Utilizing the Twisted Boundary Conditions to Determine DD^* Scattering Phase Shifts

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details at arXiv:2410.09815 [hep-lat]

Motivation for investigating DD^*

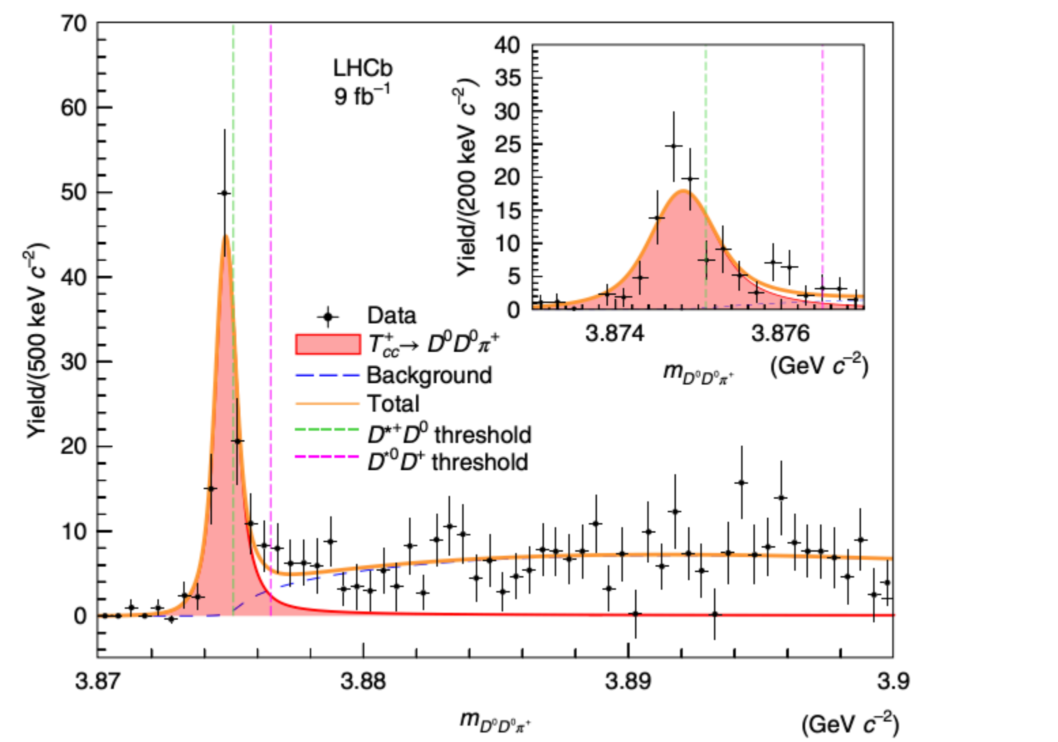
- Experimental observation in a decay

$$T_{cc}^+ \to D^{*0}\, D^+,\quad D^{*0} \to D^0\, \pi^0$$

$$T_{cc}^+ \to D^{*+}\, D^0,\quad D^{*+} \to D^0\, \pi^+$$

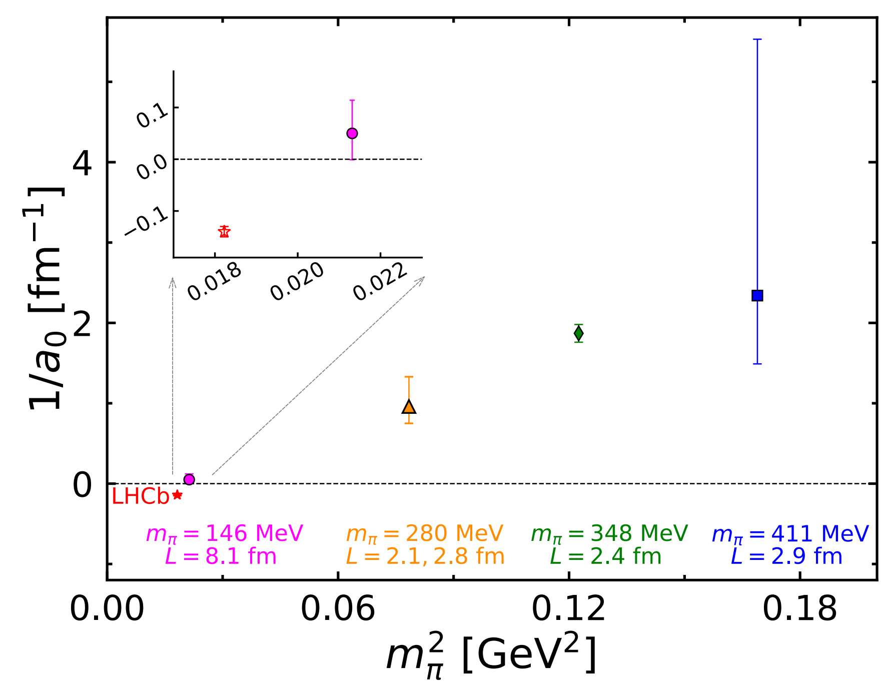
with flavor $cc\bar{u}\bar{d}$, $I\,(J^P) = 0\;(1^+)$.

