’t Hooft Anomalies and the dynamics of QFT

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Remarks

This talk is mostly a review on some theoretical developments in the category of **hep-th**.
I will try to explain some of them to people outside of hep-th. (The talk will be boring to experts...)

There are many works in the literature, but
I apologize in advance that I could not mention many people's works properly.

Introduction

Global symmetries are helpful when we study dynamics of QFT.

Suppose a theory has a global symmetry G.

Conventional use of global symmetry:

- Particles are multiplets (representations) of G, or the symmetry is spontaneously broken.
- Interaction terms must be invariant under G.

Introduction

’t Hooft introduced a new way to use global symmetries for the study of the dynamics of QCD.

['t Hooft,1979]

Let us consider a massless QCD-like theory:

- Gauge group : $SU(N_c)$ (For real QCD, $N_c = 3$)
- The number of massless quarks q : N_f
  (For real QCD, approximately $N_f = 2$ or 3)

$$L = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \sum_{i=1}^{N_f} \bar{q}_i \gamma^\mu D_\mu q_i$$

Introduction

Massless quarks are decomposed into left-handed and right-handed parts.

$$q_i \to (q_{L,i}, q_{R,i}) \qquad (i = 1, \cdots, N_f)$$

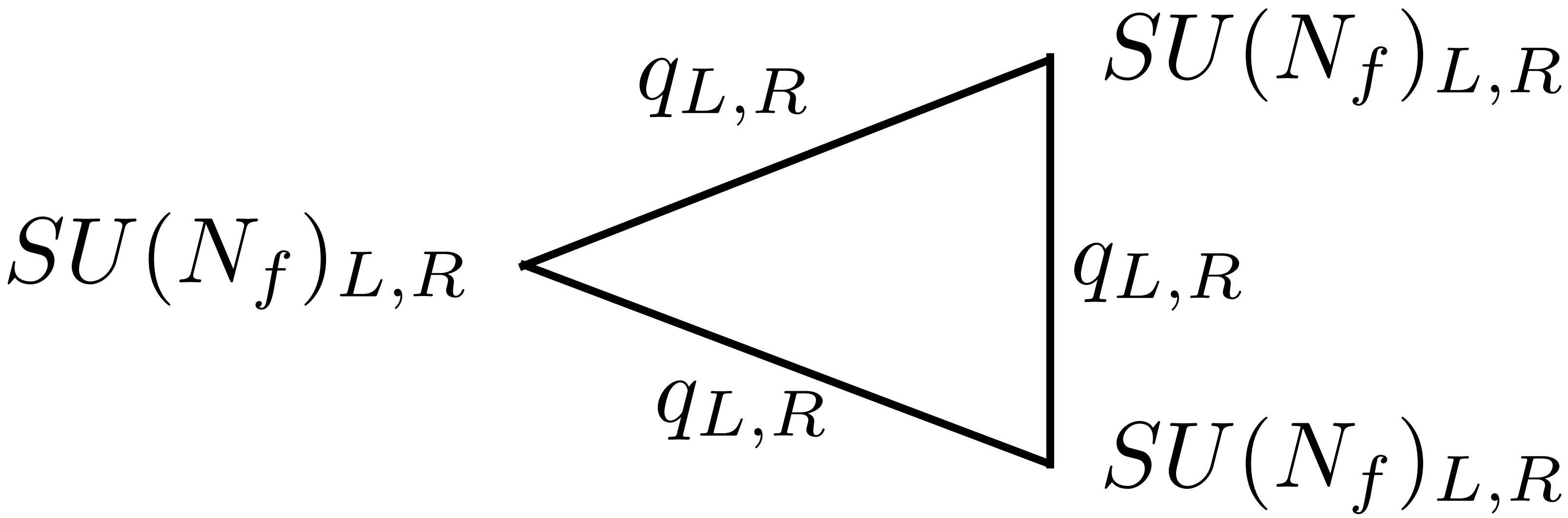
Each left- and right-handed quarks can be transformed under different flavor symmetries,

$G = SU(N_f)_L \times SU(N_f)_R$: chiral symmetry

q_L transforms under $SU(N_f)_L$.
q_R transforms under $SU(N_f)_R$.

