Formulation of $\mathrm{SU}(N)$ Lattice Gauge Theories with Schwinger Fermions

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In collaboration with
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Based on 2112.02090, 2312.17734 and work in progress

HHIQCD 2024 @YITP, Kyoto November 15, 2024

Los Alamos NATIONAL LABORATORY

LA-UR-24-29558

Overview

Part I

Formulation of $\mathrm{SU}(N)$ Hamiltonian lattice gauge theory with finite-dimensional Hilbert space using Schwinger fermions

Part II

Phase diagram of this formulation in 2d and comparison with the continuum theory

Connection to some of the previous talks

- General idea: similar to quantum link models, fermionic rishons (Uwe-Jens Wiese, Pietro Silvi)
- Technical aspect: similar to formulation of gauge link with Schwinger bosons (Indrakshi Raychowdhury)
- Numerical method: Tensor network (Akira Matsumoto, Pietro Silvi, etc.)

Outline

Hamiltonian LGT and qubit-regularization of the Hilbert space

Kogut-Susskind Hamiltonian

electric field — magnetic field: — fermion hopping

$$H = \frac{g^2}{2} \sum_{\langle i,j \rangle} \left(L_{ij}^{a2} + R_{ij}^{a2}\right) - \frac{1}{4g^2} \sum_{\square} \left(W_\square + W_\square^\dagger\right) + t \sum_{\langle i,j \rangle} \left(c_i^{\alpha\dagger} U_{ij}^{\alpha\beta} c_j^{\beta} + \text{h.c.}\right)$$

Qubit regularization: same Hamiltonian, but truncates the link Hilbert space:

	Traditional	Qubit regularization
Hilbert space	$L^2(G) = \bigoplus_{\lambda \in \widehat{\mathrm{SU}(N)}} V_\lambda \otimes V_\lambda^*$ (Peter-Weyl theorem)	$\mathcal{H}_Q := \bigoplus_{\lambda \in Q} V_\lambda \otimes V_\lambda^*$ (Symmetry is preserved)
Irreps	$\widehat{\mathrm{SU}(N)}$: Young diagrams with at most $N-1$ rows	$Q = \{\circ, \square, \begin{smallmatrix}\square\\\square\end{smallmatrix}, \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, \cdots, \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}, \overline{\square}\}$

Reasons for Q-scheme

$$Q = \left\{\circ,\ \square,\ \begin{smallmatrix}\square\\\square\end{smallmatrix},\ \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},\ \cdots,\ \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}},\ \overline{\square}\right\}$$

- Contains all N-ality: string tensions at large distance are dictated by N-ality (screening)
- Smallest quadratic Casimir among each N-ality: minimize $\frac{g^2}{2}\left(L^{a2} + R^{a2}\right)$
- Anti-symmetric representations: formulation using Schwinger fermions

Introducing Schwinger fermion representation of $\mathrm{SU}(N)$

- $c^{\alpha\dagger} : \alpha = 1, \cdots, N$ are fermion creation operators that transform in the fundamental representation of $\mathrm{SU}(N)$:

$$\{c^\alpha, c^\beta\} = \{c^{\alpha\dagger}, c^{\beta\dagger}\} = 0, \quad \{c^\alpha, c^{\beta\dagger}\} = \delta^{\alpha\beta}.$$

- The generators of $\mathrm{SU}(N)$ are defined as $Q^a := c^{\alpha\dagger} T^a_{\alpha\beta} c^\beta$, where $[T^a, T^b] = \mathrm{i} f^{abc} T^c$ $\implies [Q^a, Q^b] = \mathrm{i} f^{abc} Q^c$.
- $|0\rangle$ satisfies $c^\alpha |0\rangle = 0$ for $\alpha = 1, 2 \implies$ the trivial representation

$$|\alpha_1 \cdots \alpha_k\rangle = c^{\alpha_k \dagger} \cdots c^{\alpha_1 \dagger} |0\rangle.$$