LHCb collab. Nat. Commun. 13, 3351 (2022).



- Some lattice QCD calculations imply the existence of T_{cc}^+ as the bound state of D and D^*.

HALQCD, Phys. Rev. Lett. 131, 161901 (2023)
S. Collins et al, Phys. Rev. D 109, 094509 (2024)



Basics of Lattice QCD

$$Z = \int DA_\mu e^{iS}$$

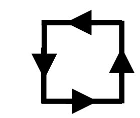
$$S = \frac{1}{4}\int d^4x \ \mathrm{tr} F_{\mu\nu}F^{\mu\nu}$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z}\int DA_\mu \mathcal{O}(A_\mu) e^{iS}$$

- Space time Discretization
- Wick rotation

$$Z = \int \prod_{n\in\mathbb{Z}^4} dU_\mu(n) e^{-S_E}$$

$$S_E = \beta \sum_{n,\mu\neq\nu} \left(1 - \frac{1}{3}\mathrm{Re}\ \mathrm{tr} U_{\mu\nu}(n)\right)$$

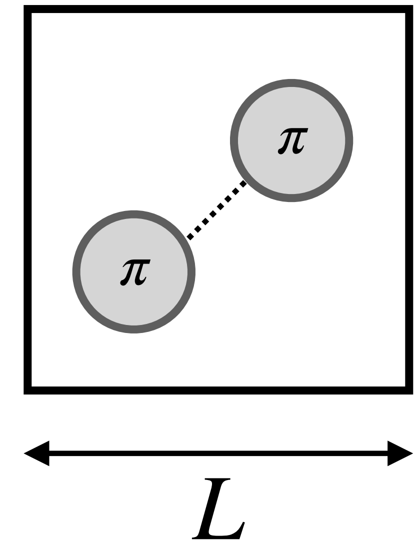


$$\langle \mathcal{O} \rangle = \int \prod_{n\in\mathbb{Z}^4} dU_\mu(n)\mathcal{O}(U_\mu)\frac{e^{-S_E}}{Z}$$

$$\approx \frac{1}{N}\sum_{i=1}^{N} \mathcal{O}(\{U_\mu\}_i)$$

- Path integral → Multiple integral
- Sample gauge configurations $\{U_\mu\}_i$ accodring to e^{-S_E}/Z to use the Monte Carlo method.

2 hadron bound state in finite size Box?



$$\langle 0|H(t)H^\dagger(0)|0\rangle = \sum_n \langle 0|e^{Ht}H(0)e^{-Ht}|n\rangle\langle n|H^\dagger(0)|0\rangle$$

$$\xrightarrow{t\to\infty} |\langle 0|H(0)|GS\rangle|^2 e^{-E_{GS}t}$$

Numerically Obtained.

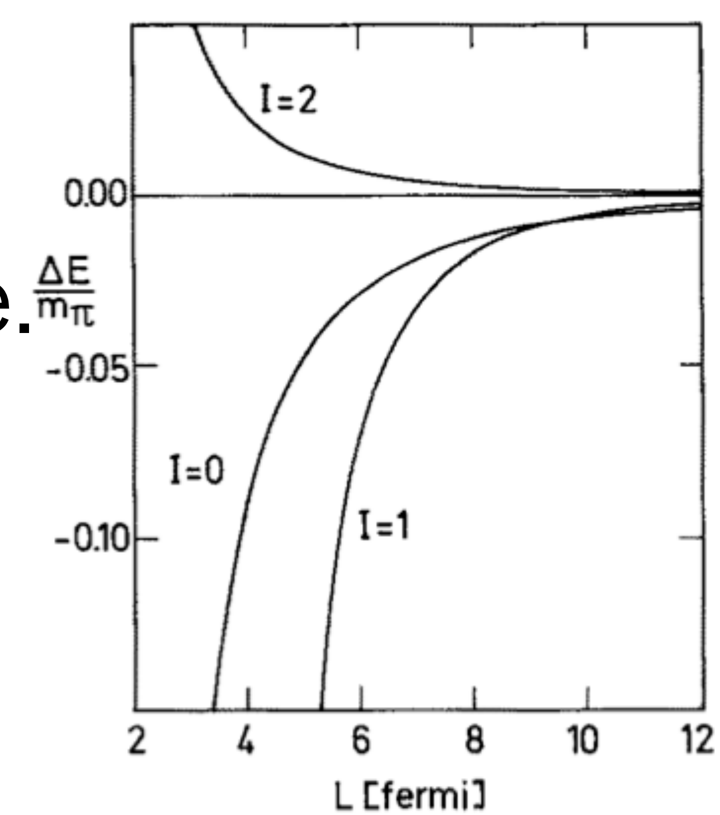
H : Hadron operator

- Energy spectra are **discretized** in a finite box and depend on the volume.
- Energy levels **below threshold** can be scattering states in infinite volume space.

$$\Delta E = -\frac{4\pi a_0}{mL^3} + O(L^{-4})$$

$$E = 2m + \Delta E$$

For two pion cases.



M.Lüscher, Commun.Math.Phys. 105 (1986).