Introduction

$SU(N_f)_L$ and $SU(N_f)_R$ have anomalies if they are coupled to background (external) gauge field.



I will later review the concept of 't Hooft anomaly more generally.

Introduction

Anomalies are known to be conserved under renormalization group (RG) flows.

UV: The quarks have the 't Hooft anomaly

↓ RG flows

IR: ??? (difficult to analyze due to strong dynamics)

No matter what happens in the IR, the IR theory must have the same anomaly as the UV theory : anomaly matching

Introduction

Application:

For the purpose of illustration, suppose for simplicity that

- The IR theory confines, in the sense that all fields (particles) are gauge-singlets (mesons, baryons,...).
- There is no massless chiral fermions in the IR.
  (This is automatically true if e.g. $N_c =$ even)

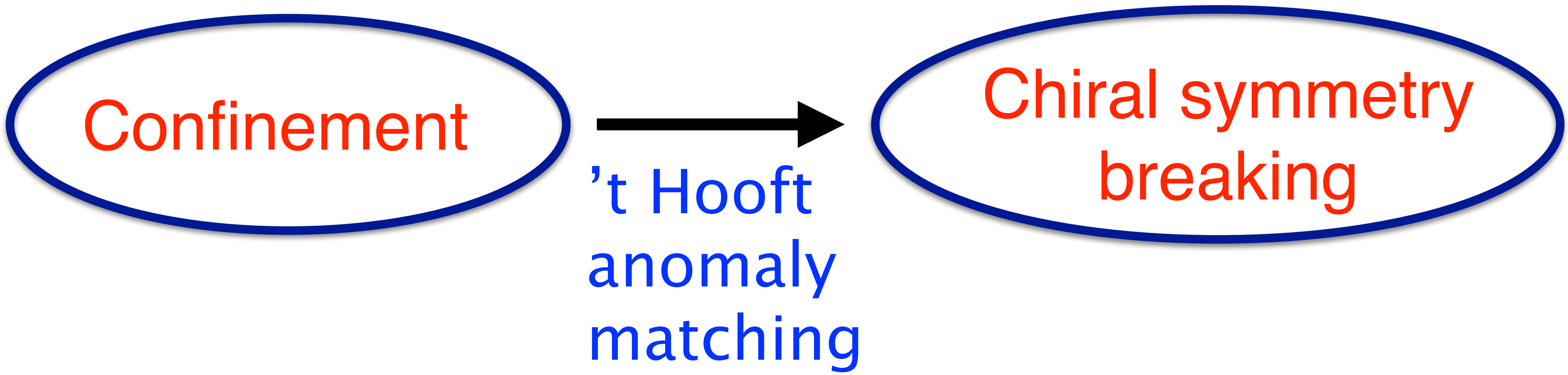
The UV has the 't Hooft anomaly, while the IR does not have massless fermions to reproduce the anomaly.

The only way to match the anomaly is that
the chiral symmetry is spontaneously broken.

(Wess-Zumino-Witten term can match the anomaly.)

Introduction

In this way, the ’t Hooft anomaly matching shows that the chiral symmetry must be spontaneously broken if the theory confines.



This is a powerful way to study the dynamics of QFT in general.

Introduction

(Recent) developments:

- Not just QCD-like theory, but more general theories (e.g. theories relevant to condensed matter, etc.)
- More anomalies (including nonperturbative ones)
- More "symmetries"

Main message of the talk:
There are many predictions on the dynamics of various QFTs from 't Hooft anomalies, waiting to be tested by numerical studies!

I will discuss a few examples.

Contents

What is anomaly?

Some comments:

’t Hooft originally considered coupling the QCD-like theory to a dynamical gauge field for the chiral symmetry, and also introduced “spectator fields”.

The gauge fields need not be dynamical.
We can consider them just as background fields.
Spectator fields are not necessary.

I would like to explain how to think about anomalies.

Background fields

QFT with a global symmetry G can be formulated in the presence of a background (external) field B for G.

Example:

In the previous QCD-like theory,

$$G = SU(N_f)_L \times SU(N_f)_R$$

$$\text{Background } B = (B_{L,\mu}, B_{R,\mu})$$

We do not perform path integrals over background fields.

