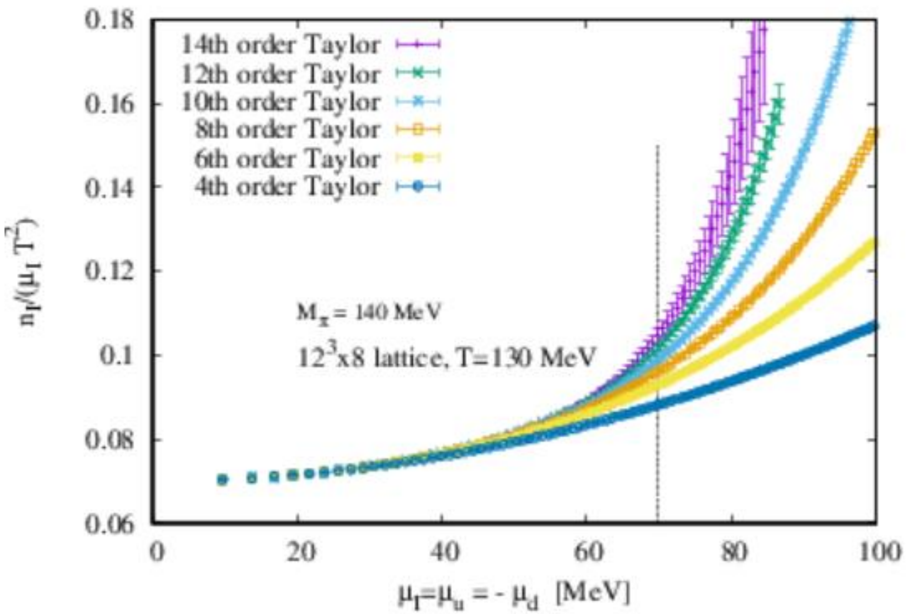
To hopefully see a divergence…

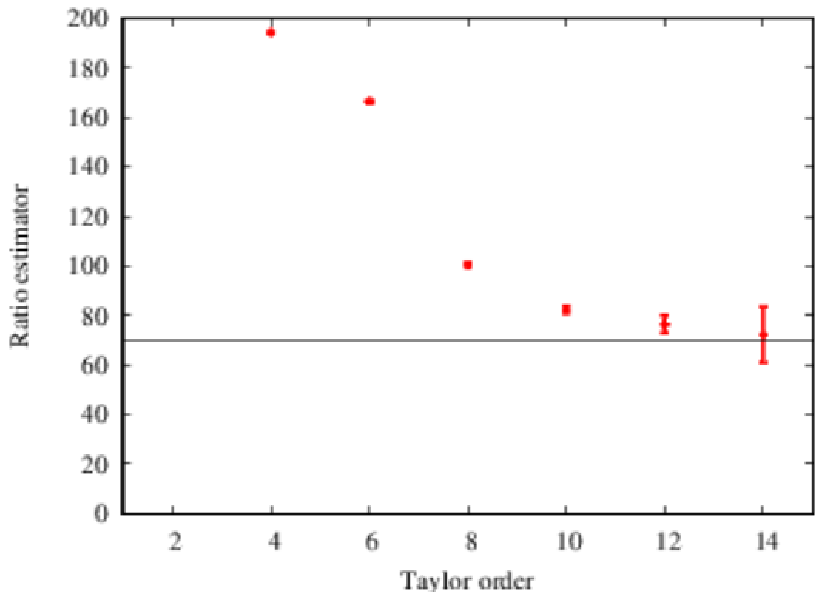
2nd order known since 2012
4th order known since 2015
6th order only now (2024) getting clean
8th maybe…

Is this even possible?

A case study: pion condensation [Wuppertal-Budapest, PRD109 (2024)]

- Instead of μ_B, introduce μ_I (prefers π^+ over π^-)
- Second order transition at low T and $\mu_I \approx m_\pi/2 \approx 70\text{MeV}$[Son, Stephanov, PRL (2001)]





Eventually finds the correct value. 6th order gives $170\text{MeV} \gg 70\text{MeV}$

Warning: the ratio estimator is not always applicable [Giordano & Pásztor, PRD99(2019)]

The HRG as a non-critical baseline

Let us start in a humble way, by comparing with a non-critical model.

Hadron resonance gas (HRG) model

$$p_{QCD} \approx \sum_H p_H^{free}$$

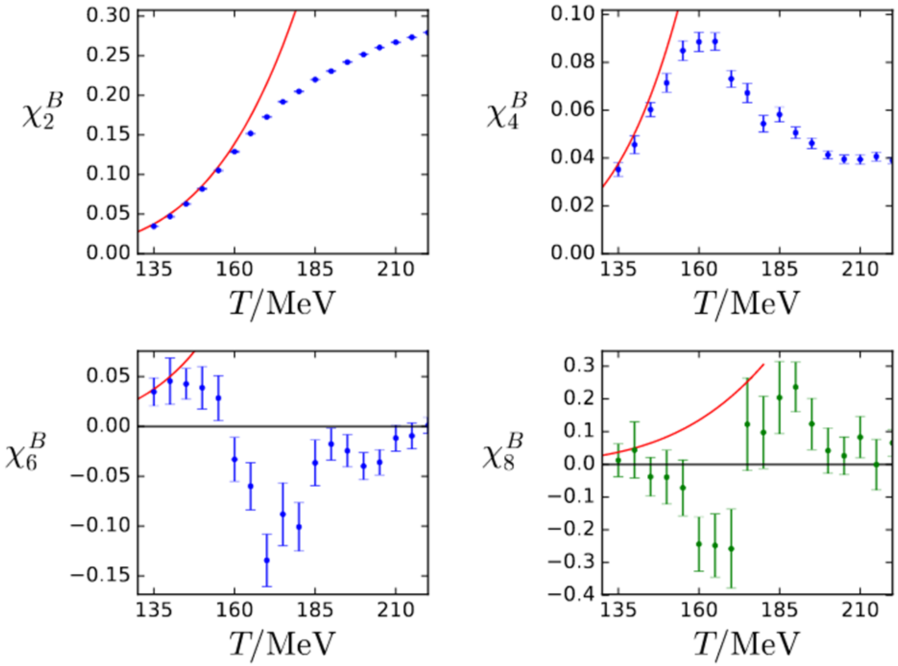
- sum over stable hadrons and resonances
- heavy ion phenomenology uses the HRG as a non-critical baseline (non-trivial: see, e.g., [Braun-Munzinger et al, NPA1008(2021)])
- in lattice QCD: can use grand canonical ensemble
- a good minimal goal: establish deviations from HRG (in lattice data with good quality control)

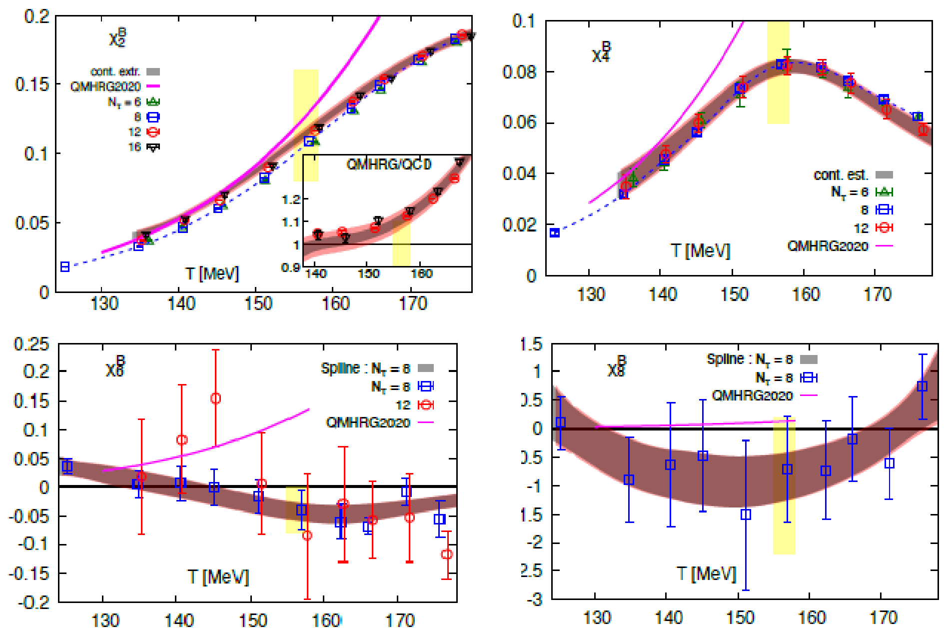
WHAT ARE THE CORRECTIONS TO THE HRG? ARE THEY LARGE?

Taylor coefficients of the pressure

From imaginary chemical potential

From zero chemical potential





[Wuppertal-Budapest, JHEP (2018)]

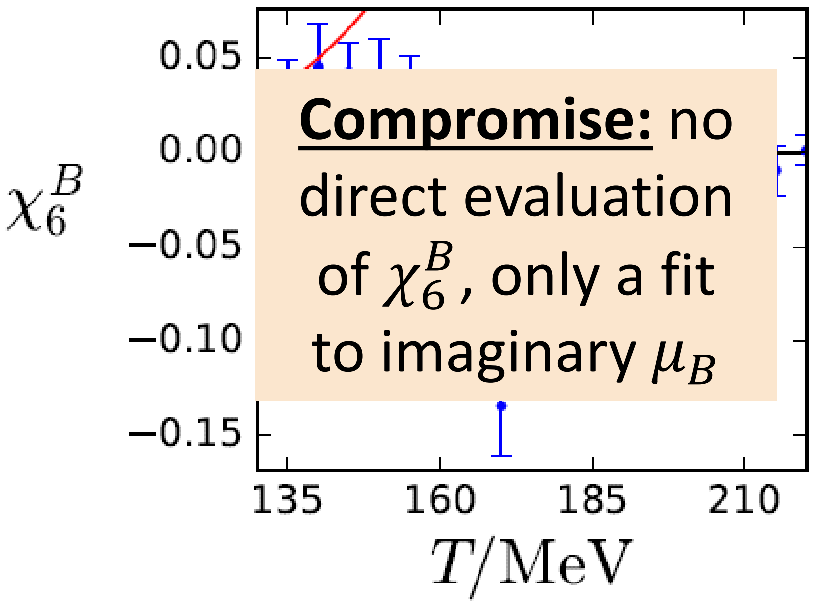
(LT=4, N_t=12)

