2+1+1 flavor QCD equation of state with Highly Improved Staggered Quarks

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Hadrons and Hadron Interactions in QCD,
Yukawa Institute for Theoretical Physics,
Oct 14 — Nov 15, 2024

Outline

- QCD phase diagram
- Lattice QCD
- Highly Improved Staggered Quark action
- Discretization effects
- Chiral crossover and the 2+1 flavor QCD equation of state
- Charm sector: fluctuations
- 2+1+1 flavor QCD equation of state
- Conclusion

QCD phase diagram

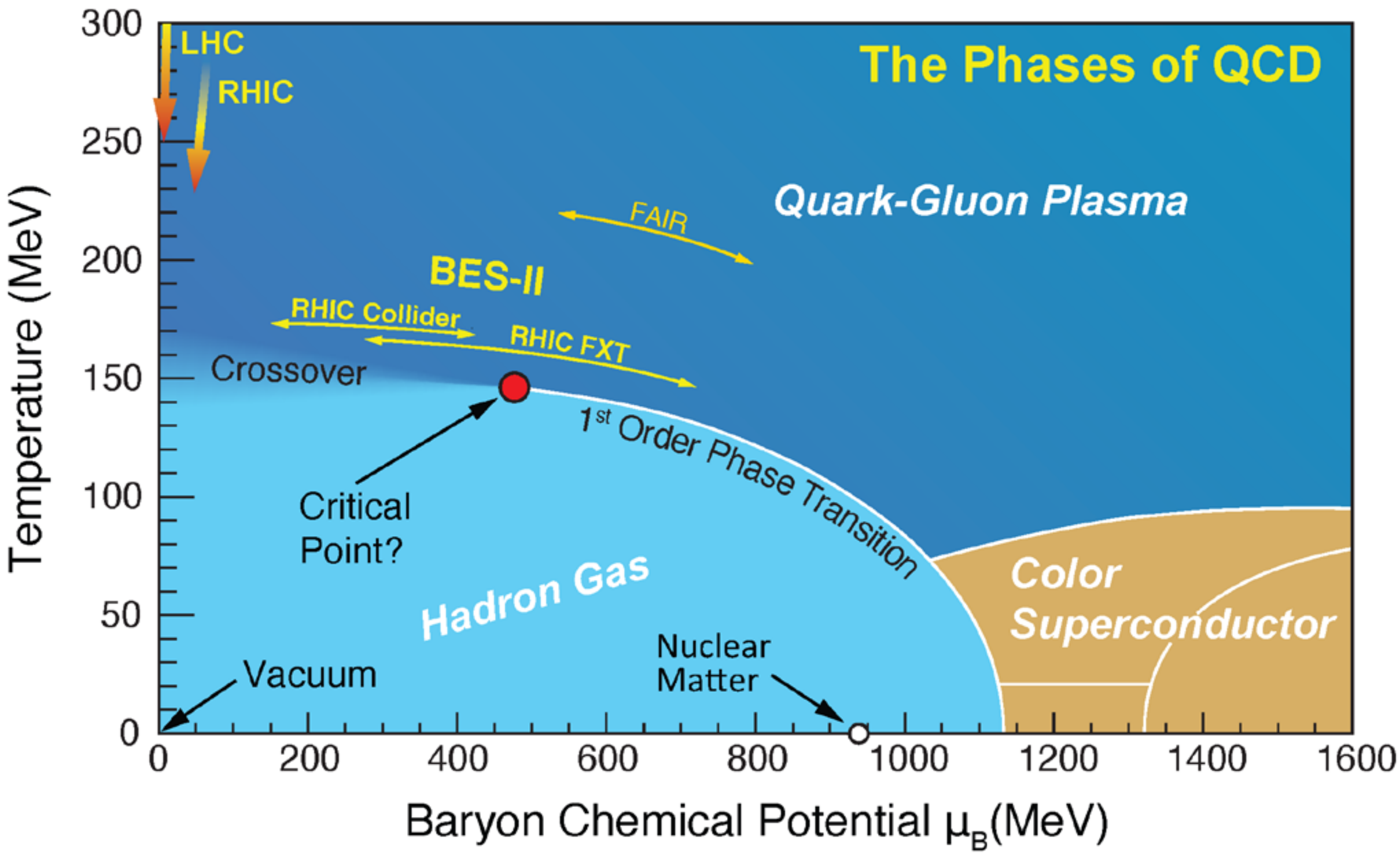
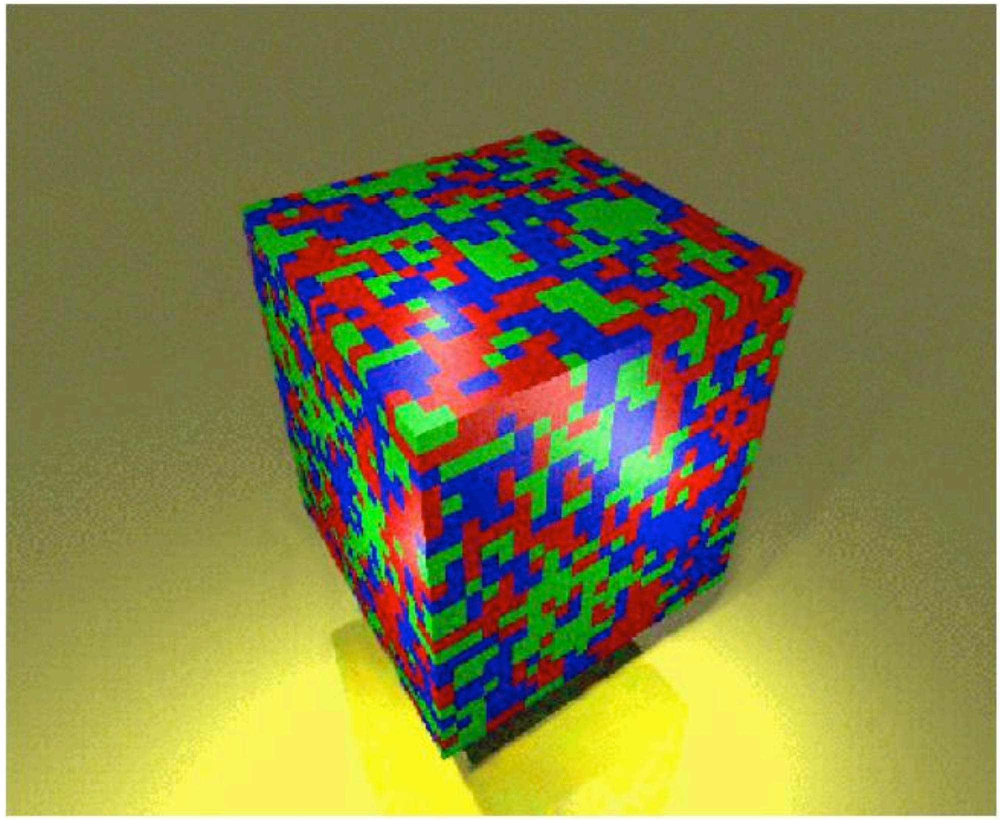


Image by Thomas Ullrich from
2023 NSAC LRP

Lattice QCD



- Euclidean space-time.
- Hypercubic lattice, momentum cutoff of the order π/a.
- Gauge-invariant regularization.
- Fermions integrated out.

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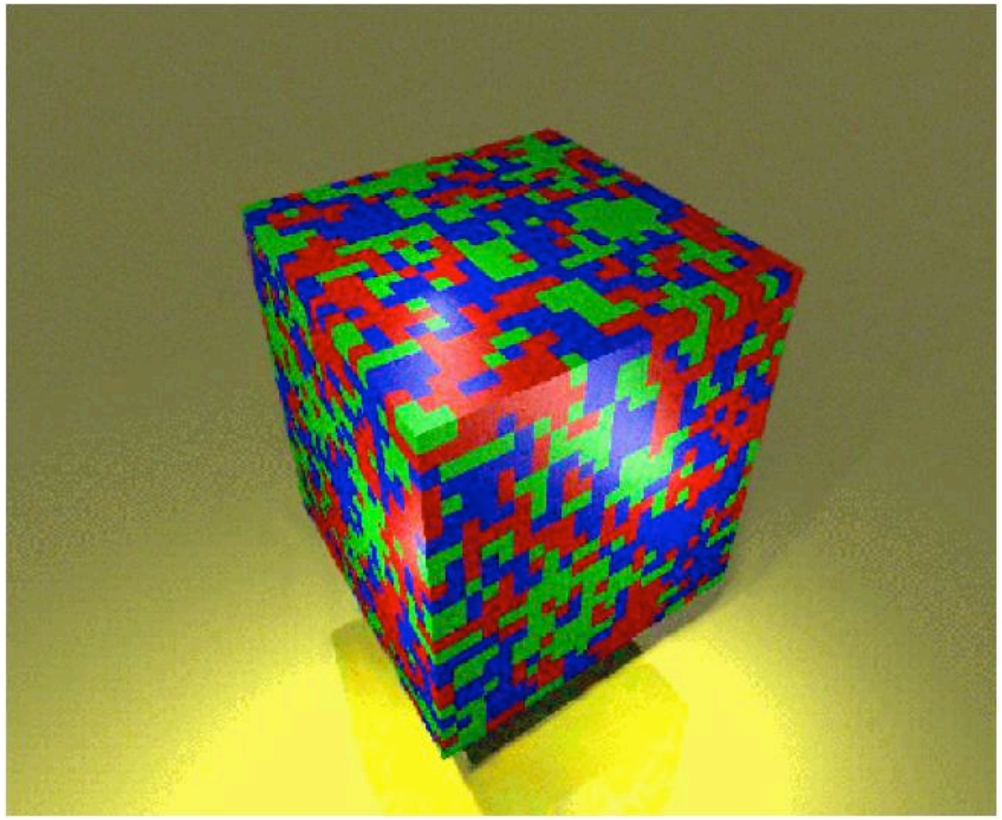

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- Gauge-invariant regularization.
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- Stochastic sampling of path integrals

$$Z = \int DU e^{-S_g[U]} \det M[U].$$

- Physics is recovered in the continuum limit.

Highly Improved Staggered Quarks (HISQ)

- Staggered Dirac operator:

$$M_{xy}[U] = 2m\delta_{xy} + \sum_{\mu} \eta_{x,\mu}(U_{x,\mu}\delta_{x,y-\hat{\mu}} - U^{\dagger}_{x-\hat{\mu},\mu}\delta_{x,y+\hat{\mu}}).$$

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- To reduce the discretization effects replace the gauge links with weighted averages over multiple paths.
- The Highly Improved Staggered Quarks action:

HPQCD, PRD75 (2007)

$V[U]$ — Fat7 smearing,

$W[V]$ — $U(3)$ projection,

$X[W]$ — Asq smearing.

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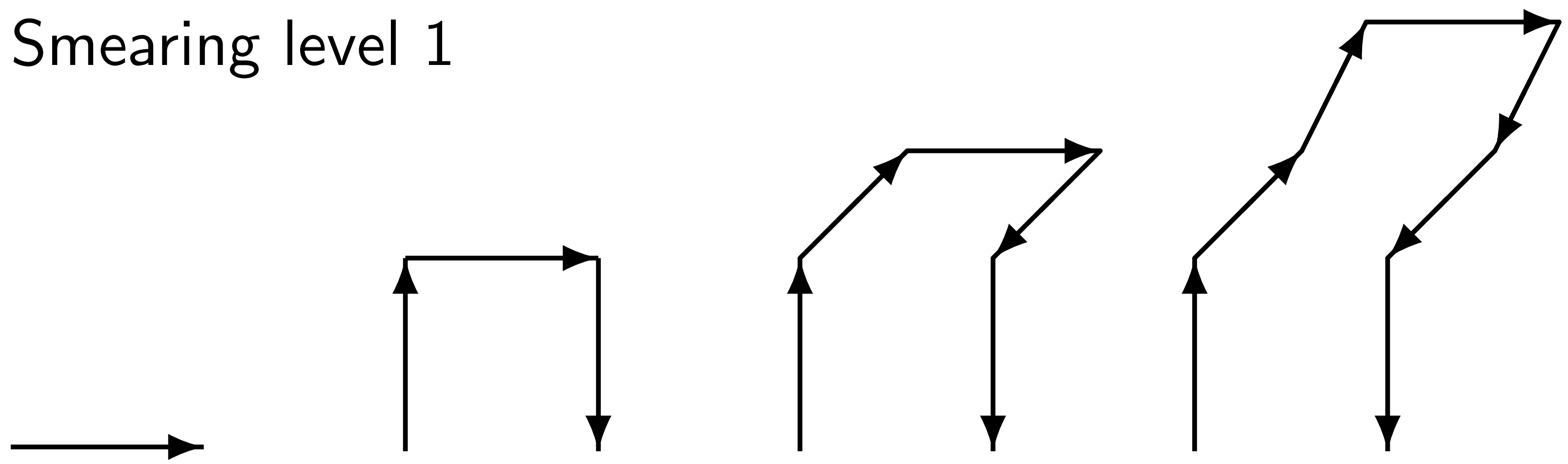