Remarks

We are not necessarily interested in the dynamics of a theory in the presence of background fields. (Sometimes we are. It depends on contexts.)

We are just using background fields as a probe or theoretical experiment of the properties of a theory.

Analogy:

If we apply electric fields to a material and get electric currents, we know that the material contains light, charged degrees of freedom.

These light charged degrees of freedom exist no matter whether we apply electric fields or not.

Partition function

Suppose

Φ : dynamical fields

B : background fields

The action

$$S = S(\Phi, B)$$

Partition function

(Generating functional of correlation functions)

$$Z(B) = \int [D\Phi] \exp(-S(\Phi, B))$$ (Euclidean, e.g. on T^4)

The partition function is a functional of B. In particular, functional derivatives of $Z(B)$ give correlation functions.

't Hooft Anomalies

In some (but not all) cases, anomalies can be understood in terms of the transformation of the partition function under gauge transformations.

$$B \to B^g \text{ : gauge transformation of } G$$

More explicitly, if $B = [B_\mu]$ is a gauge field,

$$B \to B^g = [g^{-1} B_\mu g + g^{-1} \partial_\mu g]$$

Anomaly

$$Z(B^g) = e^{i\mathscr{A}(B,g)} Z(B)$$

$\mathscr{A}(B, g)$: a violation of gauge invariance.

't Hooft Anomalies

$$Z(B^g) = e^{i\mathscr{A}(B,g)} Z(B)$$

It is known that the $\mathscr{A}(B, g)$, or more precisely its topological class in a certain mathematical sense, does not depend on RG scales.

It must be the same in both the UV and IR theories: ’t Hooft anomaly matching.

Remark:
More generally, the formulation in terms of gauge transformations is not enough. [Witten,2015]

We need more sophisticated framework.
I do not discuss it in this talk. (See e.g. [Witten-KY,2019])

Contents

Higher form symmetry

There is a concept, called higher form symmetry, which generalizes ordinary symmetry.

[Gaiotto-Kapustin-Seiberg-Willett,2014]

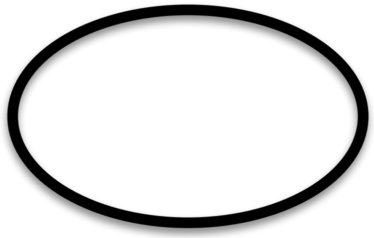
Ordinary (0-form) symmetry:

point x

$O(x)$: local operator

$$O(x) \to e^{i\alpha} O(x)$$

1-form symmetry:



curve C

$W(C)$: loop operator

$$W(C) \to e^{i\alpha} W(C)$$

1-form center symmetry

An important 1-form symmetry in gauge theories is the 1-form center symmetry.

I only discuss $SU(N)$ gauge theory for simplicity.

R : arbitrary representation of $SU(N)$

Wilson loop operator in the representation R is defined by

$$W_R(C) = \mathrm{tr}_R \, \mathrm{P} \exp\left(i \int_C A_\mu dx^\mu \right)$$

1-form center symmetry

The fact: (details omitted)

The 1-form center symmetry is a symmetry under which $W_R(C)$ transforms as

$$W_R(C) \to e^{2\pi i r k/N} W_R(C)$$

Here,

- $k \in \mathbb{Z}_N$: A symmetry transformation parameter.
  The symmetry is a $\mathbb{Z}_N$ symmetry.
- $r \in \mathbb{Z}_N$: determined by the representation R.
  The fundamental representation has $r = 1$.
  The adjoint representation has $r = 0$.

Example: Polyakov loop

For example, suppose the path integral is done on

$$S^1 \times \mathbb{R}^3 : \text{finite temperature}$$

We have a topologically nontrivial loop wrapping the S^1.

$$W_{R=\text{fundamental}}(C = S^1) : \text{Polyakov loop}$$

The 1-form center symmetry reduces to the previously known center symmetry acting on the Polyakov loop.

The condition

The $\mathbb{Z}_N$ 1-form center symmetry exists if the representation R of each dynamical matter field is such that

$$e^{2\pi rk/N} = 1$$

Rough reason:

The contribution of a matter quark virtual loop C to the path integral is roughly proportional to $W_R(C)$.
This is not invariant under the symmetry unless $e^{2\pi rk/N} = 1$.

