Higher partial waves in femtoscopy

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Outline

Introduction

- Correlation function & Koonin-Pratt (KP) formula
- Useful formulae and assumptions
- Dynamical modeling of high-energy collisions

Higher partial waves

- Spherical-source KP formula with $l > 0$
- LL formula with $l > 0$
- LL formula and optical theorem
- Regularized LL formula

High-energy nuclear collisions

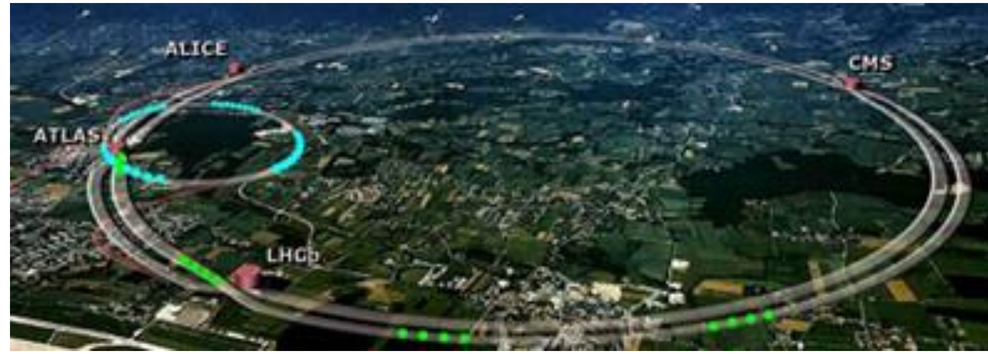
RHIC at BNL (USA)

Au+Au, etc ($\sqrt{s_{NN}}$ = 200 GeV...)

LHC at CERN (EU)

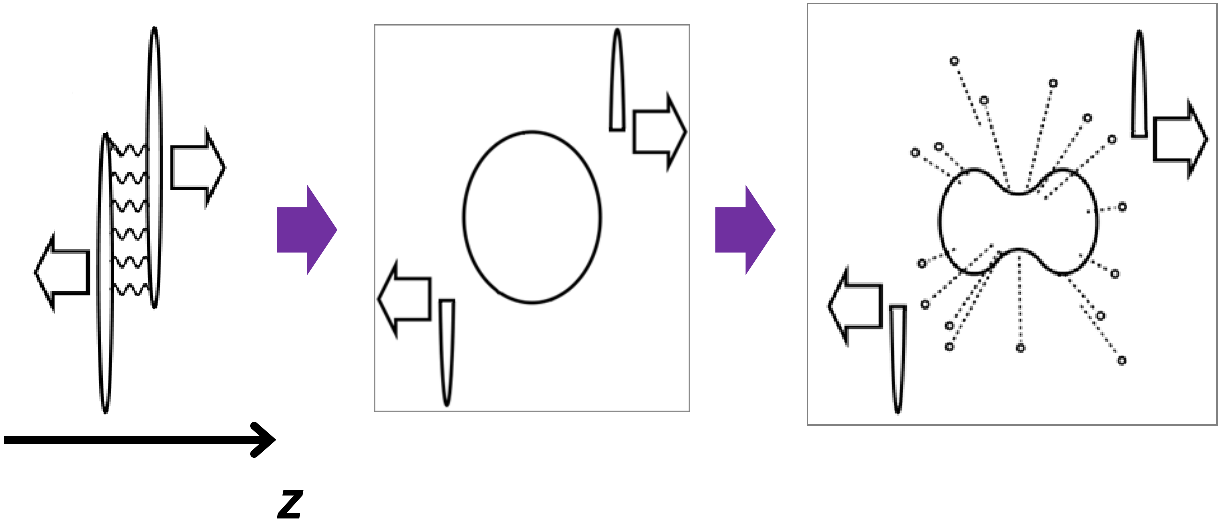
Pb+Pb, etc ($\sqrt{s_{NN}}$ = 5.02/2.76/... TeV)

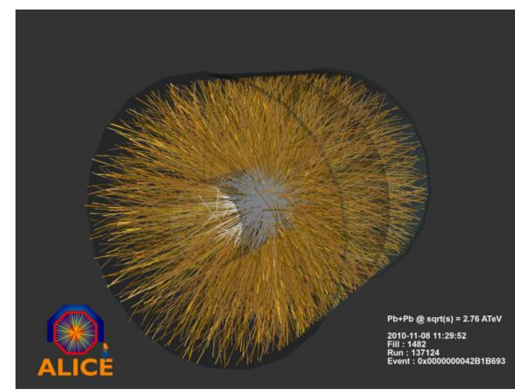
FAIR, NICA, HIAF, J-PARC-HI, …



[LHC ALICE Web site
http://alice20.web.cern.ch/alice20/ より]

Colliding nuclei (Au, Pb, Cu, U, p, d, ^{3}He, Xe, Zr, Ru, ...)

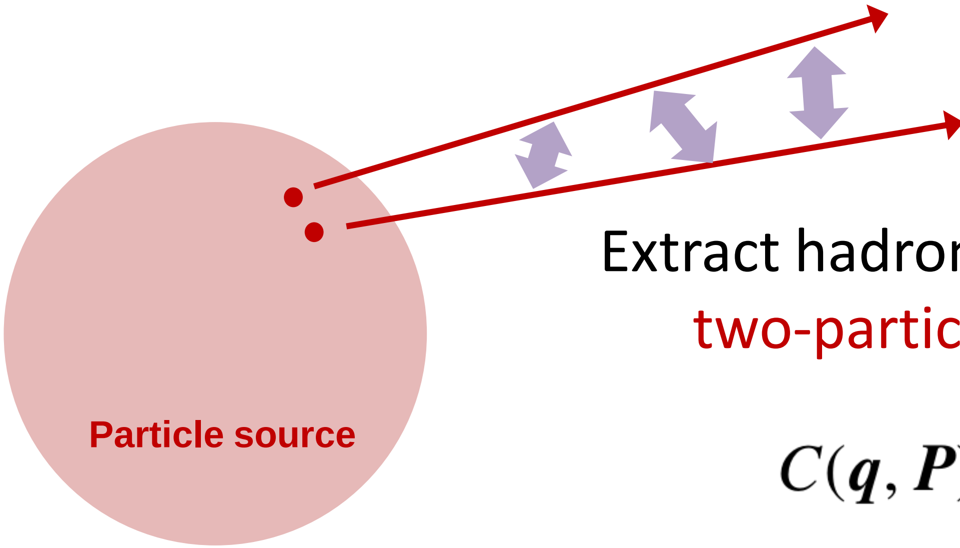




[LHC ALICE Web site
http://aliceinfo.cern.ch/Public/en/Chapter1/fstablebeams.html より]

Femtoscopy for hadron interactions

See e.g. ExHIC, Prog. Part. Nucl. Phys. 95 (2017) 279-322



Extract hadron interactions from two-particle correlations:

$$C(\boldsymbol{q},\boldsymbol{P}) = \frac{E_1 E_2 dN_{12}/d\boldsymbol{p}_1 d\boldsymbol{p}_2}{(E_1 dN_1/d\boldsymbol{p}_1)(E_2 dN_2/d\boldsymbol{p}_2)}$$

q & P: Relative & total momentum

"Koonin-Pratt (KP) formula" (widely used to relate it to physics)