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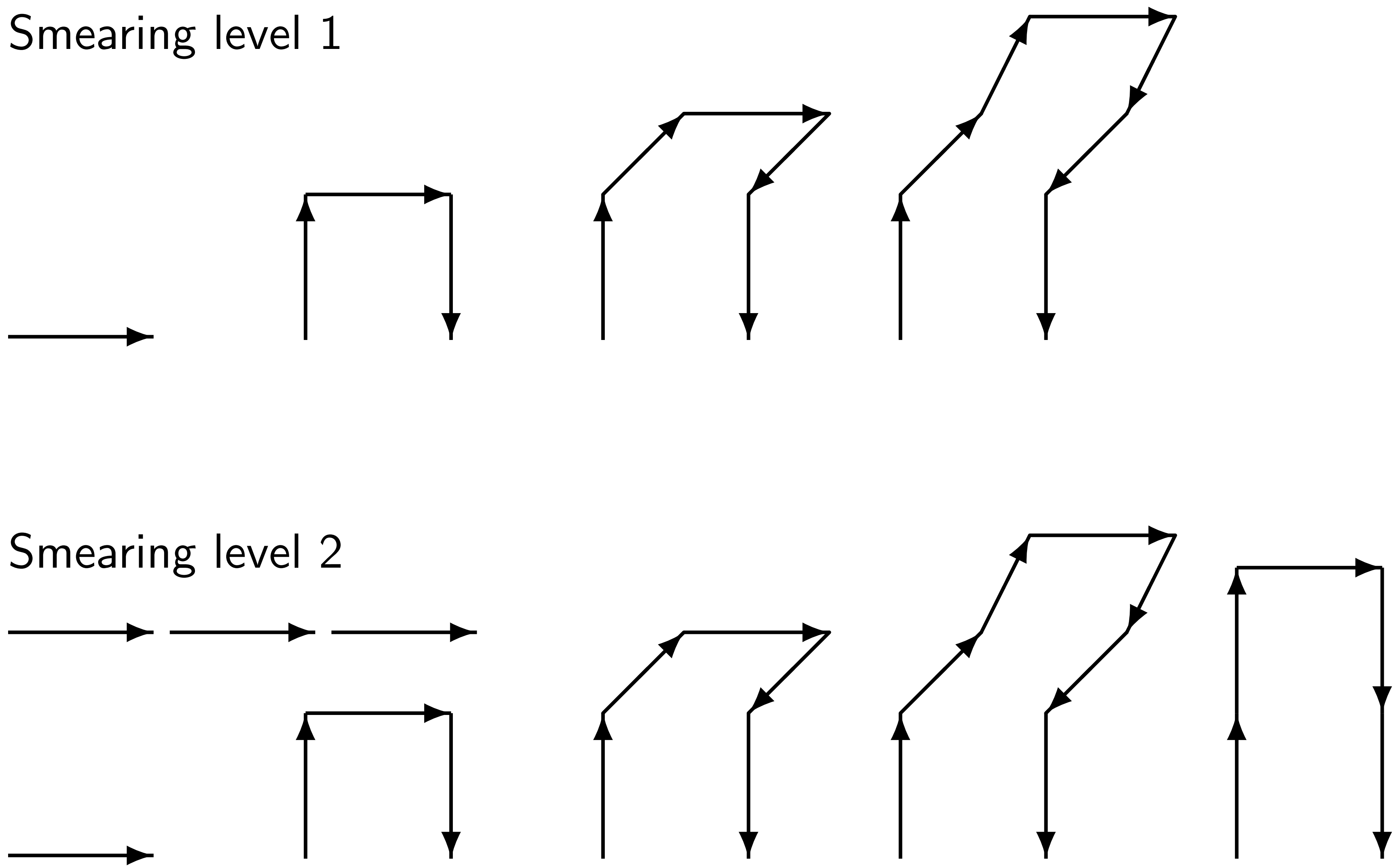

Two sets of HISQ ensembles

- MILC 2+1+1 setup:
  - One-loop Symanzik tadpole-improved gauge action.
    Lüscher, Weisz, PLB (1985)
  - The tadpole factor u_0 is tuned from the plaquette.
    Lepage, Mackenzie, PRD (1993)

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  - RHMC updating, on the finest ensembles — RHMD.
    Kennedy, Horvath, Sint, hep-lat/9809092
    Clark, Kennedy, PRL (2007)

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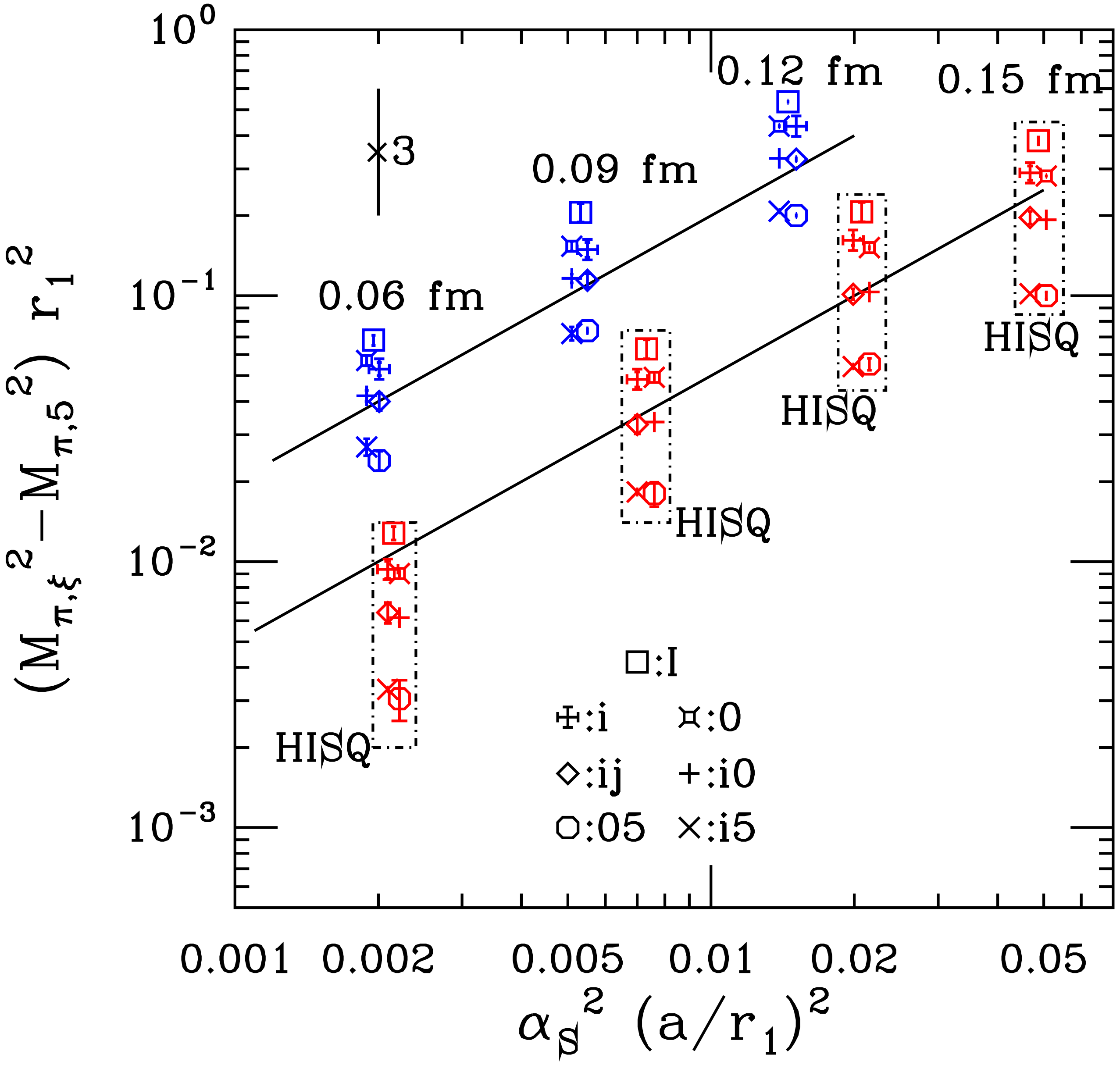
Pion taste mass splittings, 2+1+1 sea



MILC, PRD87 (2013)

- HISQ vs asqtad pion taste splittings (left).

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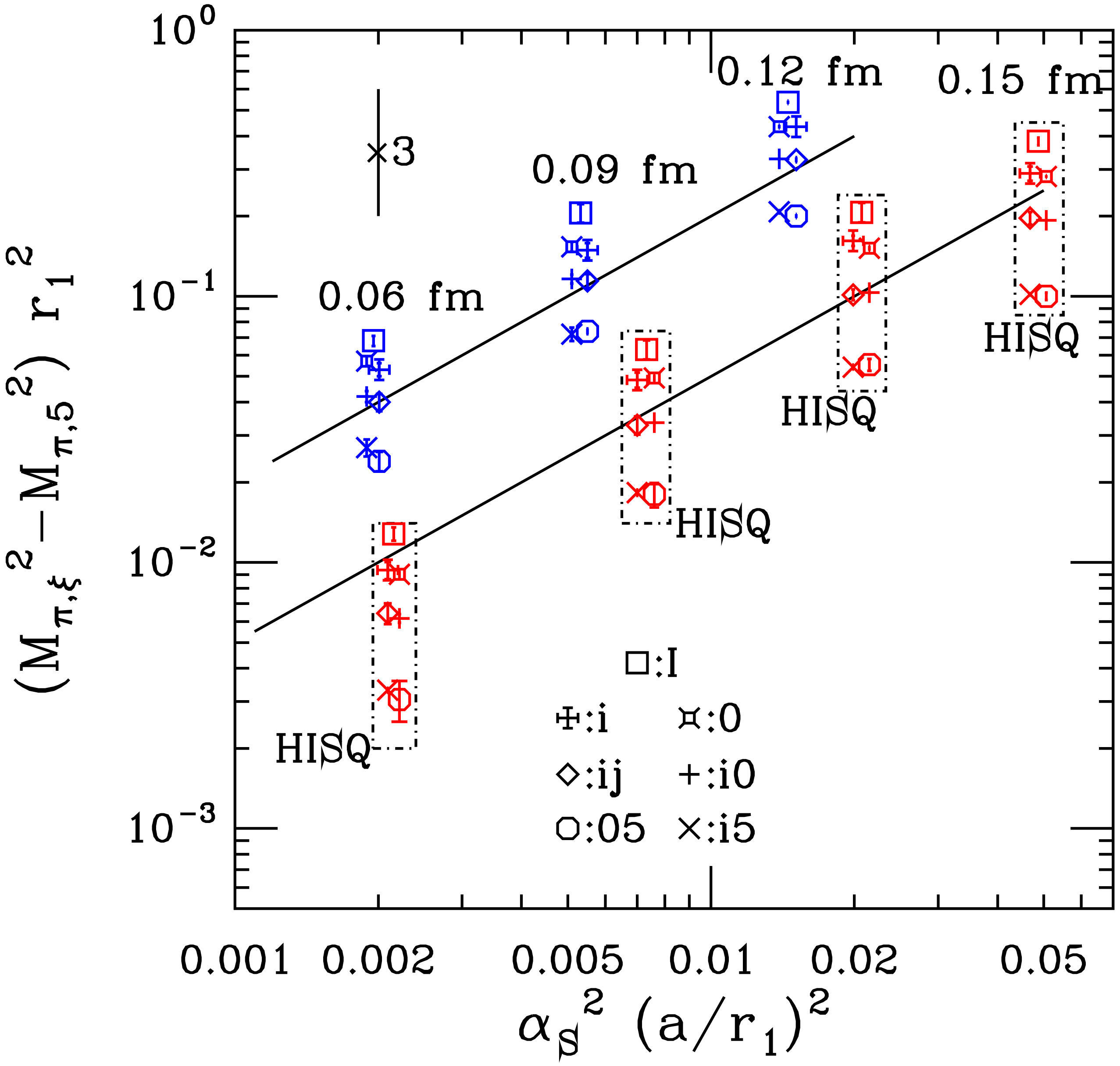
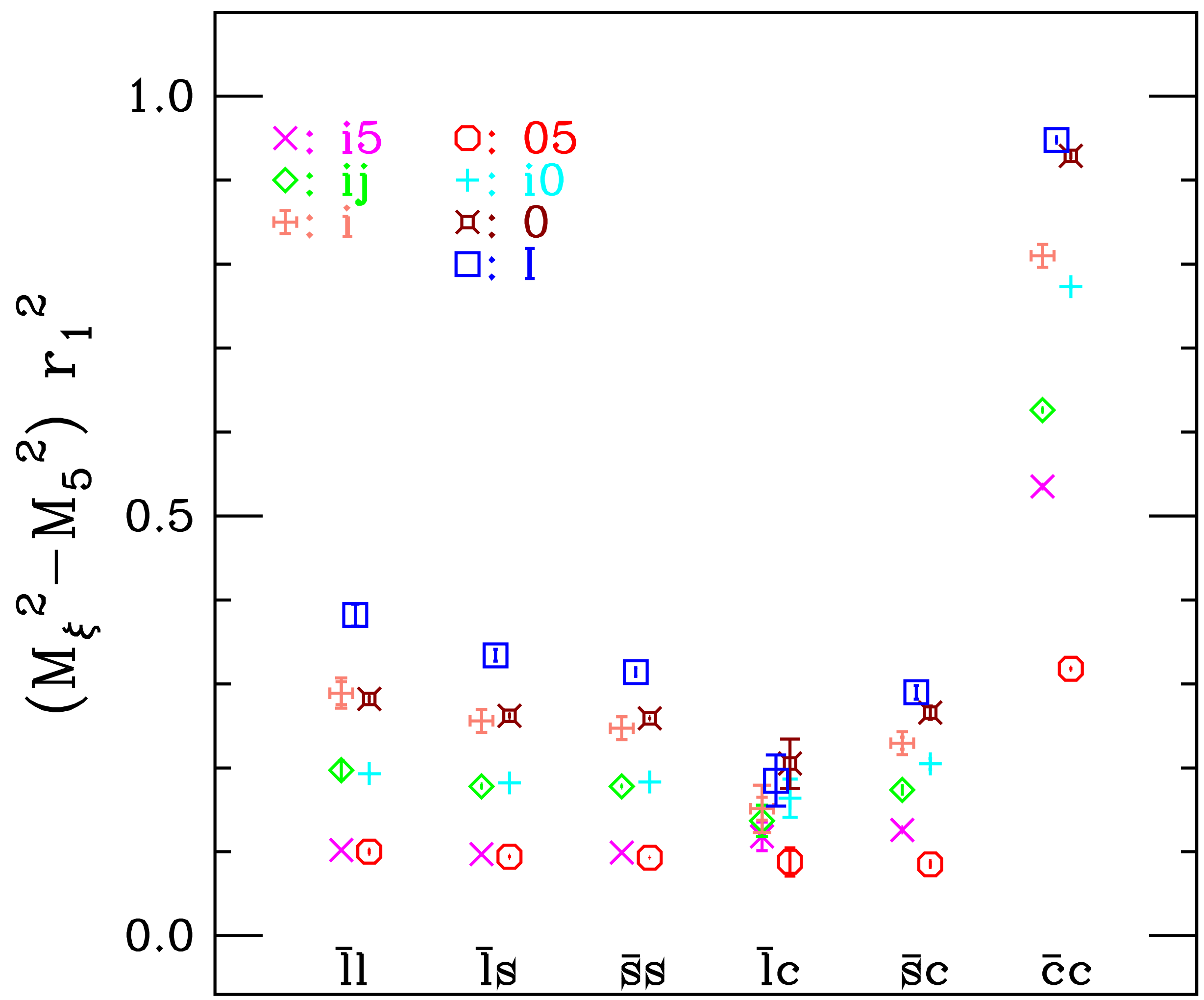
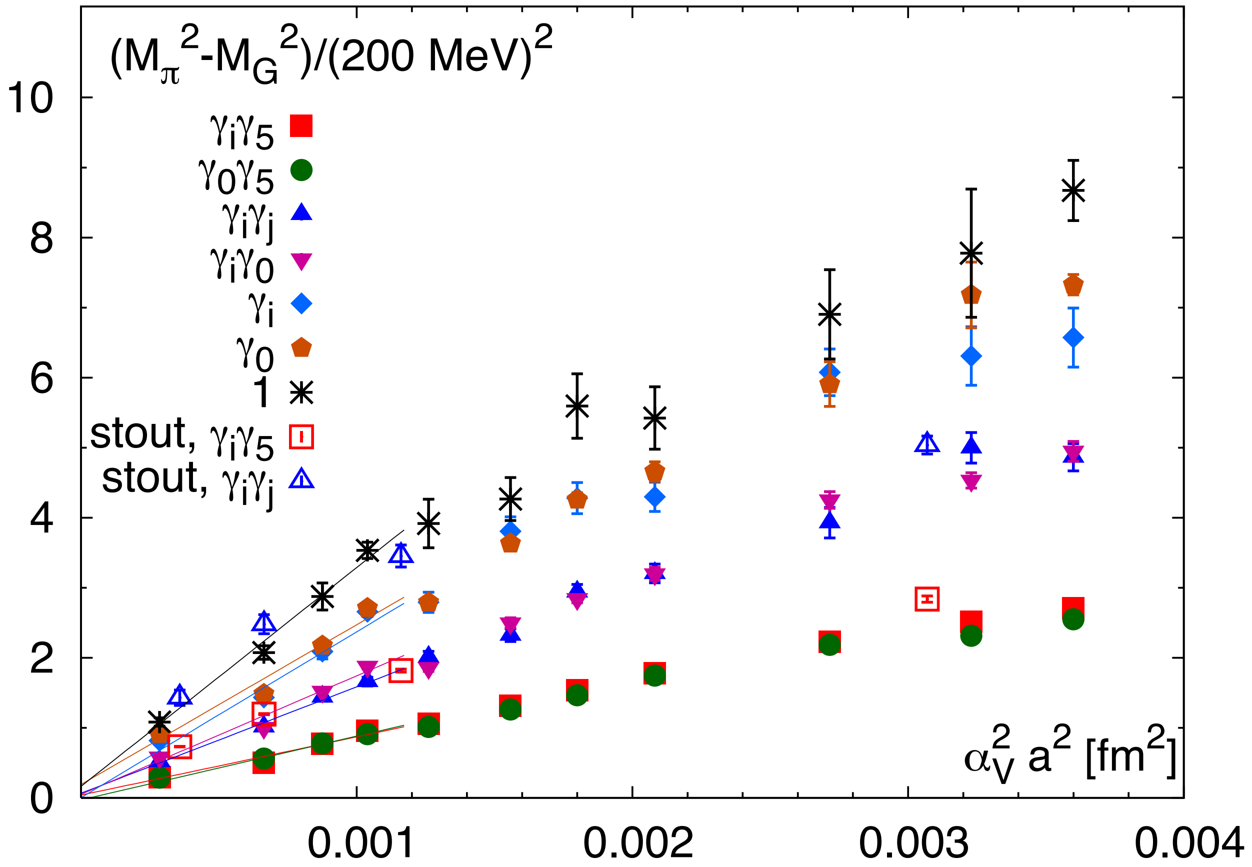




