Lattice Weyl Fermion on a single spherical domain-wall 1

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⟶ Kan's Talk is in the next session!

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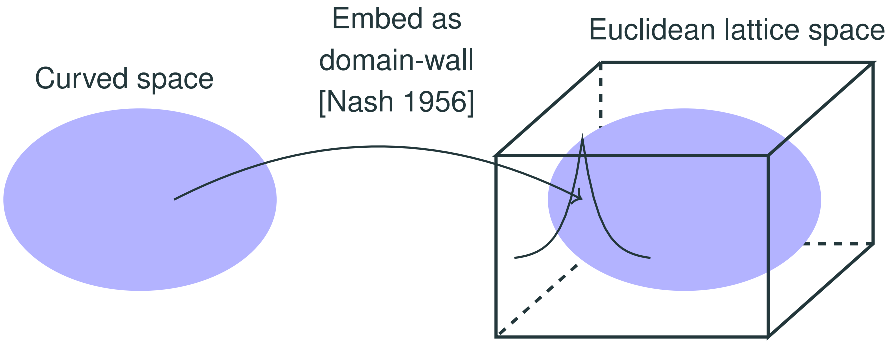
Motivation

Every curved manifold can be isometrically embedded into some higher-dimensional Euclidean spaces.



Edge modes are localized at the curved domain-wall.

= They feel "gravity" by the equivalence principle.

$\longrightarrow$ How about Weyl fermion?

Embedding a Curved Space [Nash, 1956]

For any n-dim. Riemann space (Y, g), there is an embedding $f : Y \to \mathbb{R}^m$ $(m \gg n)$ such that Y is identified as

$$x^\mu = x^\mu(y^1, \cdots, y^n) \ (\mu = 1, \cdots, m)$$

$$\left(\begin{array}{ll} x^\mu & : \text{Cartesian coordinates of } \mathbb{R}^m \\ y^i & : \text{Coordinates of } Y \end{array} \right.$$

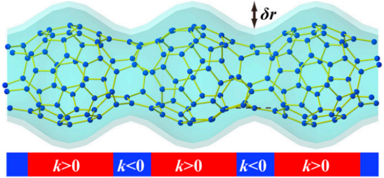
and the metric is written as

$$g_{ij} = \sum_{\mu\nu} \delta_{\mu\nu} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j}.$$

$\longrightarrow$ Vielbein and spin connection are also induced!

Any Riemannian manifold can be identified as a submanifold of a flat Euclidean space!

“Gravity" in Condensed Matter Physics



[Onoe et al., 2012] observed a gravitational effect on $1D$ uneven peanut-shaped C_{60} polymer.

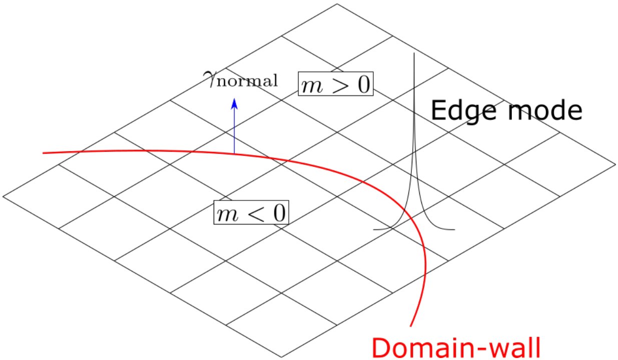
The Hamiltonian on a curved surface is

[Jensen and Koppe, 1971; da Costa, 1981]

$$H = -\frac{\hbar^2}{2m_*}\left[\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right) + h^2 - k\right], \quad \begin{cases} h\text{: Extrinsic curvature} \\ k\text{: Gaussian curvature} \end{cases}$$

$\longrightarrow$ Density of states depends on the curvatures.