Remark:
Wilson loop operators (rather than dynamical matter fields) can be considered for any R.

Confinement

Confinement is defined in terms of the 1-form center symmetry.

Definition

- Confinement means that

  the 1-form center symmetry is unbroken.

- Deconfinement means that

  the 1-form center symmetry is spontaneously broken.

For finite temperatures, Polyakov loop is the useful object to judge spontaneous breaking of center symmetry.

Background field

The 1-form center symmetry can be coupled to a background field B_{center}.

I do not discuss the details of B_{center} .
Instead, let me only mention an effect of this background on instanton numbers.

Instanton number

$SU(N)$ instanton number

$$n_{\text{instanton}} = \frac{1}{32\pi^2} \int \mathrm{tr} F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}$$

Usually $n_{\text{instanton}}$ is an integer.

However, when $B_{\text{center}} \neq 0$,

Fractional instanton number

$$n_{\text{instanton}} = \frac{\mathscr{A}(B_{\text{center}})}{N} + (\text{integer})$$

$\mathscr{A}(B_{\text{center}}) \in \mathbb{Z}_N$: determined by B_{center}

This fact will play important roles in later applications.

Contents

Adjoint fermion

Let us consider the following theory :

- Gauge group : $SU(N)$
- A massless Weyl fermion λ in the adjoint representation.

This is also known as $\mathcal{N}=1$ supersymmetric Yang–Mills, but supersymmetry is not necessary in this talk.

It is possible to generalize it to the case of many fermions.

Discrete axial symmetry

Under the axial rotation

$$\lambda \to e^{i\alpha}\lambda$$

There is an axial anomaly

$$\int [D\lambda] e^{-S} \to \int [D\lambda] e^{-S} \exp(2\pi i \cdot (2N\alpha) \cdot n_{\text{instanton}})$$

$n_{\text{instanton}}$: instanton number

Without any background fields, $n_{\text{instanton}} \in \mathbb{Z}$ and hence there is a $\mathbb{Z}_{2N}$ axial symmetry

$$\lambda \to e^{2\pi i k/2N}\lambda \quad (k \in \mathbb{Z}_{2N})$$

The anomaly

When the background B_{center} for the 1-form center symmetry is turned on, the instanton number is not integer,

$$n_{\text{instanton}} = \frac{\mathscr{A}(B_{\text{center}})}{N} + (\text{integer})$$

Then,

$\mathbb{Z}_{2N}$ axial: $\lambda \to e^{2\pi i k/2N}\lambda \quad (k \in \mathbb{Z}_{2N})$

$$\int [D\lambda] e^{-S} \to \int [D\lambda] e^{-S} \exp\left(2\pi i \cdot k \frac{\mathscr{A}(B_{\text{center}})}{N}\right)$$

: the 't Hooft anomaly for the axial and 1-form center symmetries

Consequence

The existence of the ’t Hooft anomaly implies that the
IR theory cannot have a trivial confining vacuum,
in the sense that

- The center symmetry is unbroken (confinement)
- There is only a single vacuum in which the axial symmetry is unbroken.
- (There is no light degrees freedom in the IR.)

Such a trivial IR theory cannot satisfy the anomaly matching condition. Something more nontrivial must happen.

(Some more details : [Cordova-Ohmori,2019])

Consequence

What is believed to happen:

spontaneous axial symmetry breaking $\mathbb{Z}_{2N} \to \mathbb{Z}_2$
by fermion condensation

$$\langle \lambda\lambda \rangle \sim \exp(2\pi i k/N) \qquad (k = 1,\cdots, N)$$

There are N-vacua labeled by k.
They are related by $\mathbb{Z}_{2N}$ axial transformations.

Finite temperature

The anomaly also has implications for finite temperature.

We need to check that the anomaly is nontrivial even at finite temperature. I do not discuss it in this talk.