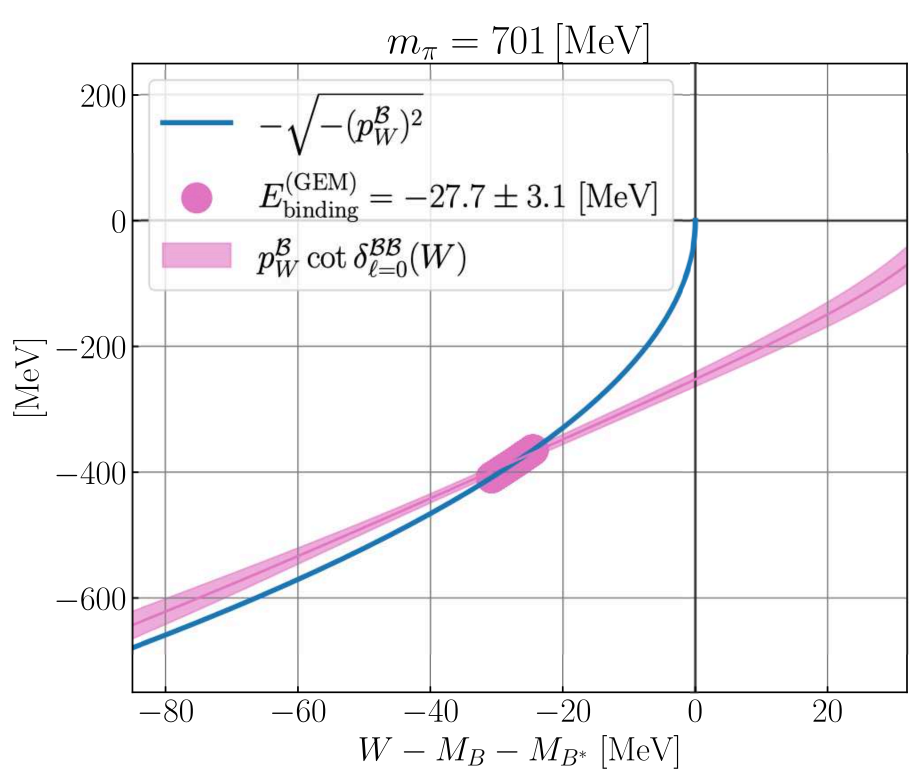
Scattering Phase Shift

$$S = e^{2i\delta(k)} = \frac{\cot\delta(k) + i}{\cot\delta(k) - i}$$

Scattering phase shift $\delta(k)$ is defined through S matrix.

Bound states

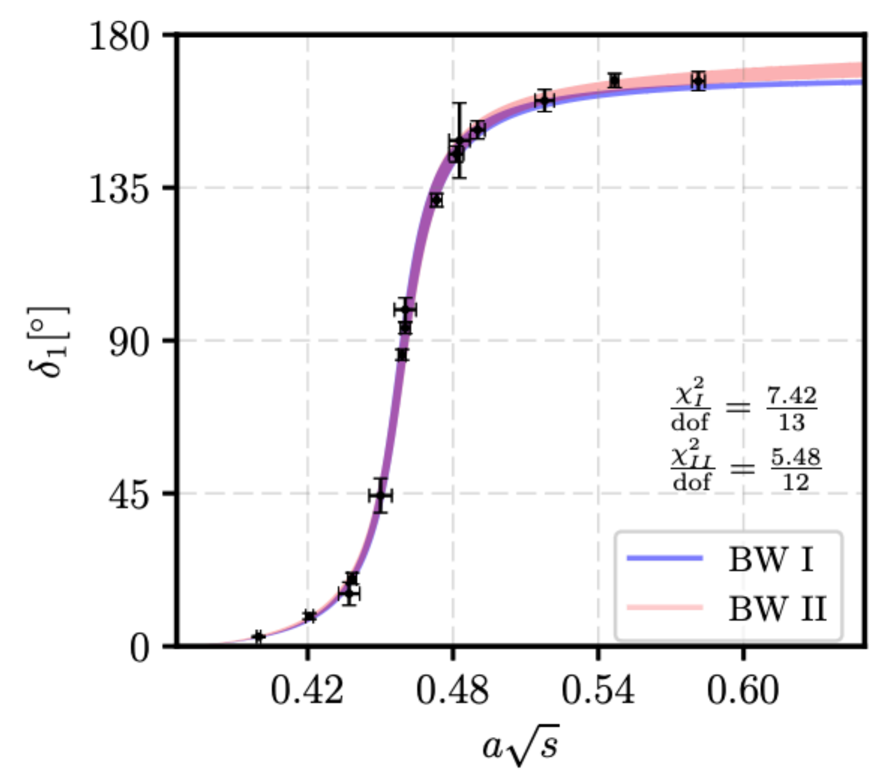
$$\cot\delta(k) = i \leftrightarrow k\cot\delta(k) = -\sqrt{-k^2}$$



S. Aoki, T. Aoki, arXiv:2212.00202 (2022).

Resonances

Rapidly cross $\delta(k) = \frac{\pi}{2}$.



