Lattice Weyl fermion on a single spherical domain-wall 2

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Based on collaborations with Shoto Aoki (U. Tokyo), Hidenori Fukaya (Osaka U.), arXiv:2402.09774 [hep-lat].

Introduction

We consider an S^2 domain-wall (DW) in flat $\mathbb{R}^3$ space.

Shoto showed in the previous presentation that the edge zero mode on the DW appears and feels gravity through the induced connection [Aoki–Fukaya–KN ('24)] (cf. [Aoki–Fukaya ('22)]).

In this talk, we turn on the $U(1)$ gauge field.

We consider a Dirac fermion with a position-dependent mass

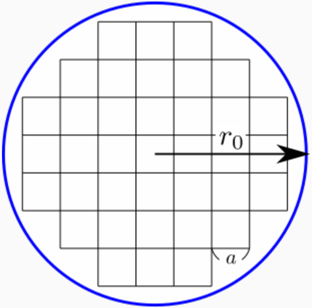
$$m(\boldsymbol{x}) = \begin{cases} -m & \text{for } |\boldsymbol{x}| \leq r_0, \\ M \to \infty & \text{otherwise,} \end{cases}$$

in $\mathbb{R}^3$, where we employ the Shamir DW fermion [Shamir ('93)].

A similar setup has been recently studied by [Kaplan ('23), Kaplan–Sen ('24), Clancy–Kaplan ('24)].

The chirality operator that anticommutes with the Hamiltonian is

$$\sigma_r = \frac{1}{|\boldsymbol{x}|} \sum_{i=1}^{3} \sigma^i x_i.$$



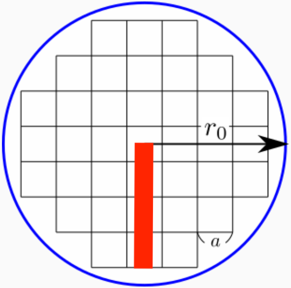
Nontrivial $U(1)$ gauge field

We turn on the nontrivial $U(1)$ gauge link variables,

$$U_j(\boldsymbol{x}) = \exp\left(\mathrm{i}\int_x^{x+\hat{j}} A_j(\boldsymbol{x}')dx'^j\right)$$

with the vector potential

$$A_x = \frac{-q_m y}{r(r+z)}, \quad A_y = \frac{q_m x}{r(r+z)}, \quad A_z = 0.$$



Here $q_m = n/2$ ($n \in \mathbb{Z}$) is the quantized topological (magnetic) charge.

This expression has a singularity at $x = y = 0$ and $z < 0$, which is the Dirac string.

The Dirac string has no physical effect.

To see this, let's compare the values of the two plaquettes $p_\pm$ in the xy-plane whose centers are located at $\boldsymbol{x}_\pm = (0,0,\pm 1/2)$:

$$\begin{aligned} p_\pm =& \exp\Bigg(\mathrm{i}\int_{-1/2}^{1/2} dx A_x(x,-1/2,\pm 1/2) + \mathrm{i}\int_{-1/2}^{1/2} dy A_y(1/2,y,\pm 1/2) \\ & - \mathrm{i}\int_{-1/2}^{1/2} dx A_x(x,1/2,\pm 1/2) - \mathrm{i}\int_{-1/2}^{1/2} dy A_y(-1/2,y,\pm 1/2) \Bigg) \\ =& \exp(\pm \mathrm{i}\pi n/3). \end{aligned}$$

Thus the singularity of the gauge field is automatically canceled by the multivaluedness of the $U(1)$ link variables.

Spectra of the Dirac operator

In the presence the nontrivial $U(1)$ gauge field, the Wilson Dirac operator is given by

$$D_W = \sum_{i=1}^{3} \sigma^i \frac{\nabla_i - \nabla_i^\dagger}{2a} + \frac{1}{2a}\nabla_i \nabla_i^\dagger - m,$$