MILC, PRD87 (2013)

- HISQ vs asqtad pion taste splittings (left).
- Splitting pattern for different quark masses (right).

Pion taste mass splittings, 2+1 sea



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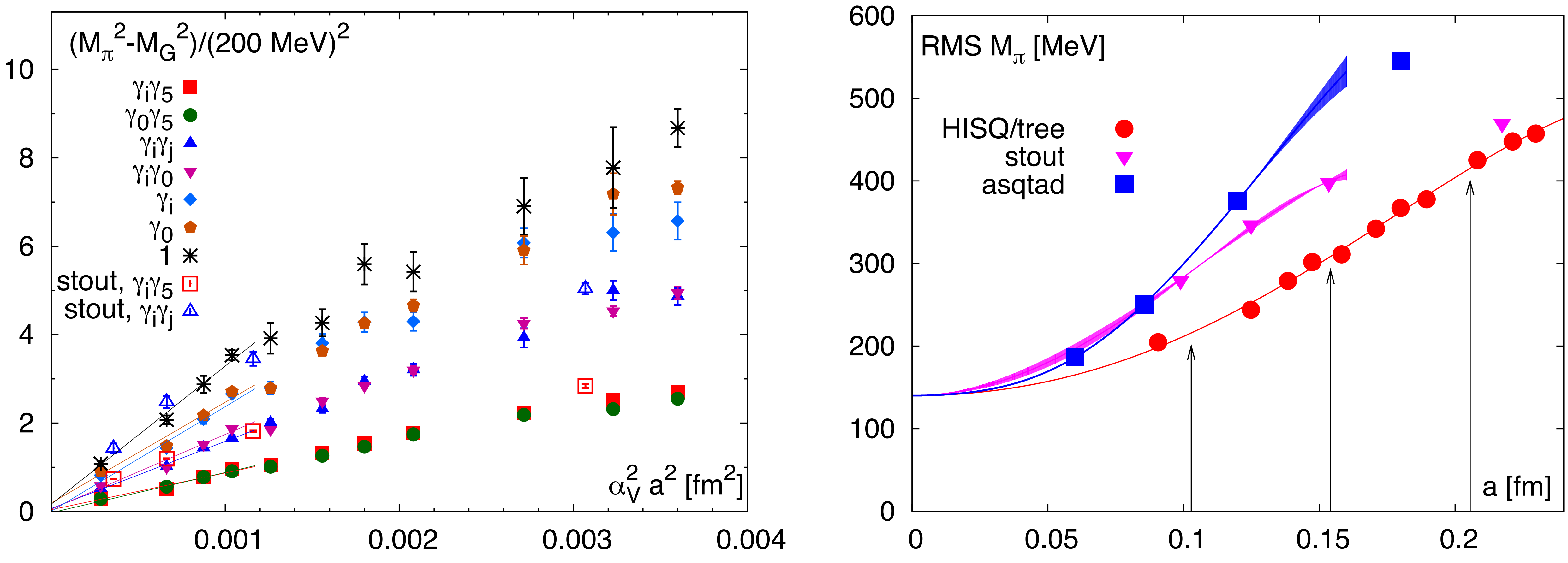

HotQCD, PRD85 (2012)

- HISQ pion taste splittings (left).
- Root-mean-squared pion mass for HISQ, stout and asqtad (right).

Lines of constant physics approach

- Staggered fermions are especially convenient for the lines of constant physics (LCP) approach to finite-temperature calculations:

$$T(a) = \frac{1}{a\,N_\tau} \quad \text{at fixed } N_\tau.$$

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- Staggered fermions are especially convenient for the lines of constant physics (LCP) approach to finite-temperature calculations:

$$T(a) = \frac{1}{a N_\tau} \quad \text{at fixed } N_\tau.$$

- The continuum limit is taken as $1/N_\tau^2 \to \infty$.
- In finite-temperature geometry we fix the aspect ratio $N_s/N_\tau = 4$.

The chiral crossover

- Chiral condensate and susceptibility

$$\Sigma = \frac{1}{f_K^4}\left[m_s\langle \bar{u}u + \bar{d}d\rangle - (m_u + m_d)\langle \bar{s}s\rangle\right],$$

$$\chi^{\Sigma} = m_s\left(\frac{\partial}{\partial m_u} + \frac{\partial}{\partial m_d}\right)\Sigma$$

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$$\chi^\Sigma = m_s\left(\frac{\partial}{\partial m_u} + \frac{\partial}{\partial m_d}\right)\Sigma$$

- Taylor expansion

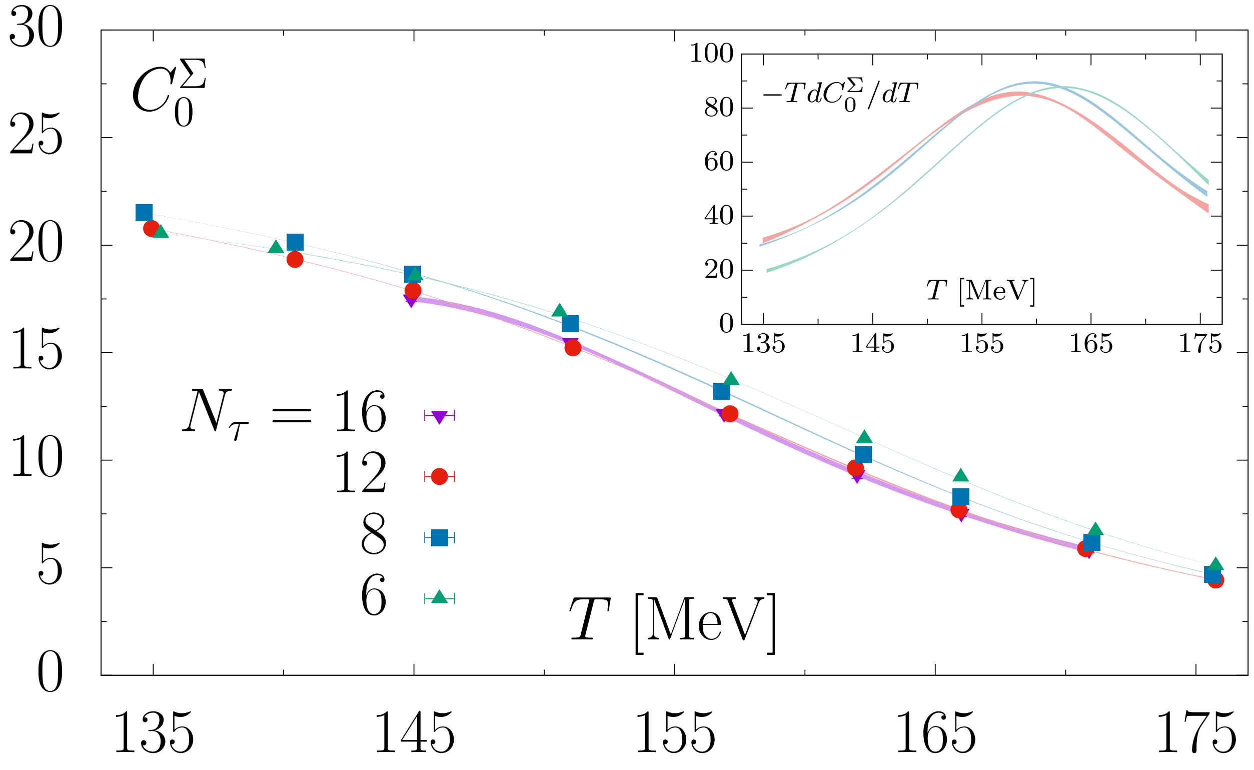
$$\Sigma(T,\mu_X) = \sum_{n=0}^{\infty} \frac{C_{2n}^\Sigma(T)}{(2n)!}\left(\frac{\mu_X}{T}\right)^{2n}, \quad C_{2n}^\Sigma(T) = \left.\frac{\partial^{2n}\,\Sigma}{\partial\left(\mu_X/T\right)^{2n}}\right|_{\mu_X=0}.$$

The chiral crossover

- Criteria to define T_c — relate to the singularities in the chiral limit

$$
\begin{aligned}
\partial_T^2 C^{\Sigma} 0(T) &= 0, \\
\partial_T C_2^{\Sigma}(T) &= 0, \\
\partial_T \chi^{\Sigma}(T) &= 0, \\
\partial_T C_0^{\chi}(T) &= 0, \\
C_2^{\chi}(T) &= 0.
\end{aligned}
$$