[HotQCD, PRD105 (2022)]

(LT=4, N_t=6,8,(12))

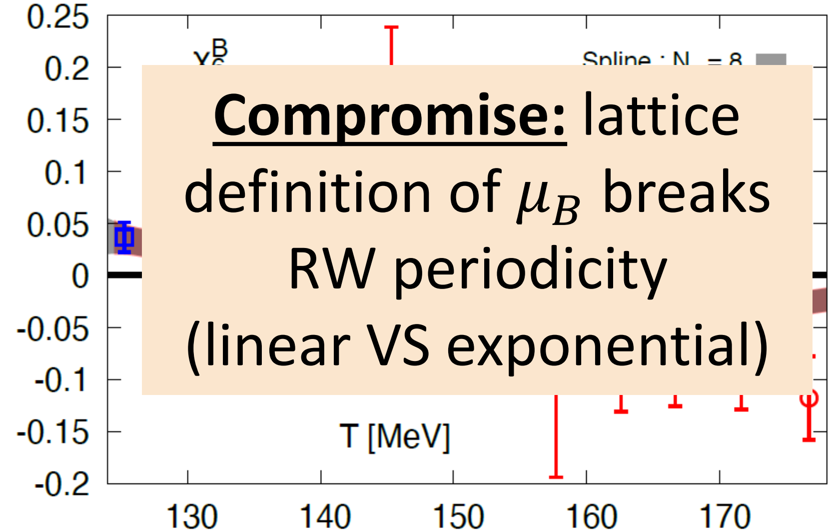
6^{th} order: zoom in to see discrepancies

From imaginary chemical potential



Compromise: no direct evaluation of χ_6^B, only a fit to imaginary μ_B

From zero chemical potential (Taylor)



Compromise: lattice definition of μ_B breaks RW periodicity (linear VS exponential)

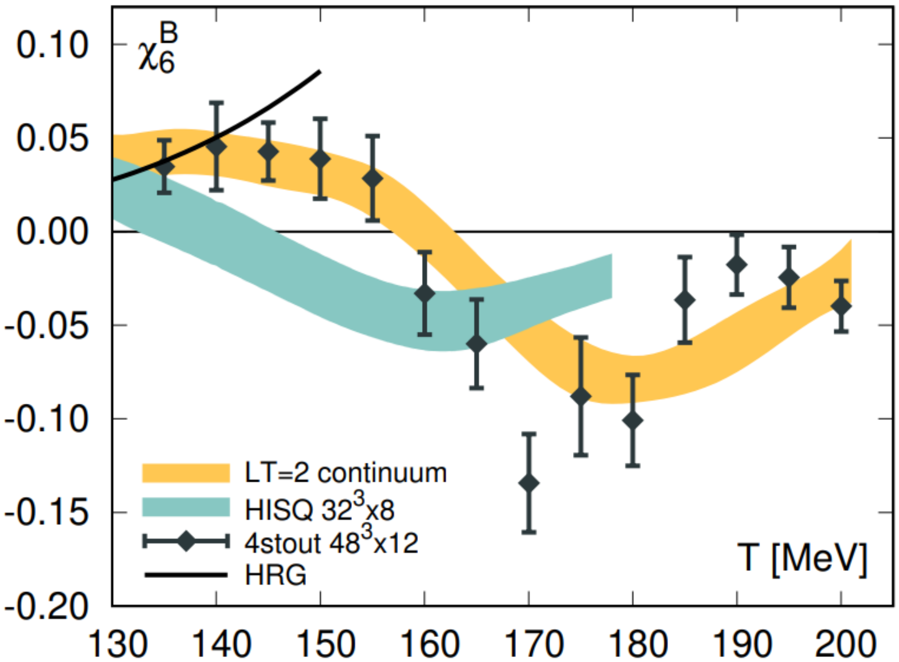
[Wuppertal-Budapest, JHEP (2018)] (LT=4, N_t=12) [HotQCD, PRD105 (2022)] (LT=4, N_t=8,(12))

- N_t=12 (left, WB) agrees with the HRG (value, slope) better than N_t=8 (right, HotQCD) at low T

- At T=145-155MeV: N_t=12>0 and N_t=8<0

6^{th} order order: new dataset

[Wuppertal-Budapest, PRD 110 (2024)]



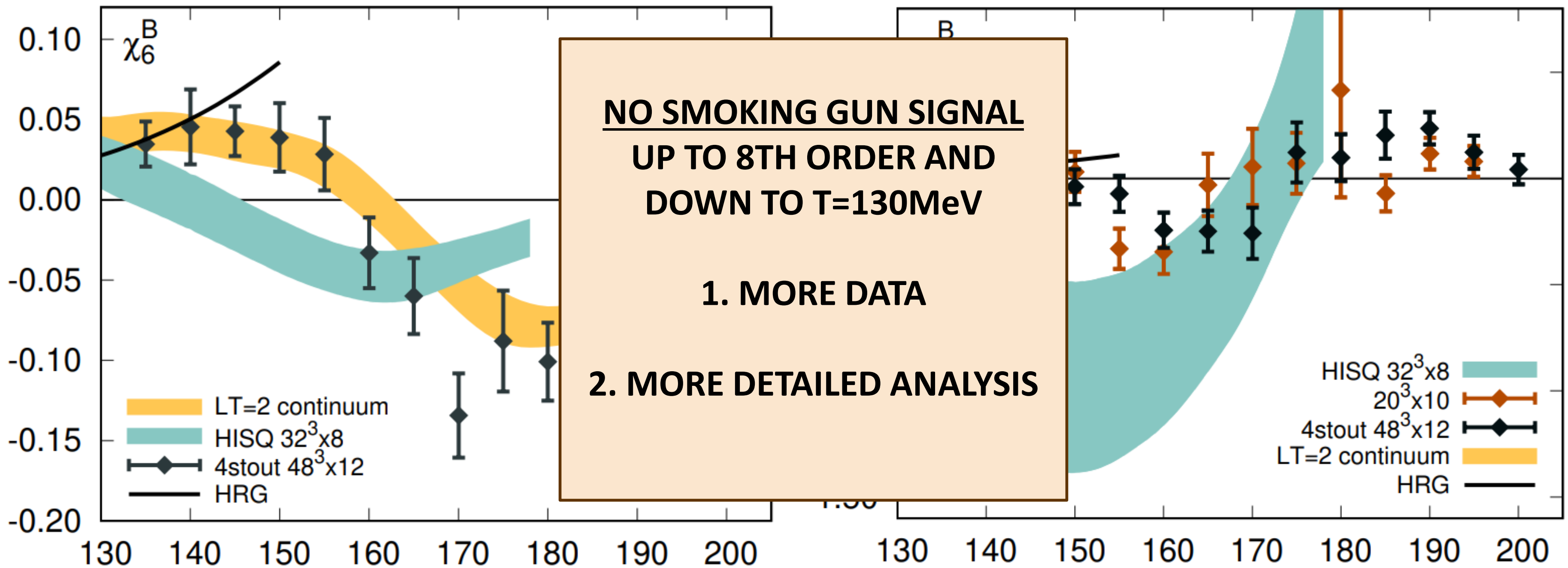
New dataset (4HEX discretization):
Taylor, (no fit to Im μ_B needed)
Improved taste breaking, allows continuum
Proper lattice definition of μ_B (w. RW periodicity)
Smaller volume: LT=2

Low T: cut-off effects dominate
Smaller T means larger a for fixed N_t
5 points at least 1σ below: $\left(\frac{1-0.68}{2}\right)^5 \approx 10^{-4}$

RELEVANT FOR CEP SEARCH ($T<T_{crossover}$)
160MeV>T>145MeV: HRG ≠ LATTICE
130MeV<T<145MeV: HRG ≈ LATTICE
(at least roughly)

6^{th} and 8^{th} order order: new dataset

[Wuppertal-Budapest, PRD 110 (2024)]



<u>NO SMOKING GUN SIGNAL</u>
UP TO 8TH ORDER AND DOWN TO T=130MeV

1. MORE DATA

2. MORE DETAILED ANALYSIS

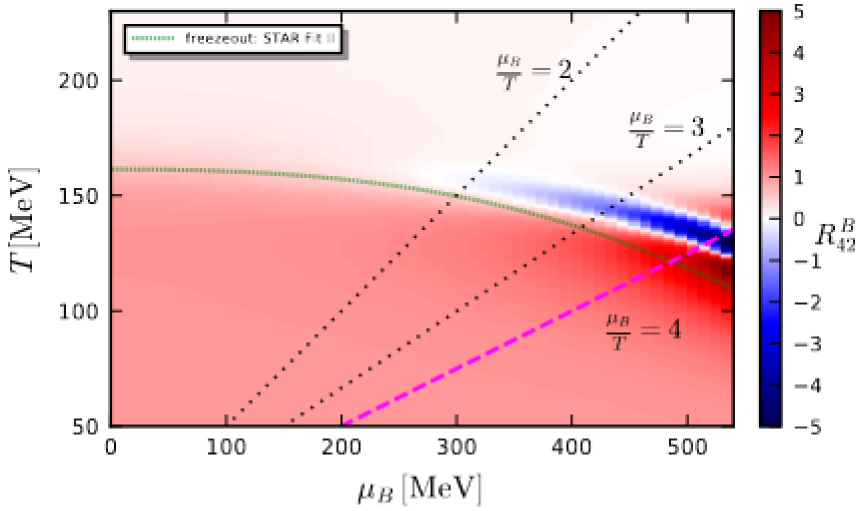
UP TO 8TH ORDER

T>145MeV: HRG ≠ LATTICE
130MeV<T<145MeV: HRG ≈ LATTICE
(at least roughly)

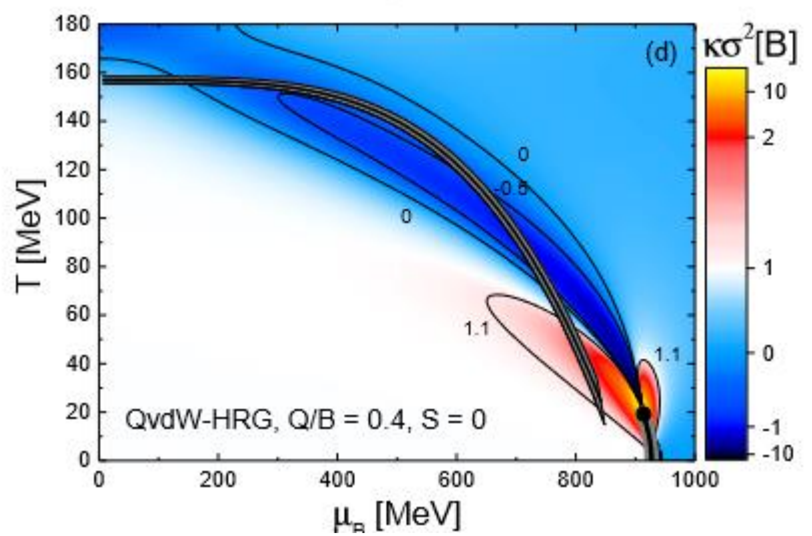
Possible scenarios for $\mu_B > 0$?