Dirac Fermion on Curved Domain-Wall [SA and H. Fukaya, 2022]



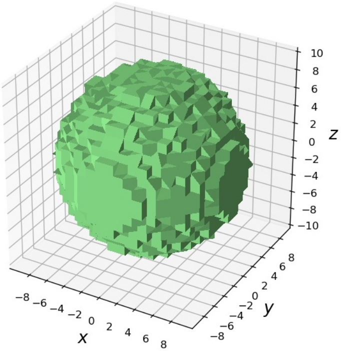
- Edge modes ($\gamma_{\text{normal}} = +1$) appear at the wall.
- They feel gravity through the induced spin connection.
- The continuum limit is unique (lattice spacing $\to 0$).
- Rotational symmetry is automatically recovered.

cf. flat case [Kaplan, 1992], spherical TI [Takane and Imura, 2013]

Weyl Fermion on Single Spherical Domain-wall

We investigate a Free fermion system with S^2 domain-wall.

$$D = \sum_{i=1}^{3} \sigma^i \frac{\partial}{\partial x^i} - m$$
$$\to \not{D}^{S^2} \frac{1}{2}(1 + \sigma^3)$$



- Spectrum
- Edge modes
- Continuum Limit
- Restoration of Symmetry

Kan will assign $U(1)$ gauge connection.

cf. S^1: [Kaplan and Sen, 2024], [Kaplan's talk]
S^2: [Clancy and Kaplan, 2024]

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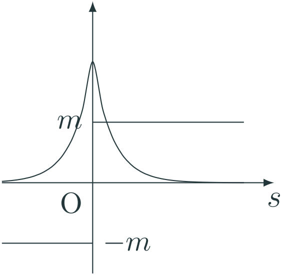
Flat Domain-wall

We consider a Hermitian massive Dirac operator in $(n+1)$-dim space:

$$H = \bar{\gamma}\left(\sum_{i=1}^{n} \gamma^i \partial_i + \gamma^s \partial_s + m(s)\right).$$



Assuming $\psi(x,s) = \eta_+(x) e^{-|m|s}$ and $\gamma^s \eta_+ = +\eta_+$, we find

$$H\psi(x,s) = \left(\bar{\gamma}\sum_{i=1}^{n} \gamma^i \partial_i\right) \eta_+(x) e^{-|m|s}$$

commutes with γ_s

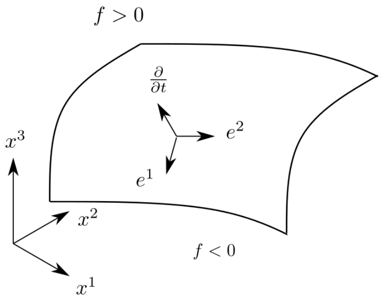
Edge states are localized at $s = 0$ and $\bar{\gamma}\sum_{i=1}^{n}(\gamma^i \partial_i) \simeq i\not{D}^n$!

Domain-wall

Hermitian Dirac operator with curved domain-wall

We analyze a Hermitian Dirac operator with a curved domain-wall:

$$H = \bar{\gamma}\left(\sum_{I=1}^{n+1} \gamma^I \frac{\partial}{\partial x^I} + m\text{sign}(f)\right)$$
$$= \bar{\gamma}(\not{D} + m\text{sign}(f))$$



The gamma matrices are

$$\gamma^I = \begin{cases} -\sigma_2 \otimes \tilde{\gamma}^I & (I = 1, \cdots, n) \\ \sigma_1 \otimes 1 & (I = n+1) \end{cases}$$

$$\bar{\gamma} = \sigma_3 \otimes 1.$$

$f = 0$ defines a domain-wall Y in $X = \mathbb{R}^{n+1}$.

cf. Takane and Imura [2013] considered the curved DW as a surface of a topological insulator.

Purpose of This Section

We solve the eigenvalue problem of $H = \bar{\gamma}(\not{D} + m\text{sign}(f))$ on $\mathbb{R}^{n+1}$. In the limit of large m, we show that eigenstates with $|E| < m$:

- Are localized at the domain-wall $Y = \{f = 0\}$,
- Exhibit positive chirality $\gamma_{\text{normal}} = \boldsymbol{n} \cdot \boldsymbol{\gamma} = +1$,
- Feel gravity through the spin connection on Y.

