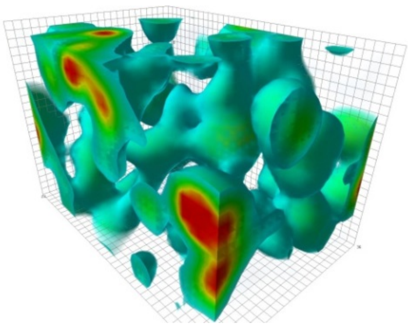
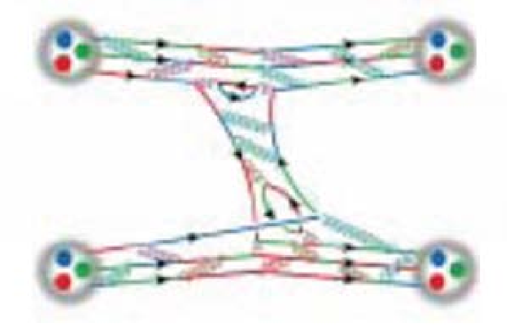
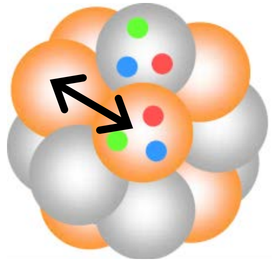
Lectures on Lattice QCD study of Hadron interactions (I)

Takumi Doi

(RIKEN iTHEMS)



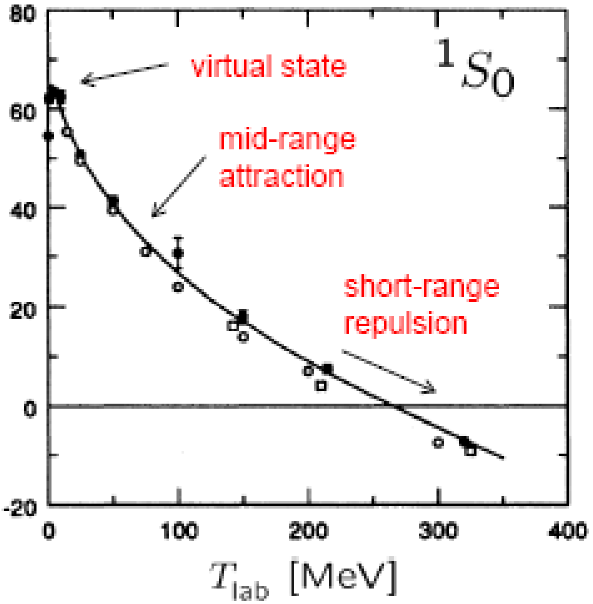




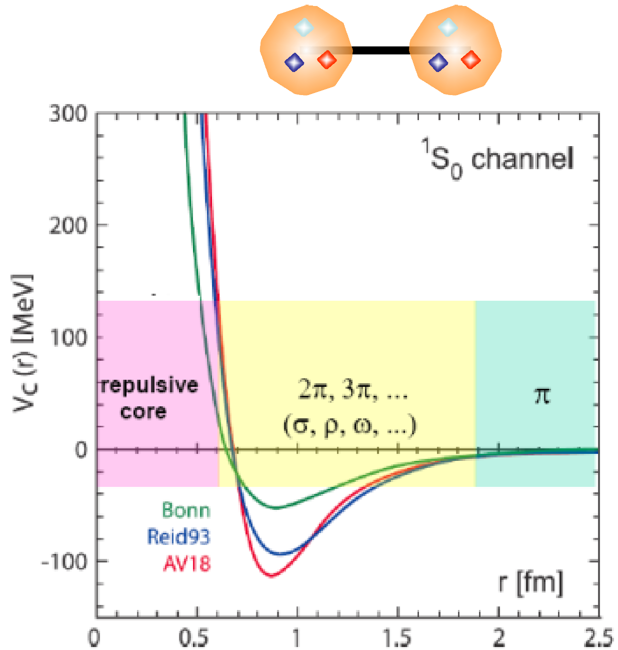
Why Hadron interactions?

One of the most important quantities which bridge particle physics and nuclear physics / astrophysics

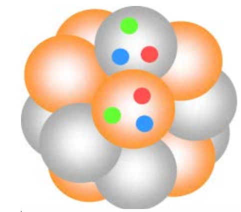
Nuclear Forces: Foundation of nuclear physics



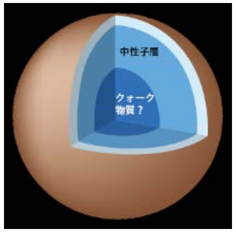
NN phase shifts from experiments



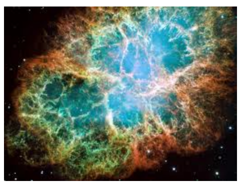
Phenomenological Nuclear Forces



Nuclei



Neutron Stars



Super Novae

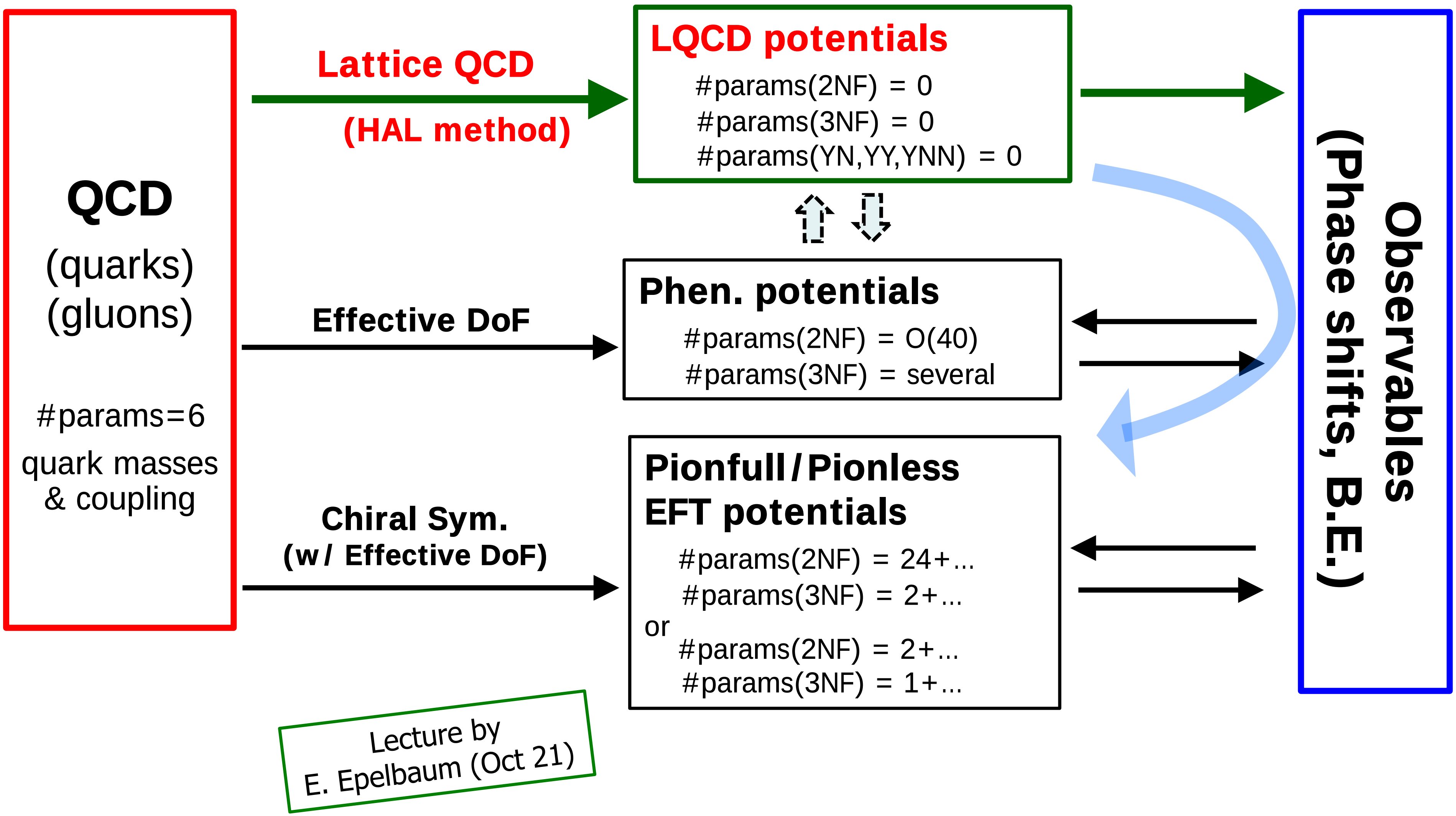
Various applications

- ***Nuclear Forces* play *crucial roles***
  - *Yet, no clear connection to QCD so far*

Phen. NN potentials: #params = 30~40

←→ QCD: #inputs = 6 : quark masses (m_u, m_d, m_s, m_c, m_b) & coupling α_s

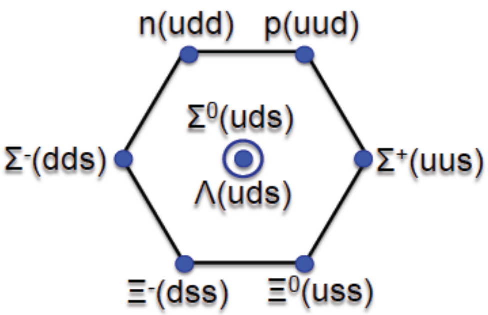
Nuclear Forces from QCD

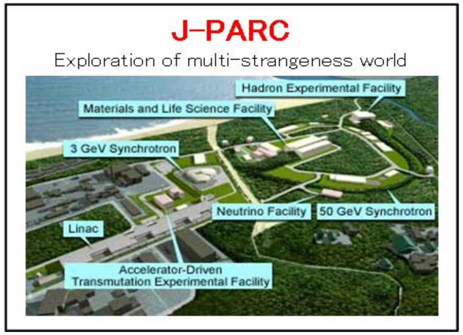


Nuclear Forces ➔ Baryon Forces (incl. Hyperons)

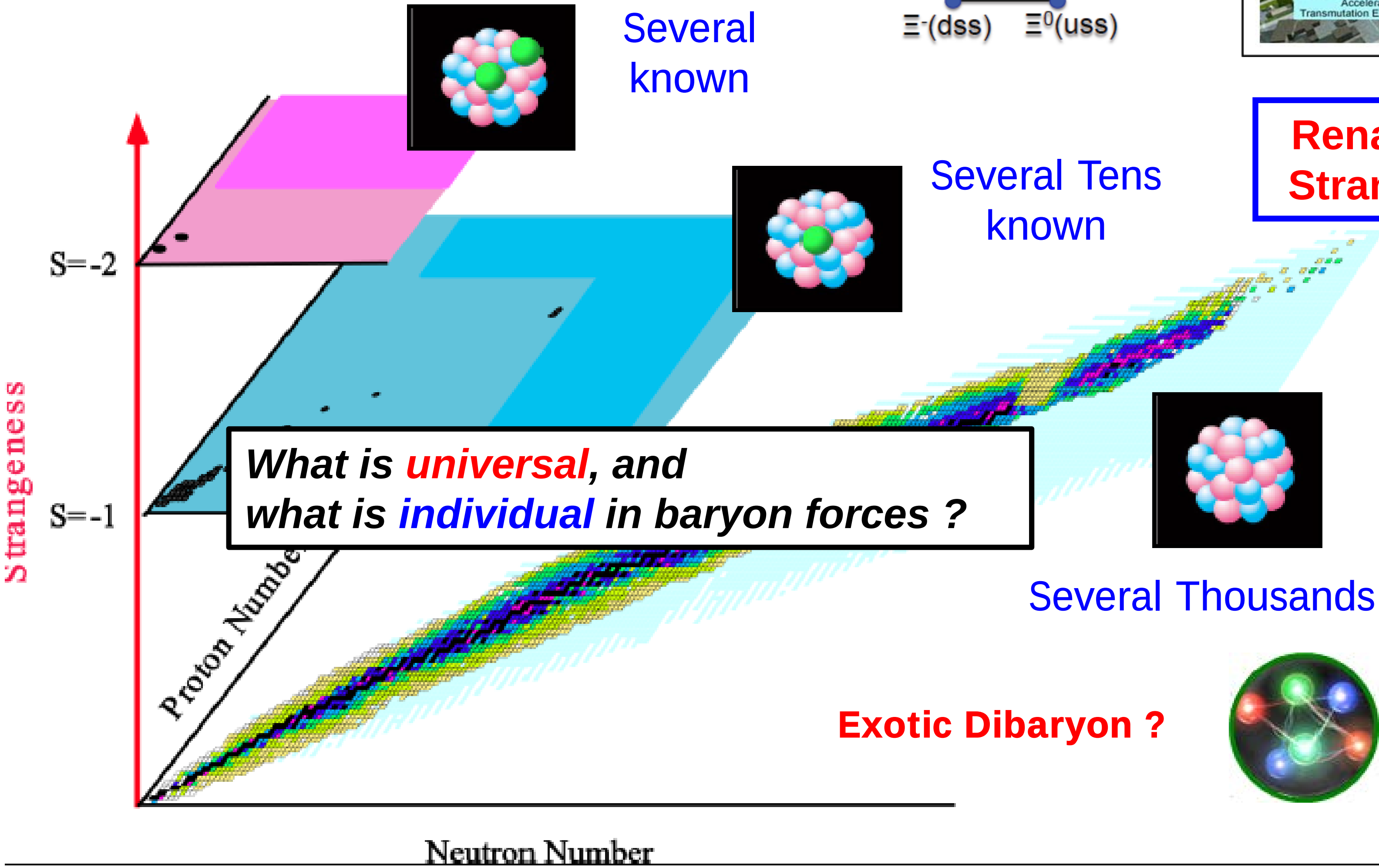
3D Nuclear Chart

Nucleons : u, d quarks
Hyperons : u, d, s quarks





Renaissance in Strange World !



Several known

Several Tens known

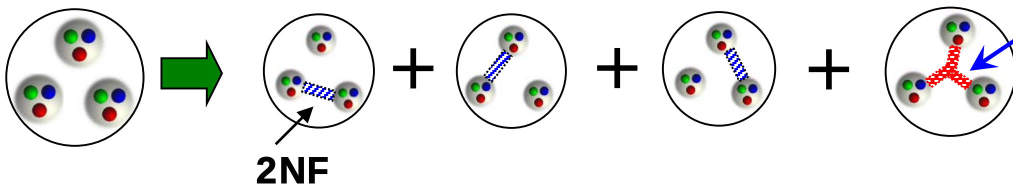
Several Thousands known

What is universal, and what is individual in baryon forces ?