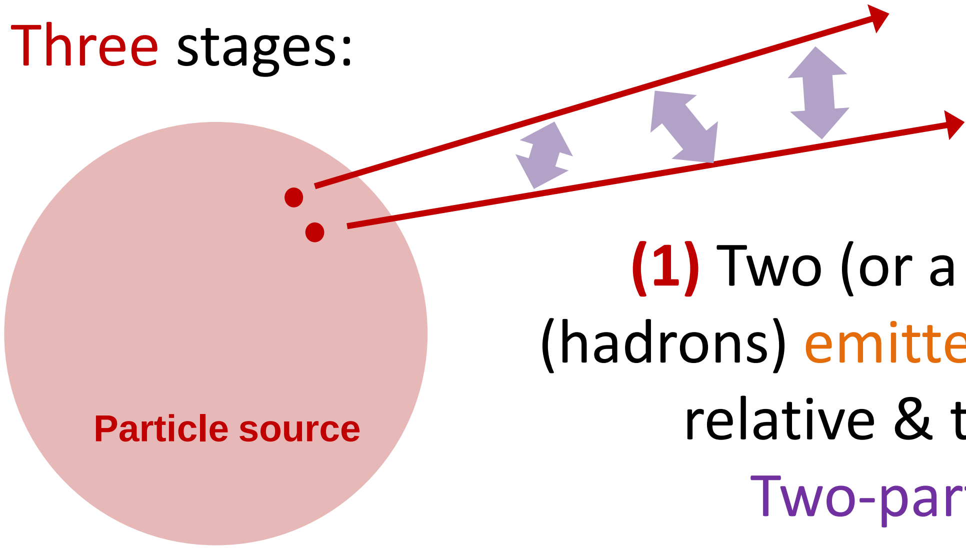
$$C(\boldsymbol{q},\boldsymbol{P}) = \int d\boldsymbol{r} S_{12}(\boldsymbol{r}) |\varphi^{(-)}(\boldsymbol{q},\boldsymbol{r})|^2.$$

$\varphi^{(-)}$: *Relative wave function*, S_{12}: two-particle source

(-) = inverse process of scattering

Picture behind Koonin-Pratt formula

Three stages:



Emmitted = isolated from the rest of the system at time *t* and evolve independently. *(Well-defined?)*

(1) Two (or a few number of) particles (hadrons) emitted from source at r_1 & r_2 with relative & total momentum q & P.

Two-particle source function

$S_{12}(q, r_1 - r_2)$

(2) Two particles interact with each other with *interactions*

assuming no change between initial and final q … "on-shell/smoothness approx."

(3) Two particles become distant from each other and observed as asymptotic plane waves p_1 & p_2

Note: *interaction* → *plane waves*
(*cf* *plane wave* → *interaction* in standard scattering)

$\varphi^{(-)} \sim \varphi^{*}_{\mathrm{scatt}}$

$$C(\boldsymbol{q}, \boldsymbol{P}) = \int d\boldsymbol{r} S_{12}(\boldsymbol{r}) |\varphi^{(-)}(\boldsymbol{q}, \boldsymbol{r})|^2.$$

Koonin-Pratt (KP) formula

S. E. Koonin (1977), Pratt, et al (1992), etc.

Based on various assumptions

Three quantities are related by KP formula

$$\underline{C(\boldsymbol{q},\boldsymbol{P})} = \int d\boldsymbol{r}\underline{S_{12}(\boldsymbol{r})}\underline{|\varphi^{(-)}(\boldsymbol{q},\boldsymbol{r})|^2}.$$

①Momentum correlation function (Experiment)

②Particle source

↑

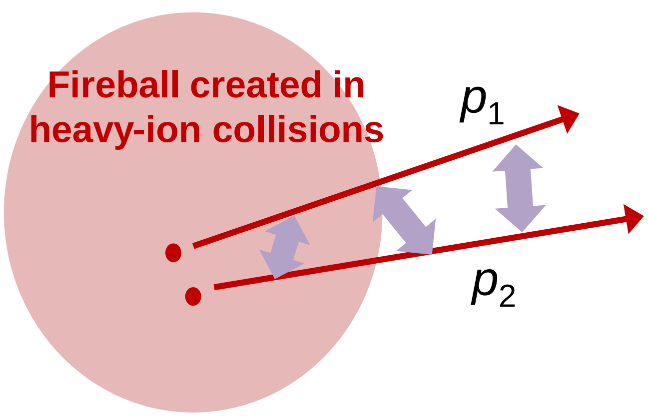
Fireball size & shape

③Relative wave function

↑

Interaction

$q \sim (p_1 - p_2)/2$, $P \sim p_1 + p_2$



$\varphi^{(-)}$: solution of Schrodinger eq.

$$\left[-\frac{1}{2\mu}\nabla^2 + \underline{V(r)}\right]\varphi^{(-)}_{\boldsymbol{q}}(\boldsymbol{r}) = E_{\boldsymbol{q}}\varphi^{(-)}_{\boldsymbol{q}}(\boldsymbol{r})$$

Boundary condition at $\boldsymbol{r} \to \infty$

$$\varphi^{(-)}_{\boldsymbol{q}}(\boldsymbol{r}) = e^{i\boldsymbol{q}\cdot\boldsymbol{r}} + \frac{f(\theta)}{r}e^{-iqr} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

Koonin-Pratt (KP) formula

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Based on various assumptions

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$$C(\boldsymbol{q}, \boldsymbol{P}) = \int d\boldsymbol{r} S_{12}(\boldsymbol{r}) |\varphi^{(-)}(\boldsymbol{q}, \boldsymbol{r})|^2 .$$

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②Particle source
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③Relative wave function
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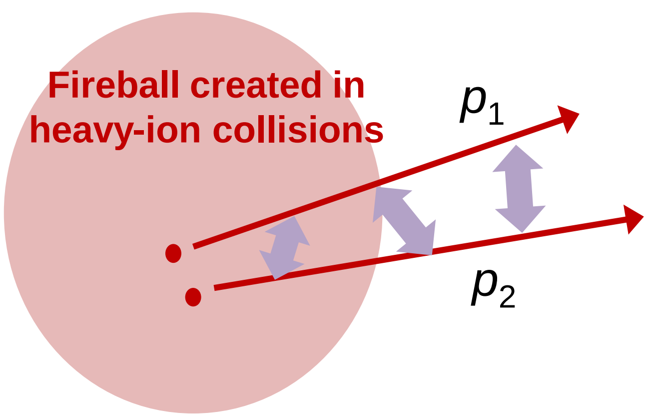


Idea: If two of them are given, the rest is constrained

①Exp & ③Wave fn → ②Source

HBT (1956), GGLP (1960)

①Exp & ②Source (assumption)
→ ③Interaction

Comparison & constraints

Theoretical models (lattice QCD, chiral EFT, etc.)

Useful formulae and many assumptions

KP formula

M. Lisa, U. Heinz, U. Wiedemann (2000), etc.

$$C_{12} \sim \mathrm{Tr}[\underline{|q\rangle\langle q|\hat{U}(\infty,t)}\,\underline{\hat{\rho}(t)}\,\underline{\hat{U}(t,\infty)}]$$
$$\sim \int dr dr' \underline{\langle q|\hat{U}(\infty,t)|r\rangle}\underline{\langle r|\hat{\rho}|r'\rangle}\underline{\langle r'|\hat{U}(t,\infty)|q\rangle}$$
$$\sim \cdots \sim \int d\boldsymbol{r} \underline{S_{12}(\boldsymbol{r},\boldsymbol{q})}|\underline{\varphi(\boldsymbol{q};\boldsymbol{r})}|^2.$$

S(r) is essentially the Wigner transform of the density matrix of 2-particle system (the rest system is traced out) at emission time *t*

+ **(1)** Two particles are separated instantly and become an isolated system
+ **(2)** Number of particles does not change

Note: particles can change in coupled-channel calculations

+ **(3)** Relative momentum q^* at emission can be replaced with the final $\boldsymbol{q}$

"on-shell approx." / ***"smoothness approx."***

Useful formulae and many assumptions

In addition to the assumptions of the picture of KP formula…

Spherical-source KP for s-wave

+ **(4)** Only *s-wave interaction modifies wave fn.*
  Higher partial waves are ignored
+ **(5)** Spherical source fn: $S_{12}(\mathbf{r}) = S(|\mathbf{r}|)$

$$C(\boldsymbol{q}) = 1 + \int d\boldsymbol{r} S(r) \left\{ |\chi_q(r)|^2 - |j_0(qr)|^2 \right\}$$

s-wave fn. s-wave of plane wave

Useful formulae and many assumptions

In addition to the assumptions of the picture of KP formula…

Lednicky-Lyuboshits (LL) formula

Lednicky, Lyuboshits (1982)

$$C_{\rm LL}(q) = 1 + \frac{|f(q)|^2}{2R^2} F_3\left(\frac{r_{\rm eff}}{R}\right) + \frac{2{\rm Re} f(q)}{\sqrt{\pi}R} F_1(2x) - \frac{{\rm Im} f(q)}{R} F_2(2x)$$

Note: F_1, F_2, F_3 are some mathematical functions

+ **(6)** $S(r)$: Gaussian with size $\sqrt{2}\,R$

← R: 1-particle source size / "chaotic source"

← Matter expansion, local equilibrium, small q, etc.
"homogeneity region"

+ **(7)** Asymptotic wave fn in entire range:

$$\varphi_{\boldsymbol{q}}^{(+)} = e^{iqz} + f(q)\frac{e^{iqr}}{r}$$

← *Source size is sufficiently larger than interaction range*

+ **(8)** $f(q)$ is parametrized by (a_0, r_0) (effective range expansion)

→ Useful for fitting experimental data to extract (a_0, r_0)

Useful formulae and many assumptions

In future: femtoscopy analysis with precision
using realistic modeling of high-energy nuclear collision reactions
(initial-state model & hydrodynamics & hadronic transport)

Toward realistic modeling

Many approximations exist behind the useful formulae (KP, LL, etc.) under corresponding assumptions.