The chiral crossover

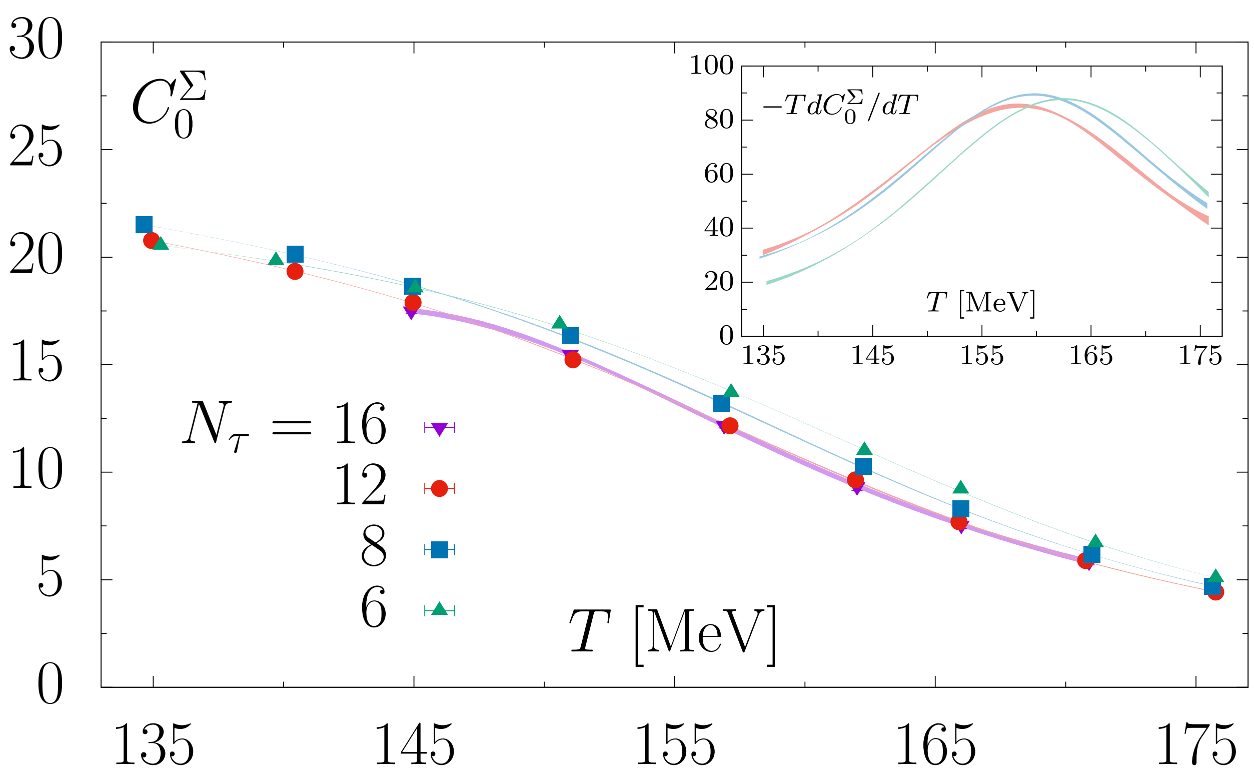


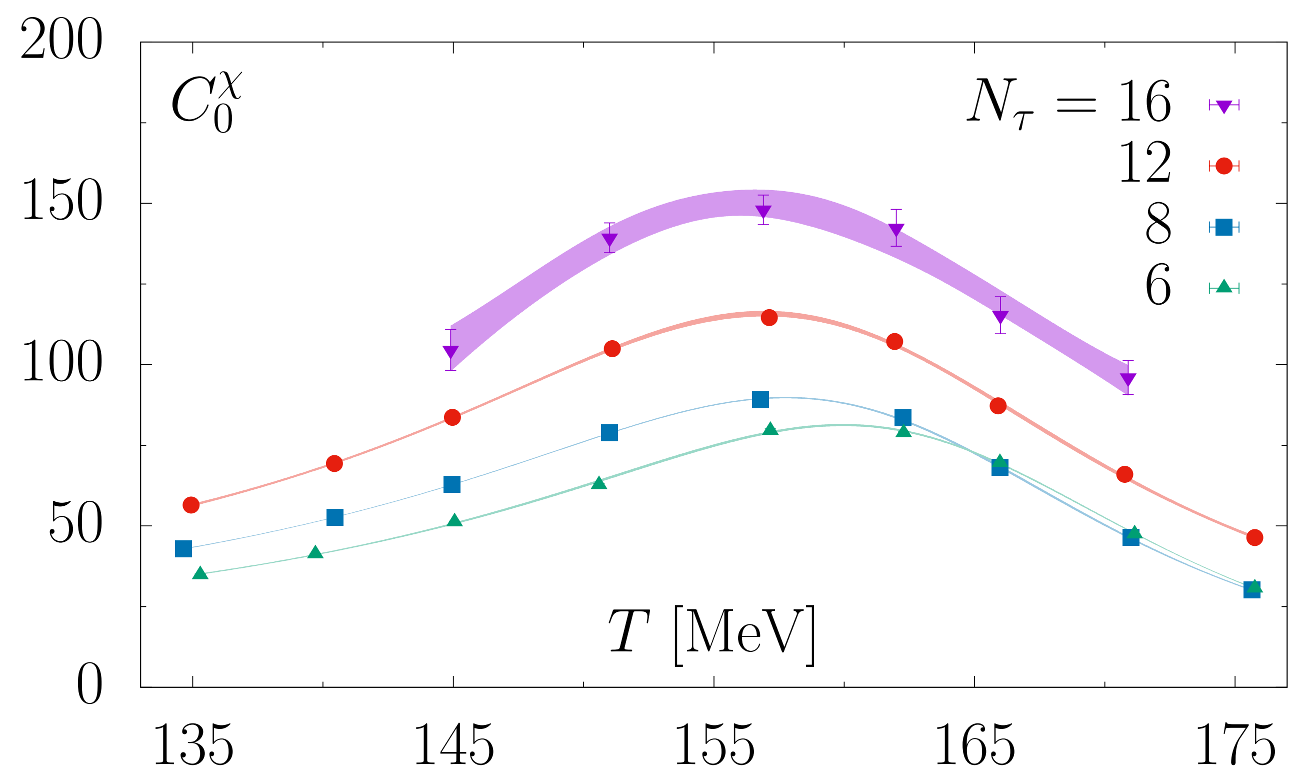
Steinbrecher, PhD thesis (2018), HotQCD, PLB795 (2019)

- Chiral order parameter Σ (left) at various cutoffs

  $N_\tau = 6, 8, 12, 16.$

The chiral crossover





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- Chiral order parameter Σ (left) at various cutoffs

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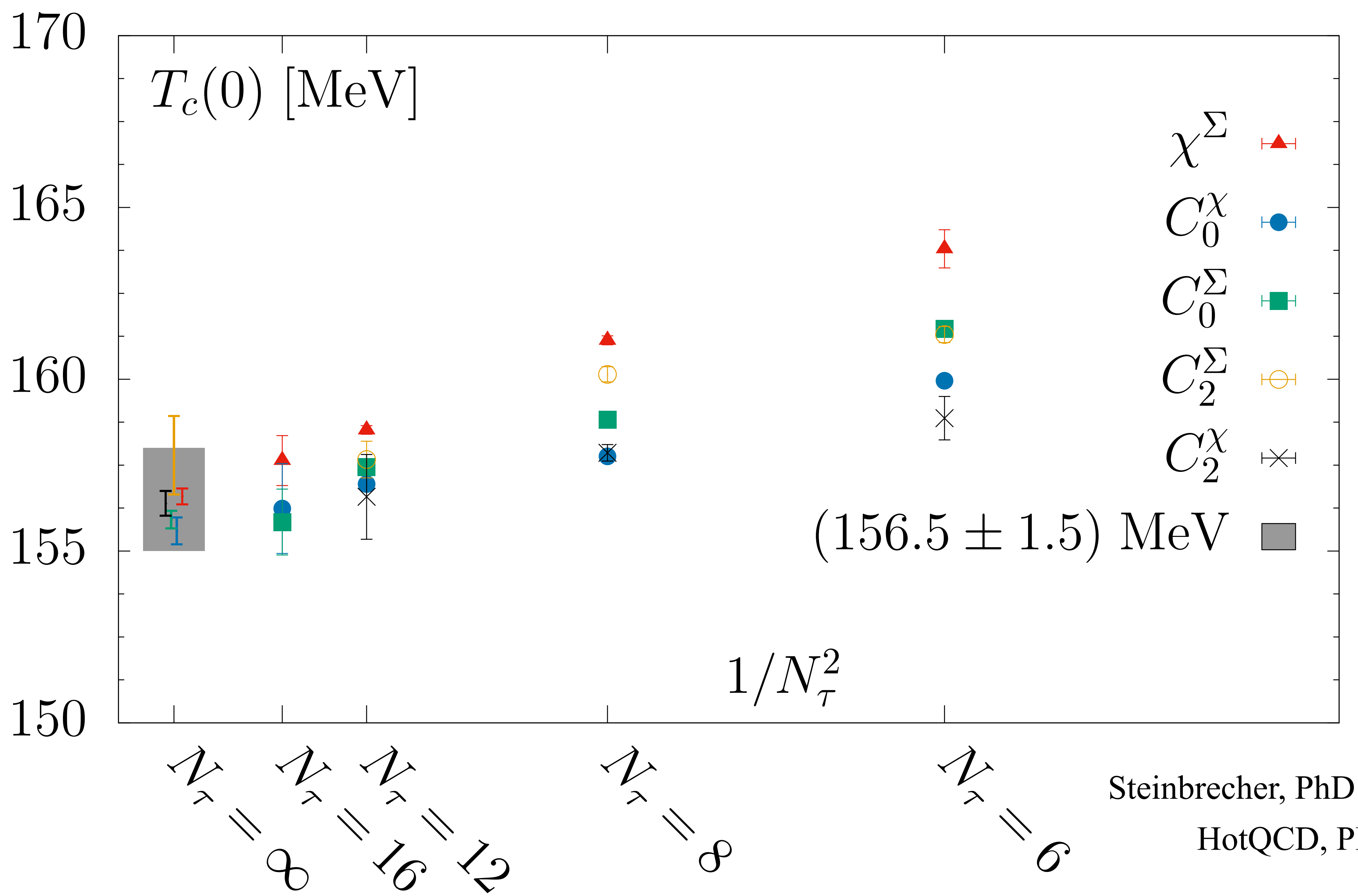
- (Quark-line) disconnected chiral susceptibility (right).

The chiral crossover



Steinbrecher, PhD thesis (2018), HotQCD, PLB795 (2019)

- The chiral crossover temperature

$$T_c = 154(9) \text{ MeV, HotQCD (2012),}$$

$$T_c = 156.5(1.5) \text{ MeV, HotQCD (2019).}$$

QCD equation of state

- The trace anomaly

$$\frac{\Theta^{\mu\mu}}{T^4} = \varepsilon - 3p = -\frac{T}{V}\frac{d\ln Z}{d\ln a},$$

$$Z = \int DUD\bar{\psi}D\psi\; e^{-S_g[U]-S_f[\psi,\bar{\psi},U]}.$$

QCD equation of state

- The trace anomaly

$$\frac{\Theta^{\mu\mu}}{T^4} = \varepsilon - 3p = -\frac{T}{V}\frac{d\ln Z}{d\ln a},$$

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- Pressure via the integral method

Boyd et al., NPB (1996)

$$\frac{p}{T^4} - \frac{p_0}{T_0^4} = \int_{T_0}^{T} dT'\,\frac{\varepsilon - 3p}{T'^5}.$$

- (T_0, p_0) is chosen where the Hadron Resonance Gas (HRG) model is applicable.

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- (T_0, p_0) is chosen where the Hadron Resonance Gas (HRG) model is applicable.
- $\dfrac{\varepsilon - 3p}{T^4}$ requires additive renormalization (vacuum subtraction) — high computational cost.

2+1 QCD equation of state

- Vacuum subtracted expectation value:

$$\Delta(X) = \langle X \rangle_\tau - \langle X \rangle_0.$$

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- The 2+1 flavor trace anomaly

$$\frac{\Theta^{\mu\mu}}{T^4} = -R_\beta(\beta)\Delta(S_g)$$

$$+R_\beta(\beta)R_{m_s}(\beta)\left[2m_l\Delta(\bar{\psi}_l\psi_l) + m_s\Delta(\bar{\psi}_s\psi_s)\right].$$

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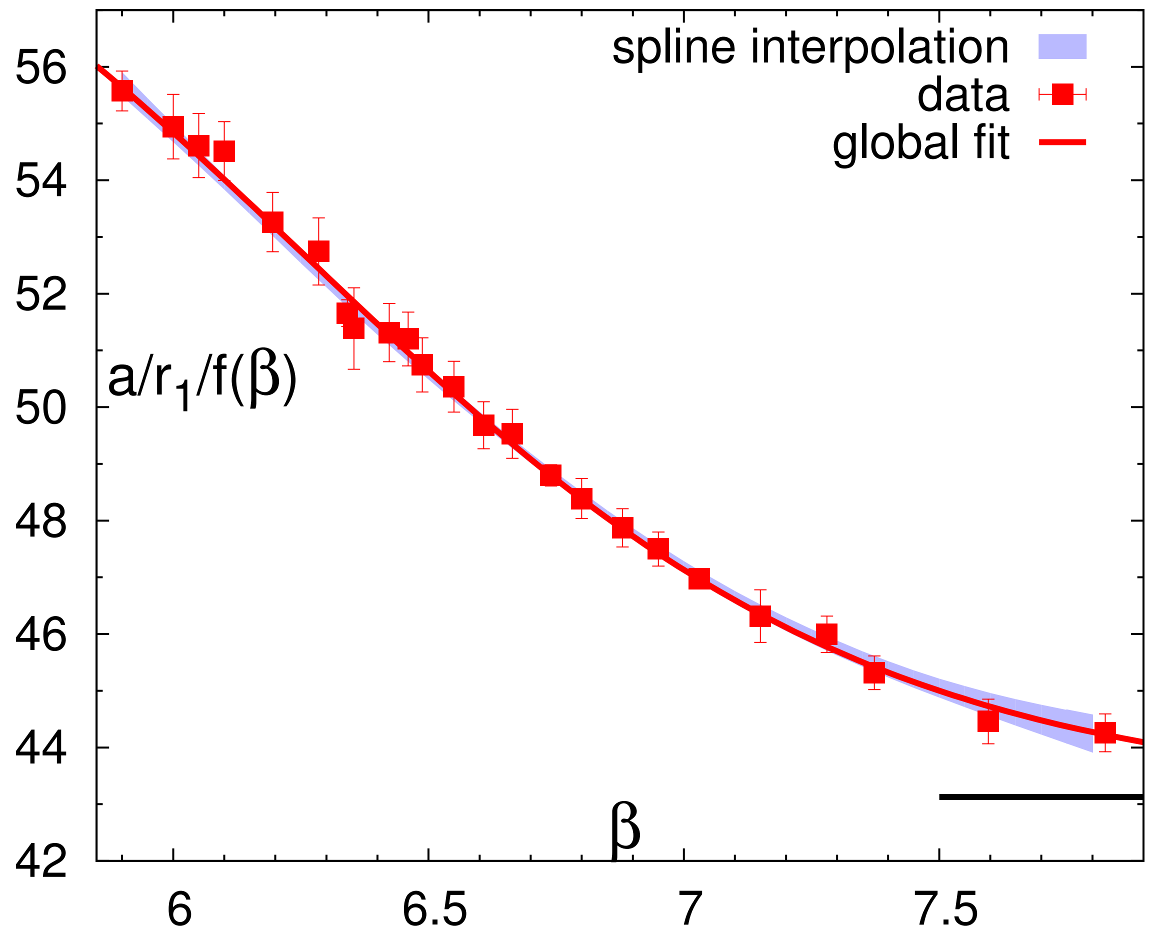
- The beta-functions ($\beta = 10/g^2$)

$$R_\beta(\beta) = T\frac{\mathrm{d}\beta}{\mathrm{d}T} = - a\frac{\mathrm{d}\beta}{\mathrm{d}a} = (r_1/a)(\beta)\left(\frac{\mathrm{d}(r_1/a)(\beta)}{\mathrm{d}\beta}\right)^{-1},$$

$$R_{m_q}(\beta) = \frac{1}{am_q(\beta)}\frac{\mathrm{d}am_q(\beta)}{\mathrm{d}\beta} \qquad \text{for } q = s\,.$$

2+1 QCD equation of state



- r_1 scale setting, Allton-style fit:

HotQCD, PRD90 (2014)

$$\frac{r_1}{a}(\beta) = \frac{c_r^{(0)} f(\beta) + c_r^{(2)}(10/\beta) f^3(\beta)}{1 + d_r^{(2)}(10/\beta) f^2(\beta)},$$

$$f(\beta) = \left(\frac{10 b_0}{\beta}\right)^{-b_1/(2b_0^2)} \exp(-\beta/20b_0).$$