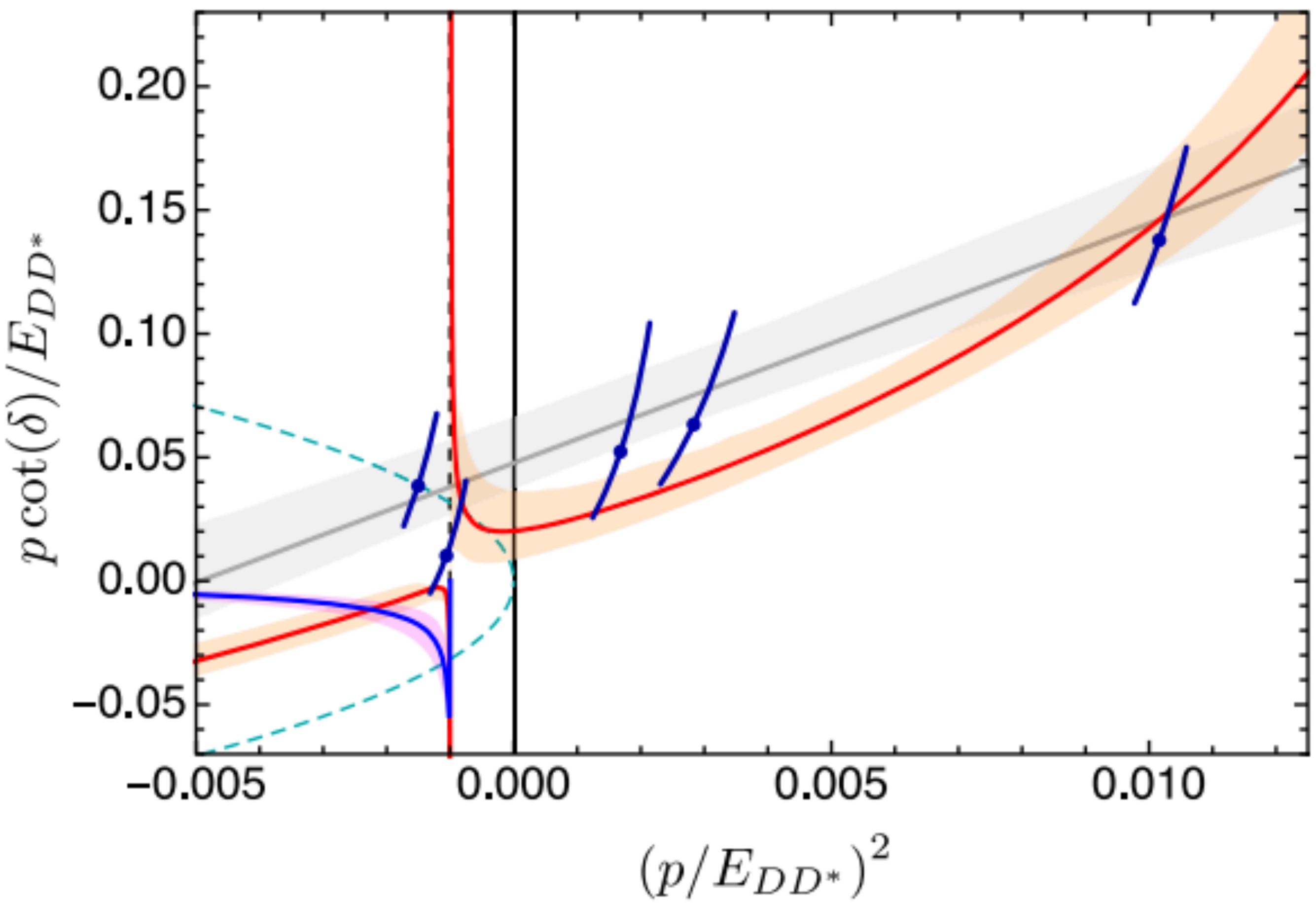
Leskovec, arXiv:1810.01927 (2018).

Search of Bound state from Lattice QCD

(Virtual) Bound states

$$\cot\delta(p) = i \leftrightarrow p\cot\delta(p) = -\sqrt{-p^2}$$

$$S = e^{2i\delta(p)} = \frac{\cot\delta(p) + i}{\cot\delta(p) - i}$$



There seems to be a singular point below threshold due to the lefthand cut.