- The number operator $\hat{k} := \sum_{\alpha=1}^{N} c^{\alpha\dagger} c^\alpha$ distinguishes different irreps of $\mathrm{SU}(N)$ (number of boxes in the Young diagram $(\square\square\cdots\square)^T$), and is related to the Casimir operator as

$$\sum_a (Q^a)^2 = \frac{N+1}{2N}(N - \hat{k})\hat{k}$$

Schwinger fermion representation of the link variables

- Link algebra:

$$[L_i^a, L_j^b] = \mathrm{i} f^{abc} L_i^c \delta_{ij}, \quad [R_i^a, R_j^b] = \mathrm{i} f^{abc} R_i^c \delta_{ij}, \quad [L_i^a, R_j^b] = 0,$$
$$[L_i^a, U_{jk}^{\alpha\beta}] = -T^a_{\alpha\gamma} U_{jk}^{\gamma\beta} \delta_{ij}, \quad [R_i^a, U_{jk}^{\alpha\beta}] = U_{jk}^{\alpha\gamma} T^a_{\gamma\beta} \delta_{ik},$$

- Each link is made of two Schwinger fermions $l^{\alpha\dagger}, r^{\alpha\dagger} : \alpha = 1, \cdots, N$

$$\{l^\alpha, l^{\beta\dagger}\} = \delta^{\alpha\beta}, \quad \{l^\alpha, l^\beta\} = \{l^{\alpha\dagger}, l^{\beta\dagger}\} = 0,$$
$$\{r^\alpha, r^{\beta\dagger}\} = \delta^{\alpha\beta}, \quad \{r^\alpha, r^\beta\} = \{r^{\alpha\dagger}, r^{\beta\dagger}\} = 0,$$
$$\{l^\alpha, r^{\beta\dagger}\} = \{l^{\alpha\dagger}, r^\beta\} = \{l^\alpha, r^\beta\} = \{l^{\alpha\dagger}, r^{\beta\dagger}\} = 0,$$

- Schwinger fermion representation of the link variables:

$$L^a = l^{\alpha\dagger} T^a_{\alpha\beta} l^\beta, \quad R^a = r^{\alpha\dagger} T^a_{\alpha\beta} r^\beta, \quad U^{\alpha\beta} = l^\alpha \frac{1}{\sqrt{\hat{k}_l \hat{k}_r}} r^{\beta\dagger}$$

- Peter-Weyl theorem, or conservation of electric flux $\implies k_l + k_r = N$.
- $|k_l = N; k_r = 0\rangle \equiv |k_l = 0; k_r = N\rangle$ (does not ruin global fermion parity).

Schwinger fermion representation of the KS Hamiltonian

Kogut-Susskind Hamiltonian

electric field — magnetic field: — fermion hopping

$$H = \frac{g^2}{2}\sum_{\langle i,j\rangle}\frac{N+1}{N}(N-\hat{k})\hat{k} - \frac{1}{4g^2}\sum_{\square}(W_\square + W_\square^\dagger) + t\sum_{\langle i,j\rangle}(c_i^{\alpha\dagger} l_i^\alpha)\frac{1}{\sqrt{\hat{k}_{l_i}\hat{k}_{r_j}}}(r_j^{\beta\dagger} c_j^\beta) + \text{h.c.}$$

where

$$W_\square = l_1^{\alpha_1}\frac{1}{\sqrt{\hat{k}_{l_1}\hat{k}_{r_2}}} r_2^{\alpha_2\dagger} l_2^{\alpha_2}\frac{1}{\sqrt{\hat{k}_{l_2}\hat{k}_{r_3}}} r_3^{\alpha_3\dagger} l_3^{\alpha_3}\frac{1}{\sqrt{\hat{k}_{l_3}\hat{k}_{r_4}}} r_4^{\alpha_4\dagger} l_4^{\alpha_4}\frac{1}{\sqrt{\hat{k}_{l_4}\hat{k}_{r_1}}} r_1^{\alpha_1\dagger}$$

Advantages:

- manifestly gauge invariant on each site. Gauss law: conservation of quark number plus Schwinger fermion number $\mod N$.
- gauge invariant operators on each site are bosonic

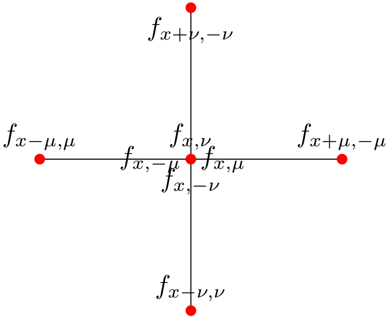
Gauge invariant operators

$$C^\dagger_{x,\mu} := c^{\alpha\dagger}_x f^\alpha_{x,\mu}, \quad F_{x,\mu\nu} := f^{\alpha\dagger}_{x,\mu} f^\alpha_{x,\nu}.$$