Here, the chirality for a spinor ψ is defined as

$$\langle \psi | \gamma_{\text{normal}} | \psi \rangle = \int_{\mathbb{R}^{n+1}} d^{n+1}x \, \psi^\dagger \gamma_{\text{normal}} \psi.$$

Appropriate vielbein

We take an appropriate vielbein on $\mathbb{R}^{n+1}$

$$\left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^{n+1}}\right) \to \Big(\underbrace{e_1, \cdots, e_n}_{\text{vielbein on } Y}, e_{n+1} = \frac{\partial}{\partial t}\Big),$$

where $e_I = e_I{}^J \frac{\partial}{\partial x^J}$ and t denotes the distance from Y.
The gamma matric in the normal direction is written by

$$\gamma_{\text{normal}} = e_{n+1}^{\ J} \gamma_J$$

and the Hermitian Dirac operator is

$$H = \bar{\gamma}\left(\gamma^I \frac{\partial}{\partial x^I} + m\,\text{sign}(f)\right) = \bar{\gamma}\big(\gamma^I (e^{-1})_I{}^J e_J + m\,\text{sign}(f)\big)$$
$$= \bar{\gamma}\big(s\gamma^J s^{-1} e_J + m\,\text{sign}(f)\big).$$

Here, $s = \exp\big(\frac{1}{4}\sum_{IJ} \alpha_{IJ}\gamma^I\gamma^J\big)$ is defined as $s^{-1}\gamma^J s = \gamma^I e_I{}^J$.
$(\alpha_{IJ} = -\alpha_{JI} \in \mathbb{R})$

Induced Spin Connection

By taking the local Lorentz trsf s, γ_{normal} and H changes to

$$s^{-1}\gamma_{\text{normal}}s = \gamma^{n+1} = \sigma_1 \otimes 1$$

$$s^{-1}Hs = \bar{\gamma}\left(\gamma^J\left(e_J + \underbrace{s^{-1}e_J(s)}\right) + m\,\text{sign}(f)\right)$$

Pure Spin Connection

$$= \bar{\gamma}\sum_{i=1}^{n}\gamma^i\left(e_i + \underbrace{\omega_{i,jk}\frac{\gamma^j\gamma^k}{4}}\right) + \bar{\gamma}\gamma^{n+1}\left(\frac{\partial}{\partial t} - \underbrace{\frac{1}{2}}\operatorname{tr}h + m\epsilon\gamma^{n+1}\right)$$

Induced Spin Connection   Extrinsic Curvature

$$= \sigma^1 \otimes i\not{D}^Y + \sigma^3\sigma^1\left(\frac{\partial}{\partial t} - \frac{1}{2}\operatorname{tr}h + \sigma^1 m\epsilon\right) \otimes 1$$

Low-energy modes must eliminate the second term.

$\longrightarrow$ $\psi \sim e^{-|m|t}$, $\gamma_{\text{normal}}\psi = +\psi$, $H\psi = i\not{D}^Y\psi$

They are localized at the wall and feel gravity through indeced spin connection!

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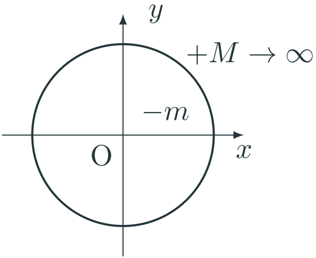
S^2 domain-wall

Domain wall:

$$m(r) = \begin{cases} -m & (r < r_0) \\ +M \to +\infty & (r \geq r_0) \end{cases}$$



Dirac operator:

$$D = \sum_{i=1}^{3} \sigma_i \frac{\partial}{\partial x^i} + m(r) = \sigma_r \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) + m(r) - \sigma_r \frac{D^{S^2}}{r},$$

$$D^{S^2} = \sum_{i=1}^{3} \sigma^i L_i + 1, \ \sigma_r = \frac{x}{r}\sigma_1 + \frac{y}{r}\sigma_2 + \frac{z}{r}\sigma_3.$$