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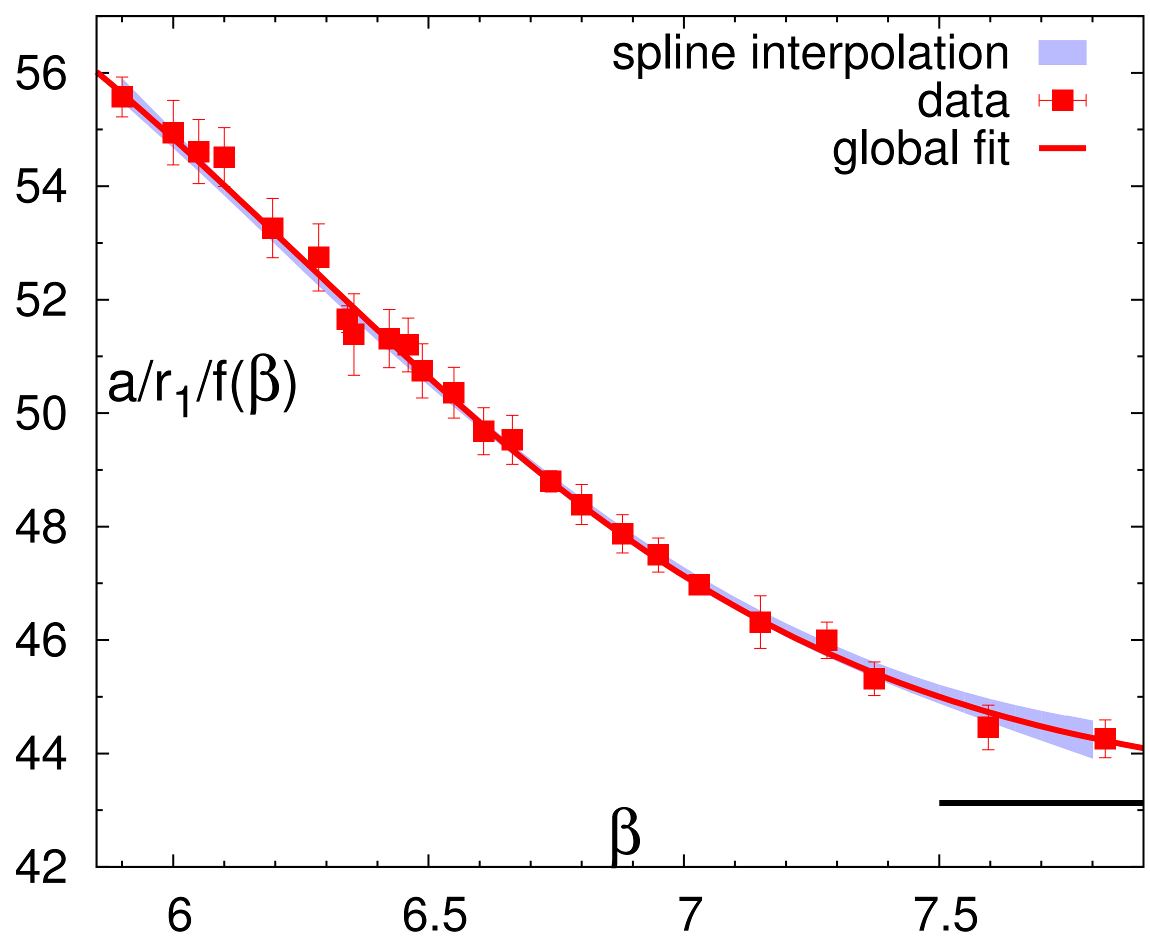

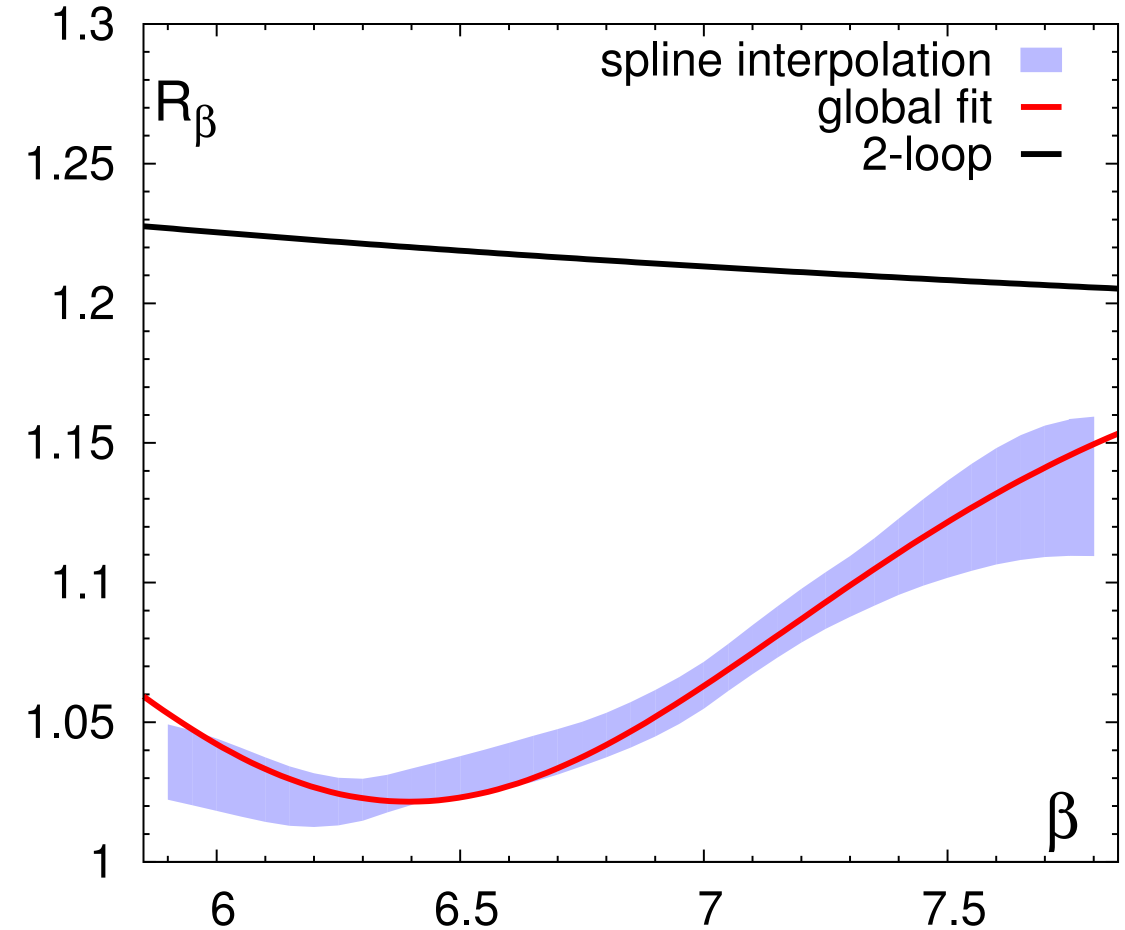

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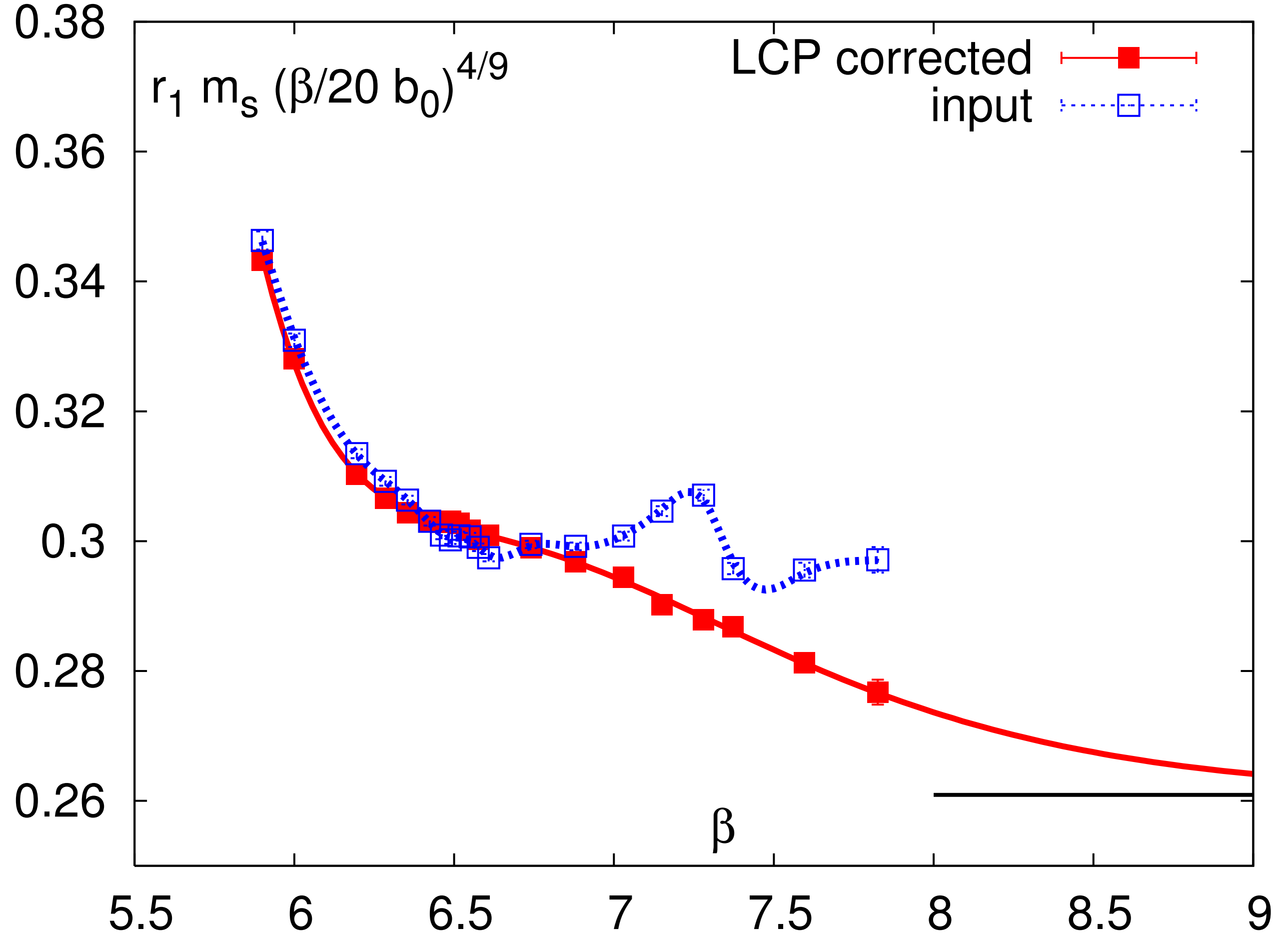