[Stephanov, PRL 107 (2011)] **Non-monotonic χ_4^B/χ_2^B near critical point**

FRG on quark-meson model
(lattice assisted model)
Chiral CEP = (93 MeV, 672 MeV)
[Fu et al, PRD 104 (2021)]



Quantum VdW gas
(Non-ideal version of HRG)
Liquid-gas CEP = (19.7 MeV, 922 MeV)
[Vovchenko et al, PRL 118 (2017)]



In both models, χ_4^B/χ_2^B is non-monotonic quite far away from the respective CEP.

The two critical points

In the simplest scenario, where we assume that deconfinement and chiral symmetry restoration are linked.

Property	Chiral CEP and 1st order	Liquid-gas CEP and 1st order
Universality class	3D Ising	3D Ising
Baryon fluctations	Diverge at CEP	Diverge az CEP
χ_4^B/χ_2^B	Non-monotonic (potentially far from CEP)	Non-monotonic (potentially far from CEP)
Lee-Yang zeros	Approach real axis at CEP	Approach real axis at CEP
Entropy	$\left(\frac{\partial T}{\partial s}\right)_{\mu_{CEP}} = 0$	$\left(\frac{\partial T}{\partial s}\right)_{\mu_{CEP}} = 0$
T	???	O(10-20 MeV)
μ_B	???	O(m_N ~ 1 GeV)
Chiral symmetry	Restored on one side	Broken on both sides
Confinement	Only on one side	On both sides

Lee-Yang zeros

$$Z = \mathrm{Tr}\left(e^{-(H-\mu_B B)/T}\right) = \sum_n e^{n\mu_B/T}\mathrm{Tr}_n\left(e^{-H/T}\right) = \sum_n e^{n\mu_B/T} Z_n$$

Up to a non-vanishing factor, a polynomial in $e^{\mu_B/T}$ → has roots **(Lee-Yang zeros)**
$p \propto logZ$ → has logarithmic branch points

If in the thermo limit an ∞ # of these tends to the real line → phase transition
Finite size scaling/density of zeros in the thermo limit → order of the transition

If in the thermo limit the Lee-Yang zeros stay away from the real line → crossover

Near a critical point, the infinite volume limit of the zeros **(Lee-Yang edge)**

$$\mathrm{Im}\,\mu_{\mathrm{LYE}} \sim \mathrm{A}|\mathrm{T}-\mathrm{T_C}|^{\beta\delta}$$

Lots of recent activity on LY zeros

[Giordan… …99(2019)]

[Gio… …apas… …Pásztor, PRD101 (2020)]

[Mu… …v, P…]

[Dimopoulo… …(202…]

[Basar, PRC110(202…]

[Parma-Bielefeld, 2405.101…]

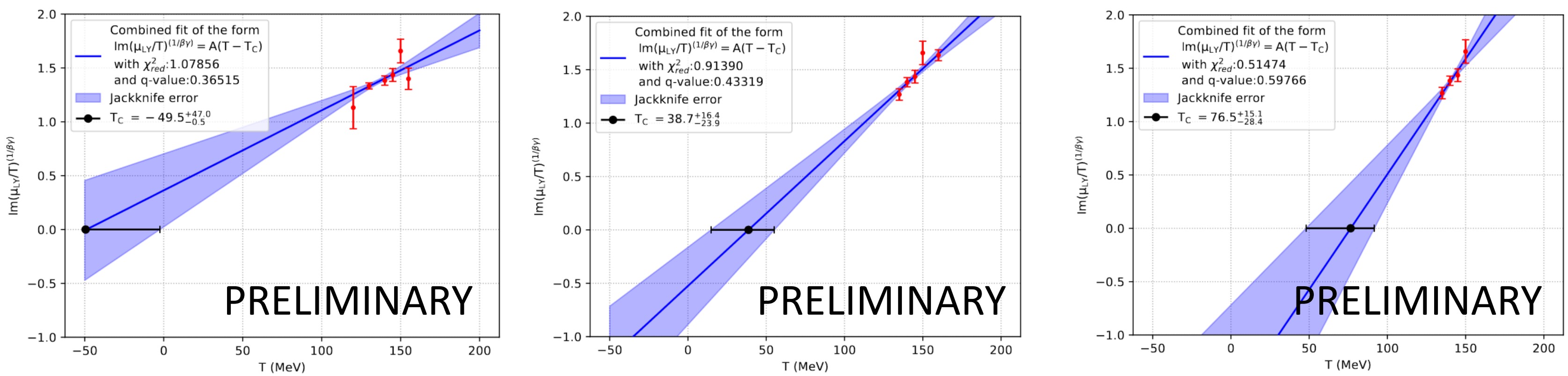
NONE OF THESE USED WELL CONTROLLED LATTICE RESULTS BELOW $T_{\text{CROSSOVER}}$

AT THE MOMENT, SYSTEMATIC ERRORS ARE AROUND 100%

Search for Lee-Yang zeros (an example)

Find poles of the rational function:

$$\chi_2^B(T,\mu_B) \approx \frac{A_0+A_1\hat{\mu}_B^2}{1+B_1\hat{\mu}_B^2+B_2\hat{\mu}_B^4} = \chi_2^B(T,0) + \chi_4^B(T,0)\frac{\hat{\mu}_B^2}{2!} + +\chi_6^B(T,0)\frac{\hat{\mu}_B^4}{4!} + \chi_8^B(T,0)\frac{\hat{\mu}_B^6}{6!}$$



Using the data of [Wuppertal-Budapest, PRD 110 (2024)] (4HEX, 16^3X8)

- A sign of a strengthening transition (the imaginary part decreases).
- Large (>100%) systematics. Here, only the fit range dependence is shown. Other sources of systematics (e.g. Padé truncation) also matter.
- Consistent with either a chiral or a liquid-gas CEP.

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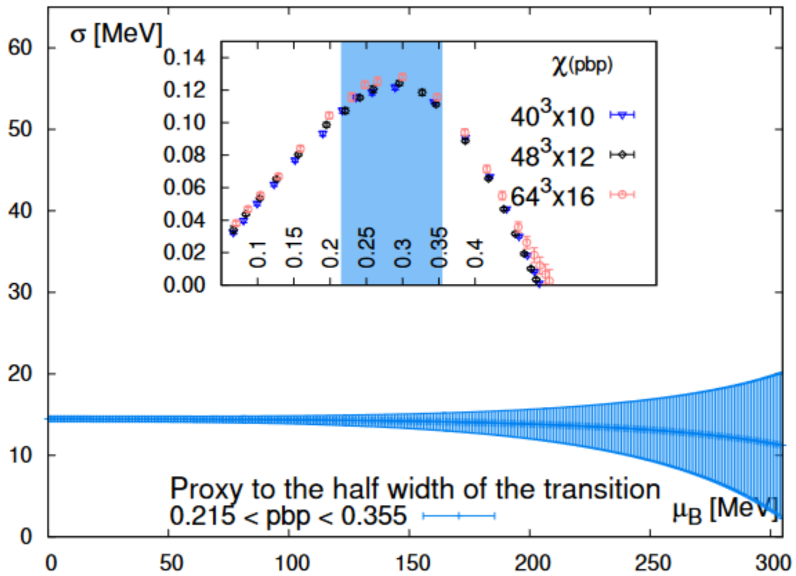
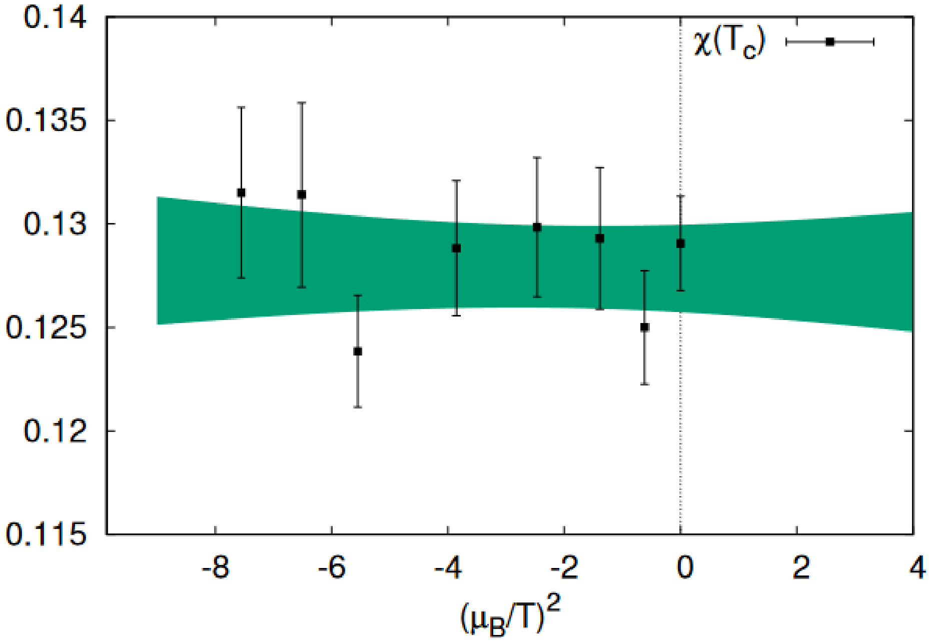
Chiral crossover ⇒ approx. constant width and strength