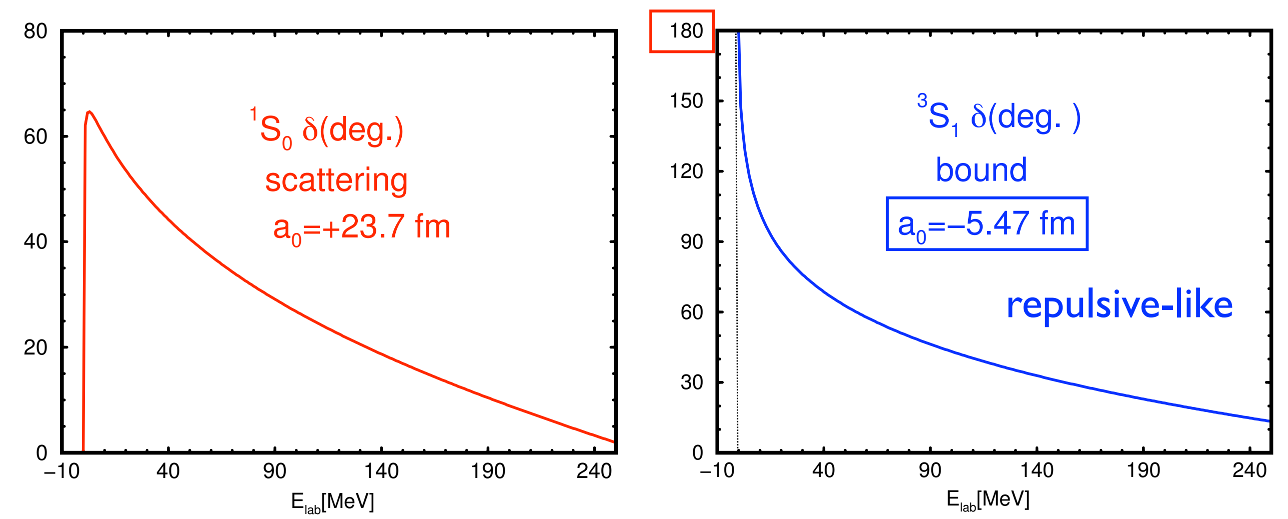
M.-L. Du, et al., Phys. Rev. Lett. 131, 131903 (2023)

Our purpose

- Levinson's theorem tells us that the scattering phase shift at zero momentum $\delta_l(0)$ and the number of bound states n are related as

$$\delta_l(0) = n\pi\,.$$

Nucleon-Nucleon scattering



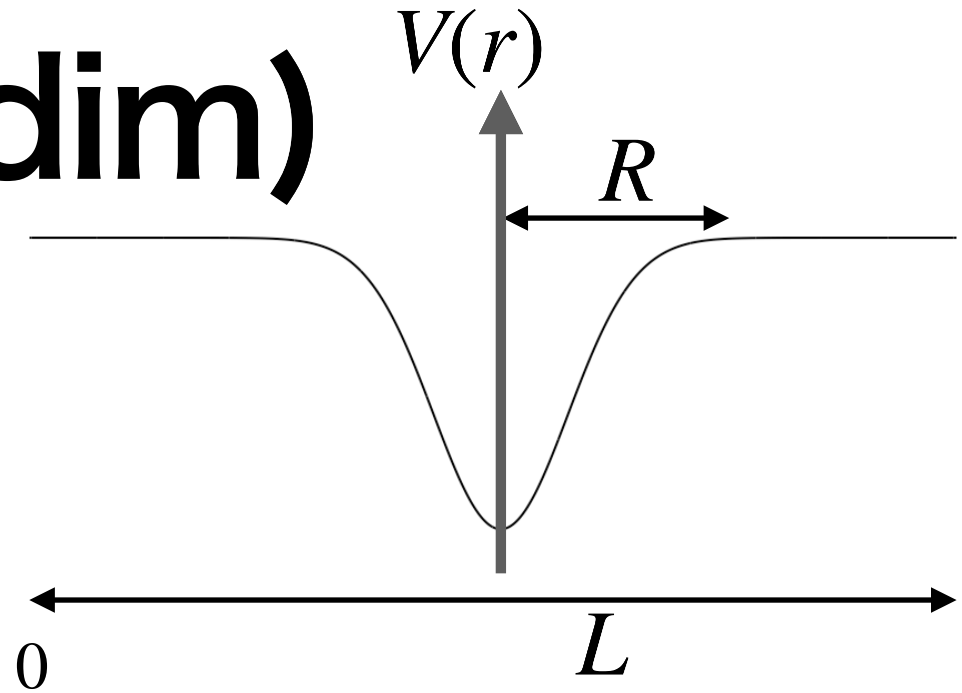
No bound state | Deuteron

- We can know n by tracing the behavior of $\delta_l(k)$.

- We calculate scattering phase shifts by **Lüscher's method**.

- We access the low energy information by employing the **twisted boundary conditions**.

Lüscher formula(1dim)



$$\left(\frac{-1}{m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x) = E\psi(x)$$

$V(r > R) = 0$

Assume the finite interaction range R.