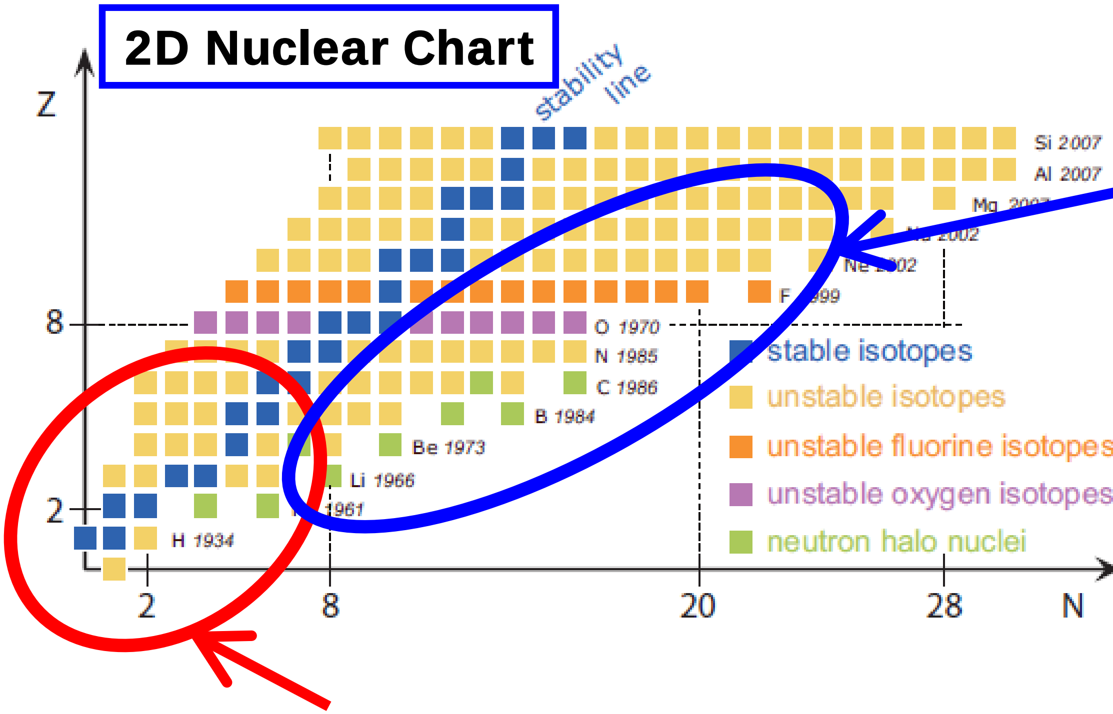
Exotic Dibaryon ?

Nuclear Forces ➔ Thee-Nucleon Forces (3NF)

What is 3NF ?



3NF: Forces which cannot be explained by pair-wise 2NF



Paradigm Shift in
Unstable Nuclei
(New Magic Numbers !)

🡐 Important role of 3NF

T.Otsuka et al., PRL105(2010)032501

➔ r-process Nucleosynthesis

RIBF/FRIB

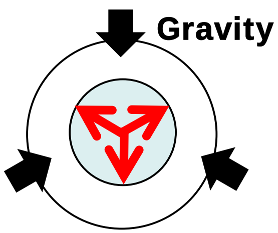
Precise ab initio calculations / experiments
→ 3NF is indispensable

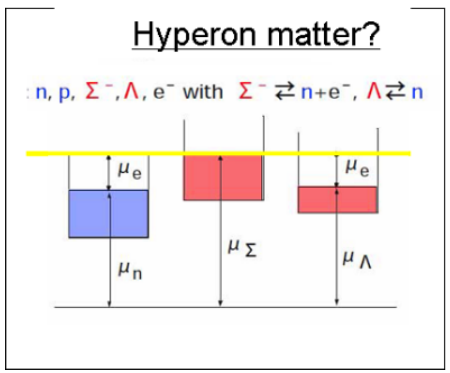
Dense Matter ← Interactions of YN, YY, + NNN, YNN,… are crucial

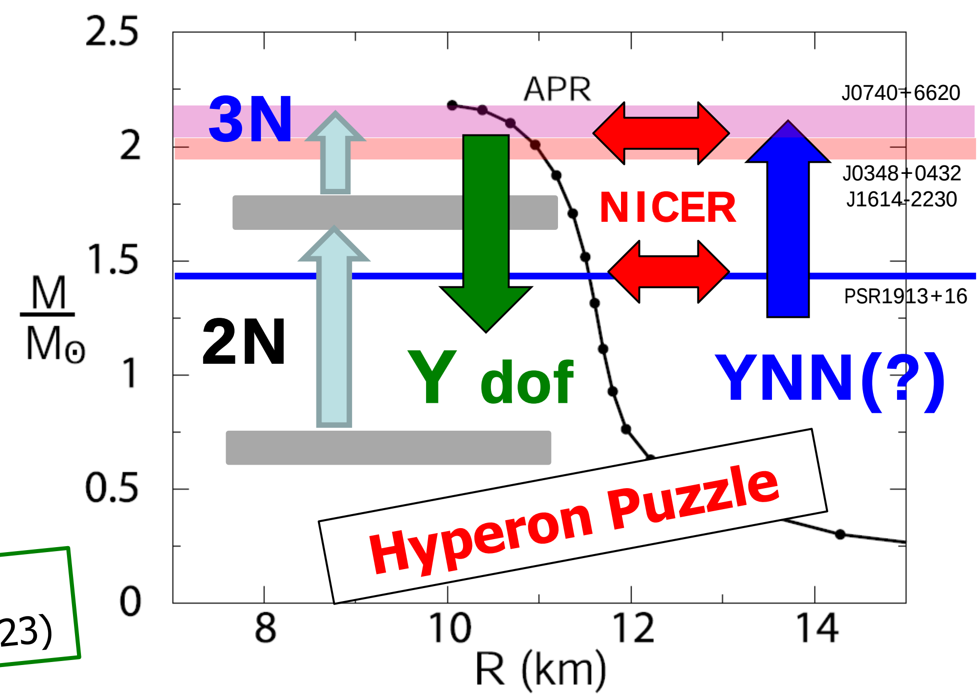
- Neutron Stars, Supernovae
  ↔ EoS of dense matter

GW

NS-NS merger   LIGO/Virgo/KAGRA







How to sustain a neutron star against gravitational collapse ?

Quark matter ?

Lecture by W. Weise (Oct 23)

G. Baym et al., Rep.Prog.Phys 81(2018)056902
Fukushima, Fujimoto, Kojo, …

Akmal et al.('98), Nishizaki et al.('02), Takatsuka et al.('08)

Nuclear/Hyperon Forces ➔ Charmed/Bottomed Forces

light ~ a few MeV ~100 MeV ~300 MeV ~1.3 GeV ~4.2 GeV heavy

up,down quarks	strange quark	Λ_{QCD}	charm quark	bottom quark	top

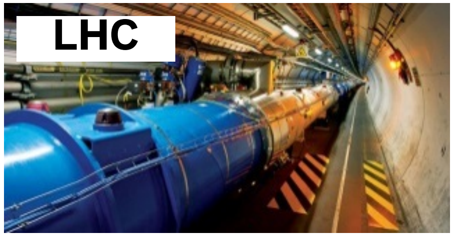
chiral symmetry

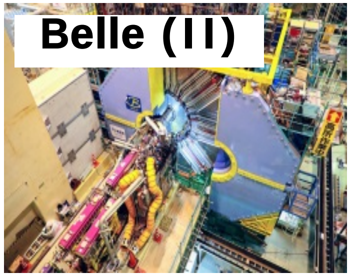
heavy quark symmetry

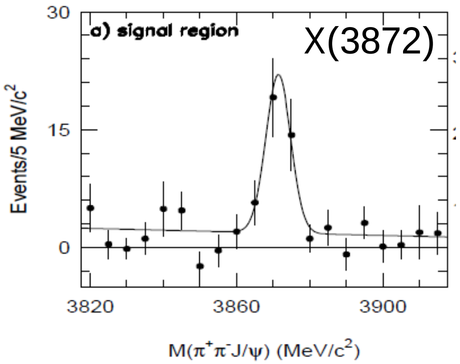
Heavy quarks: New doorway to the mysteries of QCD

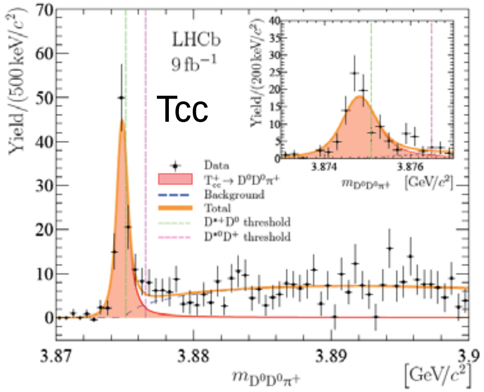
Many new exotic particles being reported!

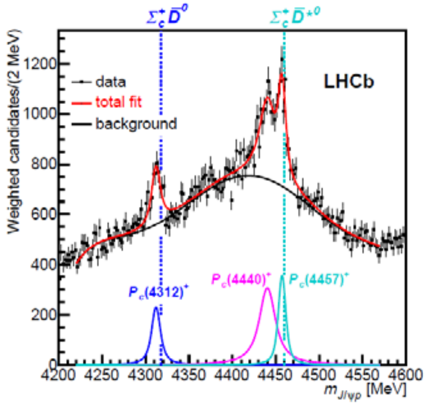
Lecture by N. Brambilla (Oct 24)











Hadron interactions crucial to understand these “signals” !

QCD (DoF=quarks/gluons)

- Formula of QCD: very simple & beautiful

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \bar{q}\left[\gamma^\mu (i\partial_\mu - g A_\mu) - m\right] q$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu$$

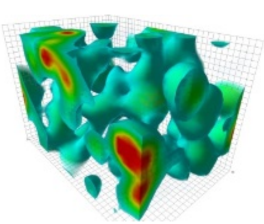
  – Only 6 parameters

quark masses (m_u, m_d, m_s, m_c, m_b, (m_t))
coupling constant $\alpha_s = g^2/4\pi$

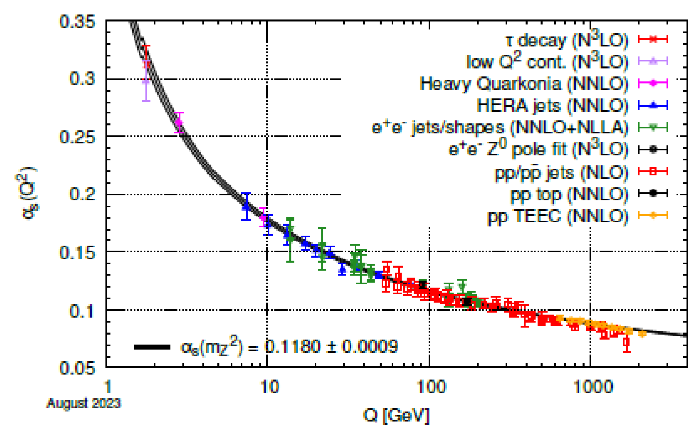
mass ($\overline{MS}$, $\mu = 2\text{GeV}$ or m_q)	m_u	m_d	m_s	m_c	m_b
[MeV]	2.16(0.07)	4.70(0.07)	93.5(0.8)	1273(5)	4183(7)

$$\alpha_s(M_Z^2) = 0.1180(9)$$

- Solving QCD: very challenging
  – Coupling is "strong" at low energy
  – Nonperturbative effects
  – Quantum effects w/ infinite # of DoF



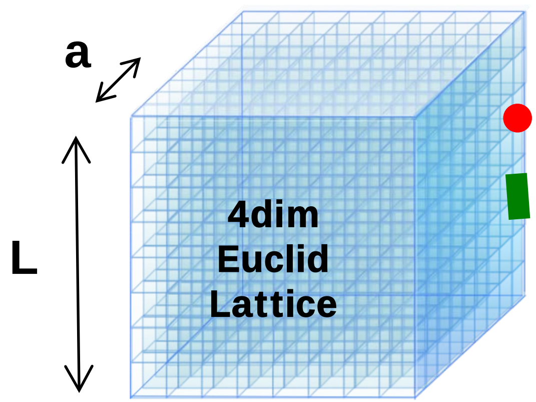
QCD vacuum



(PDG2024)

Lattice QCD
First-principle calculation of QCD

$$Z = \int dU\, dq\, d\bar{q}\; e^{-S_E}$$



quarks on sites

$$q(x)$$

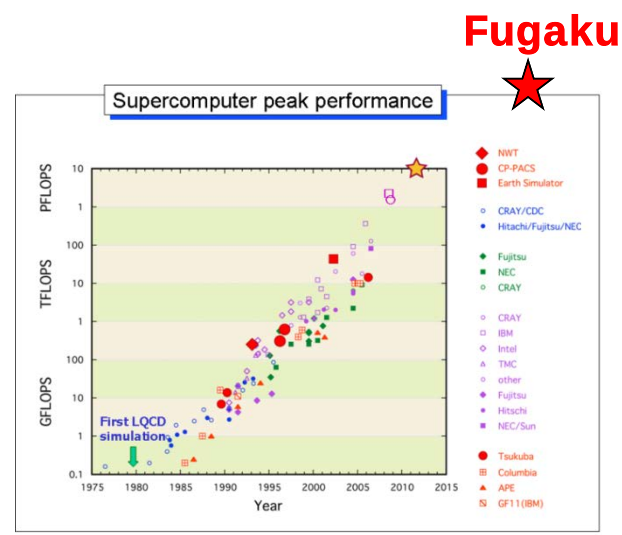
gluons on links

$$U_\mu(x, x+\mu) = \exp[-iaA_\mu]$$

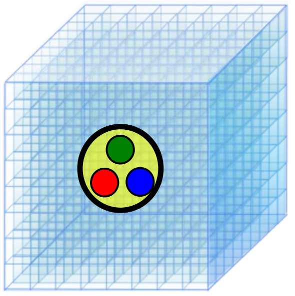
K.G. Wilson
(1974)

- Regularized system (finite a and L)
- Gauge-invariance manifest
- Fully-Nonperturbative
- DoF ~ $10^{9\text{-}10}$ ➔ Monte-Carlo w/ Euclid time
  - Numerical calc by supercomputers

Lecture by
E. Itou (Oct 16)

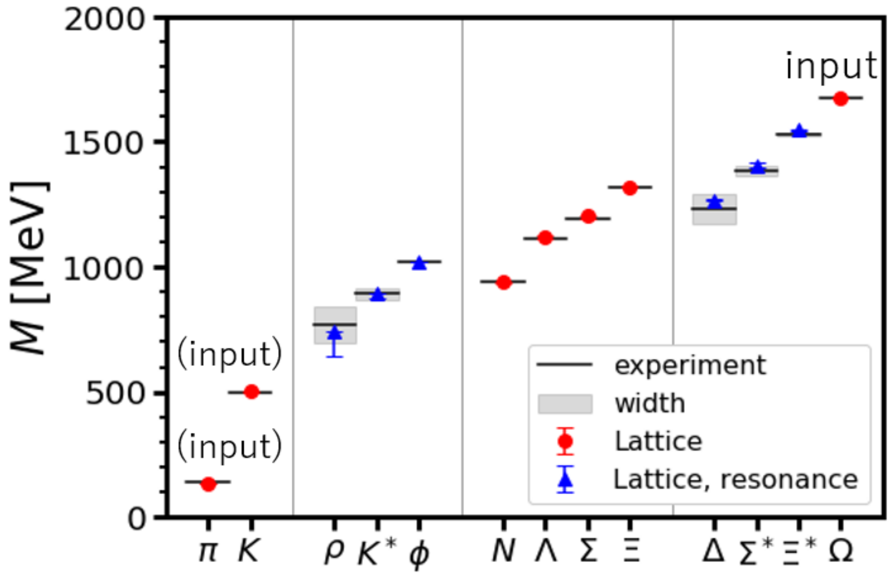


Status of Lattice QCD



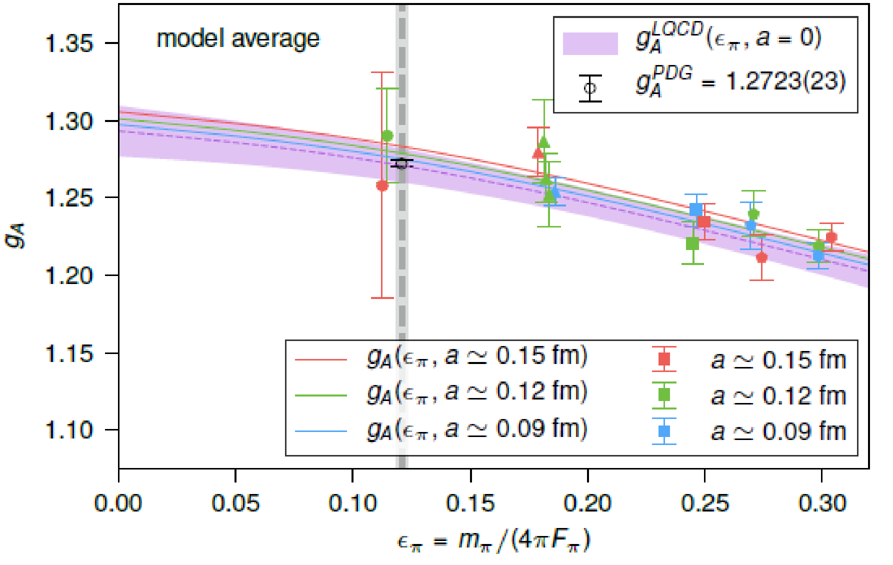
Mass & structure of single hadrons well reproduced !