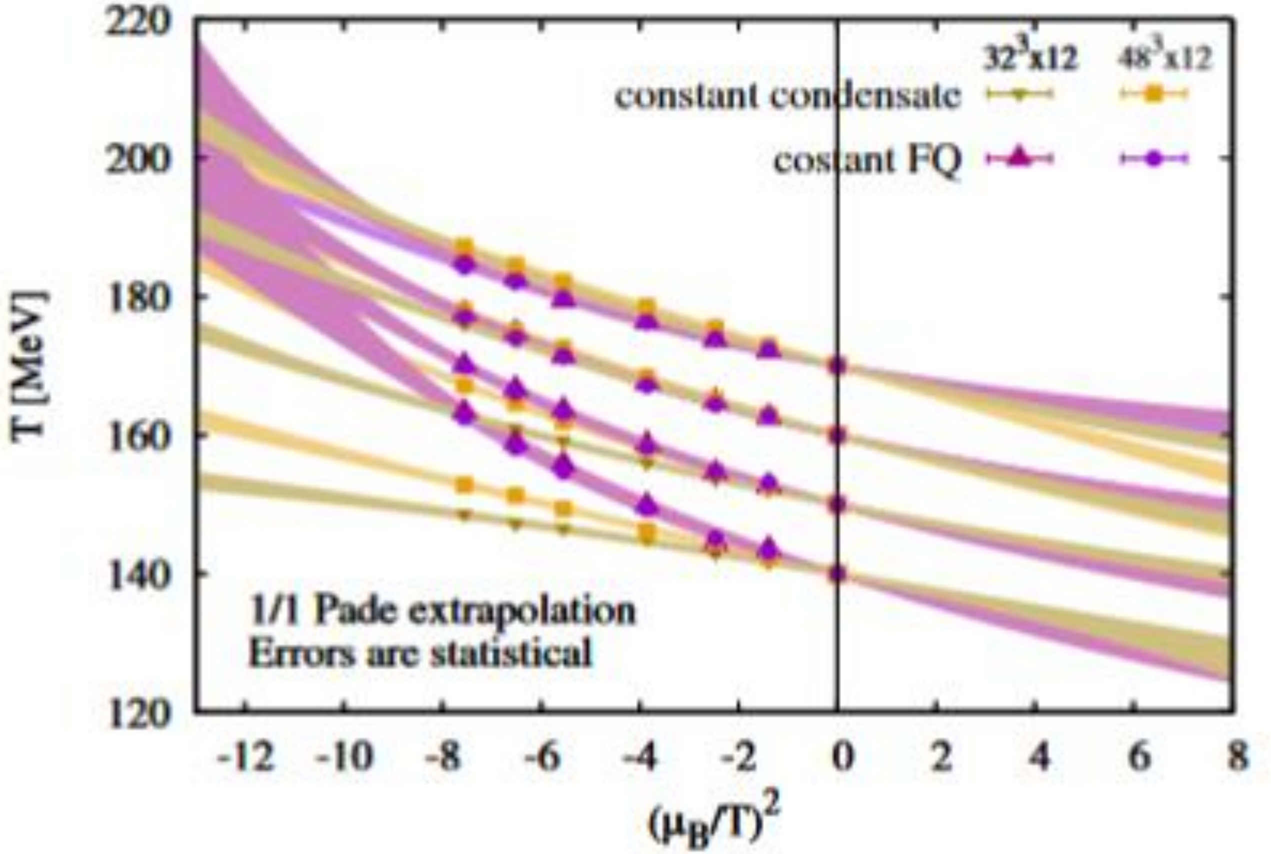
$$\sigma \approx \left(\sqrt{-\chi \left(\frac{\partial^2 \chi}{\partial T^2} \right)^{-1}} \right)_{T=T_c(\mu_B)}$$

Wuppertal-Budapest PRL125 (2020)

Slope near zero negligible. We need larger chemical potentials.

⇒ Need smaller volume, so the sign problem remains managable.

What can we do with a smaller volume?

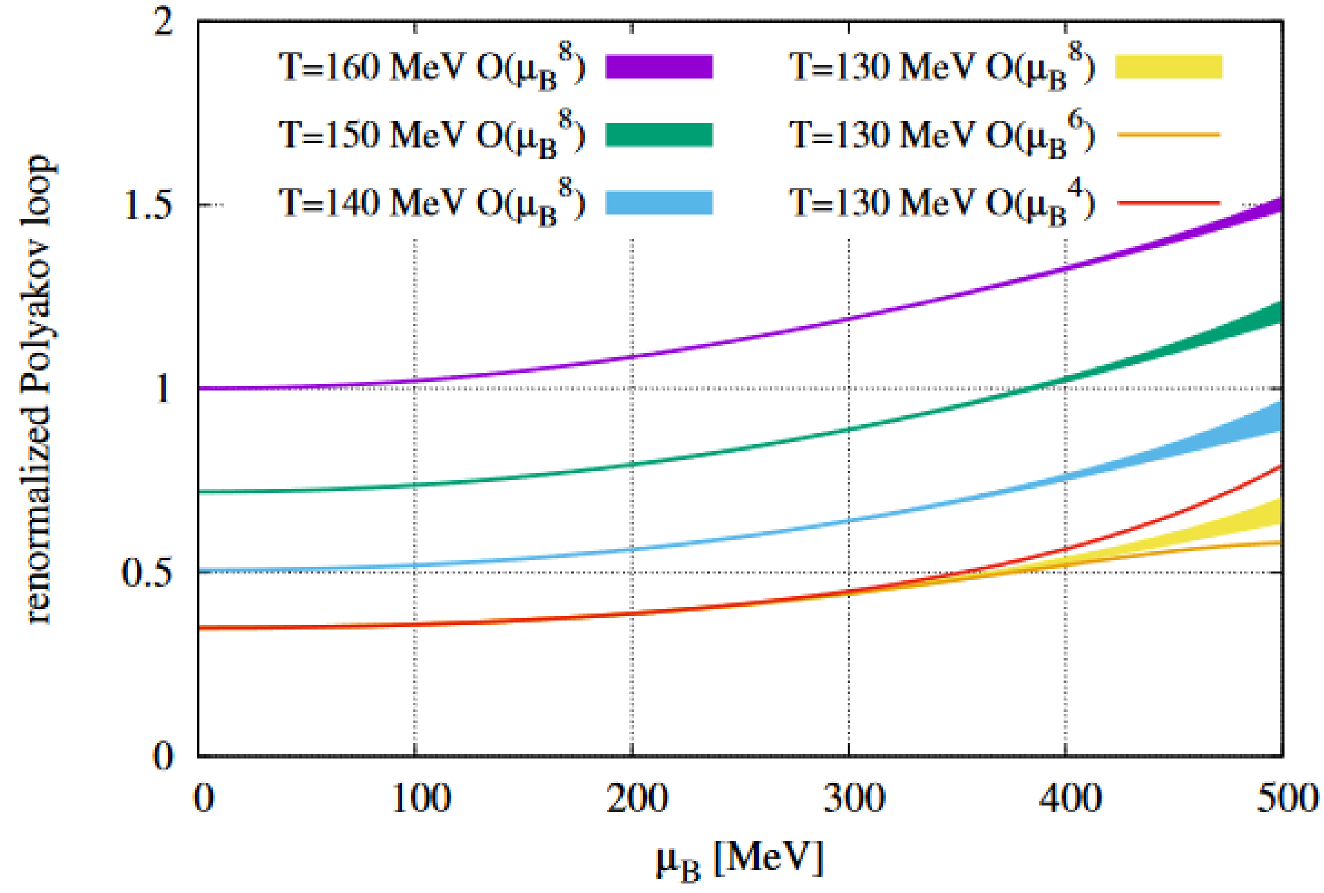


[Wuppertal-Budapest, 2405.12320, accepted by PRD]

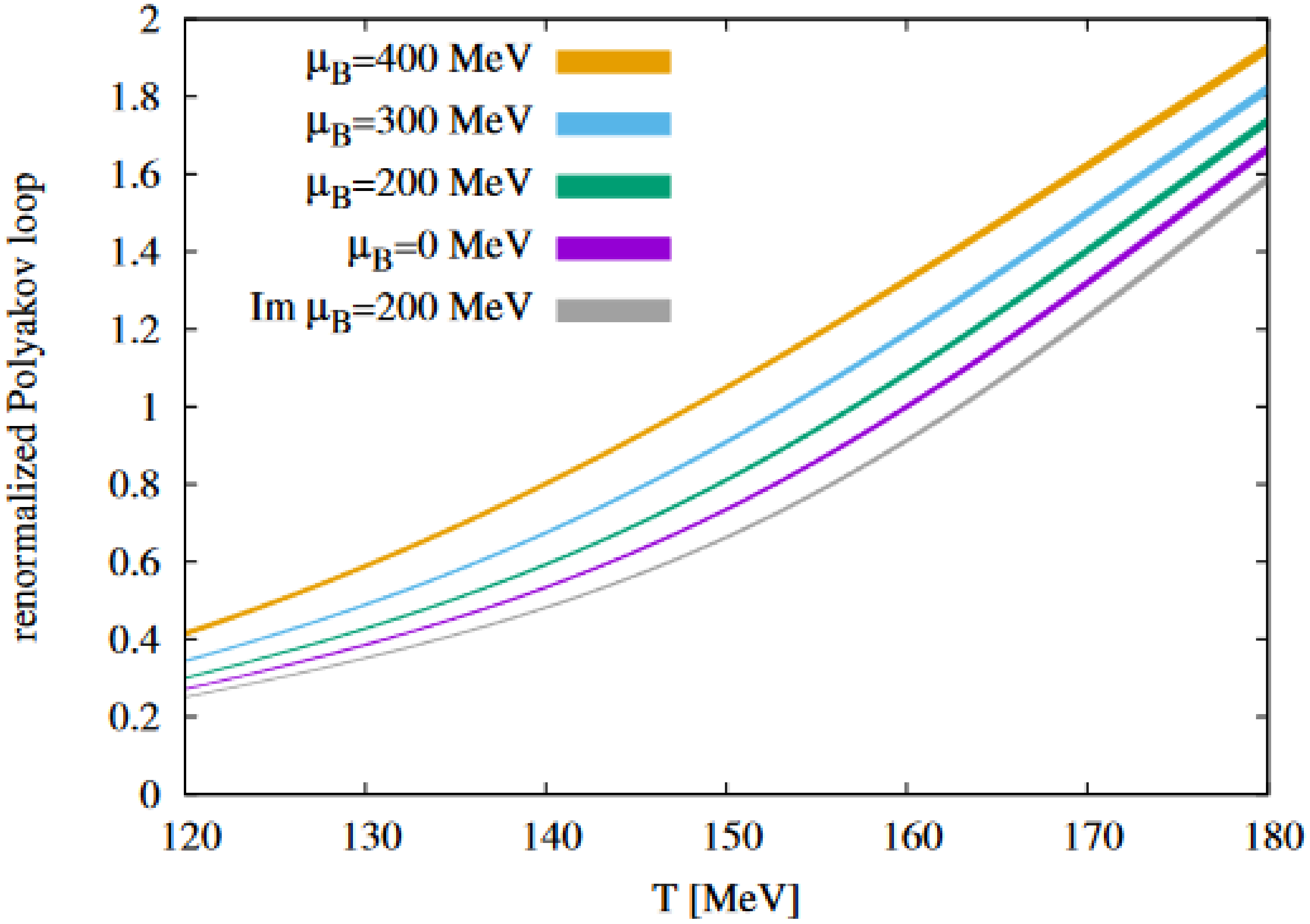
Deconfinement aspects have milder volume dependence
⇒ study deconfinement aspects at higher μ_B in a smaller volume