which depends on the gauge field through the difference operator

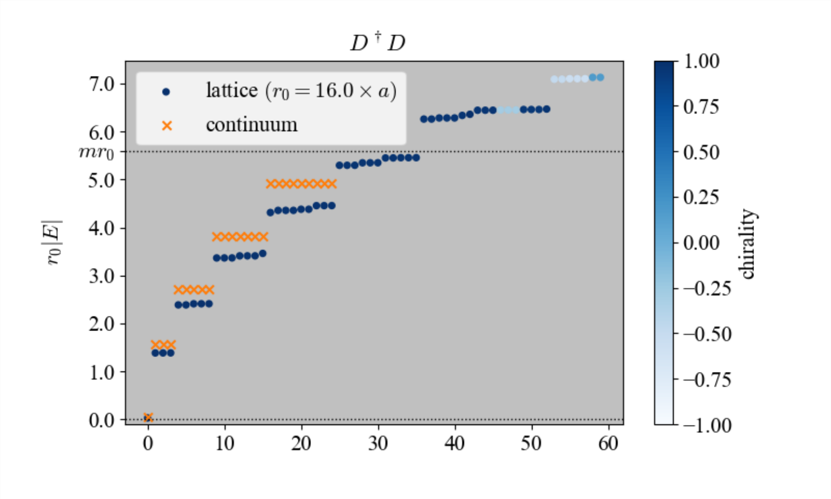
$$(\nabla_j \psi)_x = U_j(x)\psi_{x+\hat{j}} - \psi_x.$$

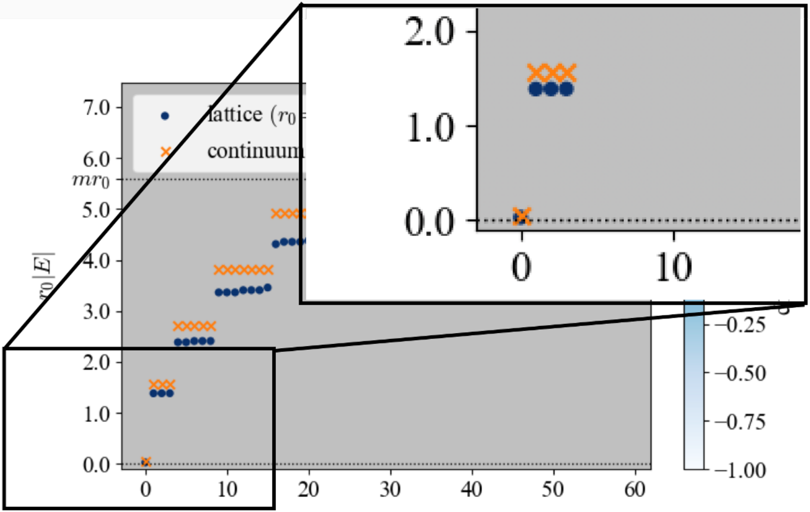
We impose $\psi_x = 0$ outside the domain-wall, i.e.,

$$\psi_x = 0 \quad \text{if} \quad |\boldsymbol{x}| \geq r_0,$$

where the center of the domain-wall is put at the origin.

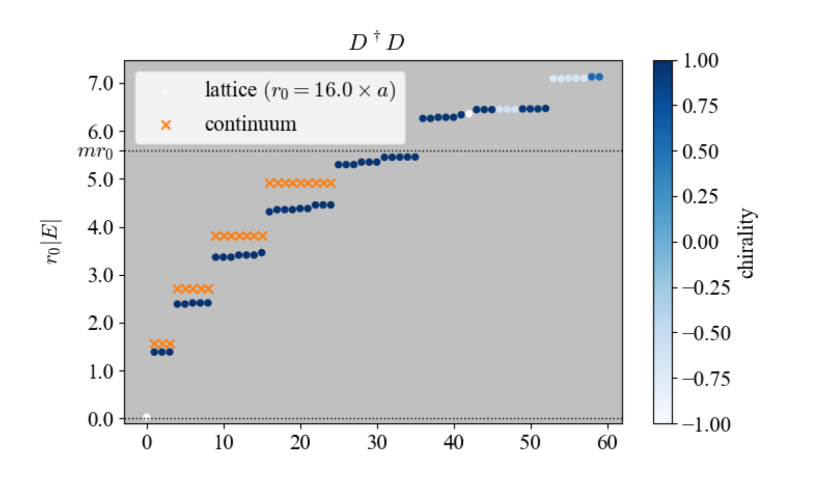
The eigenvalue spectrum of $D_W^\dagger D_W$ with $n = 1$ (at $ma = 0.35$):