Hadron (baryon, meson) masses



HAL QCD Coll., PRD in press, arXiv:2406.16665

Nucleon axial coupling



C.C. Chang et al. (CalLat Coll.), Nature 588 (2018)7708

LQCD — PACS-CS Coll., PRD81(2010)074503; BMW Coll., JHEP1108(2011)148

LQCD + LQED — BMW Coll., Science 347(2015)1452

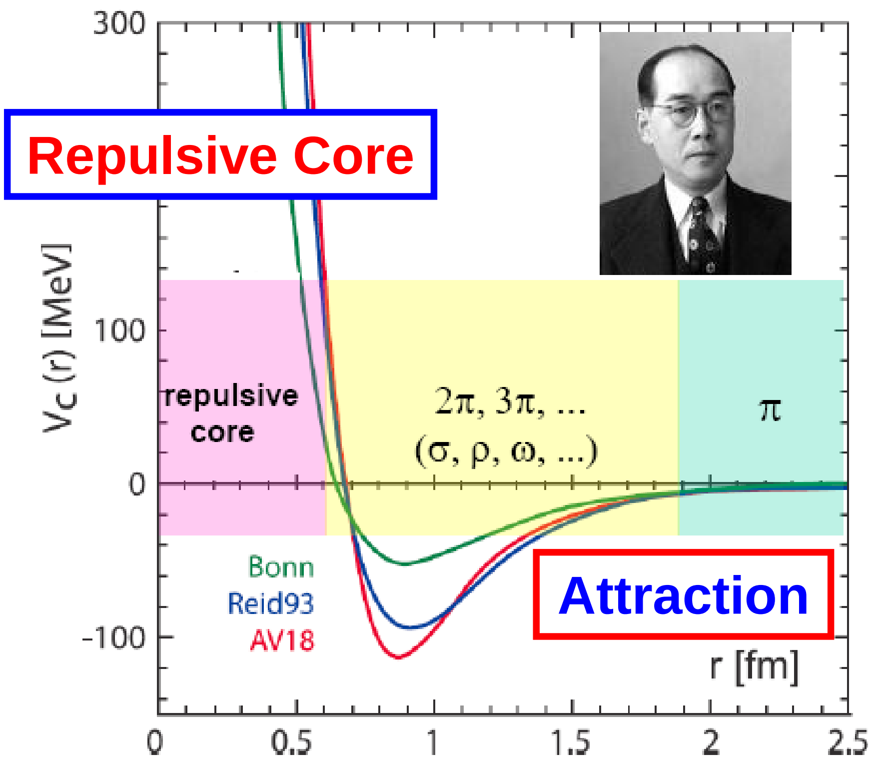
Next challenge: Interactions between 2 (& 3, …) hadrons

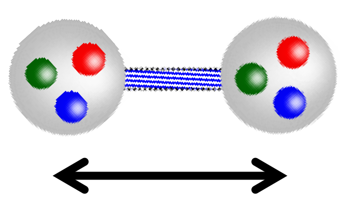
Traditional Nuclear Physics (1935(Yukawa) ~ 20th C.)

Based on

Phenomenological
Two-Nuclear Forces (2NF)

QCD

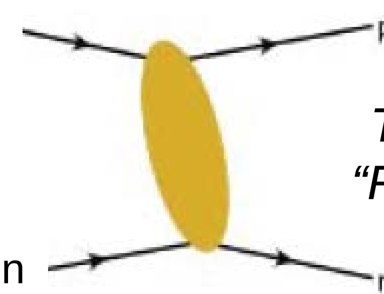




Y. Nambu:

"Even now, it is impossible to completely describe nuclear forces beginning with a fundamental equation. …, a practically impossible task."

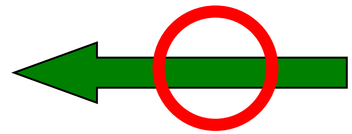
(Quarks: Frontiers in Elementary Particle Physics"
(World. Sci. (1985))



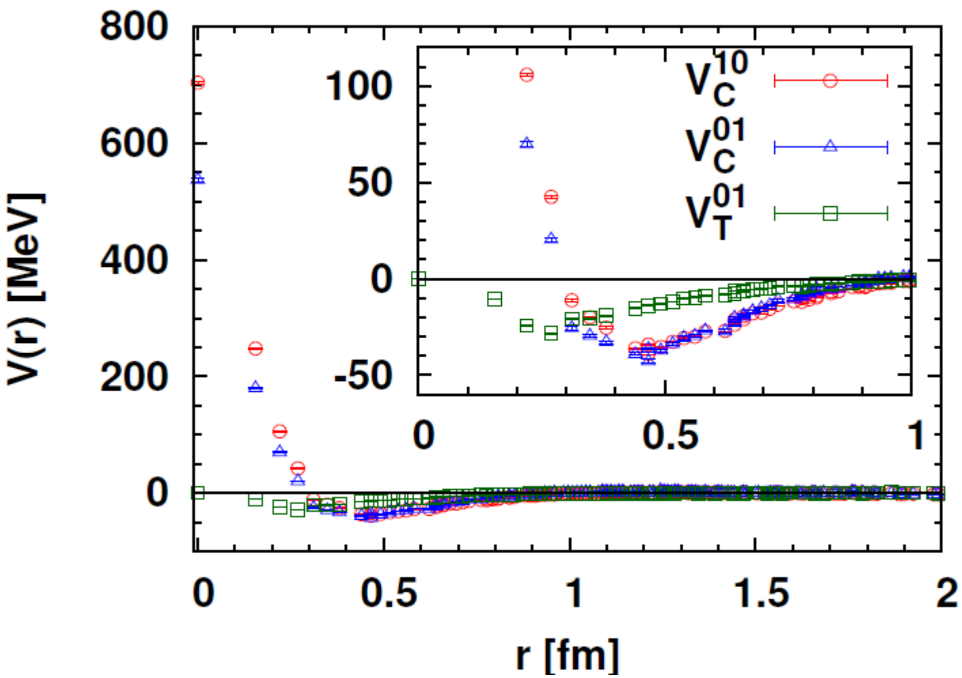
Traditional "Resolution"

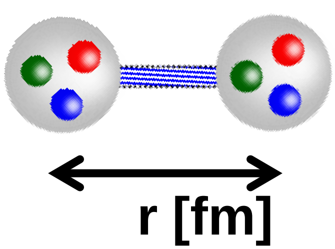
Nuclear Physics in the New Era (21th C. ~)

LQCD Two-Nuclear Forces
give theoretical basis
for nuclear physics



QCD

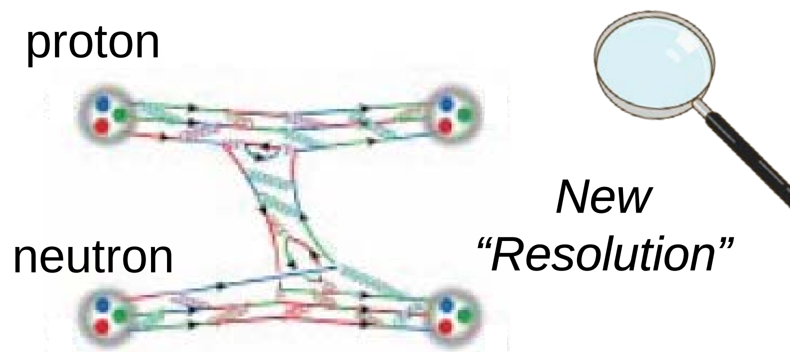




Novel theoretical framework

Massive numerical simulations

First-principles LQCD calc for 2NF becomes possible !
(Ishii-Aoki-Hatsuda, '07)



- **Outline**
  - ~~Introduction~~
  - Brief review of scattering theory
  - Scattering on the lattice
    - Luscher’s finite volume method
    - HAL QCD method
  - S/N problem
  - More on HAL QCD method
  - Reliability issue and NN controversy
  - Summary

Scattering problem

- Consider the two particle scattering by potential

$$\left[-\frac{1}{2\mu}\nabla^2 + V(\vec{r})\right]\psi(\vec{r}) = 0$$

For simplicity, we consider the central potential, $V(\vec{r}) = V(r)$

  - Incoming state:

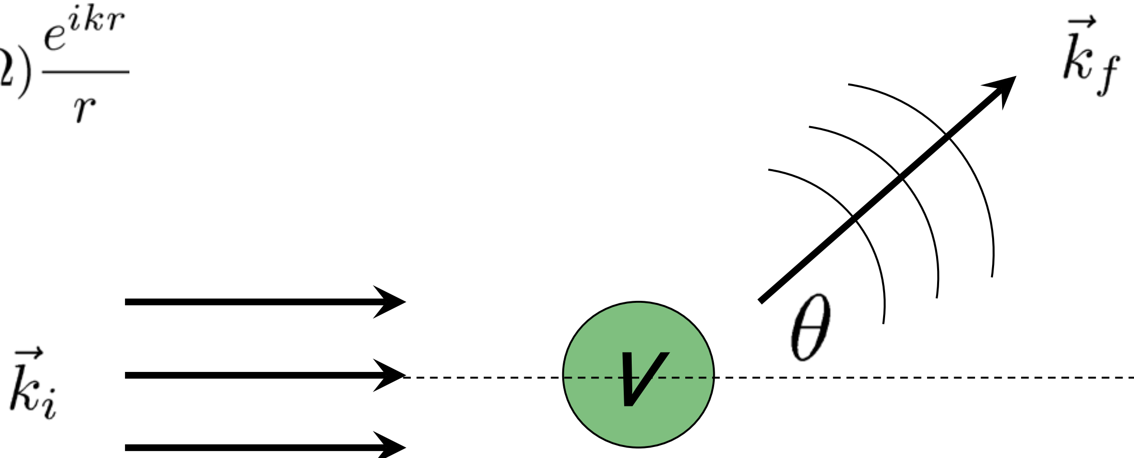
$$\psi(\vec{r}) = e^{ikz} = e^{i\vec{k}_i\cdot\vec{r}} = e^{ikr\cos\theta}$$

  - Outgoing state:

$$\psi(\vec{r}) = e^{ikz} + f(\Omega)\frac{e^{ikr}}{r}$$

  - Cross section

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2$$



Scattering problem

- Green function

$$(\nabla^2 + k^2)G(\vec{r}) = \delta(\vec{r}) \qquad G(\vec{r}) = -\frac{1}{4\pi}\frac{e^{\pm ikr}}{r}$$

  – Solution of Schrodinger eq.

$$\psi(\vec{r}) = \psi_0(\vec{r}) + \int d^3\vec{r}' G(\vec{r}-\vec{r}')U(\vec{r}')\psi(\vec{r}') \qquad U(\vec{r}) \equiv 2\mu V(\vec{r})$$

Born approximation $$f(\Omega) = -\frac{\mu}{2\pi}\int d^3\vec{r} e^{-i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}}V(\vec{r})$$

- Partial wave decomposition

$$\psi(\vec{r}) = R_l(r)Y_{lm}(\Omega) \qquad k^2 \equiv 2\mu E$$

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + U(\vec{r})\right]\phi_l(r) = k^2\phi_l(r) \qquad R_l(r) \equiv \frac{\phi_l(r)}{r} \qquad \phi_l(0) = 0$$

  – General solution is given by Bessel and Neumann func

$$R_l = \alpha_l(k)j_l(kr) + \beta_l(k)n_l(kr)$$
$(kr \gg 1)$

$$\begin{aligned} j_l(z) &\propto z^l \\ n_l(z) &\propto z^{-(l+1)} \end{aligned}$$
$(z \to 0)$

$$\begin{aligned} j_l(z) &\propto \sin(z - l\pi/2)/z \\ n_l(z) &\propto \cos(z - l\pi/2)/z \end{aligned}$$
$(z \to \infty)$