Most assumptions seem "consistent" with data *in most cases*,

but their explicit (or theoretical) justification is unclear (to me).

Recently, femtoscopy is becoming an indispensable tool for accessing hadronic interactions. We want to revisit and resort those assumptions before going to the next step.

Example: Assumptions on source function

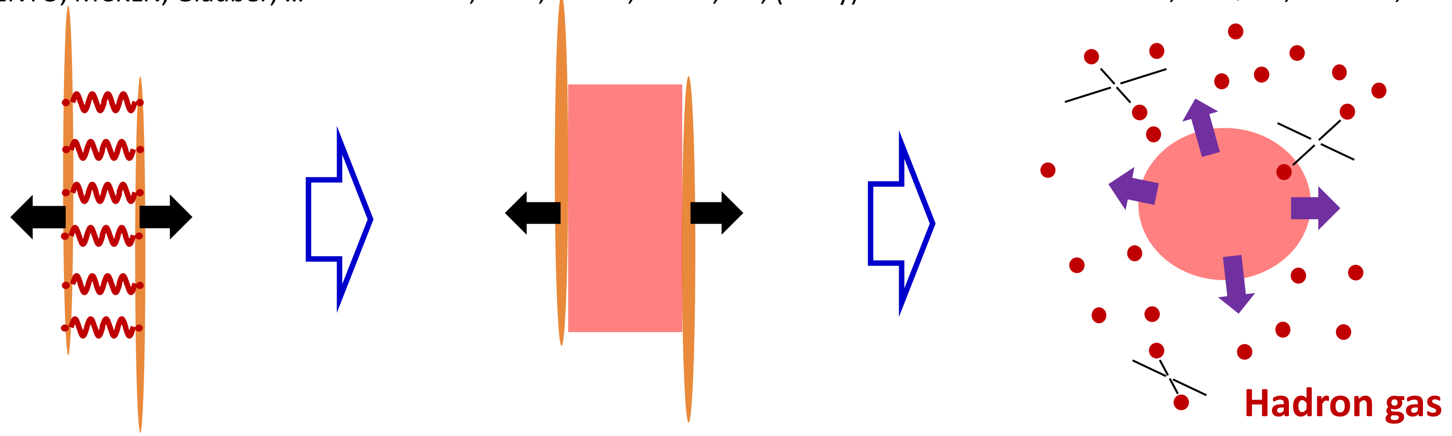
Ideally S(r) should be given by dynamical models of high-energy nuclear collision reactions

S. Pratt, PRL 102 (2009), P. Batyuk, I. Karpenko et al, PRC 96 (2017), P. Chakraborty, A.K. Pandey, S. Dash, EPJA 57 (2020), V.M. Shapoval, Y.M. Sinyukov, NPA 1016 (2021), D. Kincses, M. Stefaniak, M. Csanad, Entropy 24 (2022), K. Kuroki (2024), etc.

Modern dynamical model for HIC ~

Initial-state model + **Relativistic hydrodynamics** + **Hadronic transport**

IPGlasma, TrENTO, MCKLN, Glauber, … | MUSIC, VISH, CLVisc, vHLLE, rfh, (many)… | JAM, UrQMD, SMASH, …



What is "emission" in this more realistic picture?

(1) **End of hydro**: many hadrons still interact

(2) **End of hadronic transport**: no interaction anymore

Should emission be somewhere between (1) and (2)?

+ Realistic modeling of hadron production?

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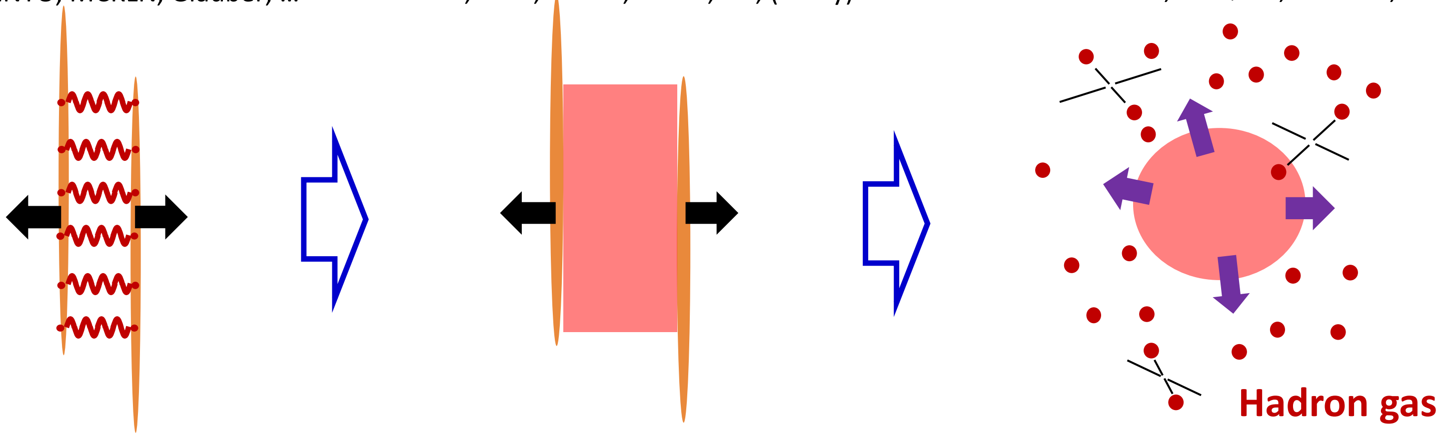


Event-by-event (EBE) fluctuations are important physics in HIC.

EBE flucts Introduce correlations between particles.

convolution

$$S_{12}(r) = S_1(r) * S_2(-r) \quad \Rightarrow \quad S_{12}(r) = \langle S_1(r) * S_2(-r) \rangle$$

Event average

→ *Non-trivial relation btw relative and 1-particle sources*

Higher partial waves (angular momentum $l > 0$)

Effect of higher partial waves ($l > 0$)

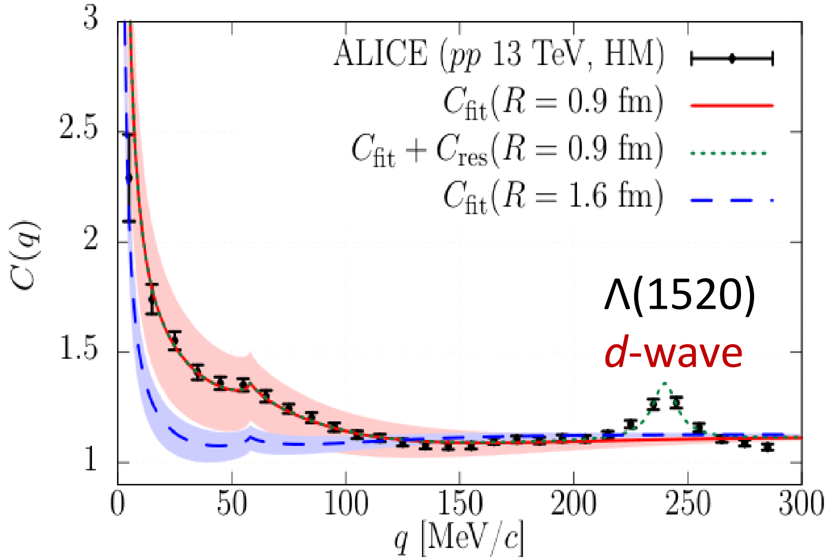
l: angular momentum

Existing analyses mainly considered **s-wave ($l = 0$)** ← **Another assumption**