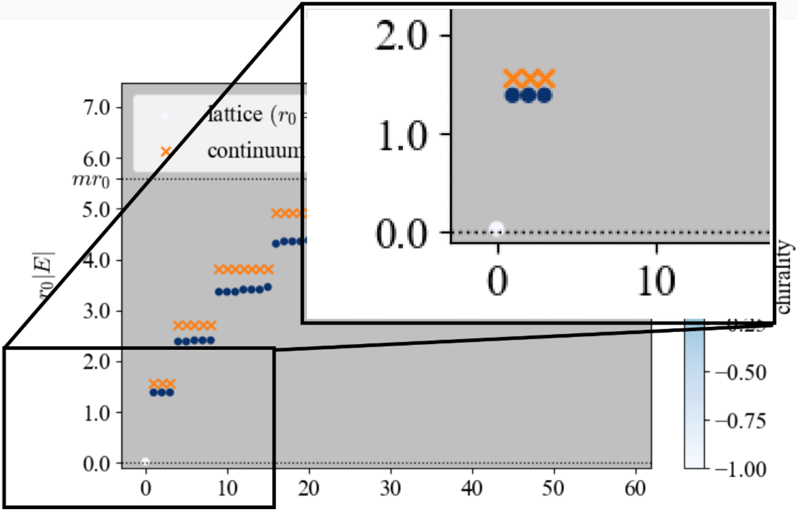




lattice (r_0
continuum
$r_0|E|$
7.0
6.0
mr_0
5.0
4.0
3.0
2.0
1.0
0.0
0
10
20
30
40
50
60
−0.25
−0.50
−0.75
−1.00

The eigenvalue spectrum of $D_W^\dagger D_W$ with $n = -1$:





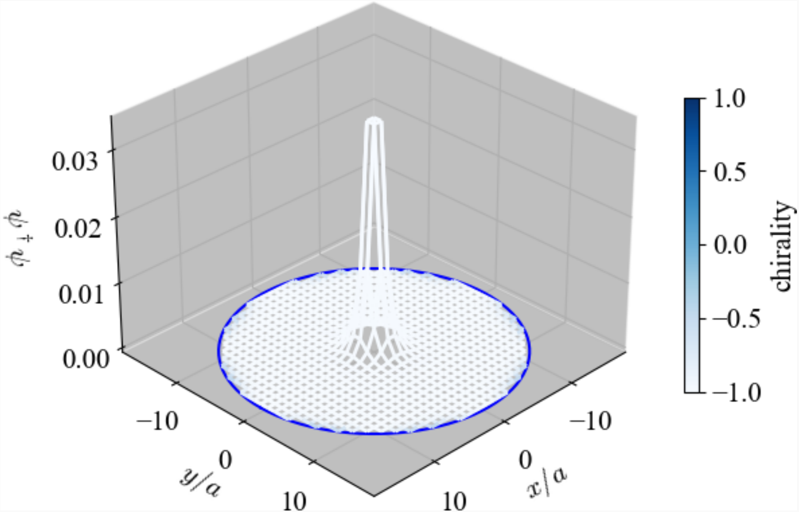
lattice (r_0
continuum
7.0
6.0
mr_0
5.0
4.0
3.0
2.0
1.0
0.0
$r_0|E|$
0
10
20
30
40
50
60
chirality
−0.50
−0.75
−1.00

We see that $D_W^\dagger D_W$ with $n = \pm 1$ has one near-zero mode with $\sim \pm 1$ chirality, respectively.

However, the near-zero mode with chirality $\sigma_r = -1$ does NOT exist in the continuum analysis.

In fact, this mode is not located at the domain-wall but localized at the center where the monopole sits.

The amplitude of the near-zero mode with $\sigma_r = -1$ chirality at the z $= 1/2$ slice



Note that the Wilson term contributes to effective mass,

$$-m_{\text{eff}} = \sum_{i=1}^{3} \frac{1}{2a} \nabla_i \nabla_i^{\dagger} - m$$

When the Wilson term becomes large, (which typically occurs in the vicinity of the singularity of the gauge field,) the sign of the effective mass flips.

Then a domain-wall is generated around the singularity and the fermion zero mode become localized on it.