Taylor expansion of the Polyakov loop (n_S=0)

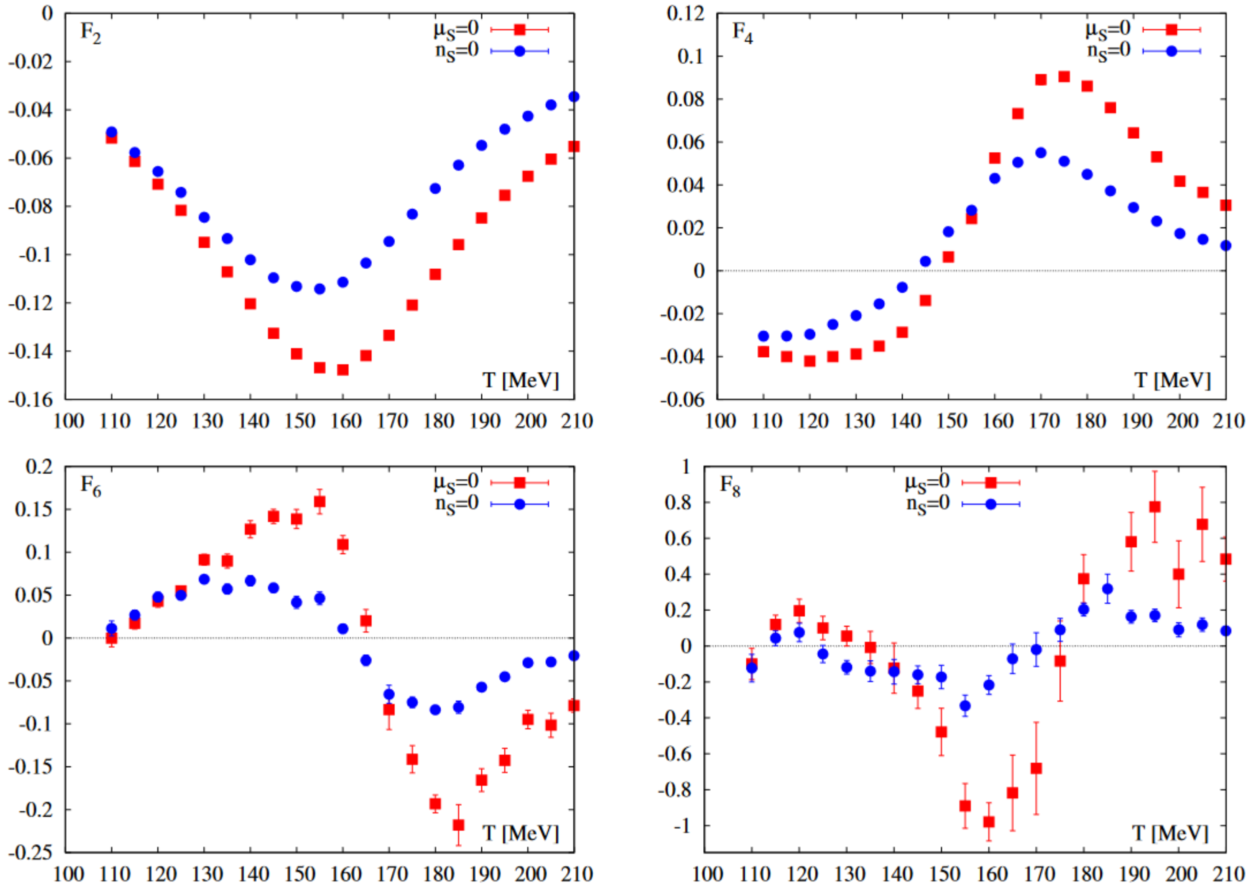
Wuppertal-Budapest, 2410.06216, 4HEX 16^3X8



Taylor expansion of the Polyakov loop (n_S=0)