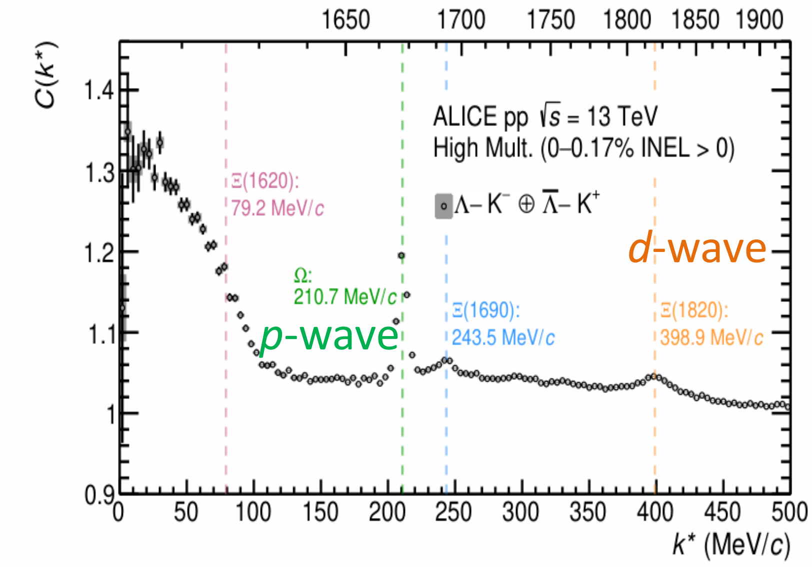
← Correlation function cannot be measured for each *l*

All-*l* contributions are mixed.

Y. Kamiya, et al, PRL **124**, 132501 (2020)



ALICE, PLB **845**, 138145 (2023)



Structures from resonances of p-wave / d-wave are observed in experimental data.

Fitting correlation function with a naive Breit-Wigner form

→ How is this justified?

Spherical-source KP with higher partial waves

Wave function can be fully expanded into partial waves

$$\varphi_q(r,\theta) = \sum_{l=0}^{\infty}(2l+1)i^l R_l(r) P_l(\cos\theta)$$

$P_l(z)$ Legendre polynomial

Apply it to KP formula (non-identical particles, single channel)

→ Correlation is sum of each partial contributions ΔC_l [←spherical $S(r)$]

$$C(q) = 1 + \sum_{l=0}^{\infty}(2l+1)\int dr 4\pi r^2 S(r) \times \left[|R_l(r)|^2 - |j_l(qr)|^2\right]$$

ΔC_l

This is a natural extension for the s-wave case

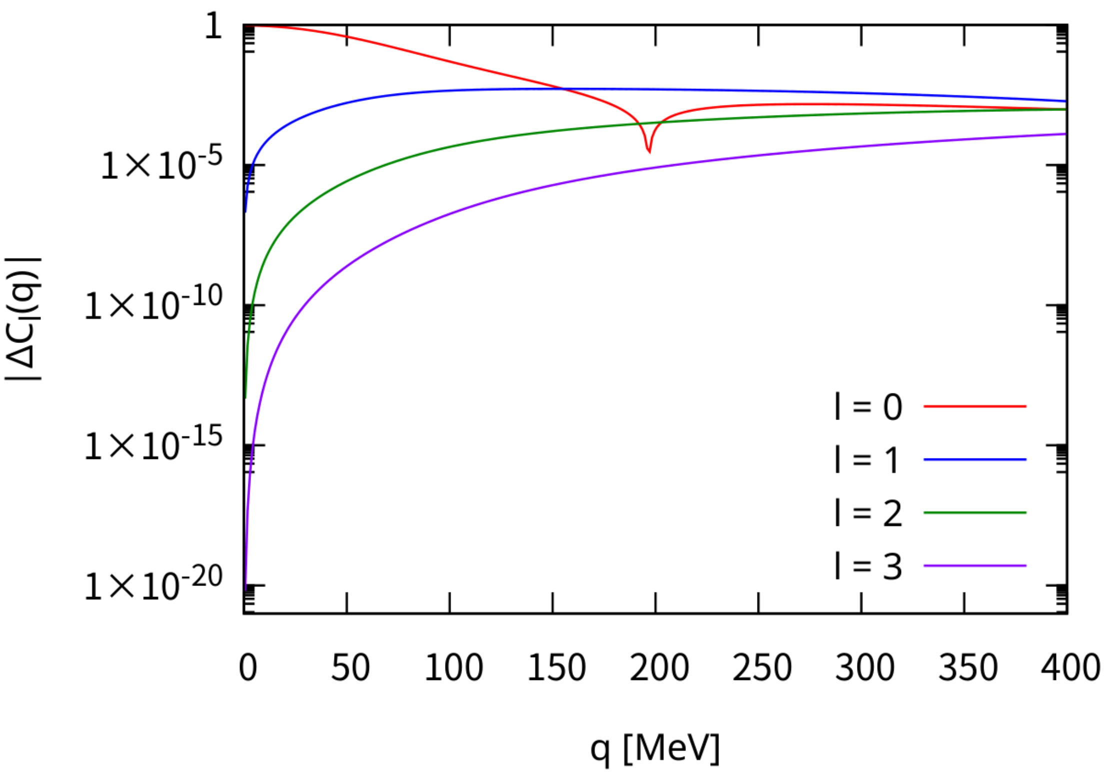
Example: Potential well

$$V(r)=\begin{cases}-V_0, & (r<b),\\ 0, & (r>b).\end{cases}$$

Typical example (no resonance)

μ = 600 MeV
V_0 = 50 MeV
b = 1 fm
Source R = 3 fm

$\Delta C_l(q)$: Contribution from l-th partial wave



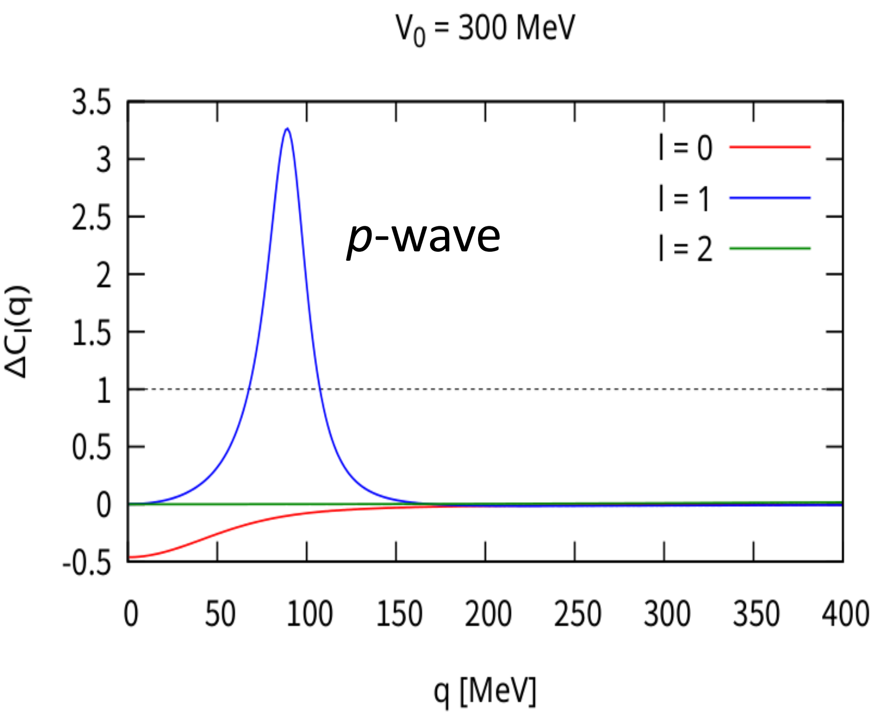
Small q → higher partial waves are suppressed
Large q → higher partial waves become relatively large