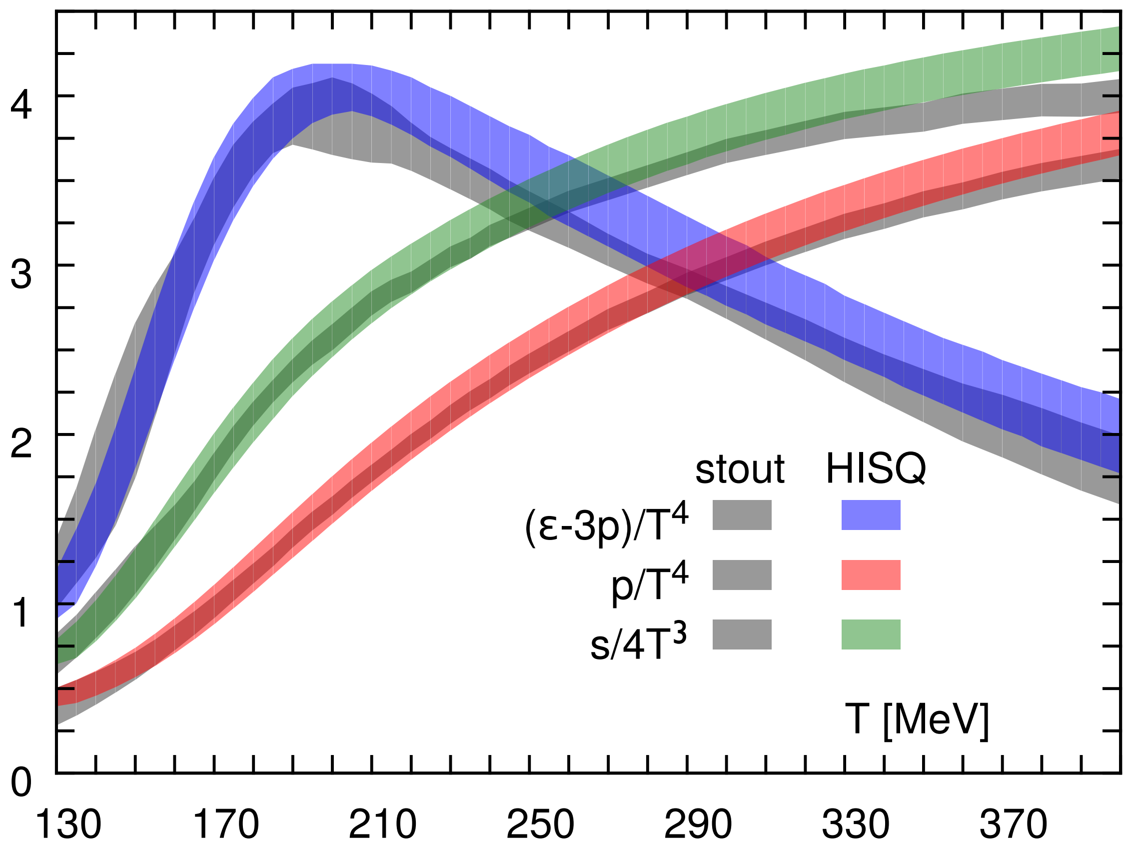
- Strange quark mass LCP

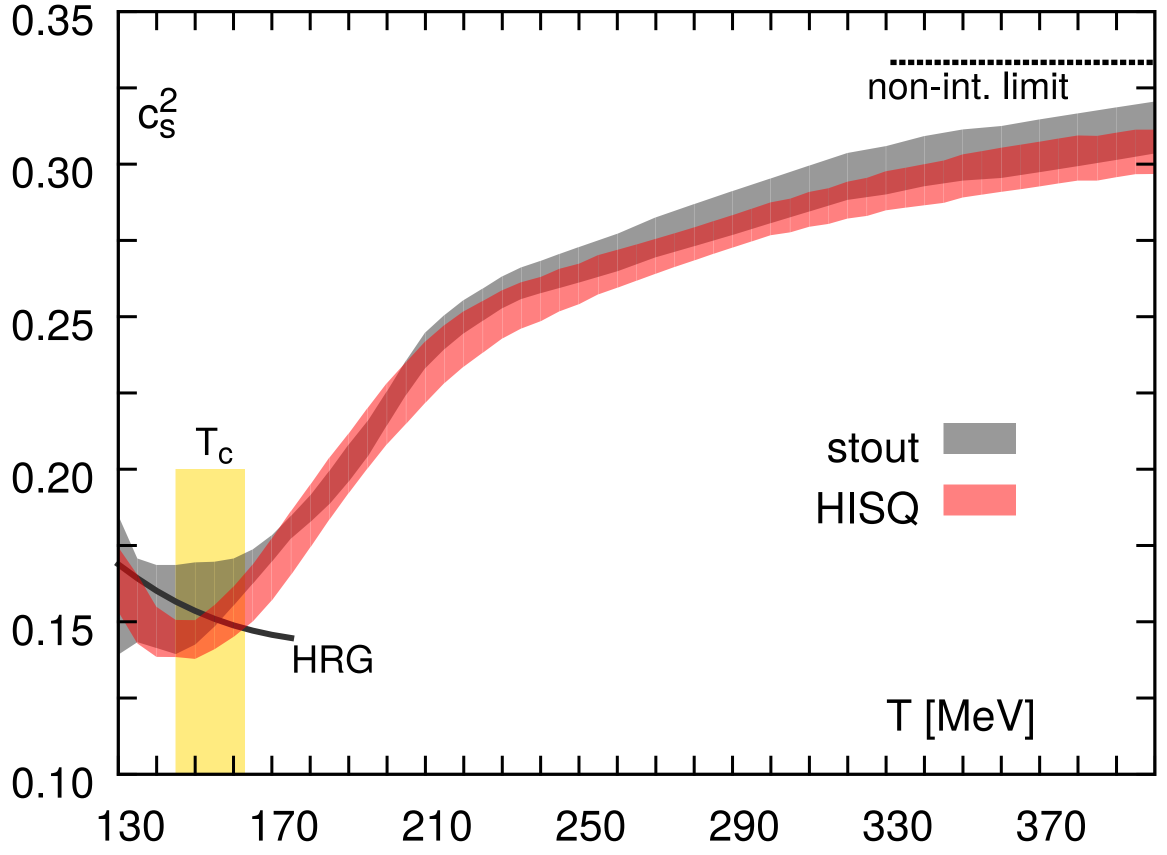
HotQCD, PRD90 (2014)

$$am_q(\beta) = \frac{c_q^{(0)} f(\beta) + c_q^{(2)}(10/\beta) f^3(\beta)}{1 + d_q^{(2)}(10/\beta) f^2(\beta)} \left(\frac{20b_0}{\beta}\right)^{\frac{4}{9}}.$$

- Light quark mass $m_l = m_s/20$.

2+1 QCD equation of state

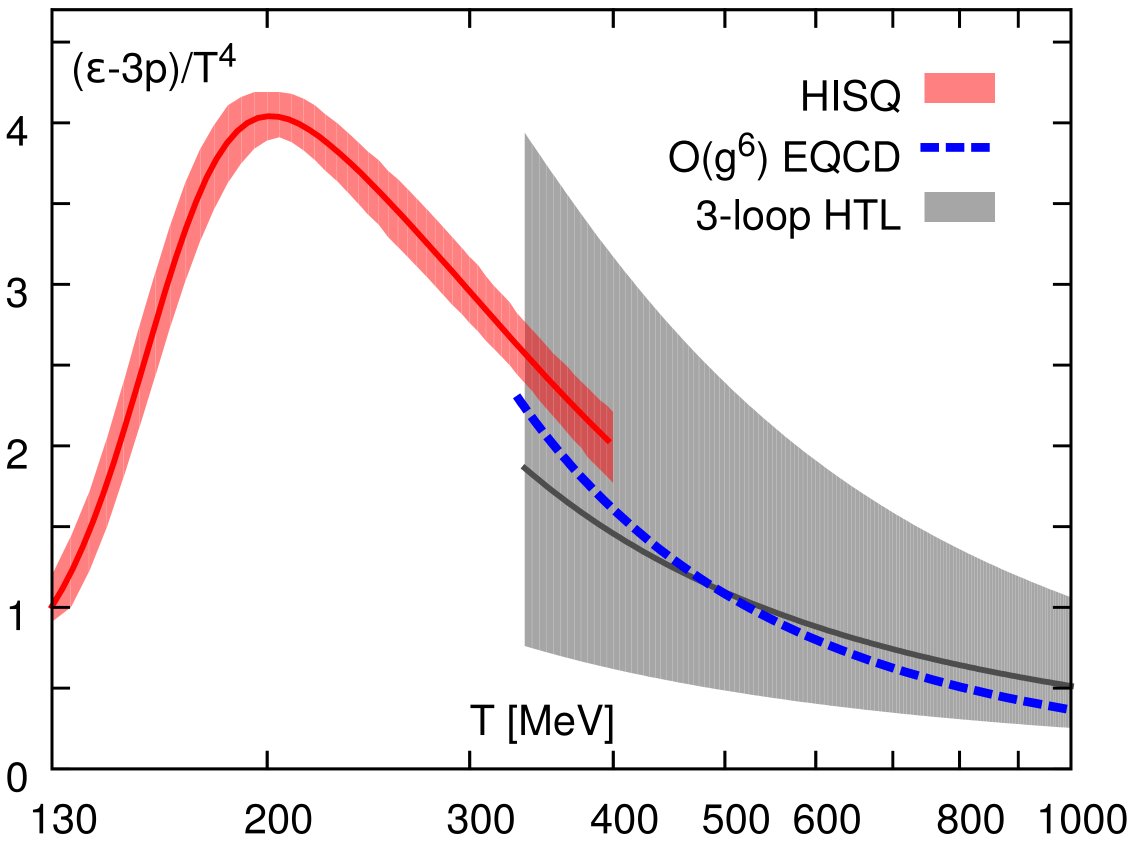


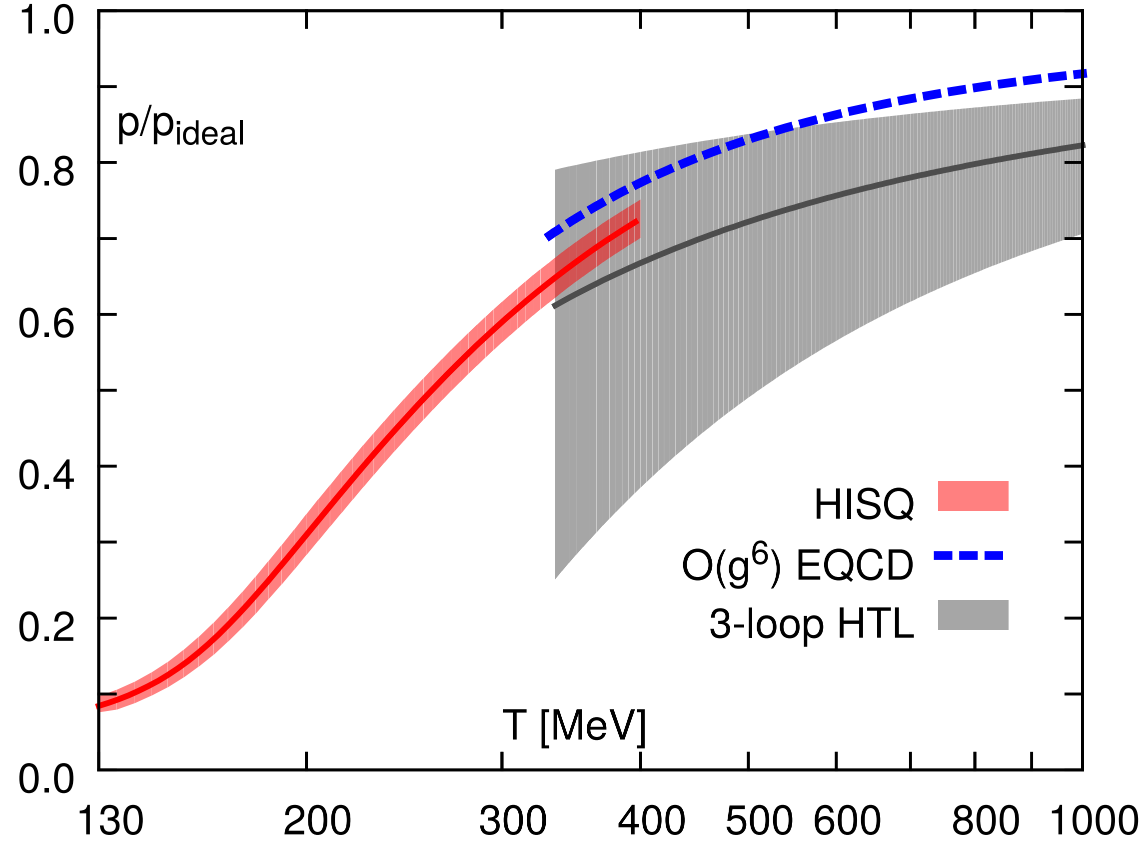


WB, PLB730 (2014), HotQCD, PRD90 (2014)

- Trace anomaly, *p*, *s* (left) and speed of sound (right) at zero baryon chemical potential.

2+1 QCD equation of state





HotQCD, PRD90 (2014)

- Comparison with perturbative calculations.

Laine and Schroeder (2006), Haque et al. (2014)

2+1 QCD EoS to high temperatures

- At high enough temperature the cutoff dependence of the pressure resembles the cutoff dependence of the second-order quark number susceptibilities.

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- Approximate the cutoff dependence as

$$\frac{p^q(T,N_\tau)}{p^q(T)} \simeq \frac{\chi_2^l(T,N_\tau)}{\chi_2^l(T)}.$$

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- Approximate the cutoff dependence as

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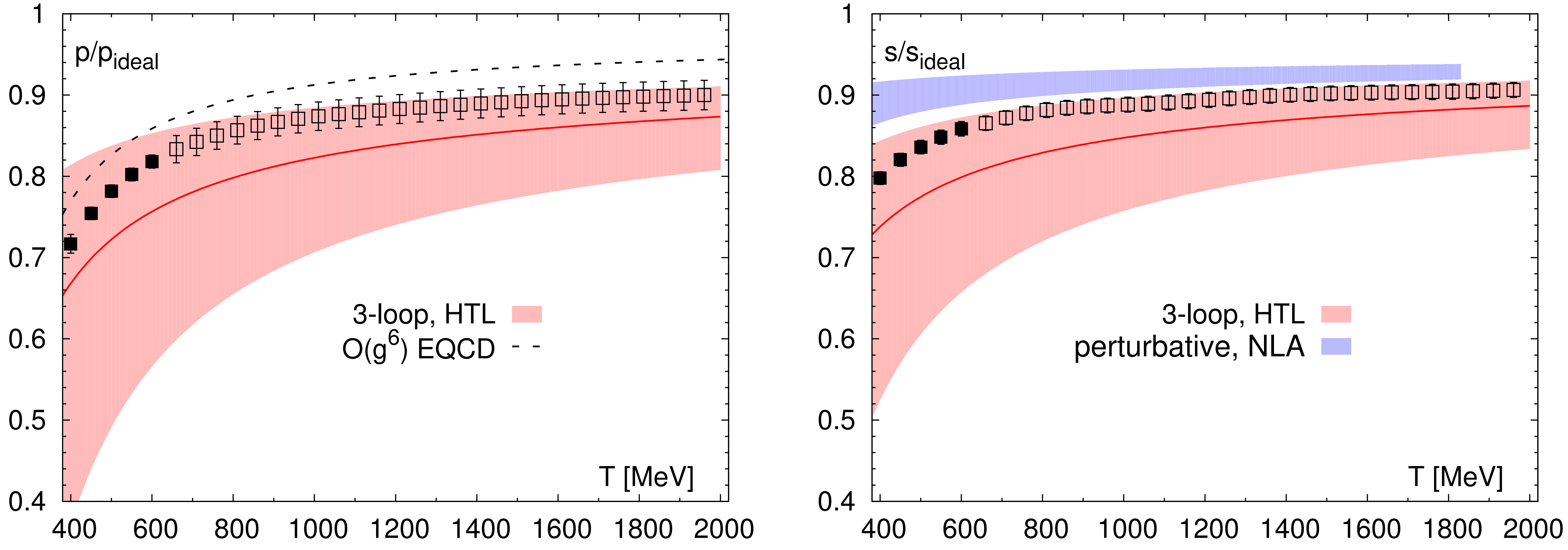
- Then the continuum pressure

$$p(T) = p(T,N_\tau) + corr(T,N_\tau),$$

$$corr(T,N_\tau) = p^q(T)\left(1 - \frac{p^q(T,N_\tau)}{p^q(T)}\right).$$

- (Cutoff effects in the gluon pressure are assumed small, if improved gluon action is used.)

2+1 QCD EoS to high temperatures



Bazavov, Petreczky, Weber PRD97 (2018)

- Pressure (left) and entropy (right) normalized by the Stefan-Boltzmann limit in 2+1 flavor QCD.
- The open symbols represent a *continuum estimate*.

Charm sector

- Generalized susceptibilities:

$$\chi_{klmn}^{BQSC} = \left. \frac{\partial^{(k+l+m+n)} \left[P\left(\hat{\mu}_B, \hat{\mu}_Q, \hat{\mu}_S, \hat{\mu}_C\right) / T^4 \right]}{\partial \hat{\mu}_B^k \; \partial \hat{\mu}_Q^l \; \partial \hat{\mu}_S^m \; \partial \hat{\mu}_C^n} \right|_{\vec{\mu}=0},$$

with $\hat{\mu}_X = \mu_X / T$.