Boundary condition: $\sigma_r \psi(x) = \pm\psi(x)$ at $r = r_0$

Effective Dirac op and Dirac op. of S^2

A local Lorentz transformation

$$R = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) & -e^{-i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) & e^{i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} e^{i\frac{\phi}{2}}$$

changes $\psi \to R^{-1}\psi$ and

$$D^{S^2} \to i\left(\sigma_1 \frac{\partial}{\partial\theta} + \frac{\sigma_2}{\sin\theta}\left(\frac{\partial}{\partial\phi} + \underbrace{\frac{i}{2} - \frac{\cos\theta}{2}\sigma_1\sigma_2}\right)\right),$$

Spin^c connection on S^2

$$\sigma_r \to \sigma_3$$

Edge states feel gravity through the induced connection!
[Takane and Imura, 2013].

Eigenstate of $D^\dagger D$ and $DD^\dagger$

Let $\chi_\pm$ satisfy

$$D^{S^2}\chi_\pm = \lambda\chi_\mp,\ (\lambda = 1, 2, \cdots)$$
$$\sigma_r\chi_\pm = \pm\chi_\pm.$$

In the large m limit, we assume $\psi_\pm = \frac{1}{r}e^{-m|r-r_0|}\chi_\pm$, then we get

$$D\psi_+ = \left(\sigma_r\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) + m\epsilon - \sigma_r\frac{D^{S^2}}{r}\right)\psi_+ \simeq \frac{\lambda}{r_0}\psi_-.$$
$$D^\dagger\psi_- = \left(-\sigma_r\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) + m\epsilon + \sigma_r\frac{D^{S^2}}{r}\right)\psi_- \simeq \frac{\lambda}{r_0}\psi_+.$$

$\longrightarrow D^\dagger D\psi_+ = \left(\frac{\lambda}{r_0}\right)^2\psi_+$ and $DD^\dagger\psi_- = \left(\frac{\lambda}{r_0}\right)^2\psi_-$

Chiral fermion appears at the wall!

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S^2 Domain-wall Fermion on Lattice

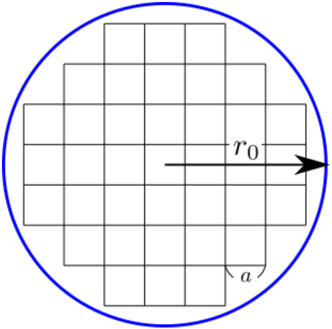
We consider a lattice on B^3 with the radius r_0.

The (Wilson) Dirac op is

$$D = \frac{1}{a}\left(\sum_{i=1}^{3}\left[\sigma^i \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2}\nabla_i \nabla_i^\dagger\right] - m\right).$$

$$(\nabla_i \psi)_x = \psi_{x+\hat{i}} - \psi_x,\ (\nabla_i^\dagger \psi)_x = \psi_{x-\hat{i}} - \psi_x$$

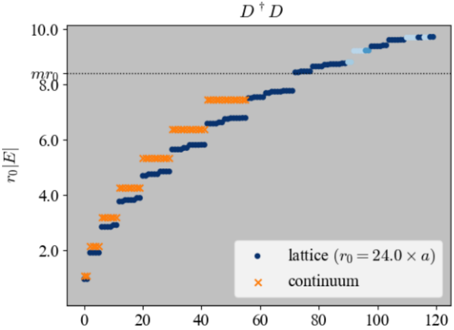
+OBC

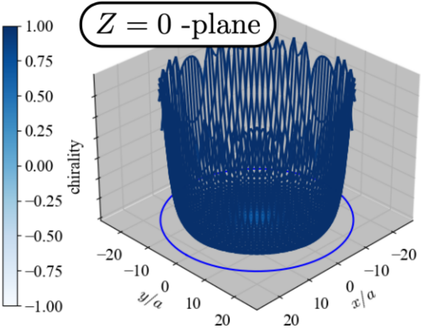


We analyze $D^\dagger D$ and $DD^\dagger$.