Scattering problem

- Asymptotic behavior and phase shift

$$\begin{aligned} R_l &= \alpha_l(k) j_l(kr) + \beta_l(k) n_l(kr) \\ &\simeq \alpha_l(k)\frac{\sin(kr - l\pi/2)}{kr} - \beta_l(k)\frac{\cos(kr - l\pi/2)}{kr} \\ &= \sqrt{\alpha_l(k)^2 + \beta_l(k)^2}\frac{\sin(kr - l\pi/2 + \delta_l(k))}{kr} \end{aligned}$$

$$\tan\delta_l(k) = -\frac{\beta_l(k)}{\alpha_l(k)}$$

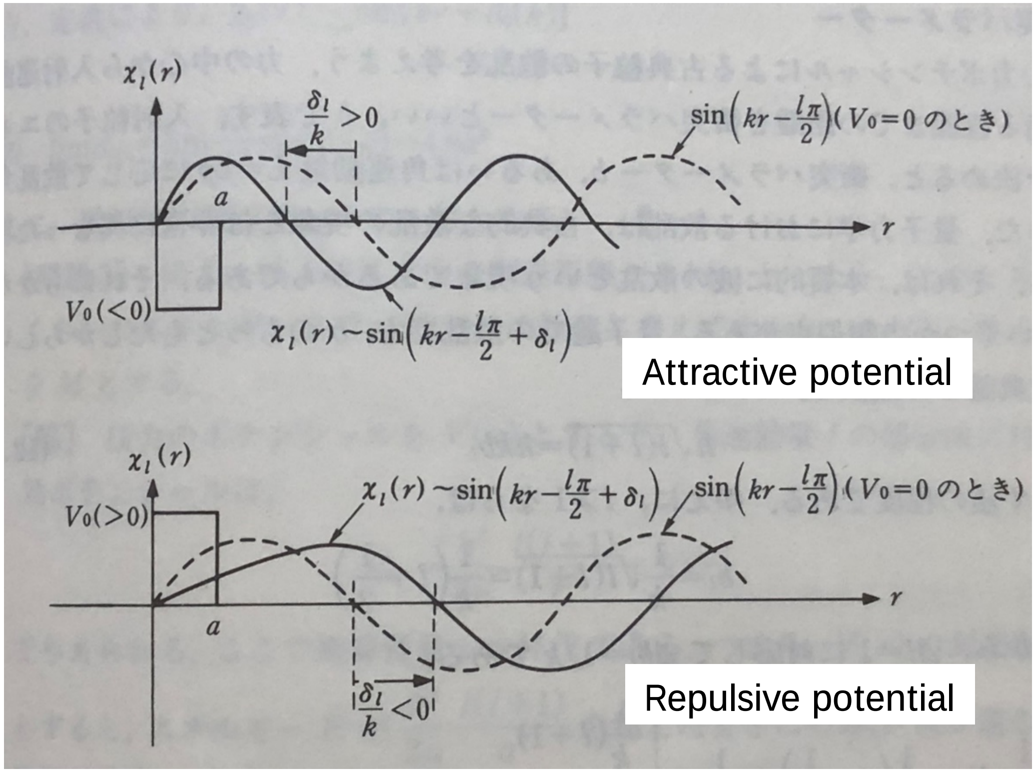
- S-matrix

$$e^{ikz} = \frac{1}{2ikr}\sum_l i^l(2l+1)\left(e^{i(kr-l\pi/2)} - e^{-i(kr-l\pi/2)}\right)P_l(\cos\theta)$$

$$\begin{aligned} \psi(\vec{r}) &= e^{ikz} + f(\Omega)\frac{e^{ikr}}{r} \\ &= \frac{1}{2ikr}\sum_l i^l(2l+1)\left(e^{2i\delta_l(k)}e^{i(kr-l\pi/2)} - e^{-i(kr-l\pi/2)}\right)P_l(\cos\theta) \end{aligned}$$

$$f(\Omega) = \sum_l (2l+1)\frac{e^{2i\delta_l(k)} - 1}{2ik}P_l(\cos\theta)$$

Scattering problem



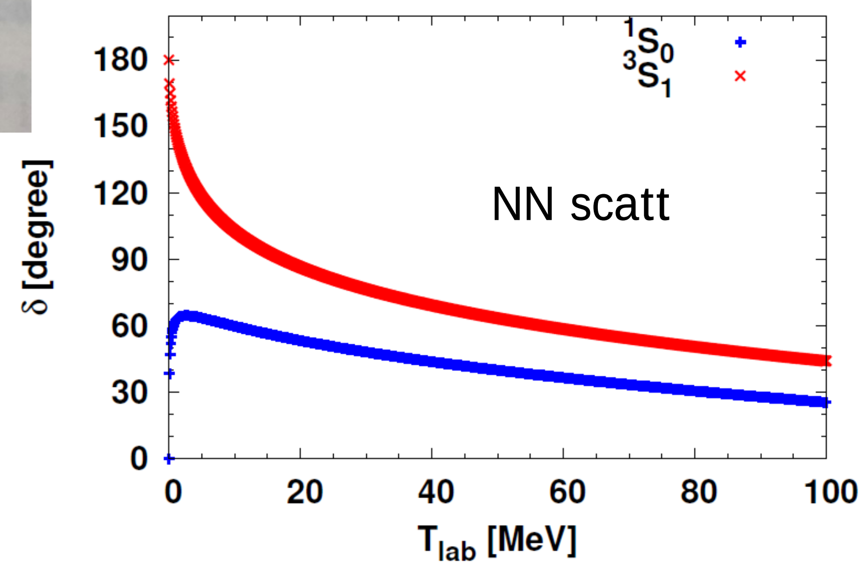
(Fig taken from Igi-Kawai book)

$$^{2S+1}L_J$$

Levinson's theorem

$$\delta_l(k=0) = N_l \pi$$

(N_l : #bound states)



Scattering problem

- Cross section

$$\sigma = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2 \delta_l(k)$$

$$f(\Omega) = \sum_l (2l+1) \frac{e^{2i\delta_l(k)} - 1}{2ik} P_l(\cos\theta)$$

- Low-energy expansion

$$\delta_l(k) \propto k^{2l+1} \quad (k \to 0)$$

  – S-wave is dominant

$$k \cot \delta_0 = +\frac{1}{a} + \frac{1}{2} r_0 k^2 + \cdots$$

a: scattering length

r_0: effective range

$$\sigma_{l=0} = 4\pi a^2 \quad (k \to 0)$$

$$\phi_0(r) \propto r \cdot \frac{\sin(kr + \delta_0(k))}{kr} \simeq r + a$$

Scattering problem

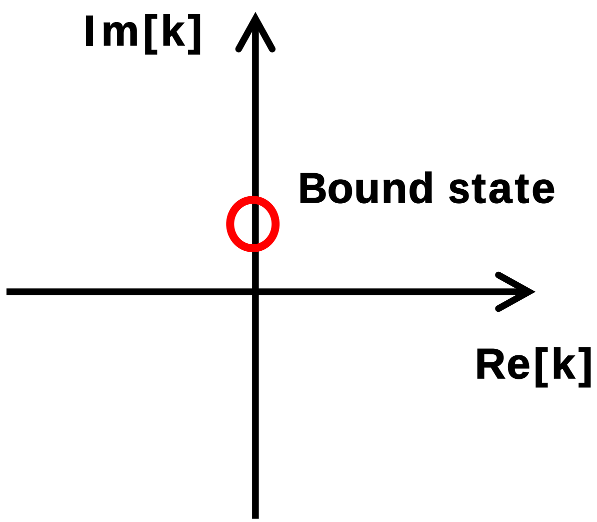
- Analytical structure

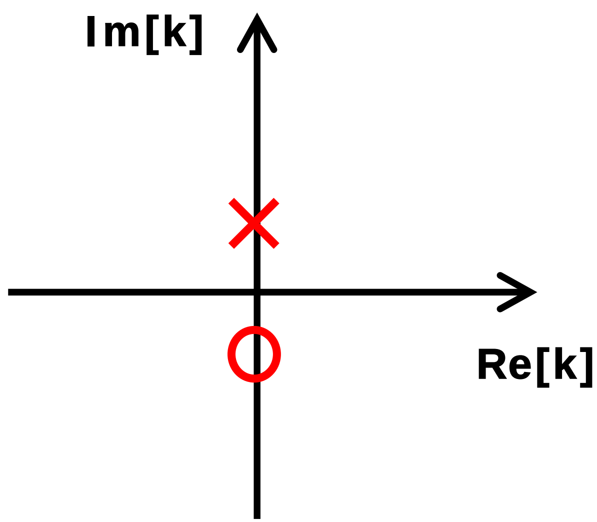
$$\phi_l(k,r) \propto \left[F_l(k)\hat{h}_l^-(kr) - F_l(-k)\hat{h}_l^+(kr)\right] \qquad (r \to \infty)$$

in-coming   out-going

Jost function: $F_l(k)$ S-matrix: $S_l(k) = \frac{F_l(-k)}{F_l(k)}$

X : pole
O : zero





If potential has IR-cut (V(r)=0 for r>R), F(k), S(k) are analytic for all k-plane

Scattering problem

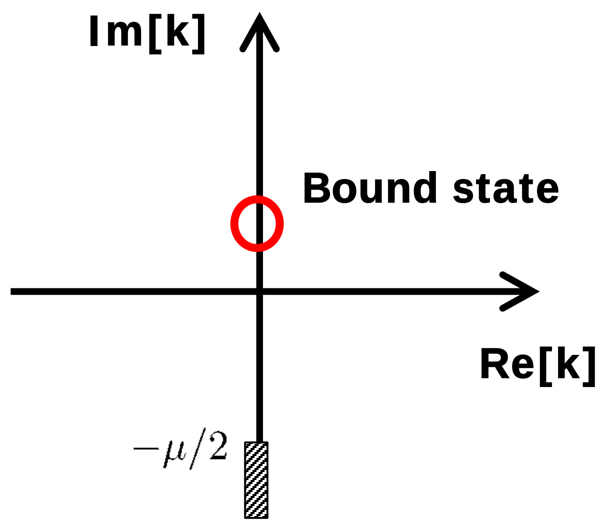
- Analytical structure

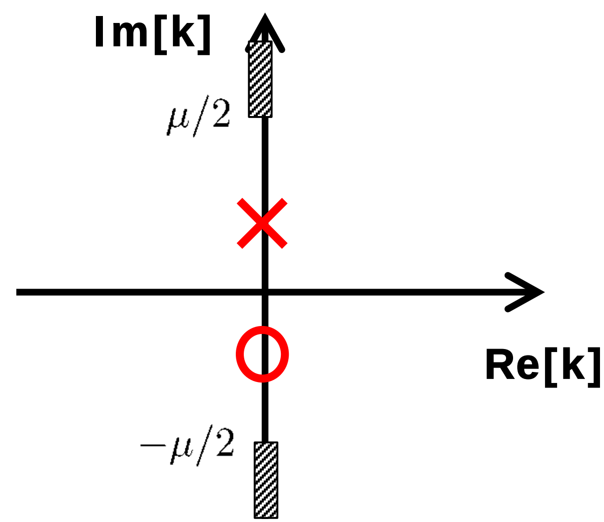
$$\phi_l(k,r) \propto \left[F_l(k)\hat{h}_l^-(kr) - F_l(-k)\hat{h}_l^+(kr) \right] \qquad (r \to \infty)$$

in-coming　　　　out-going

Jost function: $F_l(k)$　　　S-matrix: $S_l(k) = \frac{F_l(-k)}{F_l(k)}$

X : pole
O : zero





If potential is, e.g., Yukawa-type at large r,
there exists non-analytic region (left-hand cut)

Scattering problem

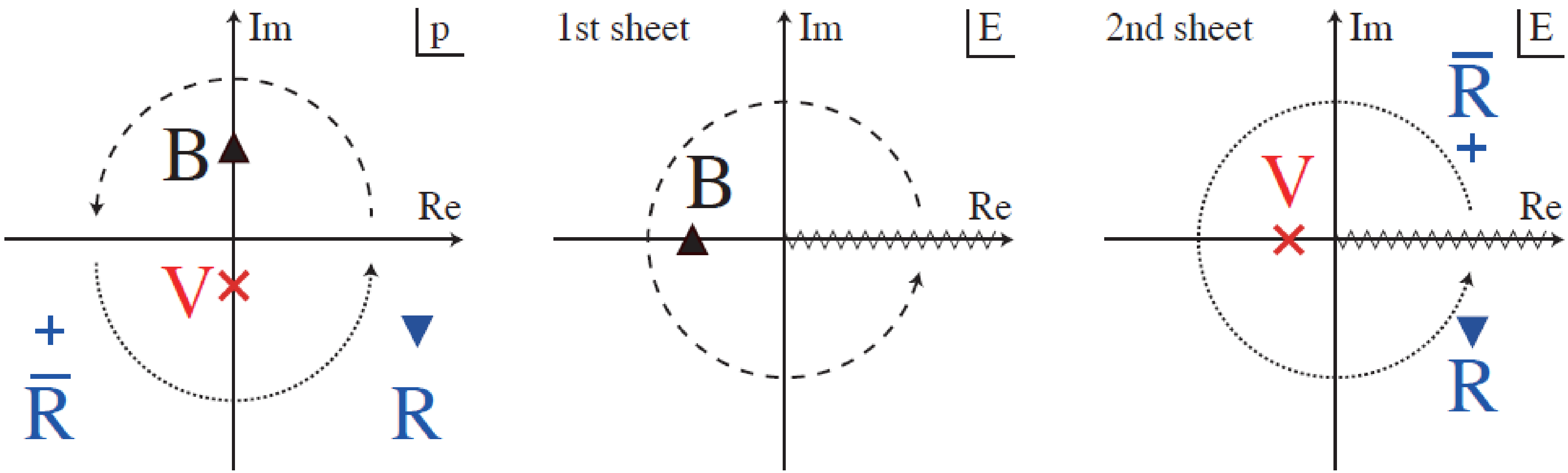


Fig. from Hyodo (原子核研究)

Poles in S-matrix for
Bound state (B), Virtual state (V), Resonance (R)

Phase shift in QFT

- Unitarity and the form of T-matrix

$$S^\dagger S = SS^\dagger = 1$$
$$S = 1 + iT,$$

$$\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle = i\sum_n \langle f|T^\dagger|n\rangle\langle n|T|i\rangle.$$

$$\langle p_a, p_b|T|k_a, k_b\rangle = (2\pi)^4\delta^4(p_a + p_b - k_a - k_b)T(p_a, p_b; k_a, k_b),$$

Elastic scattering in center of mass

$$T(p_a, p_b; k_a, k_b) = T(\boldsymbol{p}, \boldsymbol{k}).$$

$$\boldsymbol{p} = \boldsymbol{p}_a = -\boldsymbol{p}_b \qquad \boldsymbol{k} = \boldsymbol{k}_a = -\boldsymbol{k}_b \qquad |\boldsymbol{p}| = |\boldsymbol{k}|$$

Insert complete basis

$$T(\boldsymbol{p}, \boldsymbol{k}) - T^\dagger(\boldsymbol{k}, \boldsymbol{p}) = i\int \frac{d^3\boldsymbol{q}}{(2\pi)^2(2E_{\boldsymbol{q}})^2}\delta(2E_{\boldsymbol{p}} - 2E_{\boldsymbol{q}})T^\dagger(\boldsymbol{q}, \boldsymbol{p})T(\boldsymbol{q}, \boldsymbol{k})$$
$$= i\frac{|\boldsymbol{p}|}{32\pi^2 E_{\boldsymbol{p}}}\int d\Omega_{\boldsymbol{q}} T^\dagger(\boldsymbol{q}, \boldsymbol{p})T(\boldsymbol{q}, \boldsymbol{k}),$$

$$|\boldsymbol{q}| = |\boldsymbol{p}|$$

Phase shift in QFT

Using partial wave decomposition,

$$T(\boldsymbol{p},\boldsymbol{k}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} T_l(k) Y_{lm}(\Omega_{\boldsymbol{p}}) Y^*_{lm}(\Omega_{\boldsymbol{k}}), \qquad (p \equiv |\boldsymbol{p}| = k \equiv |\boldsymbol{k}|)$$

we obtain
$$T_l(p) - T_l^*(p) = i\frac{p}{8\pi E_p}|T_l(p)|^2,$$

→ T-matrix can be parametrized as

$$T_l(p) = \frac{16\pi E_p}{p} e^{i\delta_l(p)} \sin\delta_l(p), \qquad \delta_l(p) \in \mathbb{R}$$

Relation with cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} = \frac{1}{2E_{\boldsymbol{k}_a} 2E_{\boldsymbol{k}_b} |\boldsymbol{v}_a - \boldsymbol{v}_b|} \frac{|\boldsymbol{p}|}{16\pi^2 E_{cm}} |T(\boldsymbol{p};\boldsymbol{k})|^2$$
$$= \frac{|\boldsymbol{p}|}{|\boldsymbol{k}|} \cdot \frac{1}{64\pi^2 E_{cm}^2} |T(\boldsymbol{p};\boldsymbol{k})|^2$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} = |f(\Omega)|^2$$

$$f(\Omega) \equiv \sum_l (2l+1) \frac{e^{i\delta_l(p)}}{p} \sin\delta_l(p) P_l(\cos\theta)$$