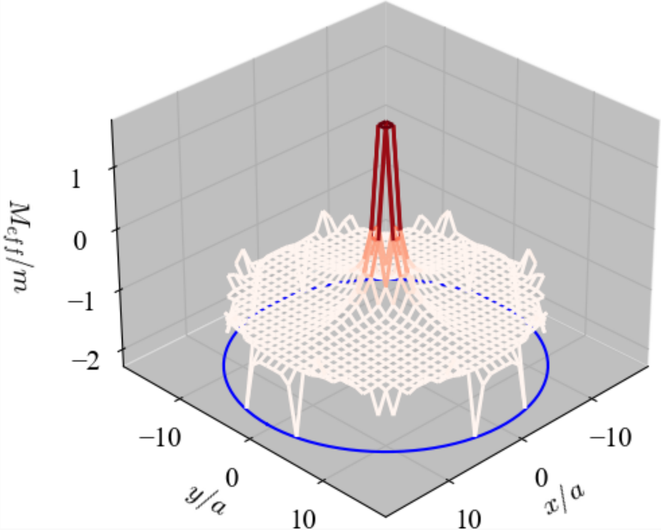
The same mechanism can explain the fractional charge of a monopole in topological insulators (i.e., Witten effect) [Aoki–Fukaya–KN–Koshino–Matsuki ('23)].

To see this explicitly, we present the distribution of the effective mass defined by

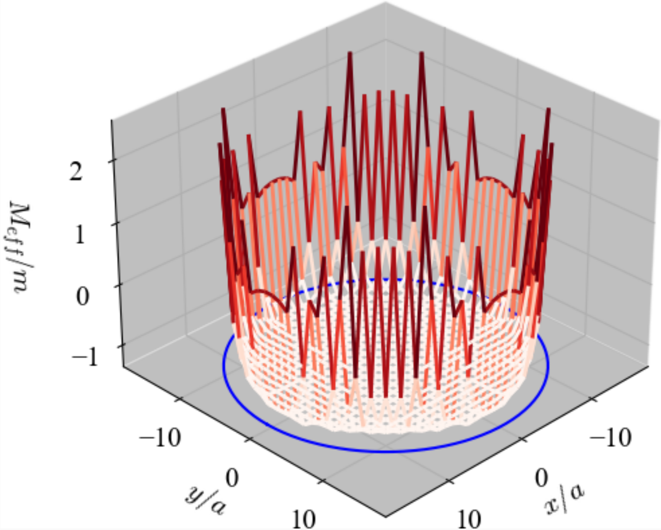
$$M_{\mathrm{eff}}(\boldsymbol{x}) = \frac{\phi_0^\dagger(\boldsymbol{x}) \left(\sum_{i=1}^{3} \frac{1}{2a} \nabla_i \nabla_i^\dagger - m \right) \phi_0(\boldsymbol{x})}{\phi_0^\dagger(\boldsymbol{x}) \phi_0(\boldsymbol{x})},$$

where $\phi_0(\boldsymbol{x})$ denotes the center-localized zero mode with $n = -1$.

$M_{\mathrm{eff}}(\boldsymbol{x})$ of the zero mode with $n = -1$ in the $z = 1/2$ slice.



$M_{\rm eff}(\boldsymbol{x})$ of the zero mode with $n = 0$ in the $z = 1/2$ slice.



Stability of opposite chiral modes

The appearance of the opposite chiral zero mode indicates that the low-energy theory is not a simple chiral theory with a single Weyl fermion on a single DW but a nontrivial vectorlike theory on the two DWs.

The center-localized mode is stable against perturbation of the gauge (and gravitational) field since it is topologically protected.

It is, therefore, crucial for constructing a chiral gauge theory in this setup, to find a formulation which eliminates such oppositely chiral zero modes.

Consider a Hermitian operator consisting of the DW Dirac operator D in continuum,

$$\hat{D} := \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}, \quad D = \sum_{i=1}^{3} \sigma^i \left(\frac{\partial}{\partial x^i} - \mathrm{i} A_i(\boldsymbol{x}) \right) + m(\boldsymbol{x})$$

with

$$m(\boldsymbol{x}) = \begin{cases} -m \ (< 0) & \text{for} \quad r_1 \leq |\boldsymbol{x}| \leq r_0, \\ +M \ (> 0) & \text{otherwise,} \end{cases}$$

where we assume that r_1 is put in the vicinity of the monopole at the inverse of the cutoff scale.