Applications of nontrivial anomalies to finite temperature is pioneered by [Gaiotto-Kapustin-Komargodski-Seiberg,2017]

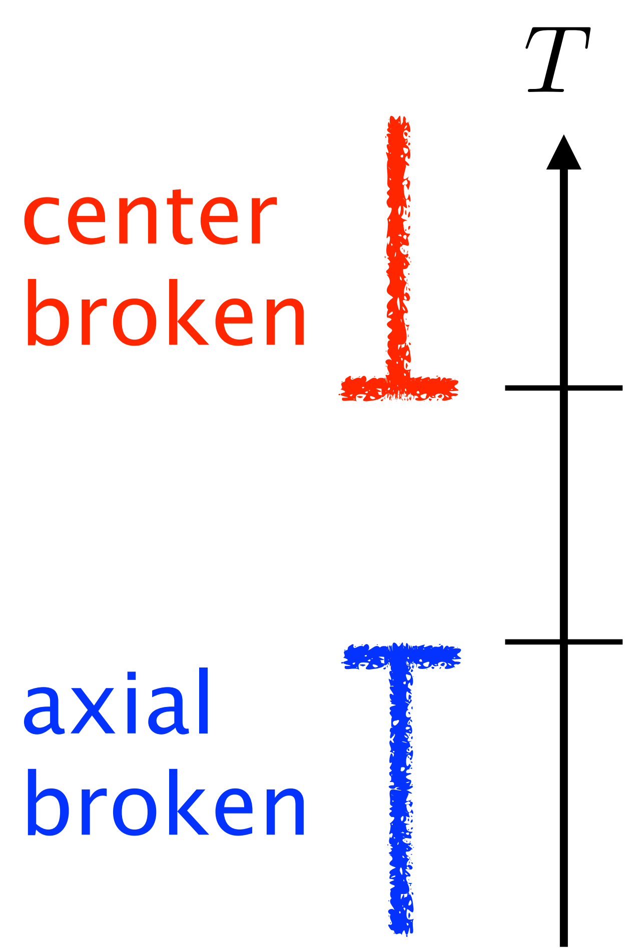
Here I discuss the case of the $SU(N)$ theory + the adjoint λ.

[Shimizu-KY,2017]
[Komargodski-Sulejmanpasic-Unsal,2017]

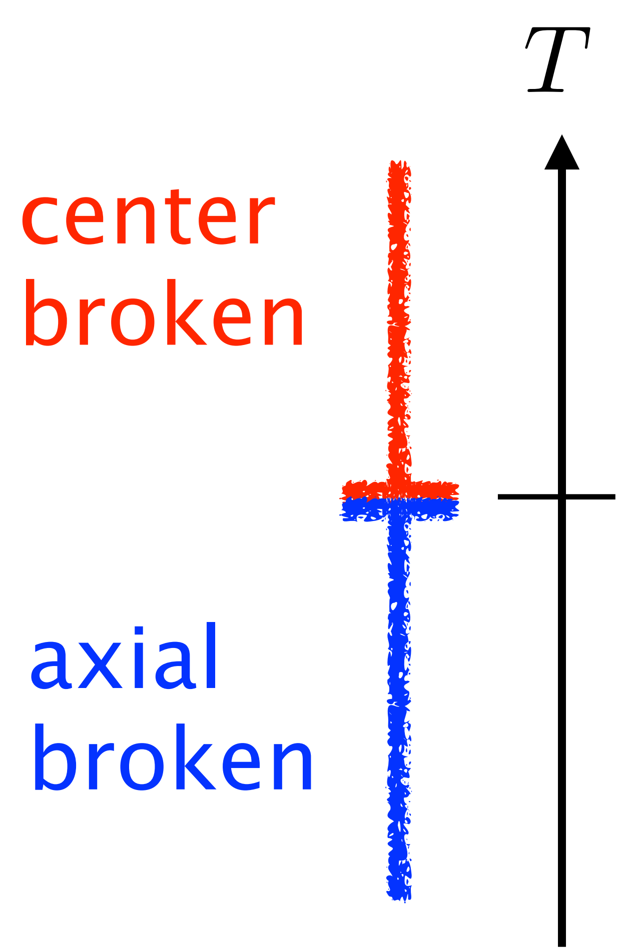
Finite temperature

Let us consider three possibilities:

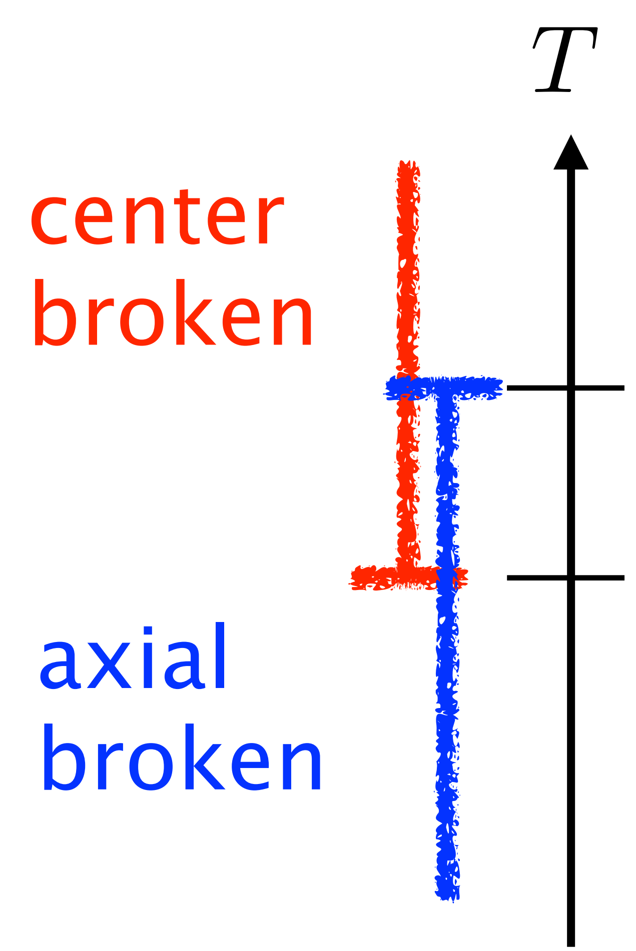
Possibility 1



Possibility 2

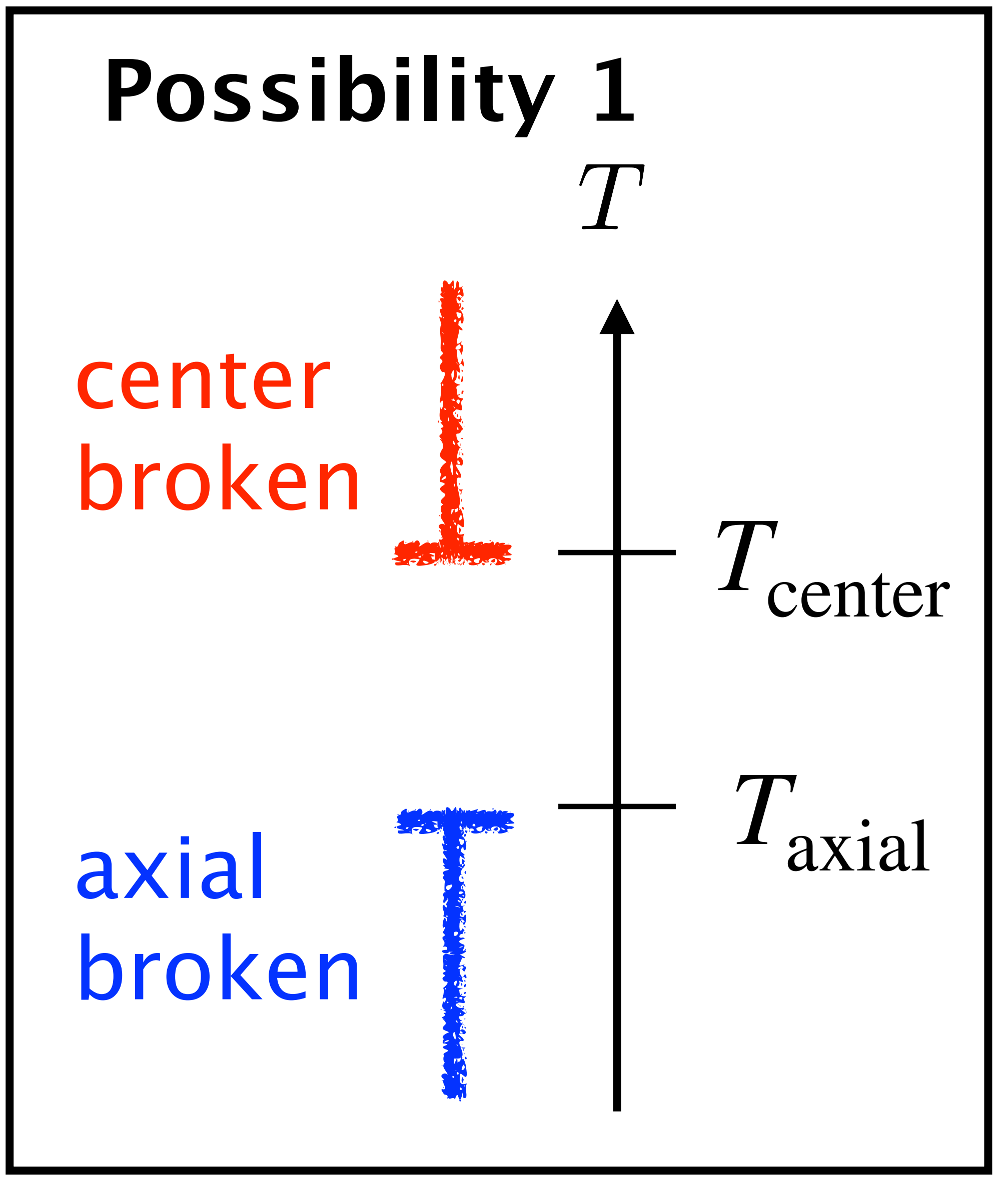


Possibility 3



Center symmetry breaking is, by definition, deconfinement.

Finite temperature



This is the possibility excluded by anomaly matching unless something very exotic happens.

Reason:
In the range

$$T_{\text{axial}} \leq T \leq T_{\text{center}}$$

both symmetries are unbroken, and it is very difficult to match the anomaly.

Finite temperature

Conclusion:

Anomaly matching strongly suggests

- The inequality $T_{\rm center} \leq T_{\rm axial}$.
- If $T_{\rm center} = T_{\rm axial}$, the phase transition is 1st order
  (I omit the details.)

Otherwise something exotic must happen.

Similar discussions are also possible in massless QCD with imaginary chemical potential at the Roberge–Weiss point.

[Shimizu–KY,2017], [KY,2019],[Furusawa–Tanizaki–Itou,2020], [Kobayashi–Yokokura–KY,2023],...

Contents

Pure Yang-Mills with θ

Let us consider $SU(N)$ pure-Yang-Mills theory with the θ angle

$$S = \int \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} - i\theta n_{\text{instanton}}$$

When the background field B_{center} for the center symmetry is zero, $n_{\text{instanton}} \in \mathbb{Z}$ and θ has a 2π-periodicity