Spectrum and Edge modes of $D^\dagger D$

We solve $D^\dagger D\psi = E^2\psi$ when $r_0 = 24a, ma = 0.35$.

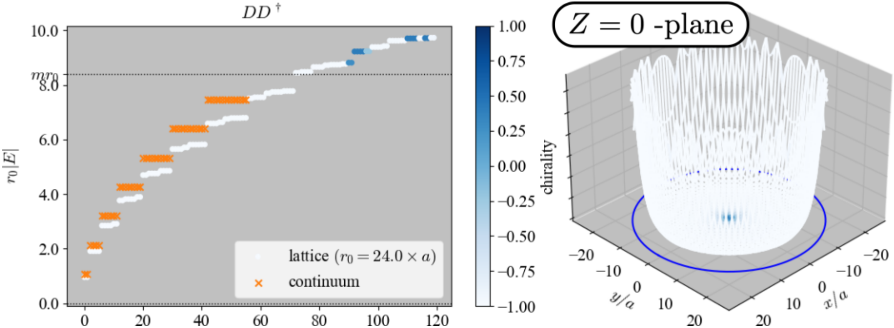




- Localized at the boundary
- $\sigma_r\psi = +\psi$
- There is a gap from zero (as a gravitational effect)
- Agrees well with the continuum prediction

Spectrum and Edge modes $DD^\dagger$

We also compute $DD^\dagger\psi = E^2\psi$.



The result is almost the same as $D^\dagger D$, but edge modes are negative chiral modes:

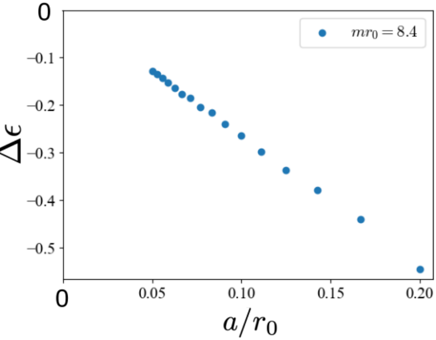
$$\sigma_r \psi = -\psi$$

It seems that a chiral theory is possible...

Continuum Limit and Finite-volume Effect

Continuum limit $a \to 0$

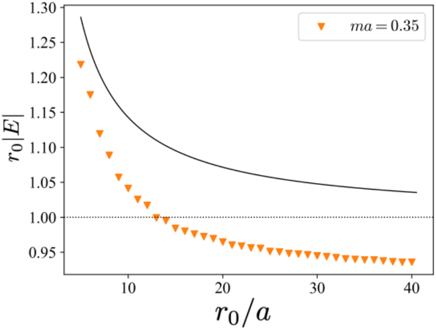
($mr_0 = 8.4$ is fixed)



Agree well with
the conti. prediction!

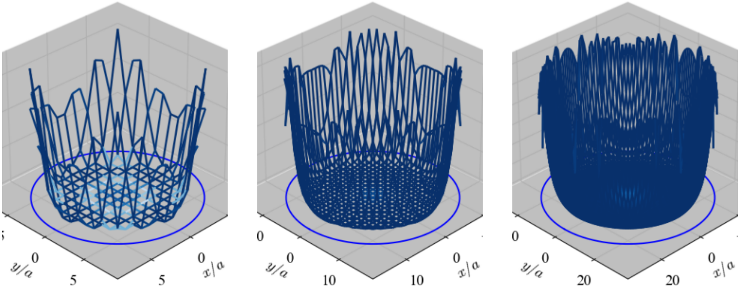
Large volume limit $r_0 \to \infty$

($ma = 0.35$ is fixed)

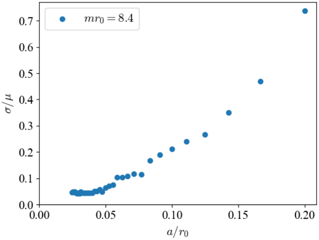


Saturates in the large r_0 limit!