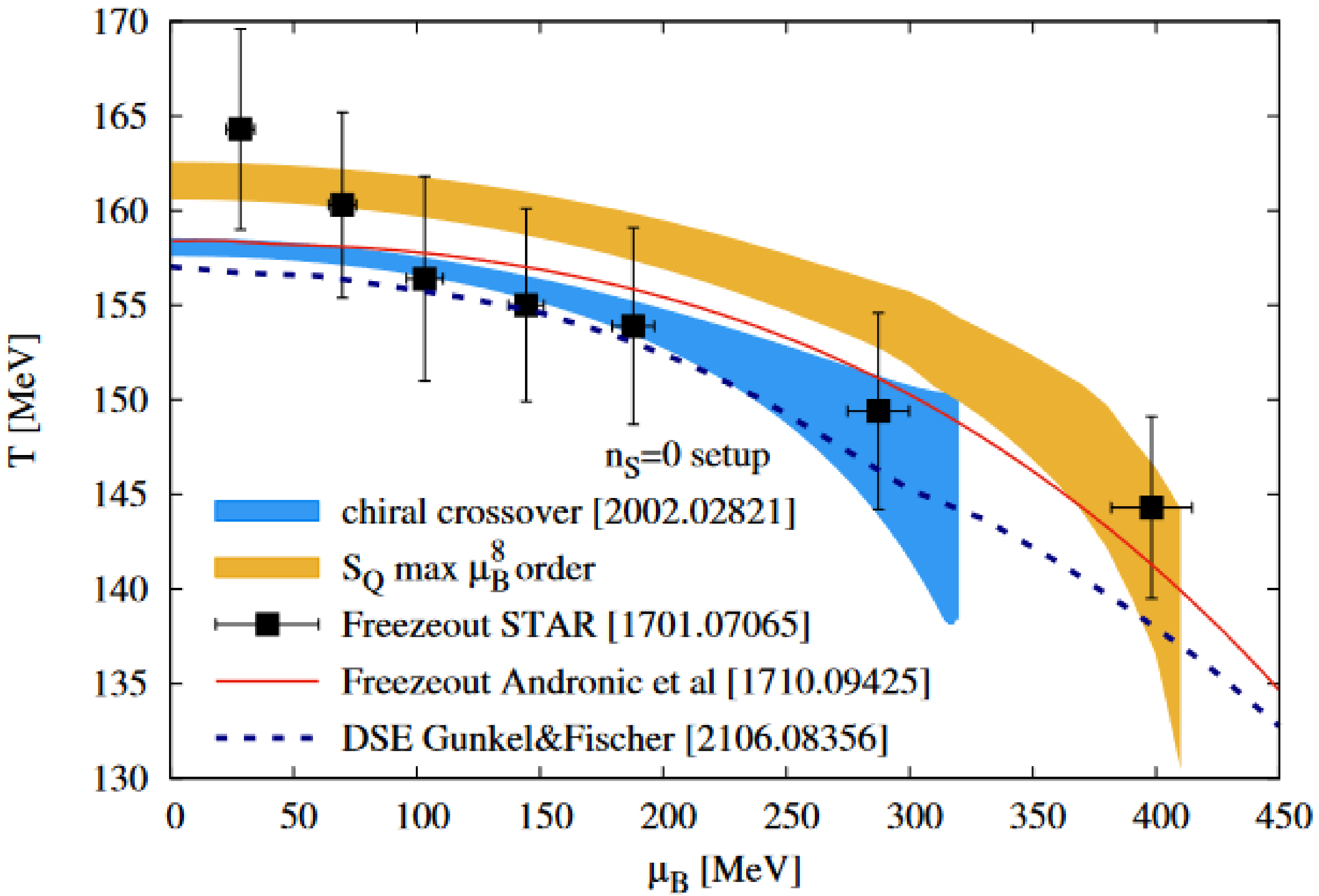


Taylor coefficients of the static quark free energy



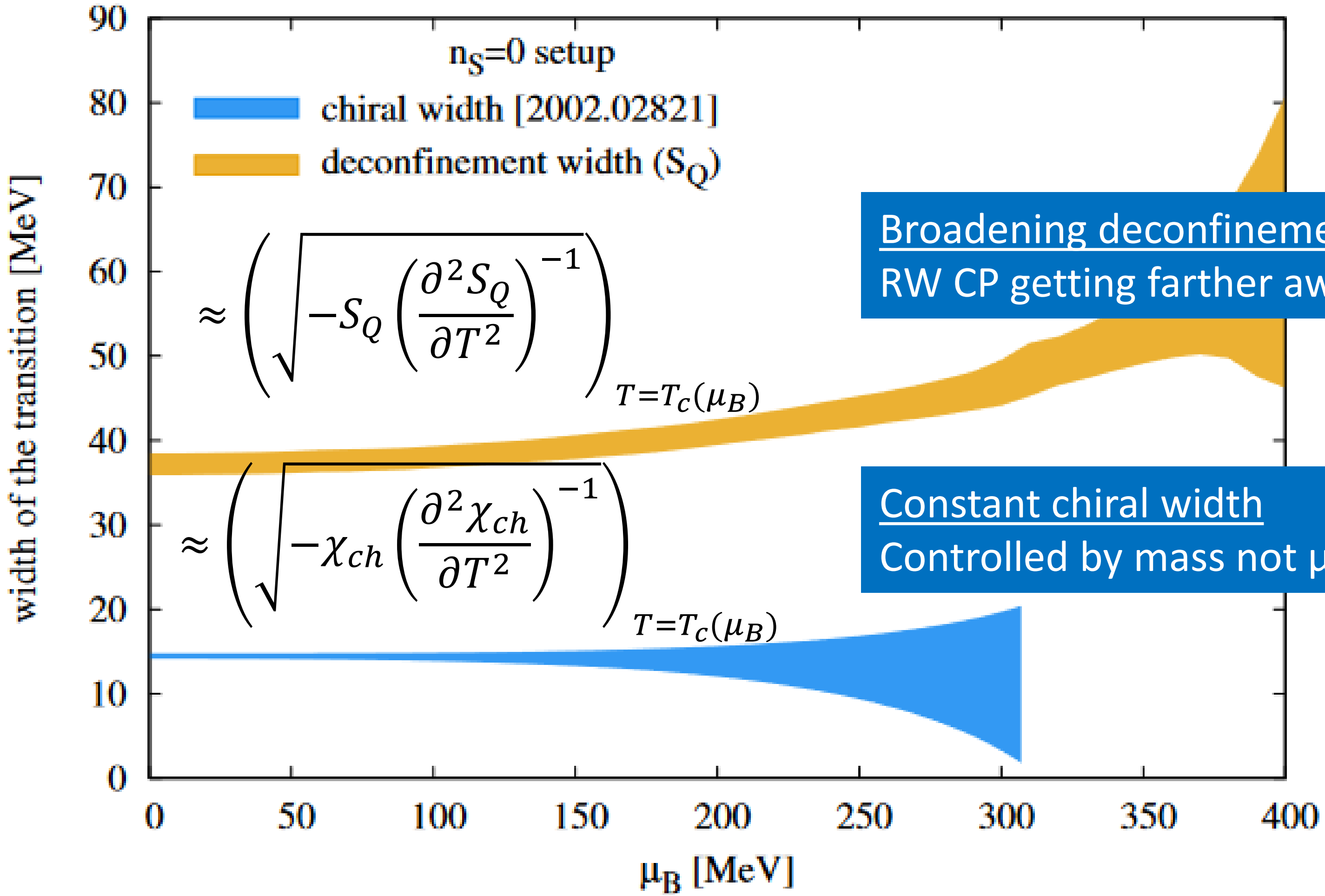
$$F_Q(T,\mu_B) = F_Q(T,0) + T \sum_{n=2,4,\ldots} \frac{F_n(T)\hat{\mu}_B^n}{n!}$$

The transition line on the phase diagram (n_S=0)



[Wuppertal-Budapest, 2410.06216]

The width of the crossover (n_S=0)

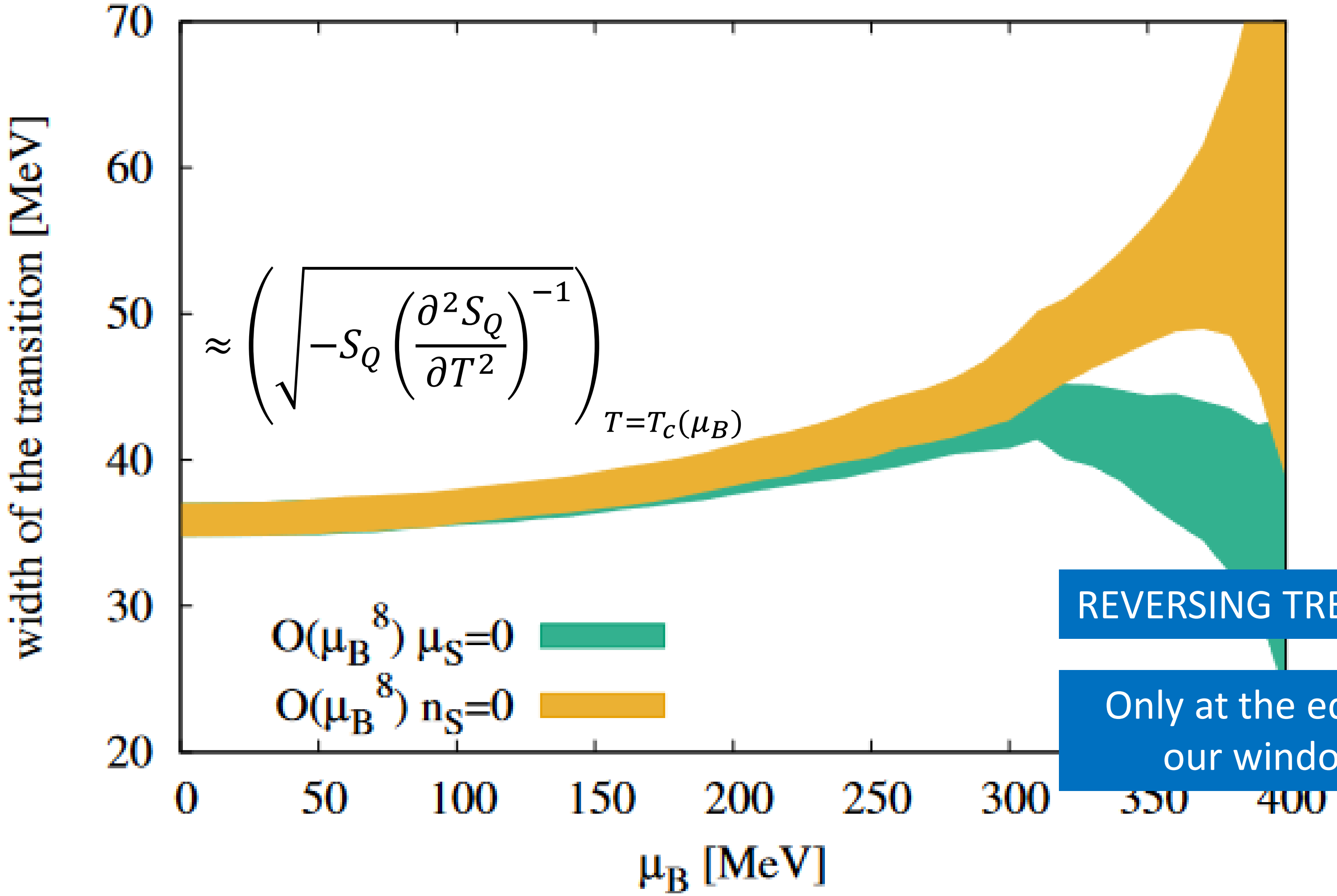


<u>Broadening deconfinement</u>
RW CP getting farther away

<u>Constant chiral width</u>
Controlled by mass not μ_B

[Wuppertal-Budapest, 2410.06216](#)

The width of the crossover (n_S=0 vs μ_S=0)



REVERSING TREND???

Only at the edge of our window.

Wuppertal-Budapest, 2410.06216

Summary

THERE IS NO CONCLUSIVE PICTURE FROM THE AVAILABLE LATTICE DATA

strengthening (Lee-Yang zero estimates) VS

constant (chiral width) VS

weakening transition (deconfinement width)

- No contradiction. For a crossover, this can happen. But this makes firm conclusions hard.
- Deconfinement CEP disfavored below μ_B=400MeV for both n_S=0 and μ_S=0
- There is a hint of the deconfinement trend reversing for μ_S=0 around μ_B=400MeV (might be a CEP signal)

BACKUP: BEYOND HRG

Corrections to the HRG (fugacity expansion)

$$\hat{p}(T,\mu_B,\mu_S) = P^{BS}_{00}(T) + \cdots + P^{BS}_{10}(T)\cosh(\hat{\mu}_B) + P^{BS}_{11}(T)\cosh(\hat{\mu}_B - \hat{\mu}_S) + \cdots + P^{BS}_{20}(T)\ \cosh(2\hat{\mu}_B) + \cdots$$

$$\chi^B_1(T,\mu_B,\mu_S) = P^{BS}_{10}(T)\sinh(\hat{\mu}_B) + P^{BS}_{11}(T)\sinh(\hat{\mu}_B - \hat{\mu}_S) + \cdots + 2P^{BS}_{20}(T)\ \sinh(2\hat{\mu}_B) + \cdots$$

$$\chi^B_2(T,\mu_B,\mu_S) = P^{BS}_{10}(T)\cosh(\hat{\mu}_B) + P^{BS}_{11}(T)cosh(\hat{\mu}_B - \hat{\mu}_S) + \cdots + 4P^{BS}_{20}(T)cosh(2\hat{\mu}_B) + \cdots$$

{ π N Λ } { N – N }

Higher order $\Rightarrow$ larger beyond HRG corr.

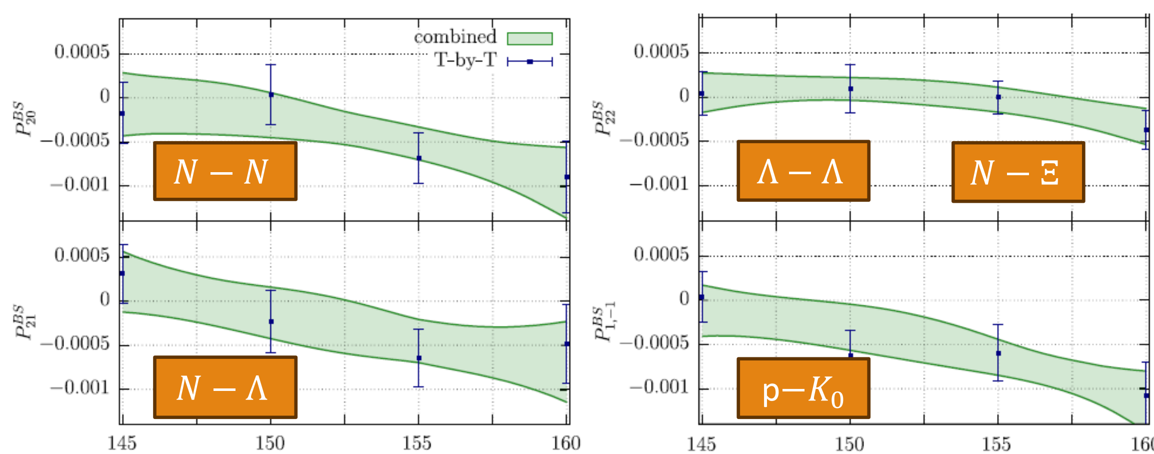
S-matrix formalism

[Dashen et al, PR187 (1969)]