In the $M \to \infty$ limit, the edge-localized states are the eigenmode of $\gamma := \sigma_3 \otimes \sigma_r$.

The positive γ modes are located at the outer domain-wall at $r = r_0$, whereas the negative γ modes are at $r = r_1$.

We have two 2D massless Dirac operators D^{S^2} at $r = r_0$ and $\bar{D}^{S^2}$ at $r = r_1$ share the same index:

$$\mathrm{Ind}\, D^{S^2} = \mathrm{Ind}\, \bar{D}^{S^2} = n = \frac{1}{2\pi} \int_{S^2} F.$$

This is a consequence of cobordism invariance of the AS index.

A comment on [Golterman–Shamir ('24)]

[Golterman–Shamir ('24)] found a superfluous conserved current in the Grabowska–Kaplan and single DW framework.

For simplicity, we employ 5D GK framework with two DWs. We have left and right-handed Weyl fermions, $\psi_L(x) = \Psi^{(L)}(x, s = 0)$ and $\psi_R(x) = \Psi^{(R)}(x, s = 0)$ on the near wall ($s = 0$).

Then we find two gauge invariant conserved $U(1)$ currents:

$$\mathcal{R}_\mu = \frac{1}{2}\sum_{s=0}^{N_5}\left(\bar{\Psi}^{(R)}_{x,s}(1+\gamma_\mu)U_{x,s,\mu}\Psi^{(R)}_{x+\hat{\mu},s} - \bar{\Psi}^{(R)}_{x+\hat{\mu},s}(1-\gamma_\mu)U^\dagger_{x,s,\mu}\Psi^{(R)}_{x,s}\right),$$

$$\mathcal{L}_\mu = \frac{1}{2}\sum_{s=0}^{N_5}\left(\bar{\Psi}^{(L)}_{x,s}(1+\gamma_\mu)U_{x,s,\mu}\Psi^{(L)}_{x+\hat{\mu},s} - \bar{\Psi}^{(L)}_{x+\hat{\mu},s}(1-\gamma_\mu)U^\dagger_{x,s,\mu}\Psi^{(L)}_{x,s}\right).$$

Alternatively, we can construct a vector and an axial current:

$$\mathcal{V}_\mu = \mathcal{R}_\mu + \mathcal{L}_\mu, \qquad \mathcal{A}_\mu = \mathcal{R}_\mu - \mathcal{L}_\mu.$$

The both are non-anomalous, which conflicts with standard QCD.

We can apply this argument to the single DW framework. Namely, we can define a fermion number conserved current for each Weyl fermion.

The stability of the zero mode that we obtained is consistent with existence of this current.

How to avoid opposite chiral modes: ideas

Note that the situation on the lattice is not exactly the same as the continuum analysis:

1. The inner domain-wall is not fixed but can appear only with singular gauge field configuration.
2. The center-localized mode is effectively a zero-dimensional object, having no θ and ϕ dependence on the small sphere, which is isolated from the bulk modes.

Taking these differences from the flat two domain-wall case into account, we propose two ideas which may be useful to avoid appearance of the opposite chiral modes.

The admissibility condition

One is to impose the admissibility condition [Lüscher ('82)]. Then the admissible link gauge fields satisfy

$$||1 - P_{ij}(\boldsymbol{x})|| < \epsilon$$

with some small real number ϵ is taken to construct the theory. Then, singular configurations are not allowed in the theory.

If ϵ is small enough, we will be able to limit the additive mass renormalization through the Wilson term and avoid dynamical creation of the domain-walls.

This would restrict the surface theory to be fixed in a topologically trivial sector where the instanton number is forced to be zero.

The symmetric mass generation

Another is the symmetric mass generation (SMG).

Recently, possibility of gaping out the edge modes for special combinations of Weyl fermions by nontrivial interactions without breaking the chiral symmetry is discussed.

This is possible only when the anomalies are canceled.

The unwanted center-localized modes in our system are essentially zero-dimensional, so the analysis may be easier than higher dimensions.