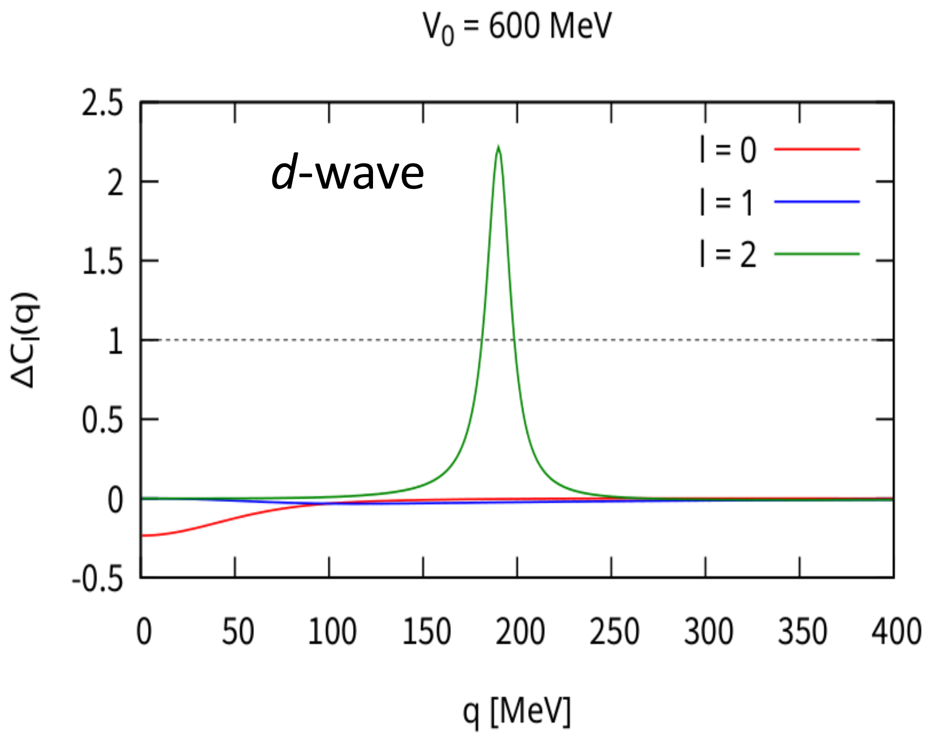
$$f_l \sim a_l q^{2l}$$

Example: Potential well (with resonances)

Change the depth of V_0

μ = 600 MeV, b = 1 fm, R = 2 fm





Contributions from higher partial wave can be significant with resonances

LL formula with higher partial waves

With the same assumptions for s-wave case,

$$C(q) = 1 + \sum_{l=0}^{\infty} \frac{(2l+1)(-1)^l e^{-4x^2} q}{2\sqrt{\pi}x^2} \left(\sqrt{\pi}\Im f_l + 2\Re f_l \int_0^{2x} dt e^{t^2} \right)$$

2 terms

Note: effective range correction is not included

Note: **3** terms in the original LL are actually essentially 2 terms due to the optical theorem:

$|f_0|^2 = q \,\mathrm{Im}\, f_0$

One can perform the sum over *l*

using $(-1)^l = P_l(\cos \pi)$

$$C(q) = 1 + \frac{4\pi}{q}\Im\left[f(\pi) \int_0^{\infty} dr S(r) e^{2iqr} \right]$$

Backward scattering amplitude: $f(\theta=\pi)$

Similar structure as optical theorem: $\sigma_{tot} = (4\pi/q)\,\mathrm{Im}\, f(0)$,

But the direction is opposite.

Optical theorem vs correlation function

Plane wave contains δ(θ) and δ(θ-π)

at the O(1/r) order of partial wave expansion

$$e^{iqz} \sim \frac{1}{iq}\delta(\theta)\frac{e^{iqz}}{r} - \frac{1}{iq}\delta(\theta-\pi)\frac{e^{-iqz}}{r}$$

Wave function

$$\varphi = e^{iqz} + f(q)\frac{e^{iqr}}{r}$$

$$= \underbrace{\left[\frac{1}{iq}\delta(\theta) + f(\theta)\right]}_{=:A}\frac{e^{iqz}}{r} - \underbrace{\frac{1}{iq}\delta(\theta-\pi)}_{=:B}\frac{e^{-iqz}}{r}$$

Optical theorem

$$\int d\Omega(|A|^2 - |B|^2)|_{r\to\infty} = 0.$$

Interference of two terms in A remains ∝ δ(θ)f(θ): **forward**

Correlation fn

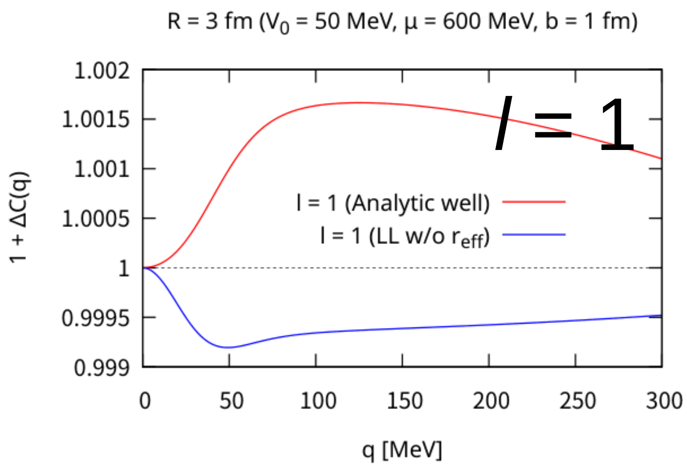
$$C(r) = \int dr S(r) \int d\Omega |A+B|^2$$

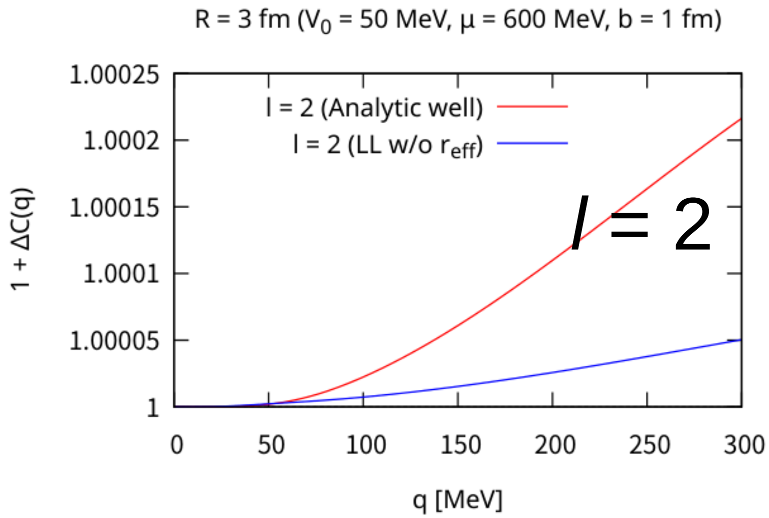
Interference of A and B remains ∝ δ(θ-π)f(θ): **backward**

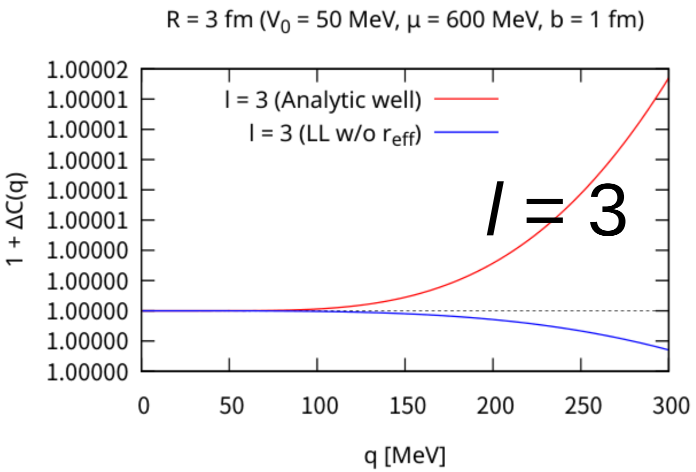
Example: Potential well

Bad: $\Delta C_l > 0$ does not match even qualitatively

Note: Analytic solution for δ_l is used (no effective range expan.)