$$\theta \sim \theta + 2\pi$$

Anomaly of 2π-periodicity

When $B_{\text{center}} \neq 0$,

$$n_{\text{instanton}} = \frac{\mathscr{A}(B_{\text{center}})}{N} + (\text{integer})$$

The partition function is defined by

$$Z(\theta, B_{\text{center}}) = \int [DA_\mu] \exp(-S(A_\mu, \theta, B_{\text{center}}))$$

Then

$$Z(\theta + 2\pi, B_{\text{center}}) = \exp\left(2\pi i \frac{\mathscr{A}(B_{\text{center}})}{N}\right) Z(\theta, B_{\text{center}})$$

The 't Hooft anomaly of the 2π-periodicity of θ.

[Cordova-Freed-Lam-Seiberg,2019]
(Early work: [Kikuchi-Tanizaki,2017])

Consequence

The ’t Hooft anomaly excludes a trivial vacuum structure, in the sense that

- The center symmetry is not broken (confinement)
- There is only a single vacuum for all values of θ.
- (There is no light degrees freedom in the IR.)

A “textbook potential (vacuum energy)” like

$$V(\theta) \sim \cos\theta$$

which assumes a single vacuum with 2π–periodicity is excluded when the theory confines.

(The same conclusion by large N analysis: [Witten,1980])

Example : adjoint fermion

Example:

The anomaly of the 2π–periodicity of θ does not change even if we add a fermion λ in the adjoint representation.