E.g., 3-4-5-0 $U(1)$ chiral fermion model in 2D [Wang–Wen ('13)],

$$S = \int dt\, dx \sum_{I=1}^{4} \psi_I^\dagger (\mathrm{i}\partial_t - \mathrm{i}v_I \partial_x)\psi_I$$

with two left-moving modes $\psi_{1,2}$ (of $v_{1,2} = +1$) and two right-moving modes $\psi_{3,4}$ (of $v_{3,4} = -1$). The fermions are charged under a chiral U(1) symmetry: $\psi_I \to e^{\mathrm{i}q_I\theta}\psi_I$ with $q_I = (3, 4, 5, 0)$.

The fermions can be gapped by the interaction

$$S_{\text{int}} = \int dt\, dx\, g_1 \left(\psi_1 \psi_2^\dagger \partial_x \psi_2^\dagger \psi_3 \psi_4 \partial_x \psi_4 + \text{h.c.}\right) + g_2 \left(\psi_1 \partial_x \psi_1 \psi_2 \psi_3^\dagger \partial_x \psi_3^\dagger \psi_4 + \text{h.c.}\right)$$

without symmetry breaking.

Summary

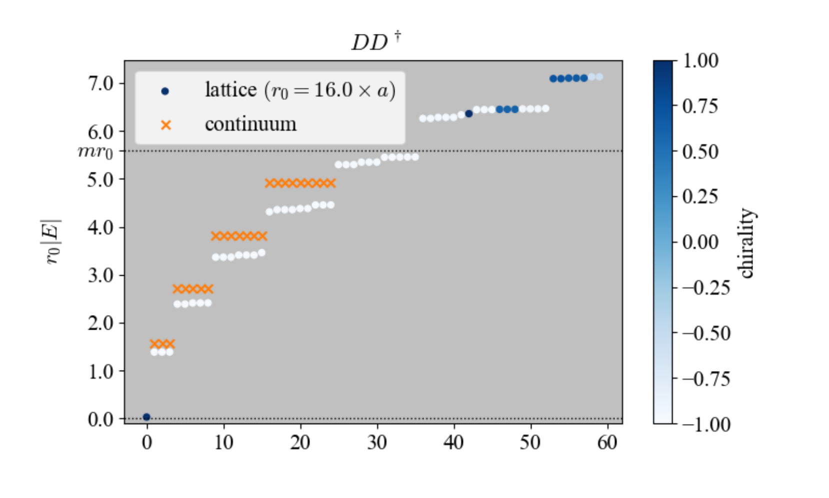
We have investigated the Shamir lattice S^2 domain-wall fermion system with the nontrivial $U(1)$ gauge field.

With the magnetic monopole-like configuration, we have observed that a stable center-localized zero mode appears in the vicinity of the monopole, having the opposite chirality to that of the edge modes.

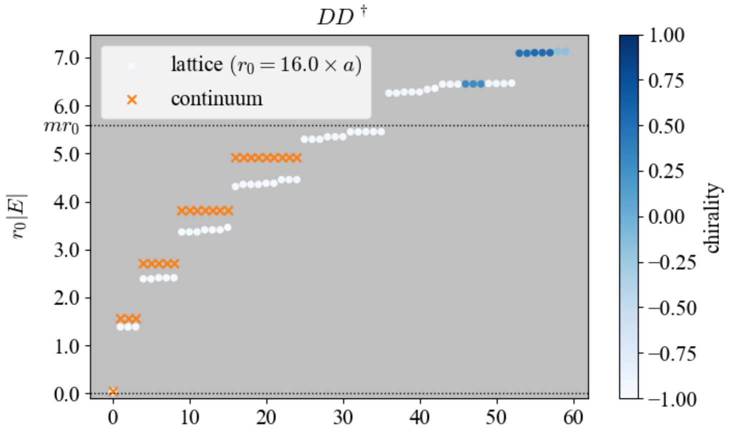
We have made two proposals to avoid appearance of these opposite chiral zero modes: the admissibility condition and the SMG.

Back-up

The eigenvalue spectrum of $D_W D_W^\dagger$ with $n = 1$:



The eigenvalue spectrum of $D_W D_W^\dagger$ with $n = -1$:



$M_{\text{eff}}(\boldsymbol{x})$ of the zero mode with $n = 0$ in the $z = 1/2$ slice.