Restoration of Rotational Symmetry



$a/r_0 = 1/8$ $\quad a/r_0 = 1/16$ $\quad a/r_0 = 1/32$



The rotational symmetry automatically recovers in the continuum limit!

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Conclusion

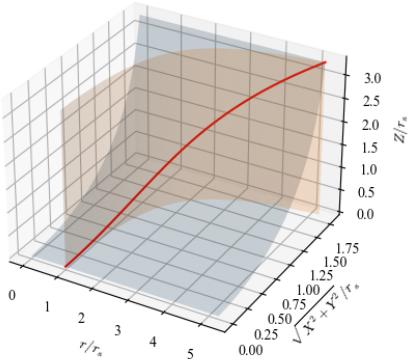
We investigate a FREE fermion system with a S^2 domain-wall.

- Weyl fermions appear at the wall.
- Continuum limit is good!
- Low-energy theory seems chiral.

But what if link variables exist?
$\longrightarrow$ Kan will explain this situation in detail.

Outlook

- Symmetric mass generation.
- Embedding
  $4D$ Schwarzschild space into
  $6D$ flat space [Kasner, 1921].



$$
\begin{aligned}
ds^2 =& dx^2 + dy^2 + dz^2 + dX(r,\tau)^2 + dY(r,\tau)^2 + dZ(r,\tau)^2 \\
=& \left(1 - \frac{\beta}{4\pi r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{\beta}{4\pi r}} + r^2 d\Omega^2
\end{aligned}
$$

Reference i

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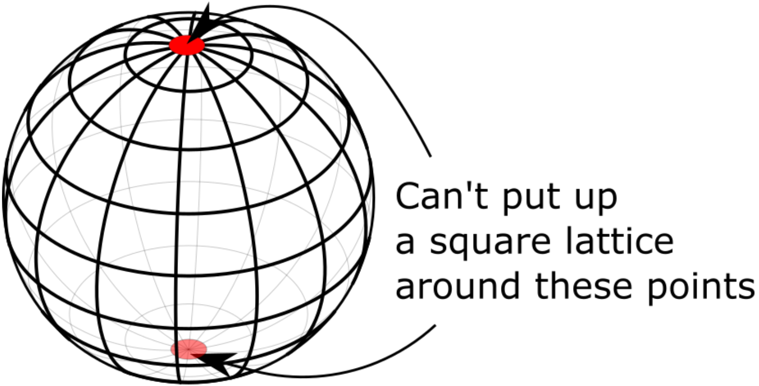
Lattice gauge theory on a curved space



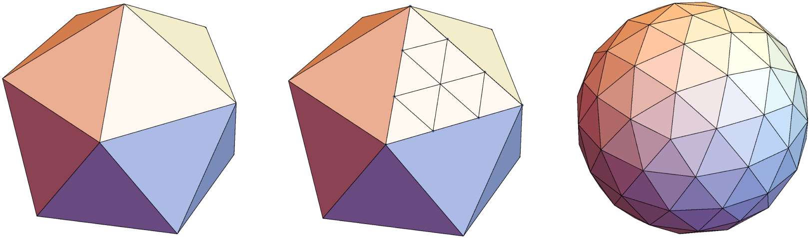
We can not approximate d-dim space with d-dim square lattice.

$\longrightarrow$ Can't handle gravity!

Triangular Lattice

Arbitrary space can be discretized by a triangular lattice.
Their lengths and angles represent the gravity.

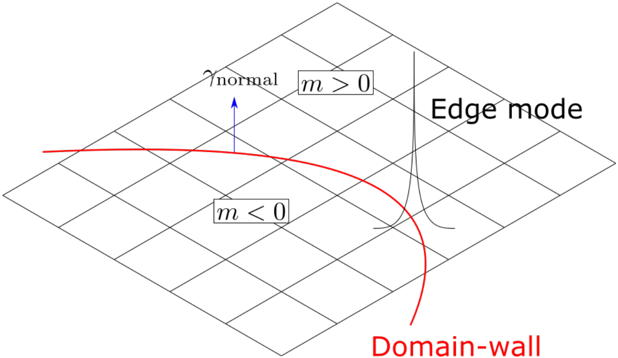
[Regge, 1961; Ambjørn et al., 2001; Brower et al., 2017]



Triangular lattice on 2-dim sphere [Brower et al., 2017]

However, continuum limit is not unique and symmetry restoration is non-trivial.