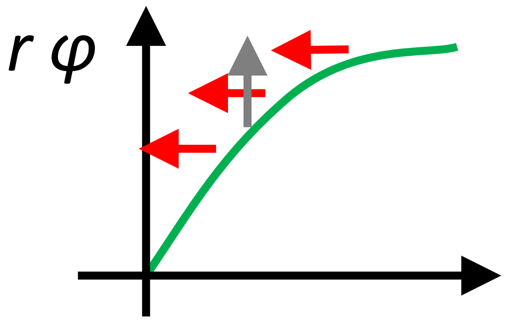
In particular, the contribution from odd *l* in the native LL formula has the opposite sign to the analytic solution.

Why alternating sign $(-1)^l$?

Asymptotic form: $R_l \sim \sin(kr - l\pi/2 + \delta_l)/kr$

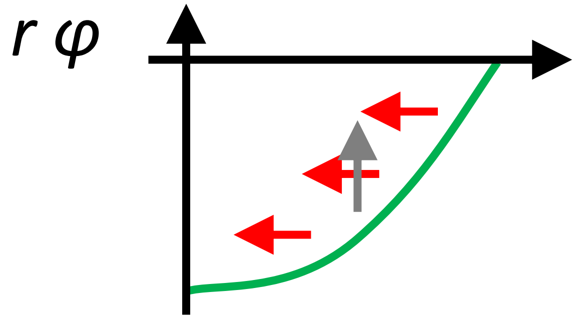
We extend this form to r → 0

$l = 0$



Increases $|\varphi|^2$ and thus
$C(q) = \int dV\, S(r)\, |\varphi|^2$

$l = 1$



Decreases $|\varphi|^2$
φ diverges as $1/r$

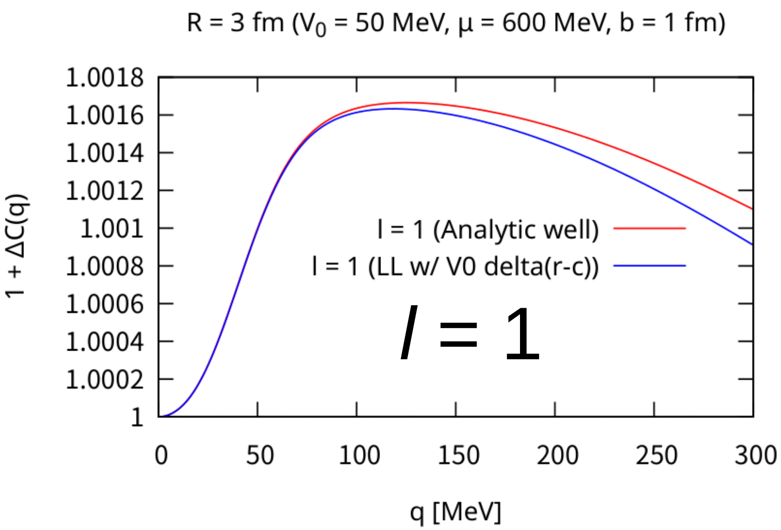
This is unphysical

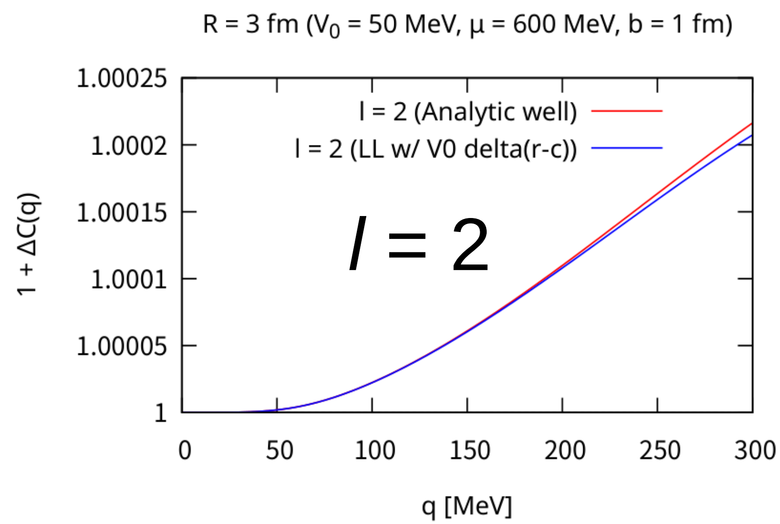
(i.e., doesn't happen in reality).

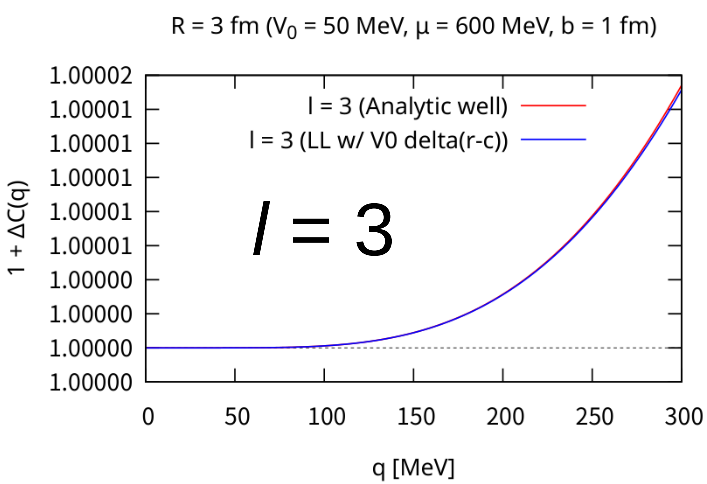
Note: If we use j_l and n_l as the asymptotic form directly,
for all l, wave fn diverges at origin more strongly and
C(q) does not converge because of $n_l \sim 1/r^{2l+1}$.

Replacing wave fn (at r～0) with regular one

✔Reproduces potential well







Wavefunction for regularized LL formula

$$\varphi(r) = \begin{cases} a j_l(qr), & (r < c), \\ \cos\delta_l j_l(qr) - \sin\delta_l y_l(qr), & (r > c). \end{cases}$$

Note: This is a solution with the δ-fn potentials

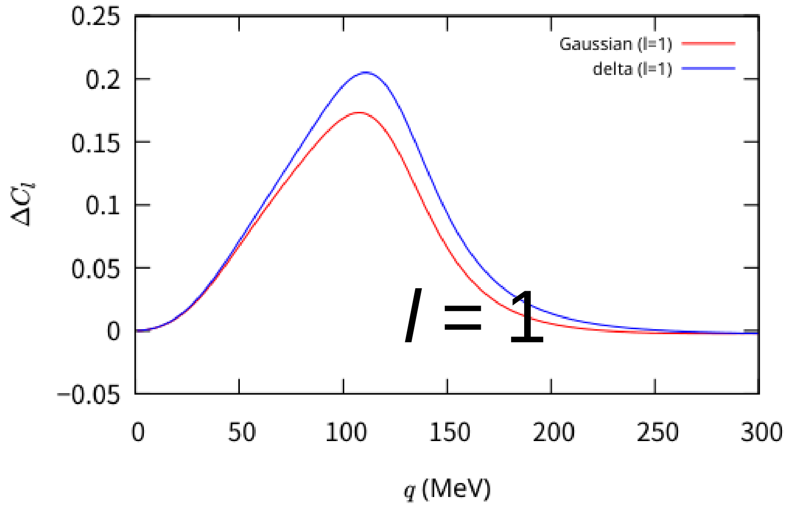
$$V(r) = V_d \delta(r - c),$$

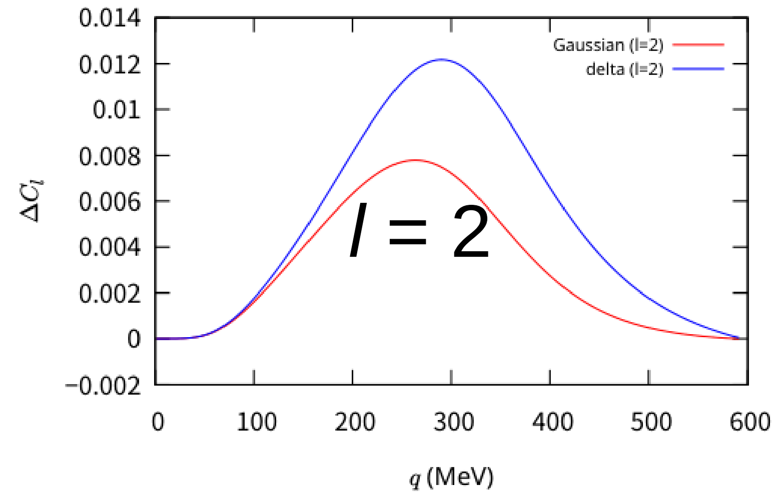
$$V_d = \frac{y_l(qc)[\partial_c \ln j_l(qc) - \partial_c \ln j_l(qc)]}{\cot\delta_l j_l(qc) - y_l(qc)}.$$