⇒ $\delta_l(p)$: phase shift

- **<u>Outline</u>**
  - Introduction
  - Brief review of scattering theory
  - Scattering on the lattice
    - Luscher’s finite volume method
    - HAL QCD method
  - S/N problem
  - More on HAL QCD method
  - Reliability issue and NN controversy
  - Summary

Maiani-Testa's No-go theorem

Maiani-Testa, PLB245(1990)245

- Consider a correlation function in infinite V limit

$$G(t_1,t_2) = \langle \hat{\pi}_{\vec{q}}(t_1)\hat{\pi}_{-\vec{q}}(t_2)J(0)\rangle \qquad (t_1 \gg t_2 \gg 0)$$

$$\propto \sum_n \langle \pi, \vec{q}|\hat{\pi}_{-\vec{q}}(0)|n, \mathrm{out}\rangle e^{-(E_n t_2 - E_q(t_2-t_1))} F_n$$

$$F_n = \langle n, \mathrm{out}|J(0)|0\rangle \ (\text{form factor})$$

↑

$$\mathrm{disconn} \times \delta_{2n} + \mathrm{conn} \times M^*_{2n}(\vec{q}, -\vec{q}; n)/(q^2 - m_\pi^2 + i\epsilon)$$

↑

$\vec{q} = \vec{0}$ Physical scattering

$$G(t_1,t_2) \propto F_2(4m_\pi^2)\left[1 + a\sqrt{\frac{m_\pi}{\pi t_2}} + \cdots\right] \qquad t_2\Delta E \ll 1$$

↑

scattering length

$\vec{q} \neq \vec{0}$ "with a coefficient proportional to an off-shell amplitude, with no direct meaning in terms of observable quantities"

In the infinite V in Euclidean time, you can access only threshold

(Information of phase (complex-ness) is lost)

"No-go theorem"

See also Bruno-Hansen, JHEP06(2021)043

Interactions on the Lattice

- Luscher's finite volume method
  - Phase shift & B.E. from temporal correlation in finite V

M.Luscher, CMP104(1986)177
CMP105(1986)153
NPB354(1991)531

- HAL QCD method
  - "Potential" from spacial (& temporal) correlation in finite V
  - Phase shift & B.E. by solving Schrodinger eq in infinite V

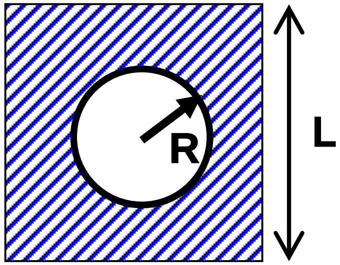
Ishii-Aoki-Hatsuda, PRL99(2007)022001, PTP123(2010)89
HAL QCD Coll., PTEP2012(2012)01A105
Aoki-Doi, Front.Phys.8(2020)307

Luscher's formula: Scatterings on the lattice

- Consider Schrodinger eq at asymptotic region

$$(\nabla^2 + k^2)\psi_k(\boldsymbol{r}) = mV_k(\boldsymbol{r})\psi_k(\boldsymbol{r})$$

$$V_k(\boldsymbol{r}) = 0 \text{ for } r > R$$

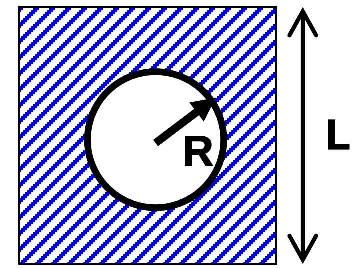


  - (periodic) Boundary Condition in finite V
    ➔ constraint on energies of the system
  - Energy E and phase shift (at E) are related

Luscher's formula: Scatterings on the lattice

– Example in two bosons in 1+1 dim QM

$$\left[-\frac{1}{2\mu}\frac{d^2}{dx^2}+V(|x|)\right]\psi(x)=E\psi(x)\equiv\frac{k^2}{2\mu}\psi(x)$$



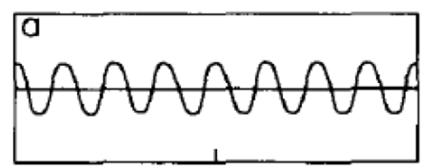
$$\psi(x)=\psi(-x)$$

In the case of free theory

Solution in infinite V
= in finite V

$$\psi(x,k)=A_k\cos(k|x|)$$

PBC poses a quantization condition ➔



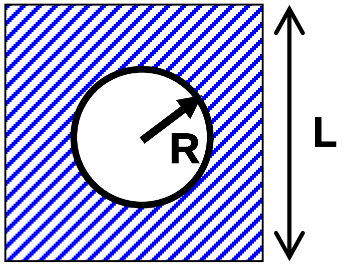
$$kL=0 \qquad (\mathrm{mod}\ 2\pi)$$

Luscher's formula: Scatterings on the lattice

– Example in two bosons in 1+1 dim QM

$$\left[-\frac{1}{2\mu}\frac{d^2}{dx^2} + V(|x|)\right]\psi(x) = E\psi(x) \equiv \frac{k^2}{2\mu}\psi(x)$$

$$\psi(x) = \psi(-x)$$



In the case of interacting theory

Consider finite V effect w/ PBC $$V_L(|x|) = \sum_{n=-\infty}^{\infty} V(|x+nL|)$$

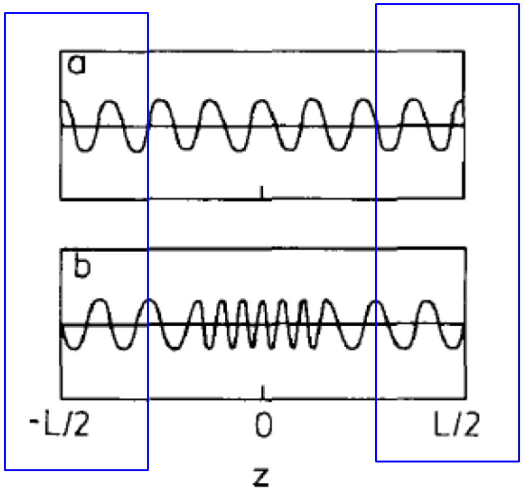
and obtain solution in infinite/finite V at asymptotic region (where potential=0)

Solution at asymptopic region $$\psi(x,k) = A_k \cos(k|x| + \delta(k))$$

PBC poses a quantization condition ➔

$$\begin{aligned} kL + 2\delta(k) &= 0 \quad (\mathrm{mod}\ 2\pi) \\ e^{2i\delta(k)} &= e^{-ikL} \end{aligned}$$

matching $\psi_{L=\infty}$ and ψ_L at asymptotic region

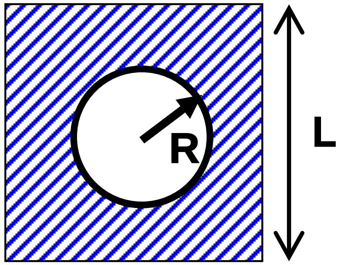


Luscher's formula: Scatterings on the lattice

– Example in two bosons in 3+1 dim QM (S-wave)

$$(\nabla^2 + k^2)\psi_k(\boldsymbol{r}) = mV_k(\boldsymbol{r})\psi_k(\boldsymbol{r})$$

$$V_k(\boldsymbol{r}) = 0 \text{ for } r > R$$



Solution in infinite V

generally complicated,
but solution at asymptotic region (r > R) is simple

$$\psi^k_\infty(r) = A_k \sin(kr + \underline{\delta(k)})/(kr)$$

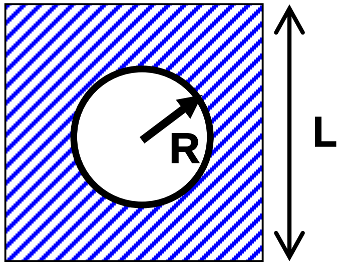
$$= \cos\delta(k)\cdot j_0(kr) + \sin\delta(k)\cdot n_0(kr)$$

Luscher's formula: Scatterings on the lattice

– Example in two bosons in 3+1 dim QM (S-wave)

$$(\nabla^2 + k^2)\psi_k(\boldsymbol{r}) = mV_k(\boldsymbol{r})\psi_k(\boldsymbol{r})$$

$$V_k(\boldsymbol{r}) = 0 \text{ for } r > R$$



Solution in finite V

PBC: $\psi(\vec{r} + \vec{n}L) = \psi(\vec{r})$

Assume that there exists asymptotic region within a finite box : R < L/2

Consider a solution at asymptotic region R < r < L/2

$$\begin{aligned}
\psi_L^k(r) &= \frac{1}{L^3}\sum_{\vec{n}\in\mathbb{Z}^3}\frac{e^{i\vec{p}_n\cdot\vec{x}}}{\vec{p}_n^2 - k^2}, \quad \vec{p}_n = 2\pi/L\cdot\vec{n} \\
&= g_{00}(k)\frac{1}{\sqrt{4\pi}}j_0(kr) + \frac{k}{4\pi}n_0(kr) + \cdots (j_{l\geq 1}(kr))
\end{aligned}$$

$$g_{00}(k) = \frac{\sqrt{4\pi}}{L^3}\sum_{\vec{n}\in\mathbb{Z}^3}\frac{1}{\vec{p}_n^2 - k^2}$$

Luscher's formula: Scatterings on the lattice

– Example in two bosons in 3+1 dim QM (S-wave)

$$(\nabla^2 + k^2)\psi_k(\boldsymbol{r}) = mV_k(\boldsymbol{r})\psi_k(\boldsymbol{r})$$

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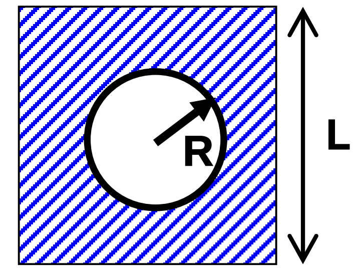
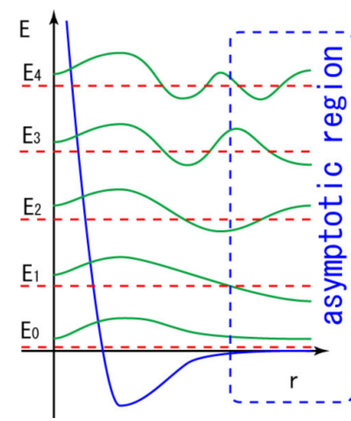


matching $\psi_{L=\infty}$ and ψ_L at asymptotic region

$$\psi^k_\infty(r) = \cos\delta(k)\cdot j_0(kr) + \sin\delta(k)\cdot n_0(kr)$$

$$\psi^k_L(r) \;=\; g_{00}(k)\frac{1}{\sqrt{4\pi}}j_0(kr) + \frac{k}{4\pi}n_0(kr) + \cdots (j_{l\geq 1}(kr))$$



⇩

Luscher's formula

$$k\cot\delta(\mathbf{k}) = \frac{2}{\sqrt{\pi}L}Z_{00}(1;q^2), \quad q = \frac{kL}{2\pi}$$

$$E = k^2/m$$

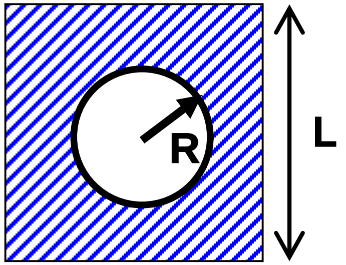
Luscher's zeta function $$Z_{00}(s;q^2) = \frac{1}{\sqrt{4\pi}}\sum_{\boldsymbol{n}\in\boldsymbol{Z}^3}\frac{1}{(\boldsymbol{n}^2 - q^2)^s}$$

Luscher's formula: Scatterings on the lattice

– Example in two bosons in 3+1 dim QM (S-wave)

$$(\nabla^2 + k^2)\psi_k(\boldsymbol{r}) = mV_k(\boldsymbol{r})\psi_k(\boldsymbol{r})$$

$$V_k(\boldsymbol{r}) = 0 \text{ for } r > R$$



A simpler formula suitable for intuitive understanding

Large V expansion

$$\Delta E = E - 2m = -\frac{4\pi \mathbf{a}}{mL^3}\left[1 + c_1\frac{a}{L} + c_2\left(\frac{a}{L}\right)^2 + \mathcal{O}\left(\frac{1}{L^3}\right)\right]$$

a: scattering length

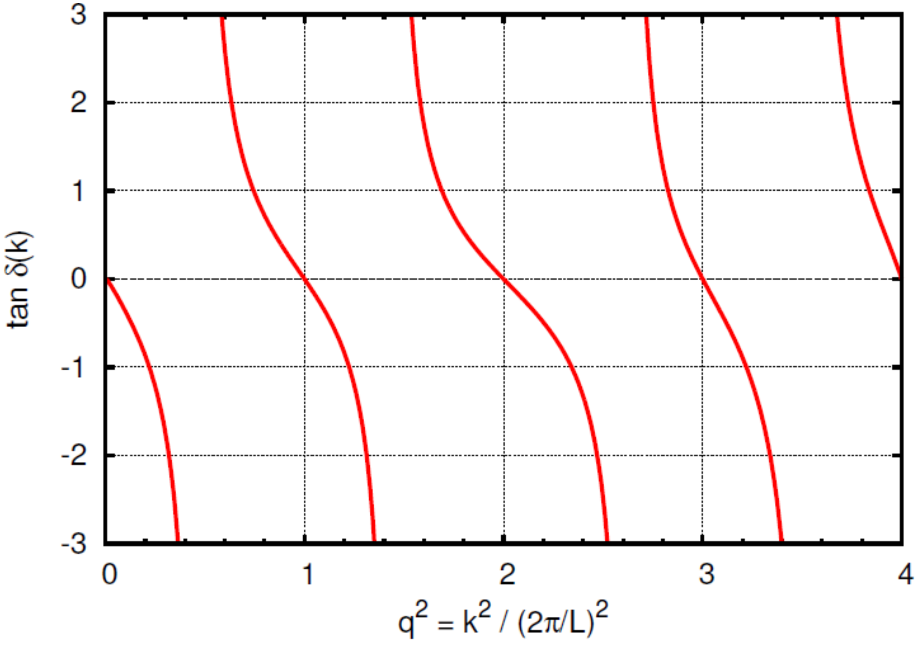
c_1, c_2: geometric constants

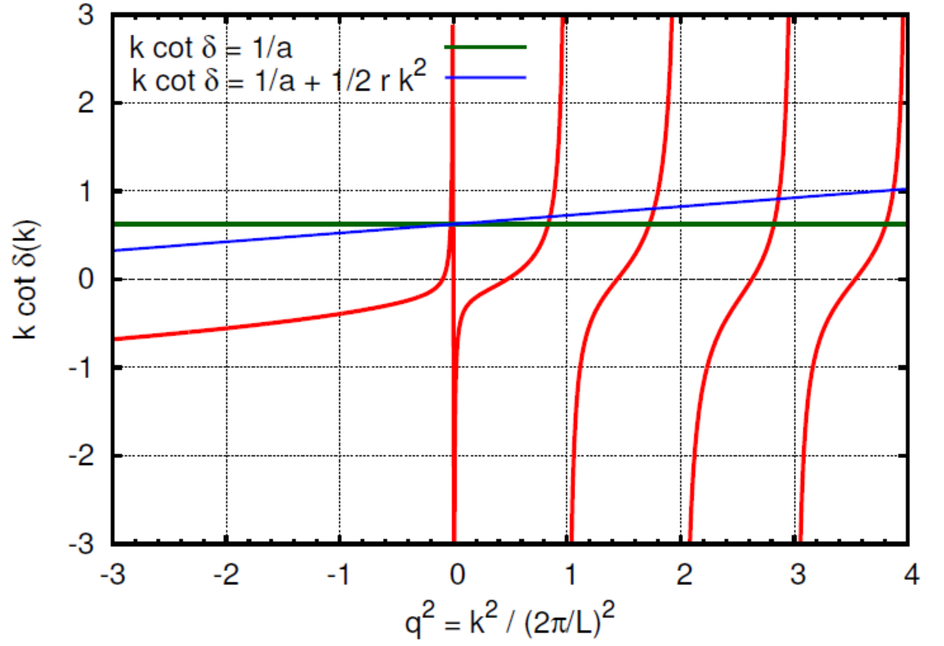
$$c_1 = Z_{00}(1;0)/\pi = -2.837$$
$$c_2 = [Z_{00}(1;0)^2 - Z_{00}(2;0)]/\pi^2 = 6.375$$