Repulsive $\Rightarrow$ - partial pressure

Attractive $\Rightarrow$ + partial pressure

B=2,3 etc. are *exp. suppressed at low T*

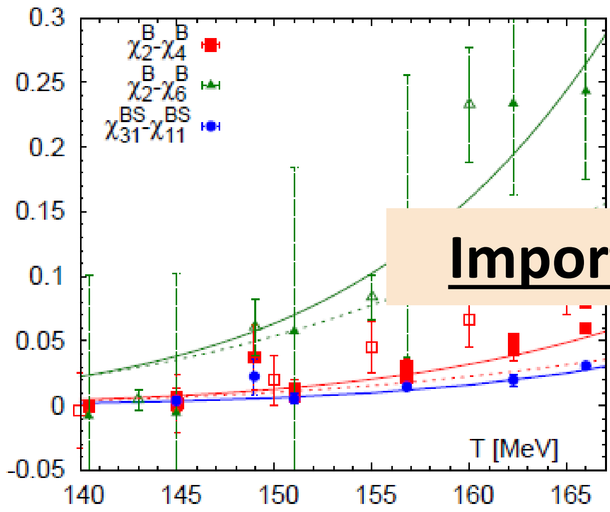


[Wuppertal-Budapest, PRD104(2021)]

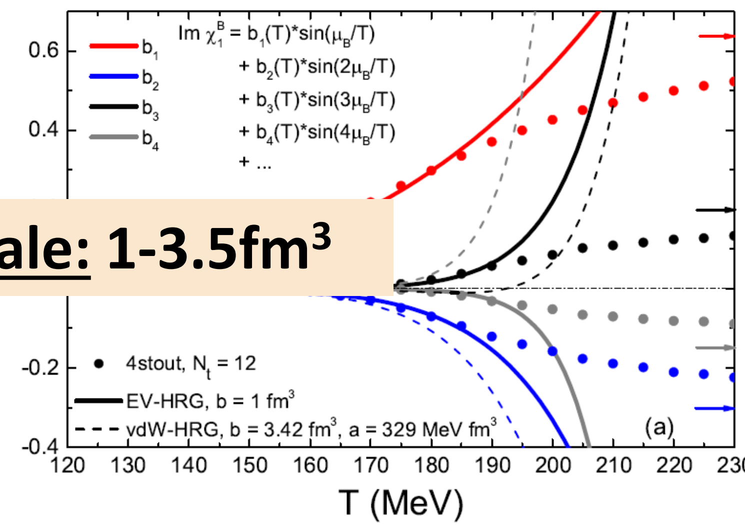
Scale of short-range repulsive interations is between 1-3.5 fm^3

Phenomenology: Short range repulsion dominates

[Huovinen, Petreczky PLB777 (2017)]



[Vovchenko, Pásztor et al, PLB 775 (2017)]



Important scale: **1-3.5fm^3**

Not all relevant phase shift are known experimentally, but the NN is known, and If one substitutes it to the S-matrix formalism, it turns out that the repulsive core of the NN interaction dominates the first correction to the HRG.

In a phenomenological model that includes both short range repulsion and long range attraction for all baryons, it is the short range repulsion that dominates the first correction.

How small is LT=2?

T=145MeV

$V \approx 20 fm^3 \approx$ 6-20X(excluded volume)
$m_\pi L \approx 1.9$

T=130MeV

$V \approx 28 fm^3 \approx$ 8-28X(excluded volume)
$m_\pi L \approx 2.2$

T=120MeV

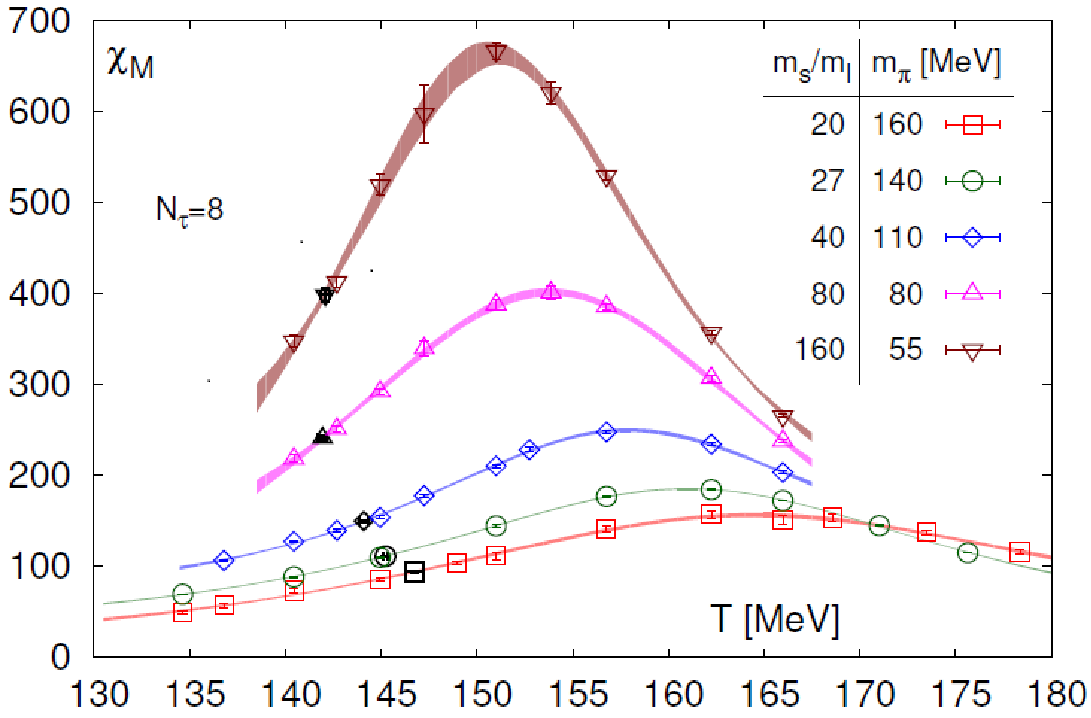
$V \approx 35 fm^3 \approx$ 10-35X(excluded volume)
$m_\pi L \approx 2.3$

Large enough to correctly capture short range repulsion between baryons accuretely but not the long range attraction.

BACKUP: O(4) IN EOS

Chiral criticality

Smaller-than-physical quark mass @ $\mu_B = 0$

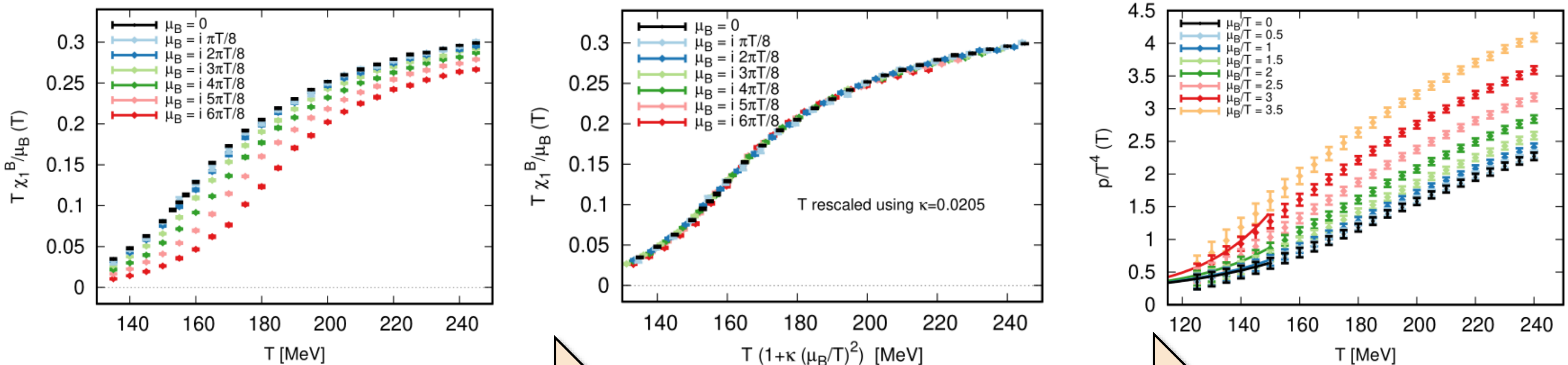


Plot from **[HotQCD, PRL123 (2019)]**

See also [Kotov, Lombardo, Trunin, PLB823 (2021)]:

scaling for heavier-than-physical quark masses

Alternative expansion scheme



Approximate collapse in the Im(μ_B) data

Account for small deviations systematically, extrapolate to real μ_B

Resummation defined by:

$$F(T,\mu_B) = F(T',0) \qquad T' = T(1 - \lambda_2(T)\hat{\mu}_B^2 - \lambda_4(T)\hat{\mu}_B^4 - \cdots)$$

continuum, $LT = 4$, $\mu_S = 0$: [Wuppertal-Budapest, PRL126 (2021)]
continuum, $LT = 4$, $n_S = 0$: [Wuppertal-Budapest, PRD105 (2022)]