$F^\dagger_{x,\mu\nu} = F_{x,\nu\mu}$ and $F_{x,\mu\mu} = F^\dagger_{x,\mu\mu} = \hat{k}_{x,\mu}$ by definition.
These operators satisfies

$$\begin{aligned}
[C_\mu, C^\dagger_\nu] &= F_{\mu\nu} - \hat{n}\delta_{\mu\nu}. \\
[F_{\mu\nu}, F_{\rho\sigma}] &= F_{\mu\sigma}\delta_{\nu\rho} - F_{\rho\nu}\delta_{\mu\sigma} \\
[F_{\mu\nu}, C_\rho] &= C_\mu \delta_{\nu\rho}, \\
[F_{\mu\nu}, C^\dagger_\rho] &= -C^\dagger_\nu \delta_{\mu\rho},
\end{aligned}$$

which is the algebra of $\mathrm{U}(2d + n_f)$. ($\mathfrak{l} = F \oplus \hat{n}$ and $\mathfrak{p} = C$ form a Cartan decomposition.)



Where is N?

Since the Hamiltonian can be written solely in terms of gauge invariant operators C_μ and $F_{\mu\nu}$ that form $\mathrm{U}(2d + N_f)$ algebra, where is the information of the original gauge group $\mathrm{SU}(N)$?

Where is N?

Since the Hamiltonian can be written solely in terms of gauge invariant operators C_μ and $F_{\mu\nu}$ that form $\mathrm{U}(2d+N_f)$ algebra, where is the information of the original gauge group $\mathrm{SU}(N)$?

- Answer: It is hidden in the representation!

$$(C_\mu^\dagger)^{N+1} = 0,\ (F_{\mu\nu}^\dagger)^{N+1} = 0 \quad \text{when } \mu \neq \nu.$$

A trivial example:

$$C_\mu = |\mu\rangle\langle 0|, \quad C_\mu^\dagger = |0\rangle\langle\mu|, \quad F_{\mu\nu} = |\mu\rangle\langle\nu|,$$

Form a representation of the $\mathrm{U}(2d+1)$ algebra in the case of $N=1$.

The continuum theory of 2d QCD

$\mathrm{SU}(N)$ Yang-Mills theory coupled to single-flavor massless Dirac fermions

$$\mathcal{L}_0 = \frac{1}{2\tilde{g}^2} \operatorname{tr} F^2 + \bar{\psi}^\alpha \mathrm{i} \not{D} \psi^\alpha$$

Symmetries:

- $\tilde{g}^2 = 0$: free fermion, $\mathrm{O}(2N)_L \times \mathrm{O}(2N)_R$ chiral symmetry. $(\psi^\alpha = \frac{1}{\sqrt{2}}(\xi^{2\alpha-1} - \mathrm{i}\xi^{2\alpha}))$
- $\tilde{g}^2 > 0$: Gauge symmetry $\mathrm{SU}(N)$, $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ $(N = 2)$; $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ $(N \geq 3)$.

Bosonization:

- $\tilde{g}^2 = 0$: $\mathrm{U}(N)_1$ or $\mathrm{SO}(2N)_1$ WZW model. Central charge: $c = N$.
- $\tilde{g}^2 > 0$: $\mathrm{SO}(2N)_1 / \mathrm{SU}(N)_1$ or $\mathrm{U}(N)_1 / \mathrm{SU}(N)_1 \cong \mathrm{U}(1)_N$ coset WZW model.
  Central charge: $c = c(\mathrm{SO}(2N)_1) - c(\mathrm{SU}(N)_1) = N - (N-1) = 1$.
  - $N = 2$: $\mathrm{SO}(4) \cong \mathrm{SU}(2)_s \times \mathrm{SU}(2)_c$, coset is $\mathrm{SU}(2)_1$ WZW model in the charge sector.