As intuitive derivation,
one can obtain LO formula from Born approximation

Interactions from Luscher's formula

(fixed L)





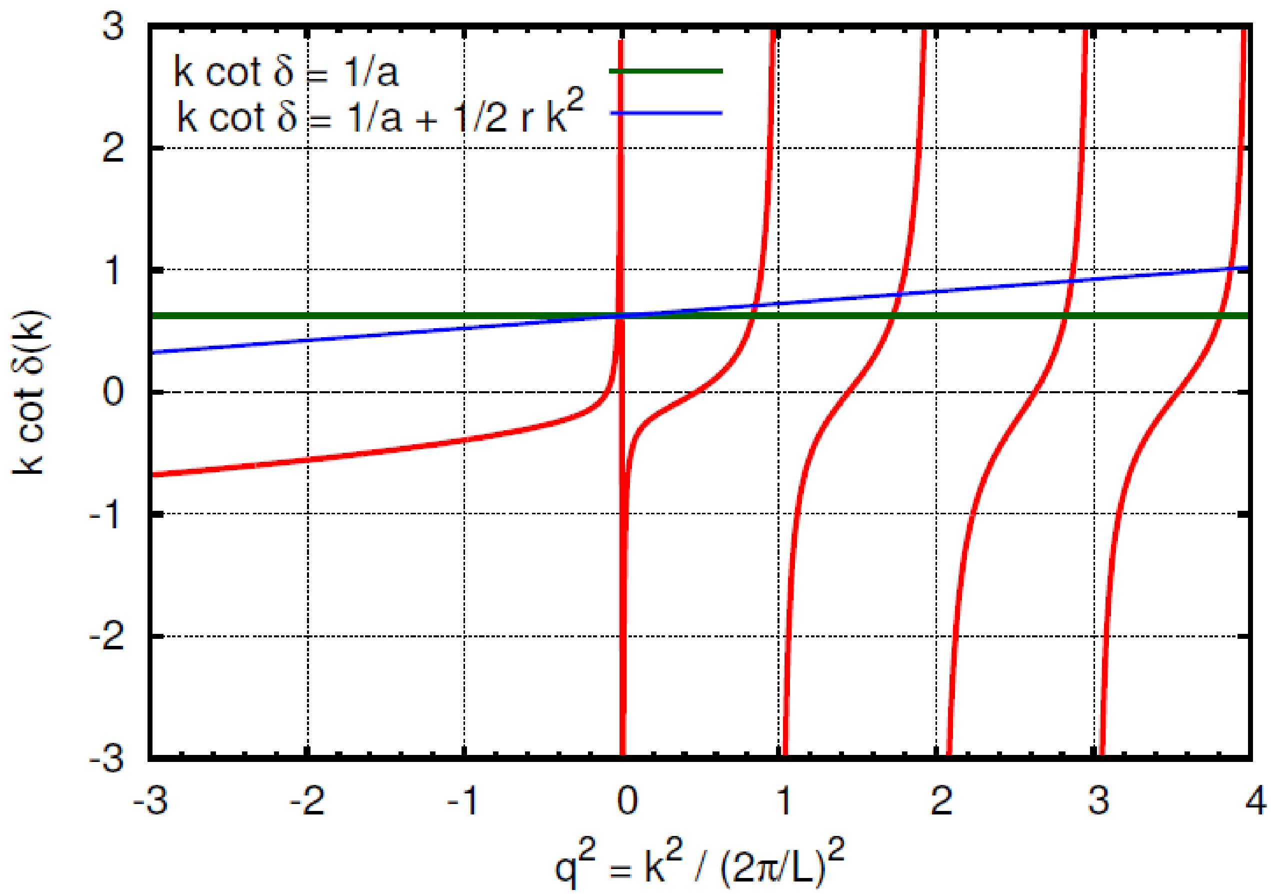
Luscher's formula (red lines) gives
quantization condition (kinematical constraint) on finite V

(high-end version of k=(2pi/L)n in free theory)

Quantization condition itself does not
have any information on dynamics

Interactions from Luscher's formula

(fixed L)



Intersections of

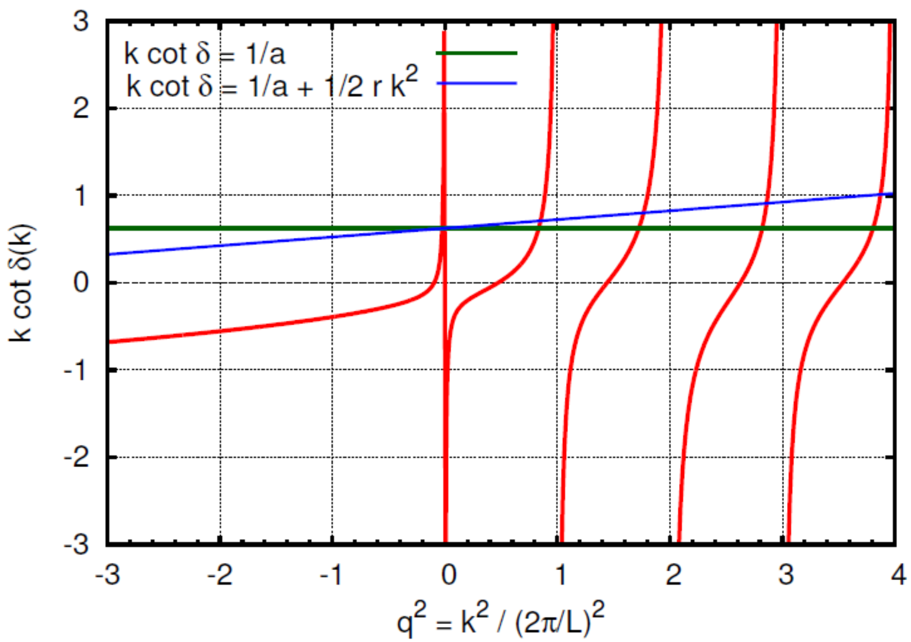
(1) Luscher's quantization condition

(2) Interaction (e.g., shown by ERE)

are realized on a lattice

$$\text{ERE:} \quad k \cot \delta_E = \frac{1}{\mathbf{a}} + \frac{1}{2}\,\mathbf{r}\,k^2 + \cdots$$

How to obtain phase shift in practice? (fixed L)



$$\text{ERE:}\quad k\cot\delta_E = \frac{1}{\mathbf{a}} + \frac{1}{2}\,\mathbf{r}\,k^2 + \cdots$$

- Calculate the energy spectrum of 2-hadron on finite V lattice
  - Temporal correlation in Euclidean time ➔ energy

$$G(t) = \langle 0|\mathcal{O}(t)\overline{\mathcal{O}}(0)|0|\rangle = \sum_n A_n e^{-E_n t} \to A_0 e^{-E_0 t} \quad (t\to\infty)$$

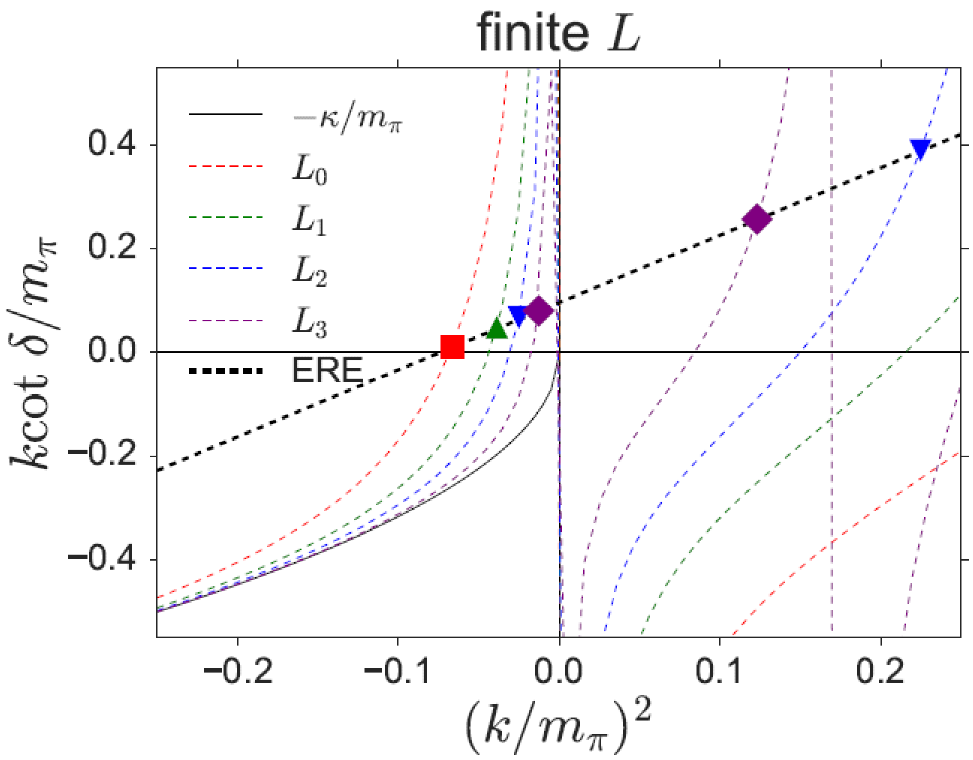
- Convert the energy shift to phase shift by Luscher’s formula

$$E \to \Delta E = E - 2m \text{ (effect of int.)} \to k \text{ (asymp. mom.)} \to \delta_E$$

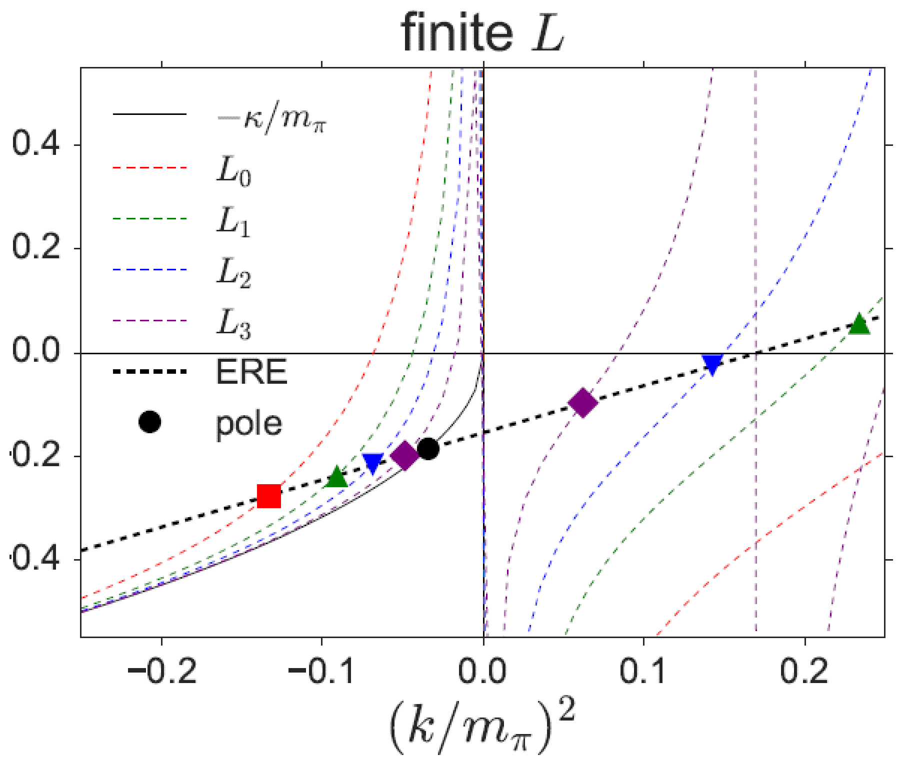
$k \cot \delta$ vs k^2 plot

(various L)

Unbound example



Bound example



$$k \cot \delta(k) = \frac{1}{a} + \frac{1}{2} r k^2 + \cdots$$

Unbound : 1/a > 0

Bound : 1/a < 0

(non-rela) QM → QFT

- Essentially the same formula can be used

- QM wave func. → Nambu-Bethe-Salpeter (NBS) wave func.
  - Interaction kernel (or so-called "potential") can be defined (see later)
  - The interaction does not become exactly zero at large r
    - Systematic error of $\sim \exp[-m_h(L/2)]$
    - N.B. To use Luscher's method, one has to check whether the volume is sufficiently large compared to the interaction range

- [Energy vs. asymptotic momentum] becomes relativistic

$$\Delta E = E - 2m : \quad k^2/m \quad \rightarrow \quad 2\sqrt{m^2 + k^2} - 2m$$

- **Outline**
  - ~~Introduction~~
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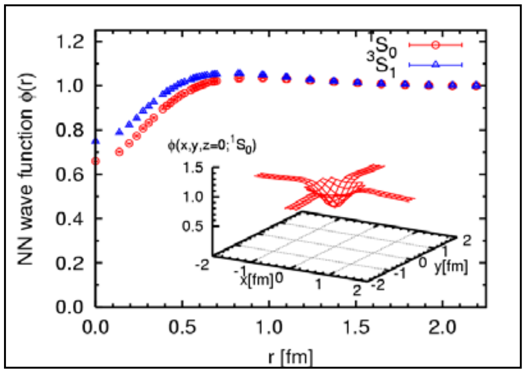
How to define/calc Hadron interactions?

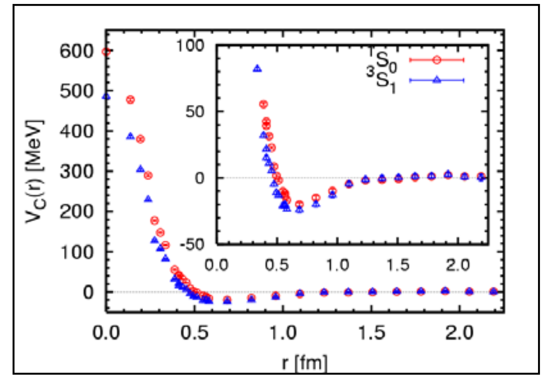
HAL QCD method

Lattice QCD

NBS wave func.