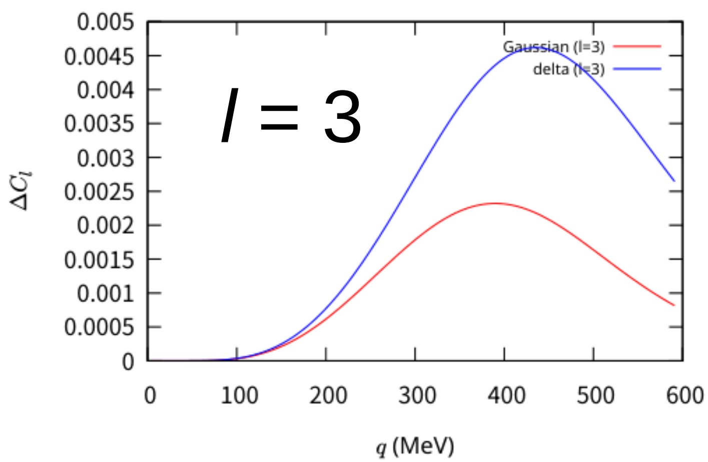
Example: Gaussian potential

Width 1 fm Depth 50 MeV

✔Reproduces Gaussian case to a good degree







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Regularized LL formula w/ reference potential

Except for the asymptotic wave function $\varphi \sim O(1/r)$,
the KP formula gives a similar behavior (for the same δ_l).

- Wave fn with the potential well
- Wave fn with the Gaussian potential
- Wave fn with the delta-function potential
- Wave fn of the asymptotic form $O(1/r)$ (traditional LL)

} Similar C(q) (first three)

Bad for $l > 0$ (last)

→ One may use one of the realistic wave functions
as **a reference model** to define the fitting form of C(q).

Suggestion: *the delta-function potential as a reference,*
which has a simple analytic wave function φ.

Summary

Femtoscopy

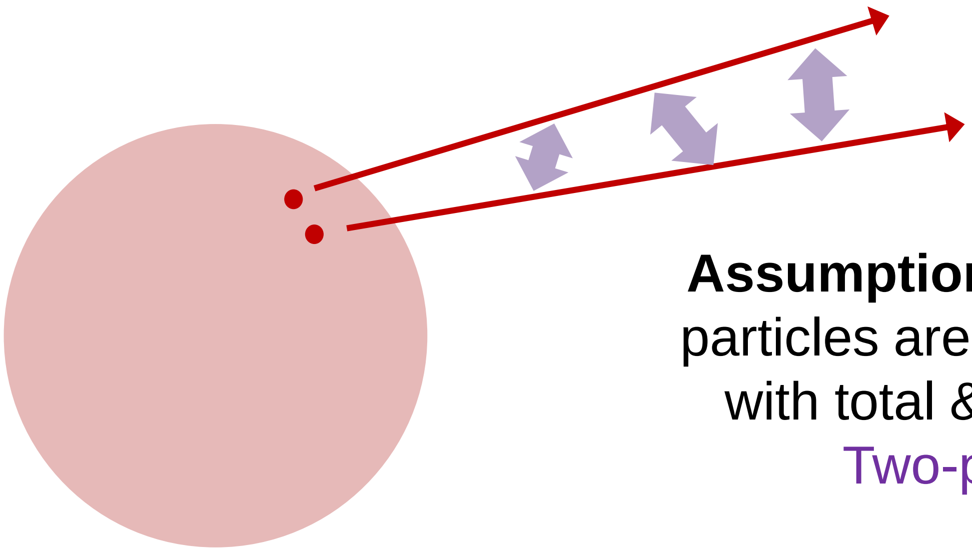
- KP and LL formulae allow us to access hadron interaction through the correlation function measured in high-energy nuclear collisions, but with many assumptions.
- Realistic dynamical modeling of high-energy collisions is important, but future development is necessary.

Higher partial waves

- With spherical source, the correction by the higher partial waves is given by sum over l
- LL formula at the full order gives an interesting structure with the backward amplitude $f(\pi)$ similar to the optical theorem, which is explained by plane-wave expansion.
- Naïve LL formula with $l > 0$ gives unphysical result, but regularization of wave function gives an improved formula.

BACKUP

Koonin-Pratt (KP) formula



Emitted = isolated from the rest of the system at time t and evolve independently. *(Well-defined?)*

Assumption 1: Two (or a few number of) particles are emitted from source at r_1 & r_2 with total & relative momentum q' & P.
Two-particle source function
$S_{12}(q', r_1 - r_2)$

Assumption 2: After interaction with $V(r)$, *no change in relative momentum*:
$q' = q$ *between emission point* q' *and final state* q

Assumption 3: Two particles become distant from each other and observed as asymptotic plane waves p_1 & p_2

Wave fn: $\varphi^{(-)} \sim \varphi^{*}_{\text{scatt}}$

interaction → plane waves
(*cf* plane wave → interaction in scattering)

$$C(\boldsymbol{q}, \boldsymbol{P}) = \int d\boldsymbol{r} S_{12}(\boldsymbol{r}) |\varphi^{(-)}(\boldsymbol{q}, \boldsymbol{r})|^2.$$

KP with s wave and spherical source

Only s-wave

Plane wave without S-wave   S-wave wave fn

$$\varphi_{\boldsymbol{q}}(\boldsymbol{r}) = \boxed{e^{i\boldsymbol{q}\cdot\boldsymbol{r}} - j_0(qr)} + \boxed{\chi_q(r)}$$

Assumption 1: Spherical two-particle source *S*(*r*)

$$C(\boldsymbol{q}) = \int d\boldsymbol{r} S(r) |\varphi_{\boldsymbol{q}}(\boldsymbol{r})|^2$$

$$= 1 + \int d\boldsymbol{r} S(r) \left\{ |\chi_q(r)|^2 - |j_0(qr)|^2 \right\}$$

j_0 : contribution from the plane wave

X_q : S-wave component of scattered wave fn

Lednicky-Lyuboshitz (LL) 公式 (S-wave)

Assumption 2: Gaussian source

$$S(r) = \frac{1}{(4\pi R^2)^{3/2}} e^{-\frac{r^2}{4R^2}}.$$

Assumption 3: φ is asymptotic form in the entire region

$$\varphi = e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

Result: Correlation function is analytically calculated

$$\begin{aligned} C_{\mathrm{LL}}(q) =& 1 + \int d\boldsymbol{r} S_{12}(r)\left(|\psi_{\mathrm{asy}}(r)|^2 - |j_0(qr)|^2\right) \\ =& 1 + \frac{|f(q)|^2}{2R^2} F_3\left(\frac{r_{\mathrm{eff}}}{R}\right) + \frac{2\mathrm{Re}f(q)}{\sqrt{\pi}R} F_1(2x) - \frac{\mathrm{Im}f(q)}{R} F_2(2x) \end{aligned}$$

Conventionally used to fit the experimental data

Spherical source w/ *higher partial wave*

Full partial-wave expansion

$$\varphi_q(r,\theta) = \sum_{l=0}^{\infty}(2l+1)i^l R_l(r) P_l(\cos\theta)$$

Correlation function

$$C(q) = 1 + \sum_{l=0}^{\infty}(2l+1)\int dr 4\pi r^2 S(r) \times \left[|R_l(r)|^2 - |j_l(qr)|^2\right]$$

Contributions from partial waves: sum of each wave

Note: if source is not spherical, there arise
all the mixtures of different partial waves $l \neq l'$

Spherical source w/ *higher partial wave*

Correlation function

$$C(q) = \sum_{l=0}^{\infty}(2l+1)\int dr\, 4\pi r^2 S(r)|R_l(r)|^2.$$

Correlation function for plane wave (LHS from defs)

$$1 = \sum_{l=0}^{\infty}(2l+1)\int dr\, 4\pi r^2 S(r)|j_l(qr)|^2.$$

By subtracting both-hand sides, we can obtain the correlation function written as sum of corrections.