Symmetries from the lattice

- Continuum symmetries forbid any relevant or marginal couplings.
- When regularizing the theory on the lattice using staggered fermions, $\mathrm{U}(1)_L \times \mathrm{U}(1)_R \to \mathrm{U}(1) \times \mathbb{Z}_2^\chi$ ($\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \to \mathrm{SU}(2) \times \mathbb{Z}_2^\chi$ for $N = 2$). $\mathbb{Z}_2^\chi$ forbids mass terms, but allows coupling between currents: $J^a_{L,R} := \frac{1}{2}\xi^T_{L,R} T^a \xi_{L,R}$

Full continuum theory

$$\mathcal{L} = \frac{1}{2\tilde{g}^2}\,\mathrm{tr}\,F^2 + \bar{\psi}^\alpha \mathrm{i}\not{D}\psi^\alpha + \sum_a \lambda^a J^a_L J^a_R$$

- $N \geq 3$: two independent couplings: λ_0 (Thirring coupling) and $\lambda_{\tilde{c}}$
- $N = 2$: one independent coupling: $\lambda_0 = \lambda_{\tilde{c}} = \lambda_c$

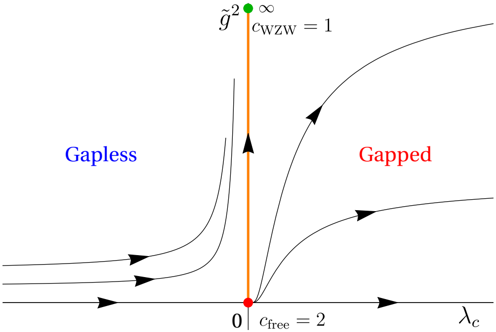
- $N \geq 3$:

$$\frac{\mathrm{d}\lambda_0}{\mathrm{d}\ln\mu} = -\frac{N-1}{2\pi}\lambda_{\tilde{c}}^2, \qquad \frac{\mathrm{d}\lambda_{\tilde{c}}}{\mathrm{d}\ln\mu} = -\frac{1}{N\pi}\lambda_0\lambda_{\tilde{c}},$$

- $N = 2$:

$$\frac{\mathrm{d}\lambda_c}{\mathrm{d}\ln\mu} = -\frac{1}{2\pi}\lambda_c^2.$$

Mixed anomaly between $\mathrm{U}(1)$ and $\mathbb{Z}_2^\chi$
$\implies$ the gapped phase should spontaneously break $\mathbb{Z}_2^\chi$, the translation-by-one-site symmetry on the lattice.



RG flow for $N = 2$

Phases in the strong coupling limit

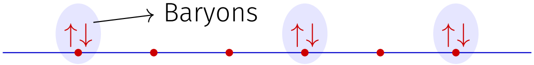
generalized Hubbard coupling

$$H = \frac{g^2}{2}\sum_{\langle i,j\rangle}\left(L_{ij}^{a2} + R_{ij}^{a2}\right) + t\sum_{\langle i,j\rangle}\left(c_i^{\alpha\dagger}U_{ij}^{\alpha\beta}c_j^{\beta} + \text{h.c.}\right) - U\sum_i n_i(N-n_i)$$

$g^2/t \gg 1$:

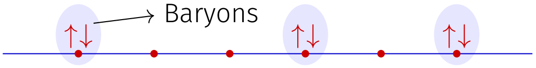
$$\frac{1}{2}\left(L_{ij}^{a2} + R_{ij}^{a2}\right)|k\rangle = \frac{N+1}{2N}k(N-k)|k\rangle,$$

gauge links prefer $k=0$ (trivial rep)



$(N=2)$

Similar analysis for $-U/t \gg 1$.



$(N=2)$

When $g^2/t \gg 1$ or $-U/t \gg 1$, treat hopping terms as a perturbation:

$$XXZ \text{ spin chain:} \quad H_{\text{eff}} = \sum_{\langle i,j \rangle} J_\perp (X_i X_j + Y_i Y_j) + J_z (Z_i Z_j - 1)$$

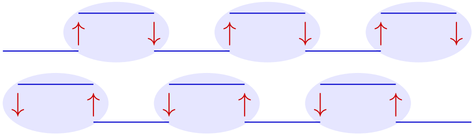
where

$$J_\perp = (-1)^{N-1} \frac{N}{2(N-1)!} \frac{t^N}{(\frac{N+1}{2N} g^2 + 2U)^{N-1}}, \quad J_z = \frac{N}{2(N-1)} \frac{t^2}{\frac{N+1}{2N} g^2 + 2U}$$

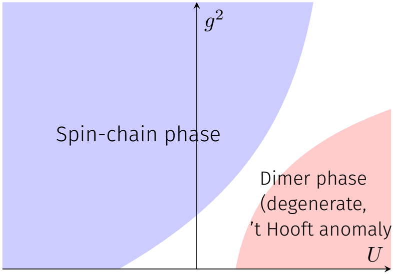
- When $N=2$, $|J_\perp| = |J_z|$ (gapless) $\implies$ $\mathrm{SU}(2)$ symmetry $\leftrightarrow$ $\mathrm{SU}(2)_1$ WZW model.
- When $N>2$, $|J_\perp| < |J_z|$ (gapped, Néel) $\implies$ $\mathrm{U}(1)$ symmetry $\leftrightarrow$ $\mathrm{U}(1)_N$ WZW model.