Lat Hadron Force



$$\psi_{\rm NBS}(\vec{r}) = \langle 0|H_1(\vec{r})H_2(\vec{0})|H_1(\vec{k})H_2(-\vec{k}), in\rangle$$
$$\simeq A_k \sin(kr - l\pi/2 + \delta_l(k))/(kr)$$

(at asymptotic region)

$$(k^2/m_N - H_0)\,\psi_{\rm NBS}(\vec{r}) = \int d\vec{r}'\, U(\vec{r},\vec{r}')\psi_{\rm NBS}(\vec{r}')$$

(Schrodinger eq.)

Potential

Faithful to phase shifts

Hadrons to Atomic nuclei HAL from Lattice QCD

Ishii-Aoki-Hatsuda PRL99(2007)022001

NBS wave func and its asymptotic behavior

Nambu-Bethe-Salpeter (NBS) wave func.

Y. Nambu, PTP5(1950)614
Hayashi-Munakata, 素粒子論研究3(1951)89, PTP7(1952)451
Salpeter-Bethe, Phys.Rev.84(1951)1232
(See also Salpeter, 0811.1050 for some history)

$$\psi(\boldsymbol{r}, t_a, t_b) = \langle 0| \mathrm{T} \left[\pi_a(\boldsymbol{r}+\boldsymbol{x}, t_a) \pi_b(\boldsymbol{x}, t_b) \right] |\boldsymbol{k}_a, a, \boldsymbol{k}_b, b; \mathrm{in}\rangle.$$

(Example for two distinguishable spinless boson system)

Consider equal time NBS w.f. in the center of mass

Insert a complete set

$$1 = \sum_c \int \frac{d^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} |\boldsymbol{p}, c; \mathrm{out}\rangle\langle \boldsymbol{p}, c; \mathrm{out}| + \sum_X \frac{1}{2E_X} |X; \mathrm{out}\rangle\langle X; \mathrm{out}|,$$

Inelastic states: neglected for simplicity

$$\psi(\boldsymbol{r}, t) = \sqrt{Z_\pi} \int \frac{d^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} e^{-iE_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot(\boldsymbol{r}+\boldsymbol{x})} \underline{\langle \boldsymbol{p}, a; \mathrm{out}|\pi_b(\boldsymbol{x}, t)|\boldsymbol{k}_a, a, \boldsymbol{k}_b, b; \mathrm{in}\rangle}.$$

By LSZ reduction formula ...

NBS wave func and its asymptotic behavior

By LSZ reduction formula ...

$$\langle \boldsymbol{p}, a; \mathrm{out} | \pi_b(\boldsymbol{x}, t) | \boldsymbol{k}_a, a, \boldsymbol{k}_b, b; \mathrm{in} \rangle$$

$$= \sqrt{Z_\pi}(2\pi)^3 2E_{\boldsymbol{p}} \delta^3(\boldsymbol{p}-\boldsymbol{k}) e^{-ik_b x}$$
$$+ \sqrt{Z_\pi} \int \frac{d^4 q}{(2\pi)^4} e^{-iqx}\, i(-p^2+m^2)\cdot i(-k_a^2+m^2)\cdot i(-k_b^2+m^2) \langle 0 | \mathrm{T} \left[\pi_a(p)\pi_b(q)\pi_a^\dagger(k_a)\pi_b^\dagger(k_b) \right] |0\rangle_{\mathrm{conn}}$$

$$= \sqrt{Z_\pi}(2\pi)^3 2E_{\boldsymbol{p}} \delta^3(\boldsymbol{p}-\boldsymbol{k}) e^{-ik_b x} + \sqrt{Z_\pi} \frac{e^{-iqx}}{m^2-q^2-i\epsilon} T(p,q;k_a,k_b) \Big|_{q=k_a+k_b-p}$$

(disconnected) (connected)

half on-shell T-matrix (p, k_a, k_b: on-shell, q: off-shell)

⇨

$$\psi(\boldsymbol{r},t) = Z_\pi e^{-2iE_{\boldsymbol{k}}t} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + Z_\pi e^{-2iE_{\boldsymbol{k}}t} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon} H(\boldsymbol{p},\boldsymbol{k}) e^{i\boldsymbol{p}\cdot\boldsymbol{r}}$$
$$\equiv \psi_{\boldsymbol{k}}(\boldsymbol{r}) \cdot Z_\pi e^{-2iE_{\boldsymbol{k}}t}$$

$$H(\boldsymbol{p},\boldsymbol{k}) \equiv \frac{E_{\boldsymbol{p}}+E_{\boldsymbol{k}}}{8E_{\boldsymbol{p}}E_{\boldsymbol{k}}} T(\boldsymbol{p},\boldsymbol{k}).$$

(integral dominated by on-shell contribution)

NBS wave func and its asymptotic behavior

Using partial wave decomposition,

$$H(p,k) \;=\; 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} H_l(p,k) Y_{lm}(\Omega_{\boldsymbol{p}}) Y^*_{lm}(\Omega_{\boldsymbol{k}}) \qquad \text{etc.}$$

We obtain

$$\psi_l(r,k) = j_l(kr) + \int_0^\infty \frac{p^2 dp}{2\pi^2} \frac{1}{p^2 - k^2 - i\epsilon} H_l(p,k) j_l(pr),$$

$$\simeq \; j_l(kr) + \frac{k}{4\pi} H_l(k,k) \{ i \cdot j_l(kr) - n_l(kr) \} \qquad (r \to \infty)$$

$$\propto \; \frac{\sin(kr - l\pi/2 + \delta_l(k))}{kr}$$

$$H_l(k,k) = \frac{4\pi}{k} e^{i\delta_l(k)} \sin \delta_l(k)$$

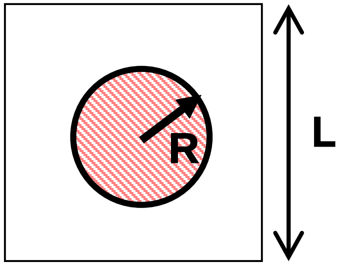
Information of phase shift is encoded in asymptotic region
with the same functional form as QM

C.-J.Lin et al., NPB619(2001)467 Ishizuka, Pos LAT2009 (2009) 119 Aoki-Hatsuda-Ishii PTP123(2010)89

"Potential" as a representation of S-matrix

- Consider the wave function at "interacting region"

$$(\nabla^2 + k^2)\psi(\boldsymbol{r}) = m \int d\boldsymbol{r}' U(\boldsymbol{r}, \boldsymbol{r}')\psi(\boldsymbol{r}'), \quad r < R$$



Probe interactions in "direct" way

- U(r,r'): faithful to the phase shift by construction
  - U(r,r'): NOT an observable, but well defined
    - Potential is NOT unique, but different potentials are phase-shift equivalent potentials
    - Choosing the pot. ←→ choosing the "scheme" (sink op.)
  - U(r,r'): E-independent, while non-local in general

Proof of Existence of E-independent potential

$$V_W(\boldsymbol{r})\psi_W(\boldsymbol{r}) = (E_W - H_0)\psi_W(\boldsymbol{r})$$ **[START] local but E-dep pot. (L^3xL^3 dof)**

- We consider the linear-indep wave functions and define

$$\mathcal{N}_{W_1 W_2} = \int d\boldsymbol{r}\,\overline{\psi_{W_1}(\boldsymbol{r})}\psi_{W_2}(\boldsymbol{r})$$

- We define the non-local potential

$$U(\boldsymbol{r},\boldsymbol{r}') = \sum_{W_1,W_2}^{W_{\rm th}} (E_{W_1} - H_0)\psi_{W_1}(\boldsymbol{r})\mathcal{N}^{-1}_{W_1 W_2}\overline{\psi_{W_2}(\boldsymbol{r}')}$$

- The above potential trivially satisfy Schrodinger eq.

$$\begin{aligned}\int d\boldsymbol{r}' U(\boldsymbol{r},\boldsymbol{r}')\psi_W(\boldsymbol{r}') &= \int d\boldsymbol{r}' \sum_{W_1,W_2}^{W_{\rm th}} (E_{W_1} - H_0)\psi_{W_1}(\boldsymbol{r})\mathcal{N}^{-1}_{W_1 W_2}\overline{\psi_{W_2}(\boldsymbol{r}')}\psi_W(\boldsymbol{r}') \\ &= \sum_{W_1,W_2}^{W_{\rm th}} (E_{W_1} - H_0)\psi_{W_1}(\boldsymbol{r})\mathcal{N}^{-1}_{W_1 W_2}\mathcal{N}_{W_2 W} \\ &= (E_W - H_0)\psi_W(\boldsymbol{r})\end{aligned}$$

Intuitive understanding

[GOAL] non-local but E-indep pot. (L^3xL^3 dof)

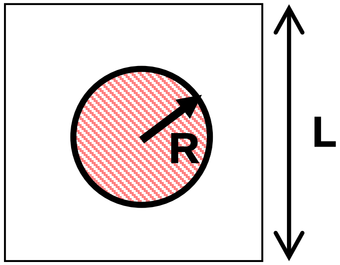
c.f. Krolikowski-Rzewuski, Nuovo Cimento, 4, 1212 (1956)

“Potential” as a representation of S-matrix

- Consider the wave function at “interacting region”

$$(\nabla^2 + k^2)\psi(\boldsymbol{r}) = m\int d\boldsymbol{r}' U(\boldsymbol{r},\boldsymbol{r}')\psi(\boldsymbol{r}'), \quad r < R$$

Probe interactions in “direct” way



– U(r,r’): faithful to the phase shift by construction
  - U(r,r’): NOT an observable, but well defined
    – Potential is NOT unique, but different potentials are phase-shift equivalent potentials
    – Choosing the pot. ←→ choosing the “scheme” (sink op.)
  - U(r,r’): E-independent, while non-local in general
    – Non-locality → derivative expansion

Okubo-Marshak(1958)

$$U(\vec{r},\vec{r}') = V_c(r) + S_{12}V_T(r) + \vec{L}\cdot\vec{S}V_{LS}(r) + \mathcal{O}(\nabla^2)$$

LO LO NLO NNLO

Most general form of the NN potential

$$V(\vec{r}_1, \vec{r}_2, \vec{\nabla}_1, \vec{\nabla}_2; \vec{\sigma}_1, \vec{\sigma}_2)$$

Okubo-Marshak(1958)

- Imposed condition
  - Hermiticity $\quad V^\dagger = V$
  - Energy/Momentum conservation $\quad V(\vec{r}, \vec{\nabla}_1, \vec{\nabla}_2; \vec{\sigma}_1, \vec{\sigma}_2), \quad \vec{r} = \vec{r}_1 - \vec{r}_2$
  - Galilei invariance $\quad V(\vec{r}, \vec{\nabla}_r; \vec{\sigma}_1, \vec{\sigma}_2)$
  - Rotational invariance $\quad V:$ scalar
  - Parity conservation $\quad V(\vec{r}, \vec{\nabla}_r; \vec{\sigma}_1, \vec{\sigma}_2) = V(-\vec{r}, -\vec{\nabla}_r; \vec{\sigma}_1, \vec{\sigma}_2)$
  - Time reversal $\quad V(\vec{r}, \vec{\nabla}_r; \vec{\sigma}_1, \vec{\sigma}_2) = V(\vec{r}, -\vec{\nabla}_r; -\vec{\sigma}_1, -\vec{\sigma}_2)$
  - Pauli principle $\quad V(\vec{r}, \vec{\nabla}_r; \vec{\sigma}_1, \vec{\sigma}_2) = V(-\vec{r}, -\vec{\nabla}_r; \vec{\sigma}_2, \vec{\sigma}_1)$
- LO

$$1 \text{ (unit operator)}, \quad (\vec{\sigma}_1 \cdot \vec{\sigma}_2), \quad S_{12} \equiv 3(\vec{\sigma}_1 \cdot \hat{r})(\vec{\sigma}_2 \cdot \hat{r}) - (\vec{\sigma}_1 \cdot \vec{\sigma}_2)$$

- NLO

$$(\vec{L} \cdot \vec{S})$$

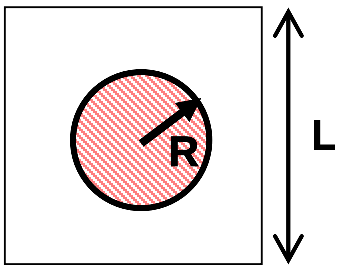
Independent DoF in Isospin space:

$$1 \text{ (unit op.)}, \quad (\vec{\tau}_1 \cdot \vec{\tau}_2)$$

"Potential" as a representation of S-matrix

- Consider the wave function at "interacting region"

$$(\nabla^2 + k^2)\psi(\boldsymbol{r}) = m\int d\boldsymbol{r'} U(\boldsymbol{r},\boldsymbol{r'})\psi(\boldsymbol{r'}), \quad r < R$$



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  - U(r,r'): E-independent, while non-local in general
    – Non-locality → derivative expansion Okubo-Marshak(1958)

– Phase shifts at all E (below inelastic threshold) obtained by solving Scrodinger eq in infinite V

Elementary particle vs composite particle?

LSZ reduction formula : elementary particle

Nishijima-Haag-Zimmermann (NHZ) reduction formula : composite particle

K. Nishijima, Phys.Rev.111(1958)995, 133(1964)B204
R. Haag, Phys.Rev 112(1958)669)
W.Zimmerman, Nuovo Cim X10 (1958) 597
(See also 西島和彦, 日本物理学会誌 47(1992)859 for history)

The same reduction formula can be used as far as
"almost-local field" B(x) is used for composite particle

(1) space-time translation like an elementary field

$$B(x+a) = e^{-iPa} B(x) e^{+iPa}$$

(2) B(x) may be expressed as (the limit of) a polynomial in the basic field A(x):

$$B(x) = h^{(0)} + \int h^{(1)}(x-y) A(y) dy + \int h^{(2)}(x-y_1, x-y_2) A(y_1) A(y_2) dy_1 dy_2 + \cdots,$$

h(r) : sufficiently smooth and decrease rapidly (stronger than any power for large r)

Example for nucleon op. $N(x) = \epsilon_{abc}(q_a^T(x) C\gamma_5 q_b(x)) q_c(x)$

Coupled Channel

(beyond inelastic threshold)

- Asymptotic behavior of NBS wave func

Ex.) A + B ←→ C + D

$$\psi_{AB}(\boldsymbol{r},\boldsymbol{k}) = 1/\sqrt{Z_A Z_B}\cdot\langle 0|\phi_A(\boldsymbol{x}+\boldsymbol{r})\phi_B(\boldsymbol{x})|W\rangle$$
$$\psi_{CD}(\boldsymbol{r},\boldsymbol{q}) = 1/\sqrt{Z_C Z_D}\cdot\langle 0|\phi_C(\boldsymbol{x}+\boldsymbol{r})\phi_D(\boldsymbol{x})|W\rangle$$

$$|W\rangle = c_{AB}|AB,W\rangle_{\rm in} + c_{CD}|CD,W\rangle_{\rm in}$$

$$W = \sqrt{m_A^2+k^2}+\sqrt{m_B^2+k^2} = \sqrt{m_C^2+q^2}+\sqrt{m_D^2+q^2}$$

$$\psi^l_{AB}(r,k) = c_{AB}\left[j_l(kr)+\frac{k}{4\pi}H_l^{AB,AB}(k,k)(n_l(kr)+ij_l(kr))\right] + c_{CD}\left[\frac{k}{4\pi}H_l^{AB,CD}(k,q)(n_l(kr)+ij_l(kr))\right]$$
$$\psi^l_{CD}(r,q) = c_{CD}\left[j_l(qr)+\frac{q}{4\pi}H_l^{CD,CD}(q,q)(n_l(qr)+ij_l(qr))\right] + c_{AB}\left[\frac{q}{4\pi}H_l^{CD,AB}(q,k)(n_l(qr)+ij_l(qr))\right]$$

where

$$H^{AB,AB(CD)}(\boldsymbol{k};\boldsymbol{k}(\boldsymbol{q})) = \frac{1}{2W}T^{AB,AB(CD)}(k_A,k_B;k_A,k_B(q_C,q_D))$$
$$H^{CD,AB(CD)}(\boldsymbol{q};\boldsymbol{k}(\boldsymbol{q})) = \frac{1}{2W}T^{CD,AB(CD)}(q_C,q_D;k_A,k_B(q_C,q_D))$$