<u>Solution for $|x| > R$</u>

$$\psi(x) = e^{-ik|x|} + e^{2i\delta(k)}e^{ik|x|}, \qquad E = \frac{k^2}{m}$$

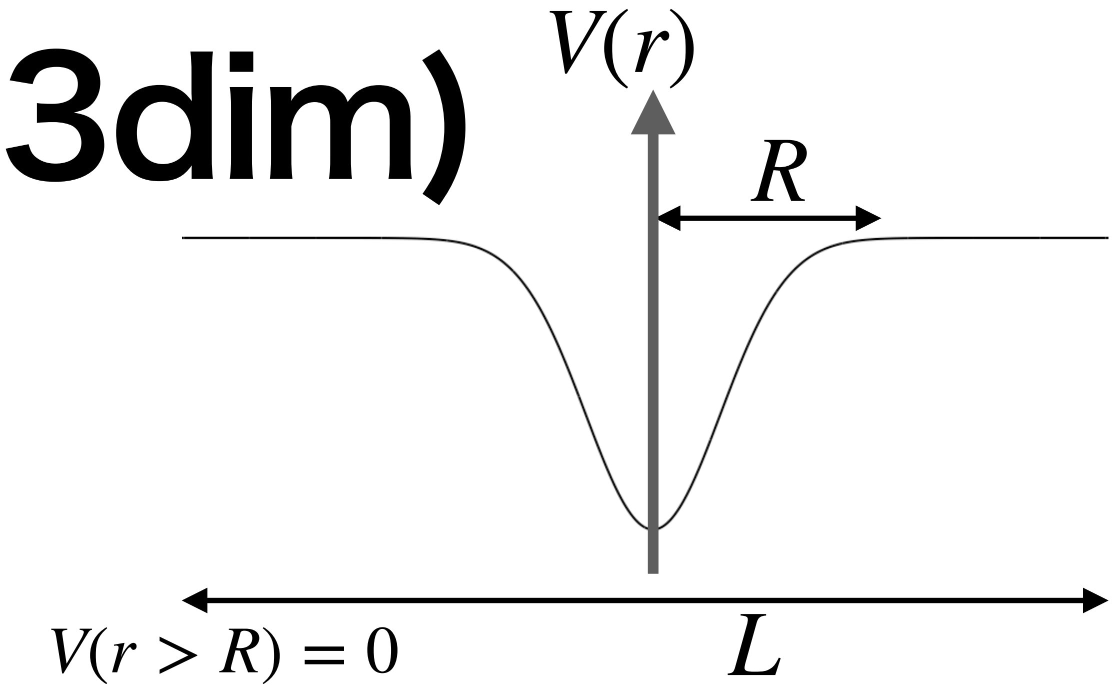
Periodic BC : $\psi(x - \frac{L}{2}) = \psi(x + \frac{L}{2})$

<u>Quantization cond.</u>

$$e^{2i\delta(k)}e^{ikL} = 1 \quad \leftrightarrow \quad \delta(k) + kL = \mathbf{2\pi n} \ (n \in \mathbb{Z})$$

- Finite volume gives rise the relation between $\delta(k)$ and E.
- k is discretized.

Lüscher formula(3dim)



$$\left(\frac{-1}{m}\Delta + V(r)\right)\psi(\vec{r}) = E\psi(\vec{r})$$

Solution for $r > R$

$$\psi(\vec{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)\psi_{lm}(r)$$

$$\psi_{lm}(r) = b_{lm}\left(\alpha_l(k)j_l(kr) + \beta_l(k)n_l(kr)\right)$$

$$E = \frac{k^2}{m}\ , \qquad \cot\delta_l(k) = \frac{\alpha_l(k)}{\beta_l(k)}$$

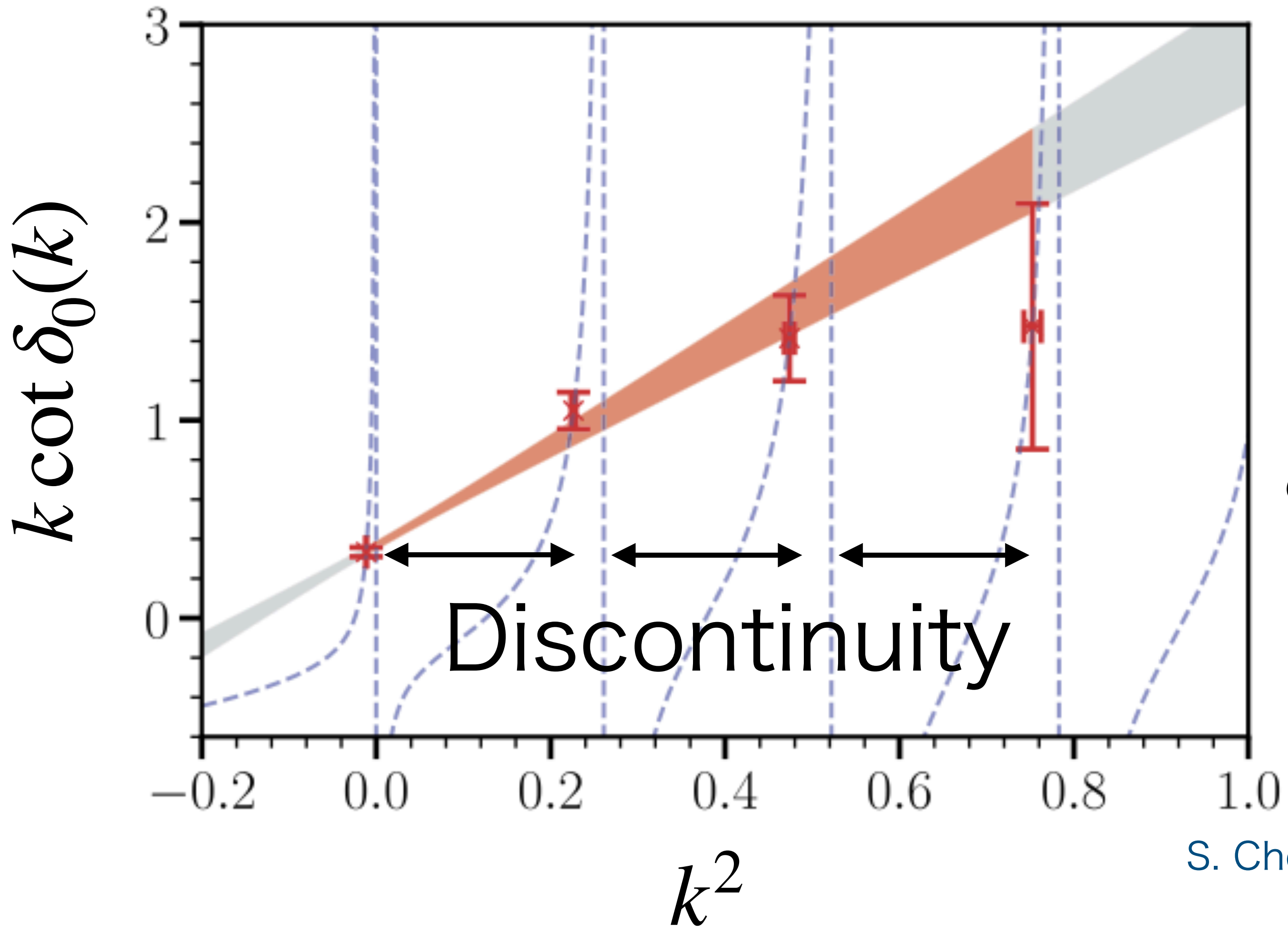
Periodic BC, ignoring $l \geq 4$.