Another version of LL formula for l > 0

Assume the following asymptotic form
[with $O(1/r^2)$ terms of plain wave retained]

$$\varphi \approx e^{iqz} + \frac{f(\theta)e^{ikr}}{r} + \underbrace{\mathcal{O}\Big(\frac{l(l+1)}{r^2}\Big)}_{\text{ignore}},$$

We may obtain the following correlation function
with an additional term

$$C(q) = 1 + \frac{4\pi}{q}\Im[f(\pi)\hat{S}(-2iq)] + \frac{4\pi}{q}\Im\int_{-1}^{1} d\cos\theta f'(\theta)\hat{S}(-iq(1-\cos\theta)).$$

Another version of LL formula for l > 0

Assume the following asymptotic form
[with O(1/r^2) terms of plain wave retained]

$$\varphi \approx e^{iqz} + \frac{f(\theta)e^{ikr}}{r} + \underbrace{\mathcal{O}\Big(\frac{l(l+1)}{r^2}\Big)}_{\text{ignore}},$$

Intermediate expression

$$\begin{aligned} C(q) &= 1 + \int d\Omega |f(\theta)|^2 \int_0^\infty dr S(r) \\ &+ \frac{4\pi}{q}\int_0^\infty dr S(r) \Im[f(\pi)e^{2iqr} - f(0)] \\ &+ \frac{4\pi}{q}\int_0^\infty dr S(r) \int_{-1}^1 d\cos\theta e^{iqr(1-\cos\theta)} f'(\theta). \end{aligned}$$

LL formula with higher-partial waves

With the same assumptions

Assumption: Gaussian source

$$S(r) = \frac{1}{(4\pi R^2)^{3/2}} e^{-\frac{r^2}{4R^2}}.$$

Assumption: φ is asymptotic form in the entire range

$$\varphi \approx e^{iqz} + \frac{f(\theta)e^{ikr}}{r} + \mathcal{O}\left(\frac{l(l+1)}{r^2}\right)$$

ignore

Note: With spherical Bessel, KP divergent

Result: Correlation function

$$C(q) = 1 + \sum_{l=0}^{\infty} \frac{(2l+1)(-1)^l e^{-4x^2} q}{2\sqrt{\pi}x^2} \times \left(\sqrt{\pi}\Im f_l + 2\Re f_l \int_0^{2x} dt e^{t^2}\right)$$

f_l: $f(\theta) =: \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) f_l.$

For l=0 (S-wave), this reproduces the original LL formula (w/o r_{eff} correction)

Q. Only two terms?

A. With the optical theorem

$|f_0|^2 = q \text{ Im } f_0$,

LL reduces to only two terms

Simpler expression for LL formula

In correlation function

$$C(q) = 1 + \sum_{l=0}^{\infty} \frac{(2l+1)(-1)^l e^{-4x^2} q}{2\sqrt{\pi} x^2} \times \left(\sqrt{\pi} \Im f_l + 2 \Re f_l \int_0^{2x} dt e^{t^2} \right)$$

The *l*-dependent part

$$\sum_{l=0}^{\infty} (2l+1)(-1)^l f_l = f(\theta = \pi)$$

Note: $(-1)^l = P_l(-1) = P_l(\cos\pi)$

Simpler representation of LL formula

$$C(q) = 1 + \frac{4\pi}{q} \Im \left[f(\pi) \int_0^{\infty} dr S(r) e^{2iqr} \right]$$

$\sim$ Im f Re S^ + Re f Im S^

Discussion

Why backward amplitude f(π)?

The plane waves contains

the delta functions at forward and backward

when expanded by the power of centrifugal force $[l(l+1)/r^2]^p$

$$e^{iqz} = \frac{2}{iqr}[\delta(1-\cos\theta)e^{iqr}$$

outgoing θ=0

$$+ \delta(1+\cos\theta)e^{-iqr}] + \mathcal{O}\left(\frac{1}{r^2}\right)$$

Incoming $\theta=\pi$

Correlation function ΔC(q) is generated by interference between outgoing $f(\theta)/r$ and incoming plane wave $\delta(\theta-\pi)$

→ $f(\theta = \pi)$

Note: **Optical theorem** comes from the normalization of the outgoing wave $|f(\theta)/r + \delta(\theta=0)/iqr|^2$ and thus $f(\theta=0)$ plays a role

Heavy-ion collisions and hadron interaction

Heavy-ion collisions can be used to constrain interactions

Production of
Hadrons & Resonances

Production of
Light nuclei & hypernuclei

Anisotropic flow v_1, v_2, …

Femtoscopy
two-particle momentum correlation

Femtoscopy

Contribution of higher partial waves are also contained in the correlation function

- What is a good fitting form of the contribution?
- What is its understanding?

Future

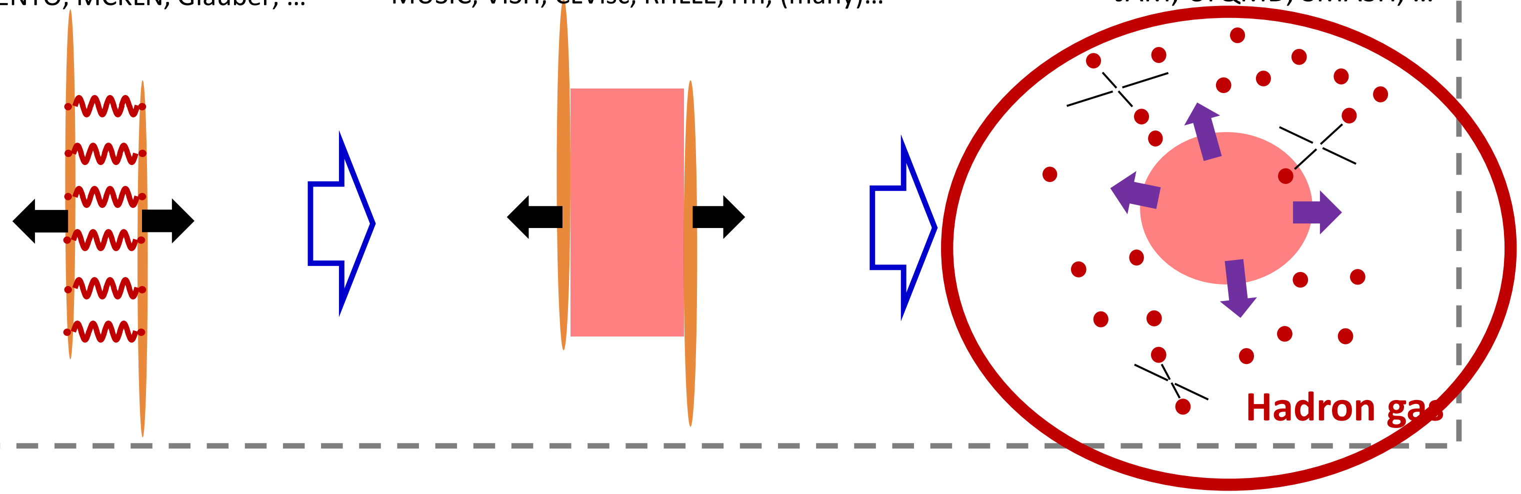
Dynamical model for HIC =

Initial-state model + **Relativistic hydrodynamics** + **Hadronic transport**

IPGlasma, TrENTO, MCKLN, Glauber, …

MUSIC, VISH, CLVisc, RHLLE, rfh, (many)…

JAM, UrQMD, SMASH, …



Further development & understanding in hadronic stage

- Hadronic transport model
  Covariant formulation of RQMD & RAMD,
  Dynamical integration with hydro,
  Dynamical formation of light nuclei, etc.
- Underlying assumptions and understanding of KP formula
  (Re-)Validation of assumptions
  What is the source function $S(r)$ in dynamical model?