Strong coupling expansion - dimer phase

when $U/t \gg 1$, each site is forced to have one fermion ($N = 2$)



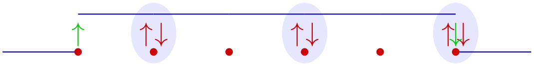
gapped, dimerized, doubly degenerate, expected from 't Hooft anomaly matching



Confinement in the strong coupling limit

Put two test quarks and pull them apart, see how the energy changes:

$-U/t \gg 1$: Raise links in-between to higher irreps, confined

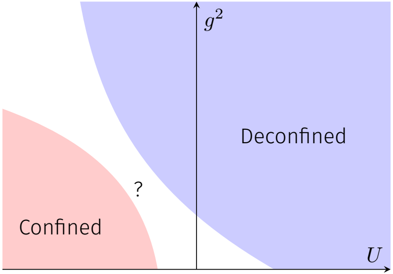


String tension: $$T_k = g^2 \frac{N+1}{2N} k(N-k).$$

$-U/t \gg 1$ or $g^2/t \gg 1$:

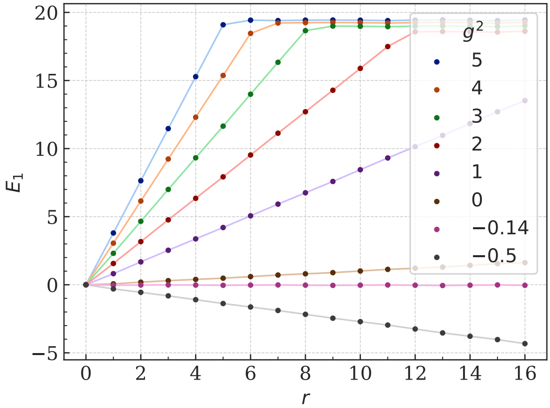
String tension: $$T_k = g^2 \frac{N+1}{2N} k(\lfloor \frac{N}{2} \rfloor - k).$$

$\implies$ Deconfined for $N = 2, 3$



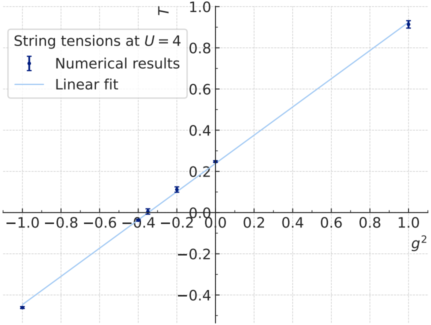
Confinement diagram for $N = 2, 3$

Confining phase



Energy as a function of the distance r between the test quarks at $N = 2$, $k = 1$ and $L = 20$ for $U = -10$.

String tensions at large U



$$T = 0.685(8)g^2 + 0.239(5)$$

- Strong coupling result: $T = 0.75g^2$
- Surprisingly, when $g^2 = 0$, $T > 0$. (In traditional theory, when $g^2 = 0$ the gauge field can be absorbed)
- In the qubit regularization, electric field term is generated by the hopping term in the RG sense:

$$H_{ij} = c_i^{\alpha\dagger} U_{ij}^{\alpha\beta} c_j^{\beta} + c_j^{\beta\dagger} (U_{ij}^{\alpha\beta})^\dagger c_i^{\alpha}$$

$$-\frac{1}{\beta} \log(\mathrm{tr}_f\, \mathrm{e}^{-\beta H_{ij}}) \begin{cases} \propto \mathbb{1} & : \text{traditional} \\ \propto L_{ij}^{a2} + R_{ij}^{a2} & : \text{qubit} \end{cases}$$

Marginal operator, level crossing and critical point

- $\mathrm{SU}(2)_1$ WZW has $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry
  Lowest 5 states: $(s_L, s_R) = (0,0)$ and $(\frac{1}{2}, \frac{1}{2})$
- On the lattice: chiral symmetry is broken
  $\lambda_c J_L \cdot J_R$ is allowed, can be tuned by U