Quantization cond.

$$\cot\delta_0(k) = \frac{Z_{00}(1;q^2)}{\pi^{3/2}q}$$

$$Z_{00}(s;q^2) = \sum_{\vec{n}\in\mathbb{Z}^3}\frac{(4\pi)^{-1/2}}{(\vec{n}^2 - q^2)^s}\ , \qquad q^2 = \left(\frac{Lk}{2\pi}\right)^2$$

Lüscher's formula with PBC.



PBC is employed.

$\delta(k)$ of DD^*.

S. Chen, et al., Phys. Lett. B 833, 137391 (2022).

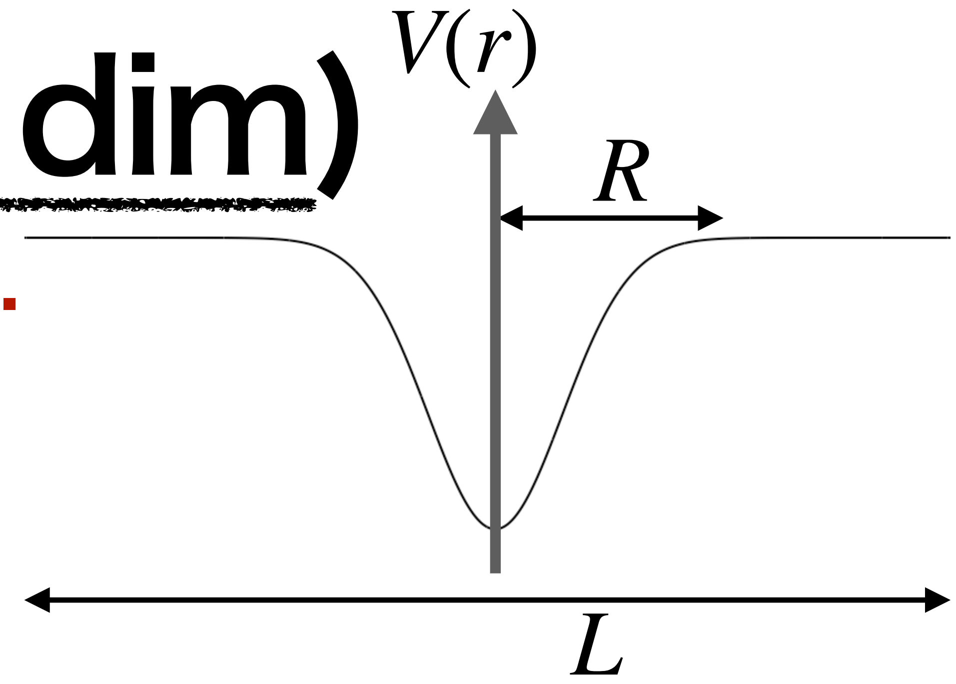
As long as PBC is employed, $\delta(k)$ is obtained discontinuously.

$$\left(\frac{L\mathbf{P}}{2\pi}\right)^2 = 0,\ 1,\ 2,\ 3,\ \cdots$$

$L \sim 2.4[\mathrm{fm}]$

then $\left(\frac{2\pi}{L}\right)^2 \sim (0.5\ \mathrm{GeV})^2$

Lüscher's formula(1dim)



Under the twisted boundary condition.

$$\left(\frac{-1}{m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x) = E\psi(x)$$

$V(r > R) = 0$

Assume the finite interaction range R.