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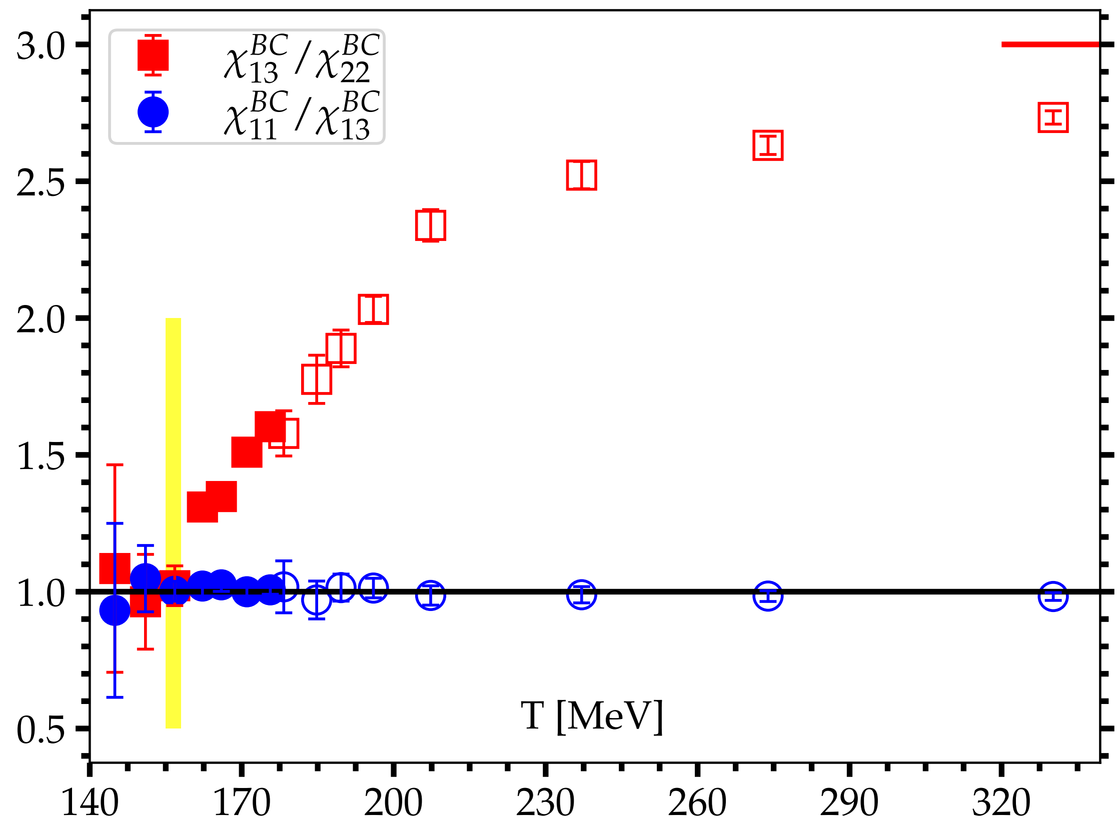
with $\hat{\mu}_X = \mu_X/T$.

- Partial pressure of the charmed hadrons in HRG:

$$P_{B/M}^{C}(T, \vec{\mu}) = \frac{1}{2\pi^2} \sum_{i \in \text{C-B/M}} g_i \left(\frac{m_i}{T}\right)^2 K_2(m_i/T)$$

$$\times \cosh(B_i\hat{\mu}_B + Q_i\hat{\mu}_Q + S_i\hat{\mu}_S + C_i\hat{\mu}_C).$$

Charm sector

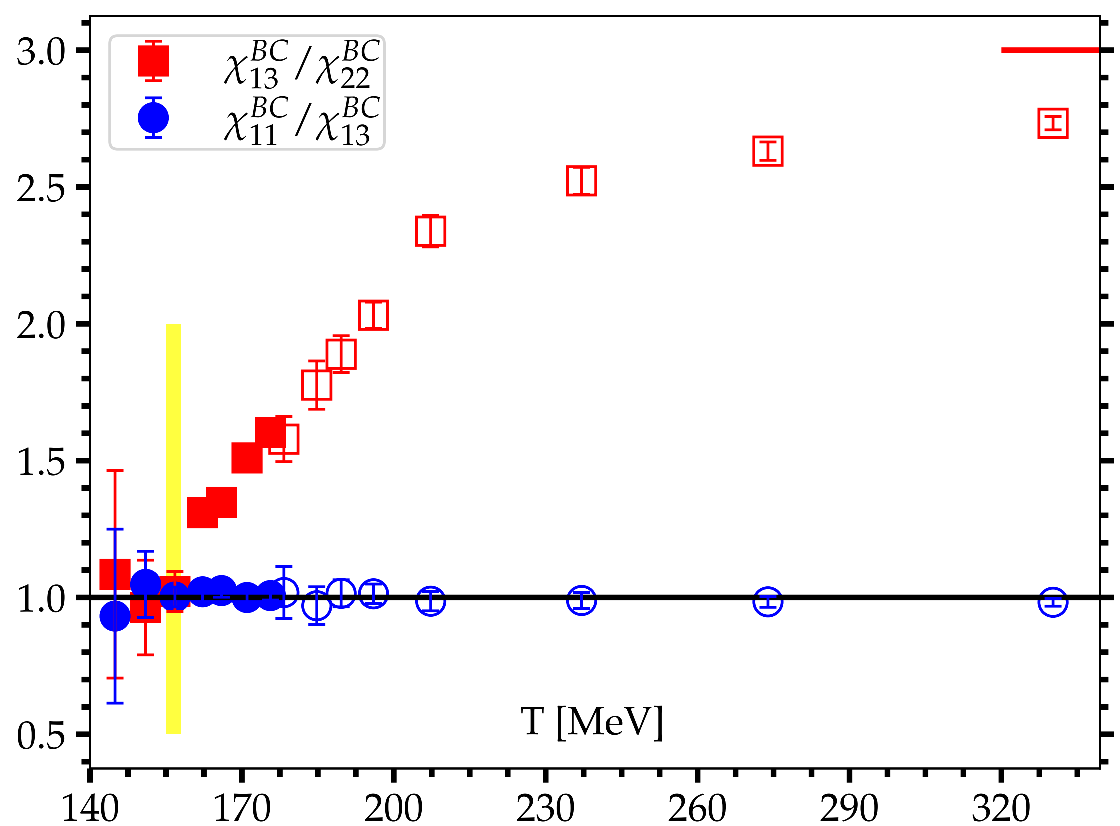


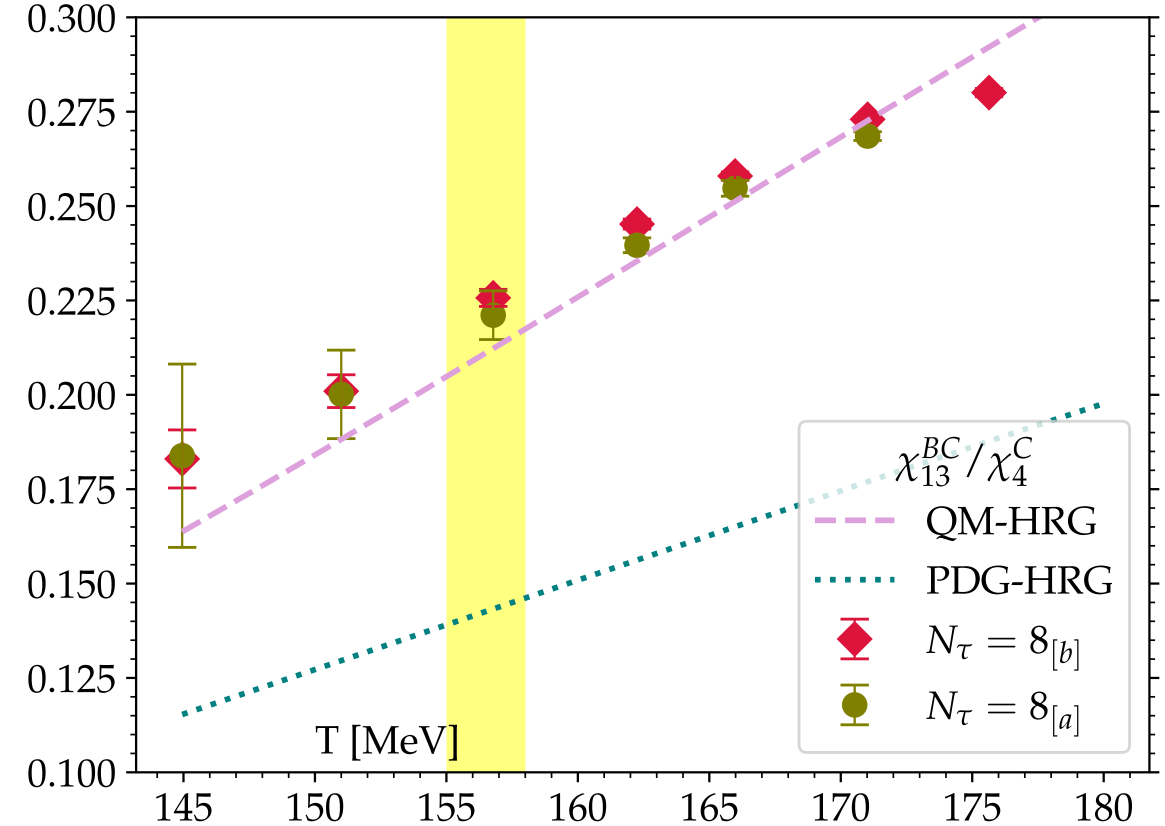
HotQCD, PLB850 (2024)

Ongoing work led by Sipaz Sharma

- Ratios of baryon-charm fluctuations (left).

Charm sector





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- Comparison with HRG (right).

Charm sector

- Extension of the HRG model:

Mukherjee, Petreczky, Sharma, PRD93 (2016)

$$P_C(T, \overrightarrow{\mu}) = P_M^C(T, \overrightarrow{\mu}) + P_B^C(T, \overrightarrow{\mu})$$
$$+P_q^C(T)\cosh\left(\frac{2}{3}\hat{\mu}_Q + \frac{1}{3}\hat{\mu}_B + \hat{\mu}_C\right).$$

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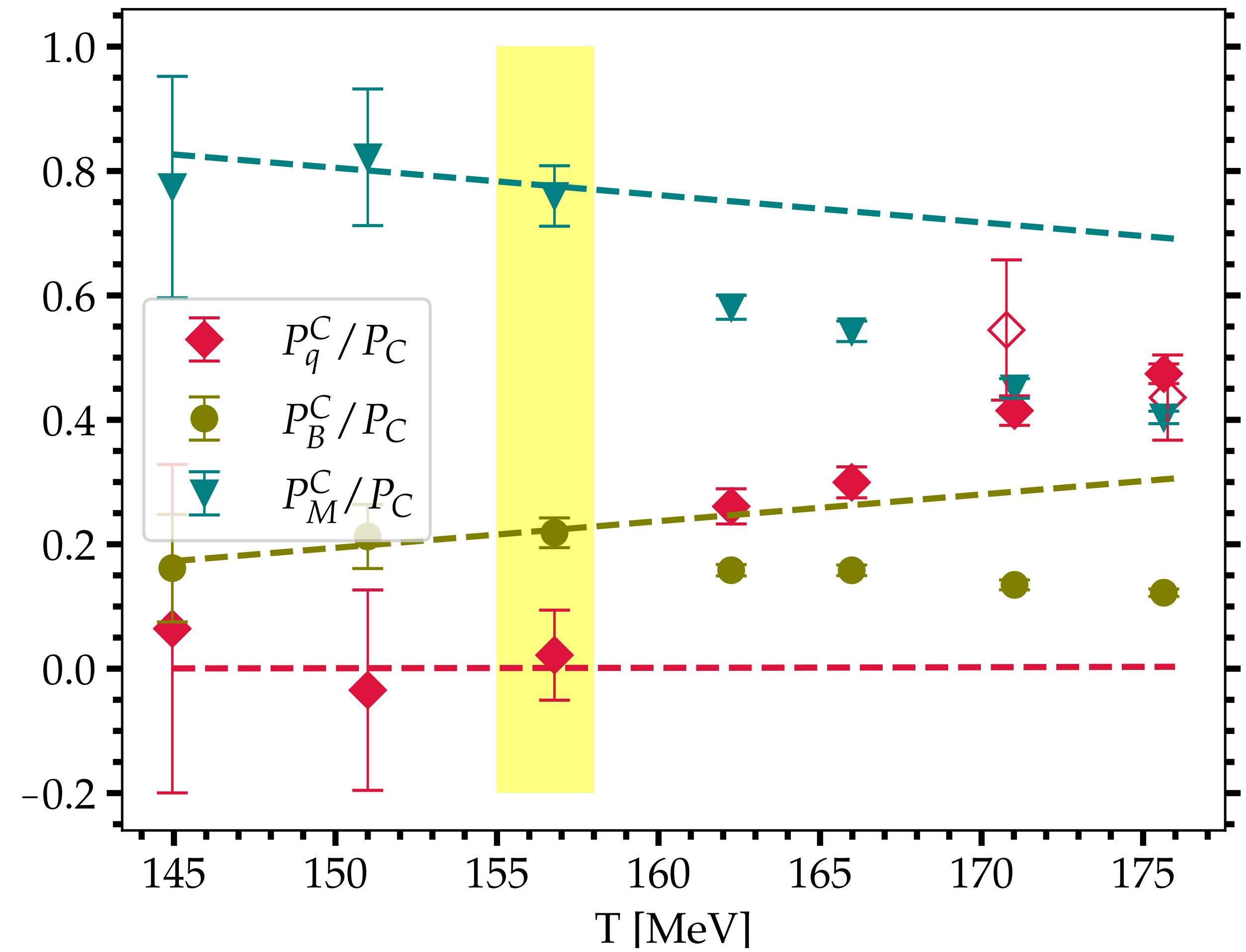
- Partial pressures:

$$P_q^C = 9(\chi_{13}^{BC} - \chi_{22}^{BC})/2,$$
$$P_B^C = (3\chi_{22}^{BC} - \chi_{13}^{BC})/2,$$
$$P_M^C = \chi_4^C + 3\chi_{22}^{BC} - 4\chi_{13}^{BC}.$$

Charm sector



HotQCD, PLB850 (2024)

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- Partial pressures of charmed mesons, charmed baryons and charm quarks.
- Dashed lines correspond to QM-HRG model.