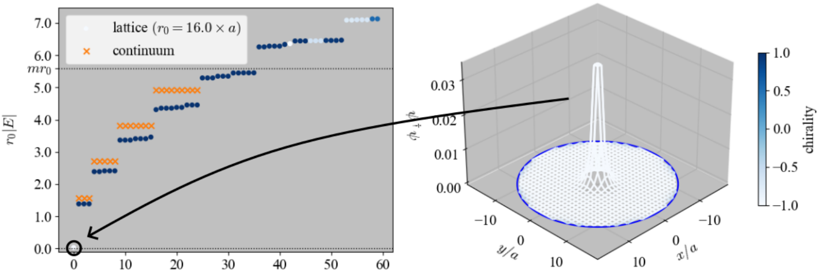
Curved Kaplan Domain-Wall [SA and H. Fukaya, 2022]



- Edge modes ($\gamma_{\text{normal}} = +1$) appear at the wall.
- They feel gravity through the induced spin connection.
- The continuum limit is unique (lattice spacing $\to 0$).
- Rotational symmetry is automatically recovered.

cf. flat case [Kaplan, 1992], spherical TI [Takane and Imura, 2013]

With nontrivial $U(1)$ link variables



When the monopole exists in the ball, another 0-mode appears around the monopole.

$\longrightarrow$ An obstacle in formulating lattice chiral gauge theory.

Induced Spin Connection

By taking the local Lorentz trsf s, γ_{normal} and H changes to

$$s^{-1}\gamma_{\text{normal}}s = \gamma^{n+1} = \sigma_1 \otimes 1$$

$$s^{-1}Hs = \bar{\gamma}\left(\gamma^J\left(e_J + \underset{\text{Pure Spin Connection}}{\underbrace{s^{-1}e_J(s)}}\right) + m\,\text{sign}(f)\right)$$

$$= \bar{\gamma}\sum_{i=1}^{n}\gamma^i\left(e_i + \underset{\text{Induced Spin Connection}}{\underbrace{\omega_{i,jk}\frac{\gamma^j\gamma^k}{4}}}\right) + \bar{\gamma}\gamma^{n+1}\left(\frac{\partial}{\partial t} - \frac{1}{2}\underset{\text{Extrinsic Curvature}}{\underbrace{\operatorname{tr} h}} + m\epsilon\gamma^{n+1}\right)$$

$$= \sigma^1 \otimes i\not{D}^Y + \sigma^3\sigma^1\left(\frac{\partial}{\partial t} - \frac{1}{2}\operatorname{tr} h + \sigma^1 m\epsilon\right)\otimes 1$$

$s^{-1}e_J(s)$ is a pure gauge field but edge states are influenced by the nontrivial spin connection on Y!

Edge mode

In the large m limit, we find an edgemode of H as

$$\psi = se^{-m|t|} \exp\left(\int_0^t dt' \frac{1}{2} \operatorname{tr} h\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \chi(y)$$

$$\left(i\not{D}^Y \chi = \sum_{i=1}^{n} \gamma^i \left(e_i + \underbrace{\omega_{i,jk}}_{} \frac{\gamma^j \gamma^k}{4} \right) \chi = \lambda \chi \right).$$

Spin connection on Y

The eigenvalue of H is obtained by

$$H\psi = se^{-m|t|} \exp\left(\int_0^t dt' \frac{1}{2} \operatorname{tr} h\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes i\not{D}^Y \chi(y) = \lambda \psi$$

$\gamma_{\text{normal}} \psi = s(\sigma_1 \otimes 1) s^{-1} \psi = +\psi \longleftarrow$ ψ is a chiral mode

Massless chiral modes appear at the wall and feel gravity through the induced spin connection!