In this case, there are many vacua labeled by k,

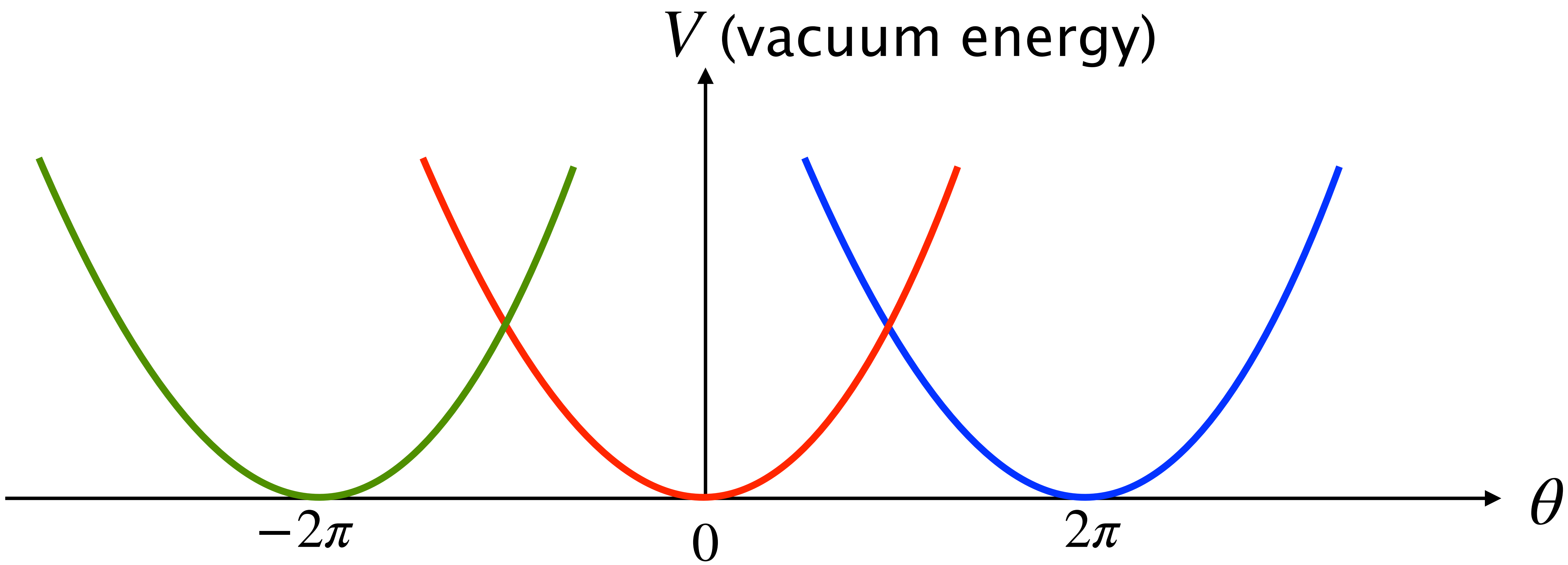
$$\langle\lambda\lambda\rangle_k \sim \exp\left(\frac{\theta + 2\pi i k}{N}\right) \qquad (k = 1, \cdots, N)$$

When θ is varied from 0 to 2π, there is a monodromy of vacua under $\theta \to \theta + 2\pi$:

$$k\text{–th vacuum} \xrightarrow{\theta\to\theta+2\pi} (k+1)\text{–th vacuum}$$

Pure Yang-Mills

What is believed to happen in pure Yang–Mills:
many metastable vacua (labeled by different colors below)



I omit the details.

Some comments

How the theory behaves as a function of θ has potential applications to cosmology:

axion-like-particle, inflaton, ...

Pure-Yang-Mills is not easy to study due to the sign problem, but there are some lattice studies.

[Kitano-Matsudo-Yamada-Yamazaki,2021]

Two-dimensional $\mathbb{CP}^{N-1}$ sigma model has a similar anomaly, and explicit computations are also possible.

[Sugeno-Yokokura-KY, in progress]

⟶ Yokokura's talk in the fifth week.

Contents

Summary

- ’t Hooft Anomalies are formulated by using background fields of global symmetries.
- Some generalized symmetries, such as 1-form center symmetry in gauge theories, are useful.
- By using new types of symmetries and anomalies, we get more information and predictions on the dynamics of various QFTs.