2+1+1 QCD equation of state

- The 2+1+1 flavor trace anomaly

$$\frac{\Theta^{\mu\mu}}{T^4} = -R_\beta(\beta)\left[\Delta(S_g) + \right]$$

$$+R_\beta(\beta)R_{m_s}(\beta)\left[2m_l\Delta(\bar{\psi}_l\psi_l) + m_s\Delta(\bar{\psi}_s\psi_s)\right]$$

2+1+1 QCD equation of state

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$$\frac{\Theta^{\mu\mu}}{T^4} = -R_\beta(\beta)\left[\Delta(S_g) + R_u(\beta)\Delta\left(\frac{dS_g}{du_0}\right)\right]$$

$$+R_\beta(\beta)R_{m_s}(\beta)\left[2m_l\Delta(\bar{\psi}_l\psi_l) + m_s\Delta(\bar{\psi}_s\psi_s)\right]$$

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- Additional beta-functions:

$$R_u(\beta)=\beta\,\frac{\mathrm{d}u_0(\beta)}{\mathrm{d}\beta},\qquad R_\epsilon(\beta)=\frac{\mathrm{d}\epsilon_N(\beta)}{\mathrm{d}\beta}.$$

2+1+1 QCD equation of state

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2+1+1 QCD equation of state

- Our strategy:
  - compute the pressure at fixed N_τ and take the continuum limit,
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  - use the $m_\pi = 300$ MeV line of constant physics,
  - connect with the 2+1 equation of state at the physical mass around $T \approx 250$ MeV.

2+1+1 EoS: $T = 0$ statistics

β	V	am_l	am_s	am_c	a, fm	TU
5.400	$16^3 \times 40$	0.0182	0.091	1.339	0.220	20K
5.469	$24^3 \times 32$	0.01856	0.0928	1.263	0.206	19K
5.541	$24^3 \times 32$	0.01718	0.859	1.157	0.192	18K
5.600	$16^3 \times 48$	0.0157	0.0785	1.08	0.181	69K
5.663	$24^3 \times 32$	0.01506	0.0753	0.996	0.170	28K
5.732	32^4	0.01394	0.0697	0.913	0.159	52K
5.800	$16^3 \times 48$	0.013	0.065	0.838	0.151	99K
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5.855	32^4	0.01216	0.0608	0.782	0.140	54K
5.925	32^4	0.01122	0.0561	0.716	0.130	55K
6.000	$24^3 \times 64$	0.0102	0.0509	0.635	0.121	111K
6.060	32^4	0.00962	0.0481	0.603	0.113	52K
6.122	32^4	0.00896	0.0448	0.558	0.106	38K
6.180	32^4	0.0084	0.042	0.518	0.100	38K
6.238	32^4	0.00784	0.0392	0.482	0.095	40K
6.300	$32^3 \times 96$	0.0074	0.037	0.44	0.089	16K
6.358	32^4	0.00682	0.0341	0.416	0.084	52K
6.445	32^4	0.00616	0.0308	0.374	0.077	95K
6.530	$36^3 \times 48$	0.0056	0.028	0.338	0.070	11K
6.632	48^4	0.00498	0.0249	0.300	0.063	3K
6.720	$48^3 \times 144$	0.0048	0.024	0.286	0.058	6K
6.875	$48^3 \times 64$	0.0038	0.019	0.228	0.050	3K
7.000	$64^3 \times 192$	0.00316	0.0158	0.188	0.045	6K
7.140	$64^3 \times 72$	0.0029	0.0145	0.172	0.039	4K
7.285	$64^3 \times 96$	0.00248	0.0124	0.148	0.034	4K

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6.180	32^4	0.0084	0.042	0.518	0.100	38K
6.238	32^4	0.00784	0.0392	0.482	0.095	40K
6.300	$32^3 \times 96$	0.0074	0.037	0.44	0.089	16K
6.358	32^4	0.00682	0.0341	0.416	0.084	52K
6.445	32^4	0.00616	0.0308	0.374	0.077	95K
6.530	$36^3 \times 48$	0.0056	0.028	0.338	0.070	11K
6.632	48^4	0.00498	0.0249	0.300	0.063	3K
6.720	$48^3 \times 144$	0.0048	0.024	0.286	0.058	6K
6.875	$48^3 \times 64$	0.0038	0.019	0.228	0.050	3K
7.000	$64^3 \times 192$	0.00316	0.0158	0.188	0.045	6K
7.140	$64^3 \times 72$	0.0029	0.0145	0.172	0.039	4K
7.285	$64^3 \times 96$	0.00248	0.0124	0.148	0.034	4K

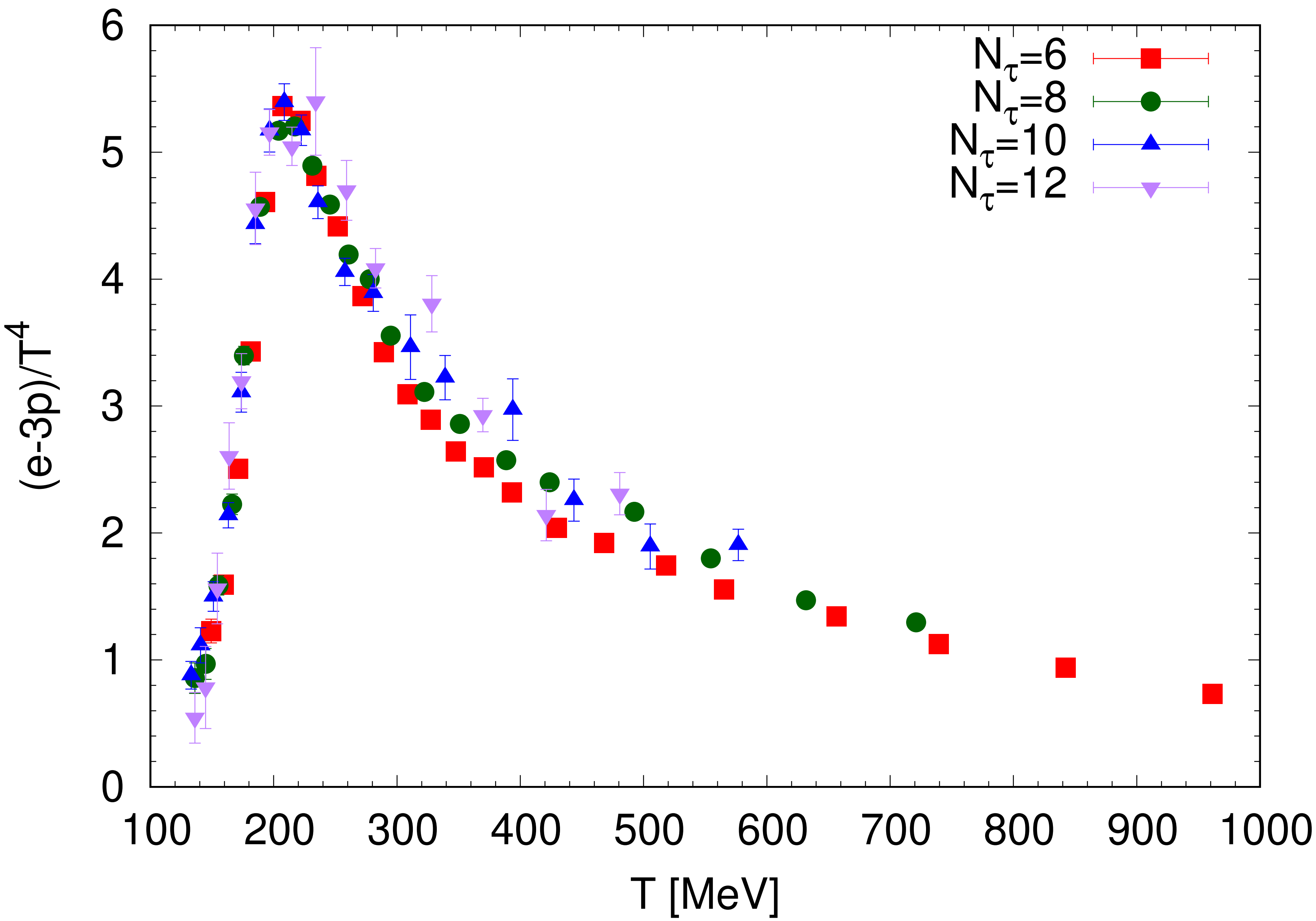
2+1+1 EoS: $T > 0$ statistics

β	$N_\tau = 6$		$N_\tau = 8$		$N_\tau = 10$		$N_\tau = 12$	
	T	TU	T	TU	T	TU	T	TU
5.400	149	50K						
5.469	160	50K						
5.541	171	50K						
5.600	182	50K	136	114K				
5.663	193	50K	145	74K				
5.732	207	50K	155	86K				
5.800	218	50K	163	81K	131	143K		
5.855	235	50K	176	105K	140	98K		
5.925	253	50K	190	105K	152	125K		
6.000	272	50K	204	105K	163	95K	136	95K
6.060	291	50K	218	99K	175	42K	145	21K
6.122	310	50K	233	101K	186	42K	155	21K

2+1+1 EoS: $T > 0$ statistics

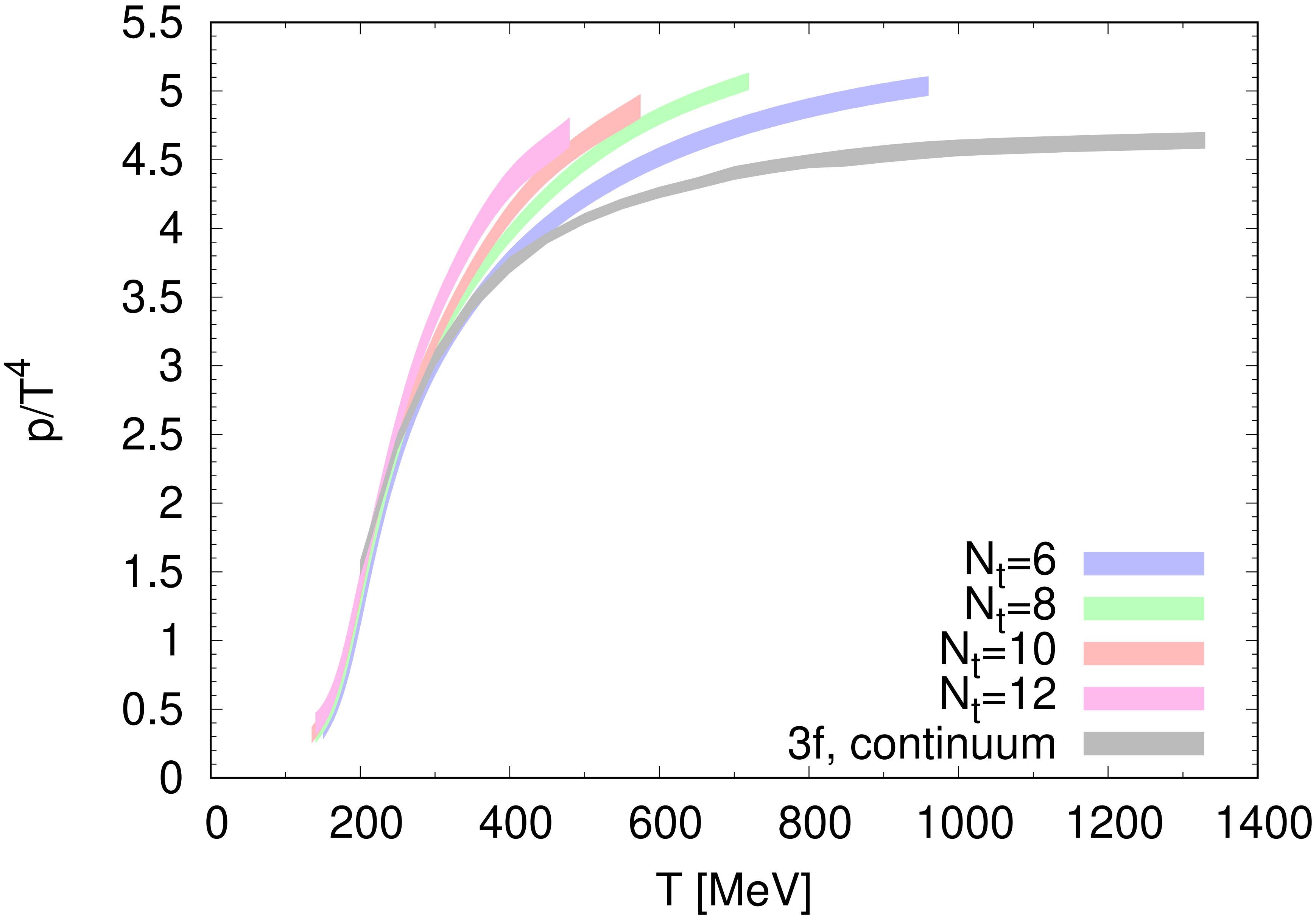
β	$N_\tau = 6$		$N_\tau = 8$		$N_\tau = 10$		$N_\tau = 12$	
	T	TU	T	TU	T	TU	T	TU
6.180	329	50K	247	99K	197	40K	165	32K
6.238	346	50K	260	96K	208	47K	173	27K
6.300	369	50K	277	98K	222	99K	184	28K
6.358	391	50K	294	96K	235	24K	196	81K
6.445	427	50K	320	96K	256	35K	214	75K
6.530	470	50K	352	99K	282	59K	235	10K
6.632	522	50K	391	96K	313	9K	261	59K
6.720	567	50K	425	100K	340	10K	284	68K
6.875	658	50K	493	108K	395	9K	329	70K
7.000	731	40K	548	110K	438	20K	366	56K
7.140	843	40K	632	101K	506	19K	422	61K
7.285	967	40K	725	101K	580	17K	483	101K

2+1+1 flavor QCD trace anomaly



- The trace anomaly in 2+1+1 flavor QCD at different cutoffs $N_\tau = 6, 8, 10$ and 12.

2+1+1 flavor QCD pressure



- The pressure in 2+1+1 flavor QCD at different cutoffs $N_\tau = 6, 8, 10$ and 12.
- The errors are purely statistical.

Conclusion

- Ongoing calculation of the 2+1+1 flavor QCD equations of state.
- The strategy is to compute the pressure with $m_\pi = 300$ MeV, take the continuum limit and stitch together with the 2+1 flavor equation of state at the physical pion mass at an appropriate temperature.
- The statistical errors for the finest lattices $N_\tau = 12$ are predominantly from zero-temperature subtraction.
- Systematic errors, in particular, scale setting need to be addressed in the final analysis.