S.Aoki et al. (HAL Coll.), Proc. Jpn. Acad. Ser. B87(2011)

Coupled Channel

- T-matrix parametrization by unitarity

$$T_l^{I,J}(W) = \frac{8\pi W}{p_I}\left[O(W)\begin{pmatrix} e^{i\delta_l^1(W)}\sin\delta_l^1(W) & 0 \\ 0 & e^{i\delta_l^2(W)}\sin\delta_l^2(W)\end{pmatrix}O^{-1}(W)\right]^{I,J}$$

$$O(W) = \begin{pmatrix} \cos\theta(W) & -\sin\theta(W) \\ \sin\theta(W) & \cos\theta(W)\end{pmatrix} \qquad (p_1 = k, p_2 = q)$$

- Asymptotic behavior ←→ $\delta_l^1(W), \delta_l^2(W), \theta(W)$

$$\begin{pmatrix}\psi_{AB}(r,k)\\ \psi_{CD}(r,q)\end{pmatrix} \simeq \begin{pmatrix} j_l(kr) & 0 \\ 0 & j_l(qr)\end{pmatrix}\begin{pmatrix} c_{AB} \\ c_{CD}\end{pmatrix}$$
$$+ \begin{pmatrix} n_l(kr)+ij_l(kr) & 0 \\ 0 & n_l(qr)+ij_l(qr)\end{pmatrix} O(W)\begin{pmatrix} e^{i\delta_l^1(W)}\sin\delta_l^1(W) & 0 \\ 0 & e^{i\delta_l^2(W)}\sin\delta_l^2(W)\end{pmatrix}O^{-1}(W)\begin{pmatrix} c_{AB} \\ c_{CD}\end{pmatrix}$$

- → Coupled channel potentials can be defined

$$(E_{k_i}^{AB} - H_0^{AB})\psi_{AB}(\boldsymbol{r},k_i) = \int d\boldsymbol{r}' U_{AB,AB}(\boldsymbol{r},\boldsymbol{r}')\psi_{AB}(\boldsymbol{r}',k_i) + \int d\boldsymbol{r}' U_{AB,CD}(\boldsymbol{r},\boldsymbol{r}')\psi_{CD}(\boldsymbol{r}',q_i)$$
$$(E_{q_i}^{CD} - H_0^{CD})\psi_{CD}(\boldsymbol{r},q_i) = \int d\boldsymbol{r}' U_{CD,AB}(\boldsymbol{r},\boldsymbol{r}')\psi_{AB}(\boldsymbol{r}',k_i) + \int d\boldsymbol{r}' U_{CD,CD}(\boldsymbol{r},\boldsymbol{r}')\psi_{CD}(\boldsymbol{r}',q_i)$$

Coupled Channel

S.Aoki et al. (HAL Coll.), PRD87(2013)034512

- Proof of Existence of E-indep potential

W_{th}^2

$\Delta_1 = [W_{th}^1, W_{th}^2)$

$W_{th}^1 = m_C + m_D$

$\Delta_0 = [W_{th}^0, W_{th}^1)$

$W_{th}^0 = m_A + m_B$

NBS wave func.

$$\begin{aligned}
\psi_{AB,AB}(\boldsymbol{r}) &= 1/\sqrt{Z_A Z_B}\cdot \langle 0|\phi_A(\boldsymbol{x}+\boldsymbol{r})\phi_B(\boldsymbol{x})|AB,W\rangle_{\text{in}} \\
\psi_{AB,CD}(\boldsymbol{r}) &= 1/\sqrt{Z_A Z_B}\cdot \langle 0|\phi_A(\boldsymbol{x}+\boldsymbol{r})\phi_B(\boldsymbol{x})|CD,W\rangle_{\text{in}} \\
\psi_{CD,AB}(\boldsymbol{r}) &= 1/\sqrt{Z_C Z_D}\cdot \langle 0|\phi_C(\boldsymbol{x}+\boldsymbol{r})\phi_D(\boldsymbol{x})|AB,W\rangle_{\text{in}} \\
\psi_{CD,CD}(\boldsymbol{r}) &= 1/\sqrt{Z_C Z_D}\cdot \langle 0|\phi_C(\boldsymbol{x}+\boldsymbol{r})\phi_D(\boldsymbol{x})|CD,W\rangle_{\text{in}}
\end{aligned}$$

Vector of NBS

$$\Psi_{XY} = (\psi_{AB,XY}, \psi_{CD,XY})^T, \quad (XY = AB \text{ or } CD) \quad \text{for } W \in \Delta_1$$

$$\Psi_{XY} = (\psi_{AB,XY}, \psi_{CD,XY})^T, \quad (XY = AB \text{ only}) \quad \text{for } W \in \Delta_0$$

Norm

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}_{AB,AB}(\Delta_0,\Delta_0) & \mathcal{N}_{AB,AB}(\Delta_0,\Delta_1) & \mathcal{N}_{AB,CD}(\Delta_0,\Delta_1) \\ \mathcal{N}_{AB,AB}(\Delta_1,\Delta_0) & \mathcal{N}_{AB,AB}(\Delta_1,\Delta_1) & \mathcal{N}_{AB,CD}(\Delta_1,\Delta_1) \\ \mathcal{N}_{CD,AB}(\Delta_1,\Delta_0) & \mathcal{N}_{CD,AB}(\Delta_1,\Delta_1) & \mathcal{N}_{CD,CD}(\Delta_1,\Delta_1) \end{pmatrix} \qquad \mathcal{N}_{XY,X'Y'} = (\Psi_{XY}, \Psi_{X'Y'})$$

E-indep pot.

$$U = \sum (E - H_0)\Psi\mathcal{N}^{-1}\overline{\Psi}$$

- Generalization to A+B ←→ C+D+E, etc. possible
  - 2-body relativistic, otherwise non-rela approx. necessary

Extension to multi-particle systems (n>=3)

- Unitarity of S-matrix

S.Aoki et al. (HAL Coll.), PRD88(2013)014036
Gongyo-Aoki PTEP2018(2018)093B03

$$T^\dagger - T = i T^\dagger T$$

Hyper-spherical func in D=3(n-1) dim

$$T([\boldsymbol{q}^A]_n, [\boldsymbol{q}^B]_n) = \sum_{[L],[K]} T_{[L][K]}(Q_A, Q_B) Y_{[L]}(\Omega_{Q_A}) \overline{Y_{[K]}(\Omega_{Q_B})}$$

$$[L] = L, M_1, M_2, \ldots$$

diagonalization

$$T_{[L][K]}(Q,Q) = \sum_{[N]} U_{[L][N]}(Q) T_{[N]}(Q) U^\dagger_{[N][K]}(Q)$$

$$(Q = Q_A = Q_B)$$

$$T_{[L]}(Q) = -\frac{2n^{3/2}}{mQ^{3n-5}} \boxed{e^{i\delta_{[L]}(Q)} \sin\delta_{[L]}(Q)}$$

c.f. R.B. Newton (1974) for n = 3

Similar formula to 2-body system
(w/ diagonalization matrix U which includes dynamics)

(non-rela approx.)

Extension to multi-particle systems (n>=3)

- NBS wave function

S.Aoki et al. (HAL Coll.), PRD88(2013)014036

Gongyo-Aoki PTEP2018(2018)093B03

$$\psi_\alpha([\boldsymbol{x}]) = {}_{\text{in}}\langle 0|\phi([\boldsymbol{x}])|\alpha\rangle_{\text{in}} = {}_{\text{in}}\langle 0|N(\vec{x}_1)N(\vec{x}_2)\cdots N(\vec{x}_n)|\alpha\rangle_{\text{in}}$$

Lippmann-Schwinger eq.

$$\psi_\alpha([\boldsymbol{x}]) = {}_{\text{in}}\langle 0|\phi([\boldsymbol{x}])|\alpha\rangle_0 + \int d\beta \frac{{}_{\text{in}}\langle 0|\phi([\boldsymbol{x}])|\beta\rangle_0 T_{\beta\alpha}}{E_\alpha - E_\beta + i\epsilon}$$

Expansion w/ hyper-coordinate

$$\psi(\boldsymbol{R}, \boldsymbol{Q}_A) = \sum_{[L],[K]} \psi_{[L][K]}(R, Q_A) Y_{[L]}(\Omega_R) \overline{Y_{[K]}(\Omega_{Q_A})}$$

$$\psi_{[L],[K]}(R, Q_A) \propto \sum_{[N]} U_{[L][N]}(Q_A) e^{i\delta_{[N]}(Q_A)} \frac{\sin(Q_A R - \Delta_L + \delta_{[N]}(Q_A))}{(Q_A R)^{(D-1)/2}} U^\dagger_{[N][K]}(Q_A)$$

Similar asymptotic behavior to 2-body system

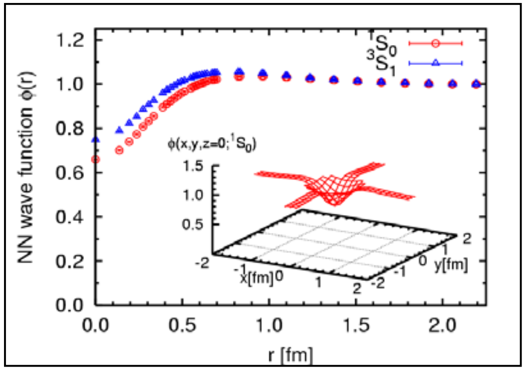
(non-rela approx.)

c.f. Finite V spectrum, n=3 only, relativistic: Hansen, Sharpe, Briceno, ...

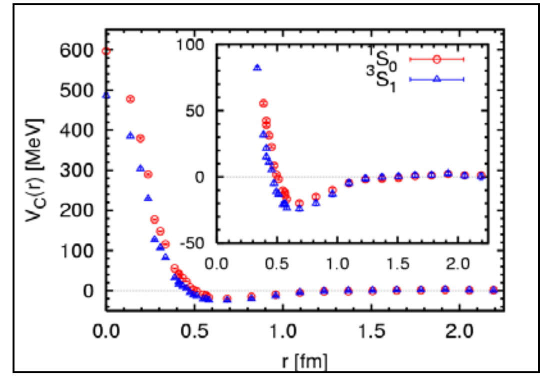
HAL QCD method

Lattice QCD

NBS wave func.



Lat Hadron Force



$$\psi_{\rm NBS}(\vec r) = \langle 0|H_1(\vec r)H_2(\vec 0)|H_1(\vec k)H_2(-\vec k), in\rangle$$
$$\simeq A_k \sin(kr - l\pi/2 + \delta_l(k))/(kr)$$

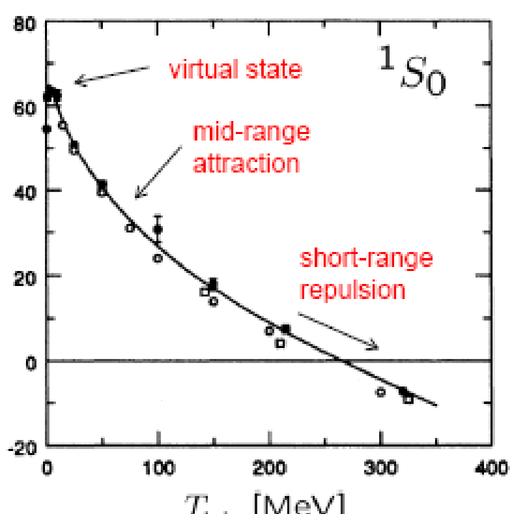
$$(k^2/m_N - H_0)\,\psi_{\rm NBS}(\vec r) = \int d\vec r'\, U(\vec r, \vec r')\psi_{\rm NBS}(\vec r')$$

E-indep (& non-local) Potential

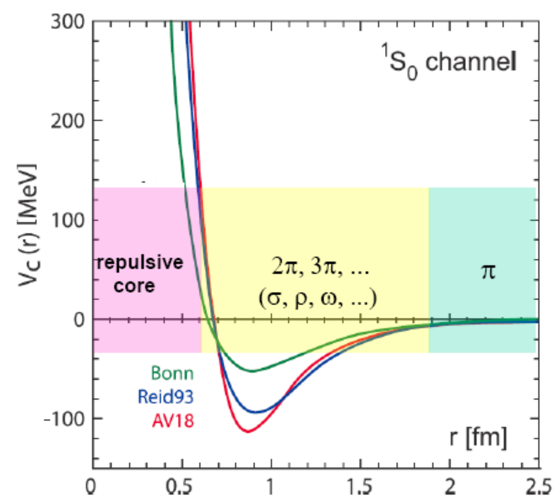
Analog to …

Scattering Exp.

Phase shifts



Phen. Potential



Digression: Nishijima's thesis

(Photo from Wikipedia)

He wrote his Ph.D. thesis
on the subject of field theory of composite particle
(not on the strangeness!)
「場の理論に於ける多体問題」(1955, Osaka U.)

束縛状態の話の方が、一番時間を食われました。ストレンジネスの仕事というのは、わかってしまえば書くのは１日で書けるという種類の仕事です。だから bound state の論文を学位論文にしたわけです。

たのですが、当時、大阪市大では学位を出していませんでした。そこで何処か学位を出す大学はないかと思って探したら、目の前にあったわけです。大阪市大は当時梅田にありました。阪大が中之島にありまして、walking distance にありましたので、阪大に行って学位が欲しいと言った訳です。そしたら、他にチョイスがなかったんですが、内山さんが主査になりました。で、内山さんが言うには、「俺はこんな理論はすっかりわかってんだけど、他の審査員がわかるかどうかわからないから、他の審査員にわかるように説明したら通してやる。」、実際はこんな表現じゃなかったんです、もっとひどい表現だったんです（笑）。とにかく審査員が大変勉強して下さいまして、わかっていただけましたので通していただきました。

西島和彦　素粒子論研究81巻3号(1990)168

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The discussion about bound states took up the most time. Once you understand it, the work involving strangeness is the kind of task that can be written in a day. That's why I made the paper on bound states my dissertation.

At that time, Osaka City Univ. did not confer degrees. So, I started looking for a university that did, and there it was right in front of me. Osaka Univ. was within walking distance, so I went there and said I wanted to get a degree. There were no other choices, and Ryoyu Utiyama san became the chief examiner. Utiyama-san said, 'I completely understood this kind of discussion, but I don’t know if the other examiners will. If you explain it in a way the other examiners can understand, I’ll pass you.' In reality, it wasn’t phrased like that; the way he said it was much harsher (laughs). Anyway, the examiners studied very hard, they understood, and I was able to pass.

K. Nishijima, Soryushiron Kenkyu, Vol. 81, Issue 3 (1990)168
+ English translation by ChatGPT