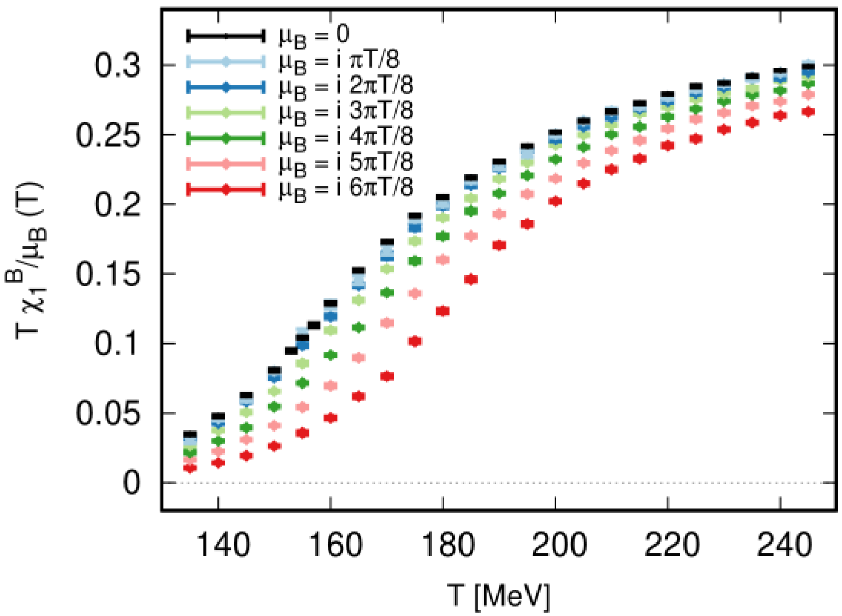
Also, small nonzero n_S

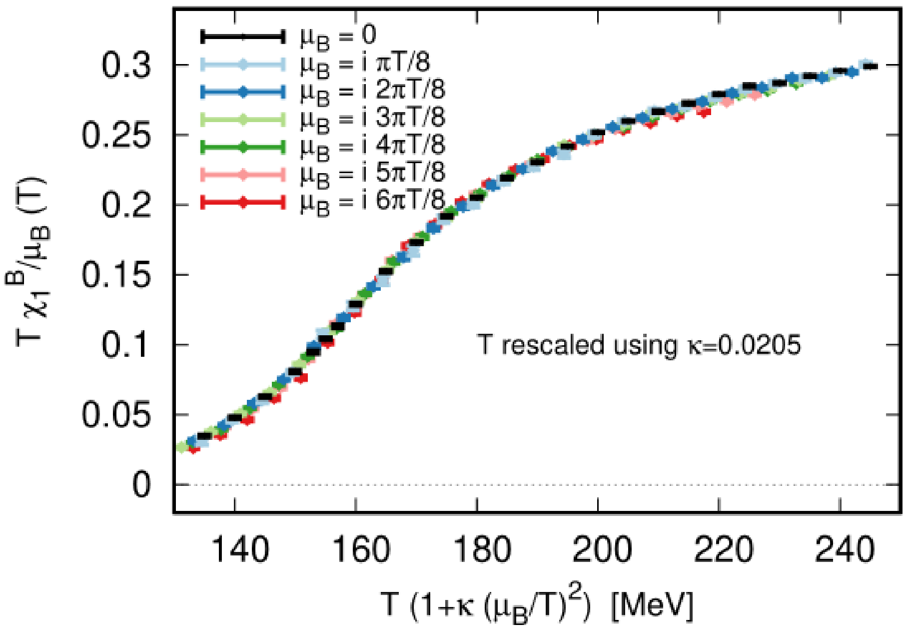
Chiral criticality and the equation of state

T and μ_B dependence with physical masses

- Empirically: approximate scaling variable $T(1+\kappa_2\hat{\mu}_B^2)$

  $\Rightarrow$ transition not sharpening for small $\hat{\mu}_B^2$





[Wuppertal-Budapest, PRL126 (2021)]

I strongly suspect that the mechanism behind this collapse is chiral scaling.

O(4) scaling and collapse plots at μ_B>0

Observation: $\chi_1^B/(\hat{\mu}_B)$ collapses as a function of $T\left(1+\kappa\hat{\mu}_B{}^2\right)$ but χ_2^B does not

Why? One possibility: scaling near the chiral limit (**Kadanoff scaling ansatz**)

$$p_{QCD}(T,\mu_B,m)-p_{QCD}(0,0,m)\sim f_{sing}(h,t)\sim t^{2-\alpha}F\left(\frac{h}{t^{\beta\delta}}\right)$$

where $h\sim m$ and t $\sim T-T_{ch}(1-\kappa\,\hat{\mu}_B^2)$

$$\Rightarrow\frac{1}{\hat{\mu}_B}\frac{\partial}{\partial\hat{\mu}_B}f_{sing}=(2-\alpha)t^{1-\alpha}F\left(\frac{h}{t^{\beta\delta}}\right)(2\kappa)+t^{1-\alpha-\beta\delta}F'\left(\frac{h}{t^{\beta\delta}}\right)(-\beta\delta)(2\kappa)$$

$\Rightarrow$ a function of the scaling variables h and t only

$$\frac{\partial^2}{\partial\hat{\mu}_B{}^2}f_{sing}=(2\kappa)G(h,t)+(2\,\kappa\,\hat{\mu}_B)^2\frac{\partial G}{\partial t}$$

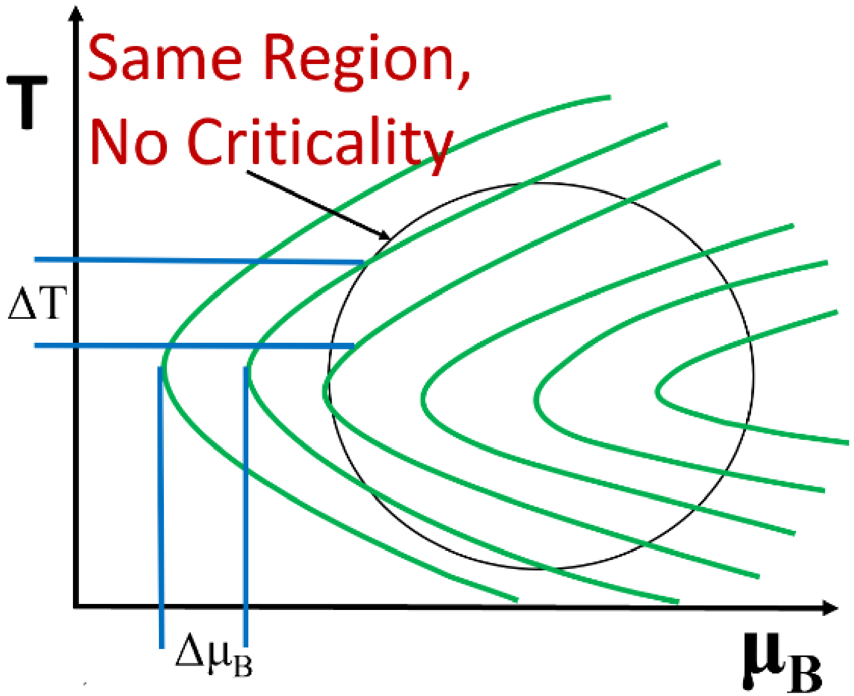
$\Rightarrow$ not a function of h and t only

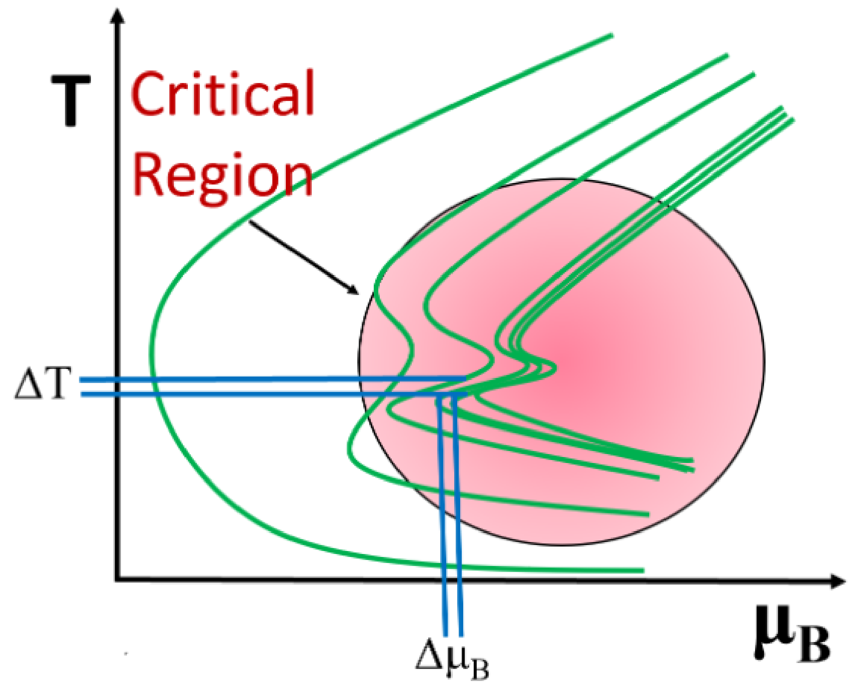
Similar for the chiral condensate: here $\left\langle\bar{\psi}\,\psi\right\rangle^R/f_\pi^4$ collapses but $\left\langle\bar{\psi}\,\psi\right\rangle^R/T^4$ doesn't

STRUCTURES IN THE BARYON NUMBER SUSCEPTIBILITIES AROUND THE CROSSOVER ARE LIKELY RELATED TO CHIRAL SYMMETRY

BACKUP: ISENTROPES

Critical lensing

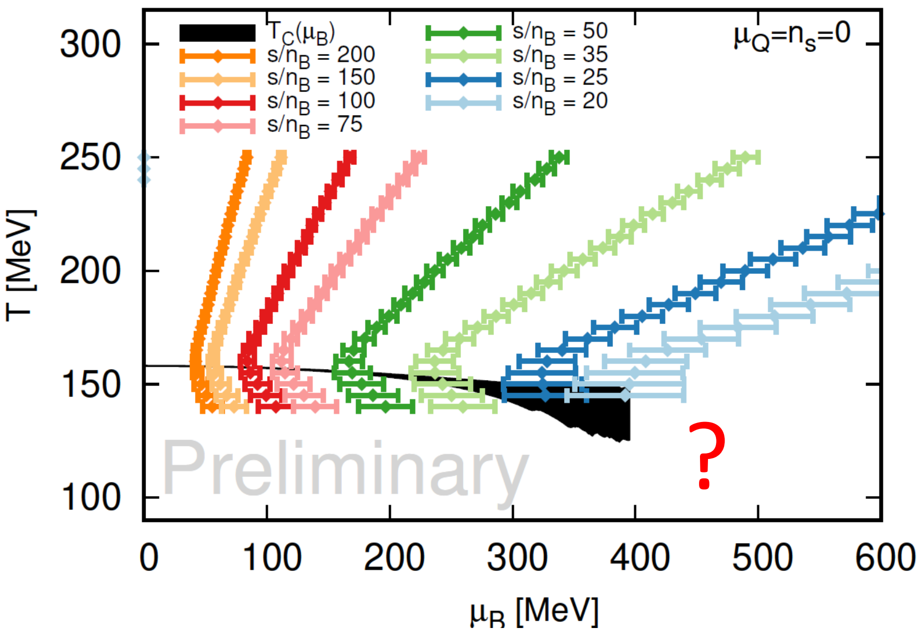




Sketch from [Dore et al, PRD106 (2022)]

EoS from extrapolations

Isentropes (resummation)



RHIC freeze-out [STAR, PRC96 (2017)]

$$\sqrt{s} = 19.6\text{GeV} \leftrightarrow \mu_B \approx 200\text{MeV}$$

$$\sqrt{s} = 11.5\text{GeV} \leftrightarrow \mu_B \approx 300\text{MeV}$$

$$\sqrt{s} = 7.7\text{GeV} \leftrightarrow \mu_B \approx 400\text{MeV}$$

No sign of critical lensing within errors

New preliminary dataset.
Improvement compared to last year comes from more accurate EoS at $\mu_B = 0$

BACKUP: n_S=0 VS μ_S=0

The crossover line (n_S=0 vs μ_S=0)