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \xrightarrow{\text{broken}} \mathrm{SU}(2)_{\mathrm{diag}}$$

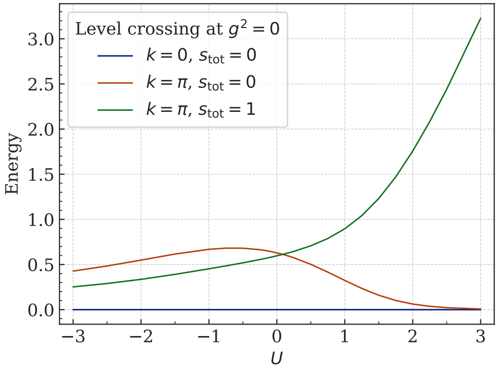
$$(s_L, s_R) = (\frac{1}{2}, \frac{1}{2}) \longrightarrow s_{\mathrm{tot}} = 1, 0$$

$$\langle J_L \cdot J_R \rangle = \frac{1}{2}\langle (J_L + J_R)^2 - J_L^2 - J_R^2 \rangle$$

$$= \frac{1}{2}\left(s_{\mathrm{tot}}(s_{\mathrm{tot}}+1) - s_L(s_L+1) - s_R(s_R+1)\right)$$

λ_c is marginal, β-function:

$$\frac{\mathrm{d}\lambda_c}{\mathrm{d}\ln\mu} = -\frac{1}{2\pi}\lambda_c^2$$

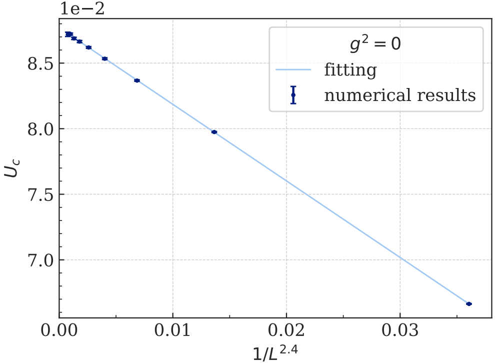


DMRG: ITensor[a]

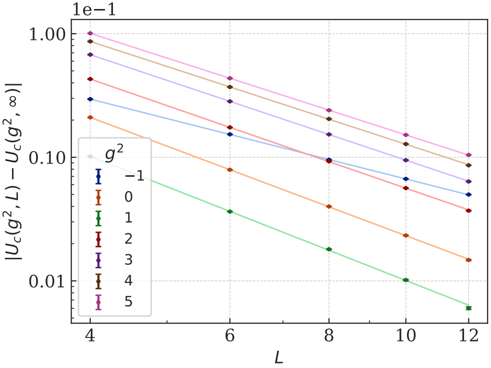
[a] M. Fishman et al., 2022, *SciPost Phys. Codebases*

Critical point extrapolation in L

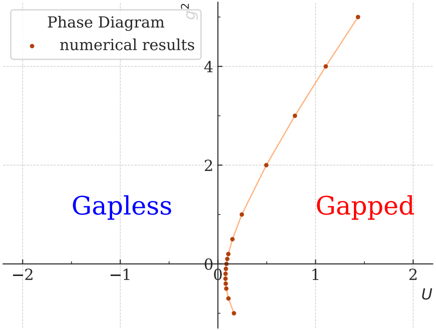
$$U_c(L) = U_c(\infty) + AL^{-a}$$



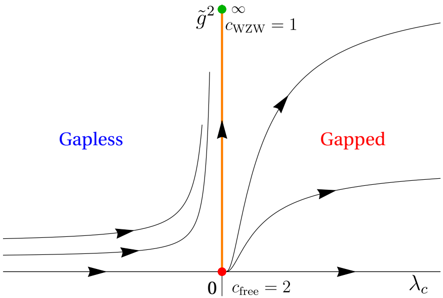
$$U_c(\infty) = -0.08769(3)$$



Phase Diagram

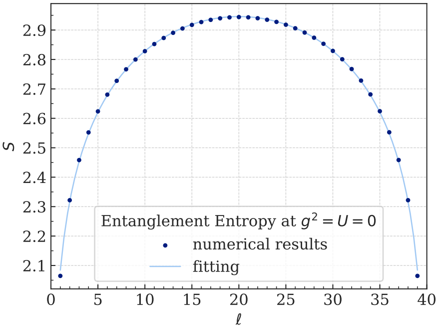


Phase diagram



Flow diagram

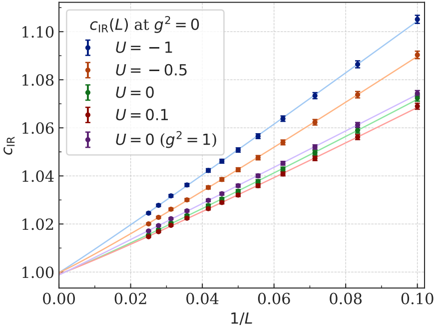
Central charge in the conformal phase



IR central charge via entanglement entropy:

$$S = \frac{c_{\rm IR}}{3} \log\left(\frac{L}{\pi a} \sin\frac{\pi \ell}{L}\right) + \text{const.}$$

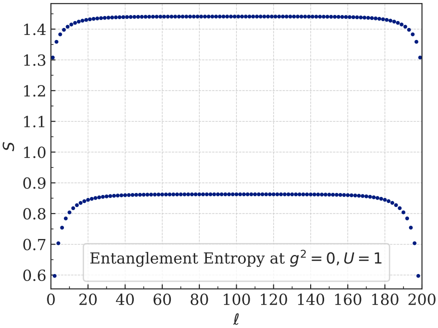
between two subsystems with size ℓ and $L - \ell$.

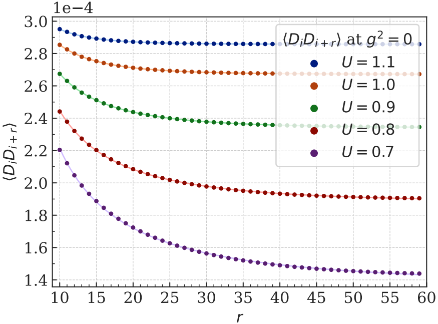


Central charge extrapolation

$c_{\rm IR}(\infty)$ ranges from $0.9988(7)$ to $0.9998(9)$.

Central charge by approaching critical points from gapped phase





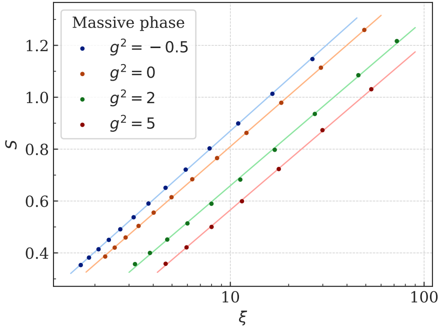
UV central charge via entanglement entropy

$$S = \frac{c}{6} \log \frac{\xi}{a} + \text{const.}$$

ξ is correlation length.

$$D_i := (-1)^i \frac{1}{2} (Q_i^z Q_{i+1}^z - Q_{i-1}^z Q_i^z),$$

$$\langle D_i D_{i+r} \rangle = \frac{A}{r} \mathrm{e}^{-\frac{r}{\xi_c}}$$



$$c = 1.737(6), 1.693(4), 1.66(1), 1.66(1)$$

Multiple sectors that decouple at the critical point and become critical simultaneously, entanglement entropy is given by the sum

$$S_0 = \sum_i S_i = \sum_i \frac{c_i}{6} \ln \frac{\xi_i}{a} + \text{const.},$$

If ξ_c diverges, and ξ_s diverges as $\frac{\xi_s}{a} \propto (\frac{\xi_c}{a})^\alpha$, this simplifies to

$$S_0 = \frac{c_{\text{eff}}}{6} \ln \frac{\xi_c}{a} + \text{const.},$$

where $c_{\text{eff}} = 1 + \alpha$. According to [a], $c_{\text{eff}} = 1 + \alpha$ is achieved with $j_{\text{max}} = \frac{3}{2}$ links.

[a] M. C. Bañuls et al., 2017, *Phys. Rev. X* arXiv: `1707.06434` (hep-lat)

Conclusions

- Formulated $\mathrm{SU}(N)$ lattice gauge theories using Schwinger fermions.
- Remarkably, the resulting theory can be expressed purely in terms of gauge-invariant operators, which form a $\mathrm{U}(2d + N_f)$ algebra.
- This formulation applies to any $\mathrm{SU}(N)$ gauge group in any spacetime dimension.
- Reproduced the IR phases of 2d QCD using finite-dimensional local Hilbert space, as well as signal of the free fermion fixed point.

Thanks for attention!