Solutions in $|x| > R$

$$\psi(x) = e^{-ik|x|} + e^{2i\delta(k)}e^{ik|x|}, \qquad E = \frac{k^2}{m}$$

Twisted BC : $e^{i\theta}\psi(x - \frac{L}{2}) = \psi(x + \frac{L}{2})$

Quantization cond.

$$e^{2i\delta(k)}e^{ikL} = e^{i\theta} \longleftrightarrow \delta(k) + kL = 2\pi n + \theta \ (n \in \mathbb{Z})$$

θ is a real value.

Lüscher's formula(3dim)

Under the twisted boundary condition (up to P wave).

One equation with two (δ_0, δ_1) unknown parameters.

$$\begin{vmatrix} \cot\delta_0(k) - M^{\vec{\theta}}_{SS}(q) & M^{\vec{\theta}}_{SP}(q) \\ M^{\vec{\theta}}_{SP}(q) & \cot\delta_1(k) - M^{\vec{\theta}}_{PP}(q) \end{vmatrix} = 0$$

$M^{\vec{\theta}}_{SS}(q)$ etc. are numerically calculable

For some special $\vec{\theta}$

$\vec{\theta} = (0,0,0),\ (\pi,0,0),\ (\pi,\pi,0),\ (\pi,\pi,\pi)$

$$\cot\delta_0(k) = \frac{Z_{00}(1;q^2)}{\pi^{3/2}q}$$

$$0 < \left(\frac{\pi}{L}\right)^2 < 2\left(\frac{\pi}{L}\right)^2 < 3\left(\frac{\pi}{L}\right)^2 < \left(\frac{2\pi}{L}\right)^2$$

↑ Minimum in Moving frame

Strategy to obtain $\delta_0(k)$, $\delta_1(k)$

S. Ozaki, S. Sasaki, Phys. Rev. D87 (2013) 014506

$$\vec{\theta} = (0,0,0),\ (\pi,0,0),\ (\pi,\pi,0),\ (\pi,\pi,\pi)$$

Step.1

$$\cot\delta_0(k) = \frac{Z_{00}(1;q^2)}{\pi^{3/2}q}$$

Effective range expansion

$$k^{2l+1}\cot\delta_l(k) = \frac{1}{a_l} + \frac{r_l}{2}k^2 + \mathcal{O}(k^4)$$

Input by fit

Using $\vec{\theta} = (\theta,\theta,\theta)$

Step.2

$$\cot\delta_1(k) = M_{PP}^{(111)}(q) + \frac{|M_{SP}^{(111)}(q)|^2}{\cot\delta_0(k) - M_{SS}^{(111)}(q)}$$

Input by fit

Using $\vec{\theta} = (0,0,\theta)$

Step.3

$$\cot\delta_0(k) = M_{PP}^{(001)}(q) + \frac{|M_{SP}^{(001)}(q)|^2}{\cot\delta_1(k) - M_{SS}^{(001)}(q)}$$

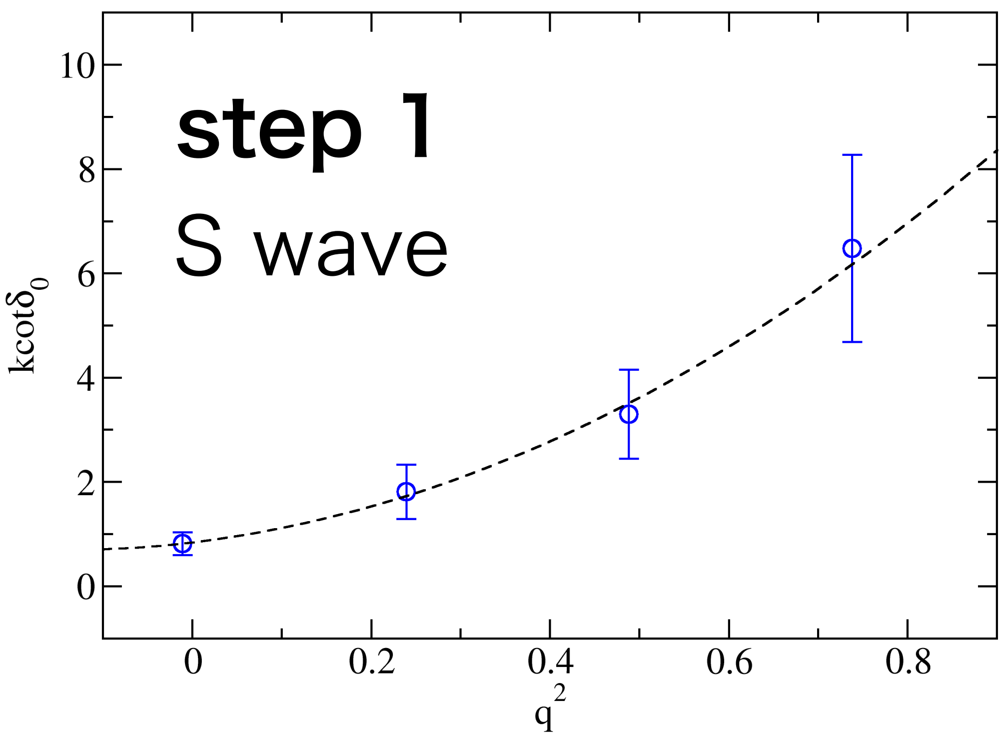
Simulation setup

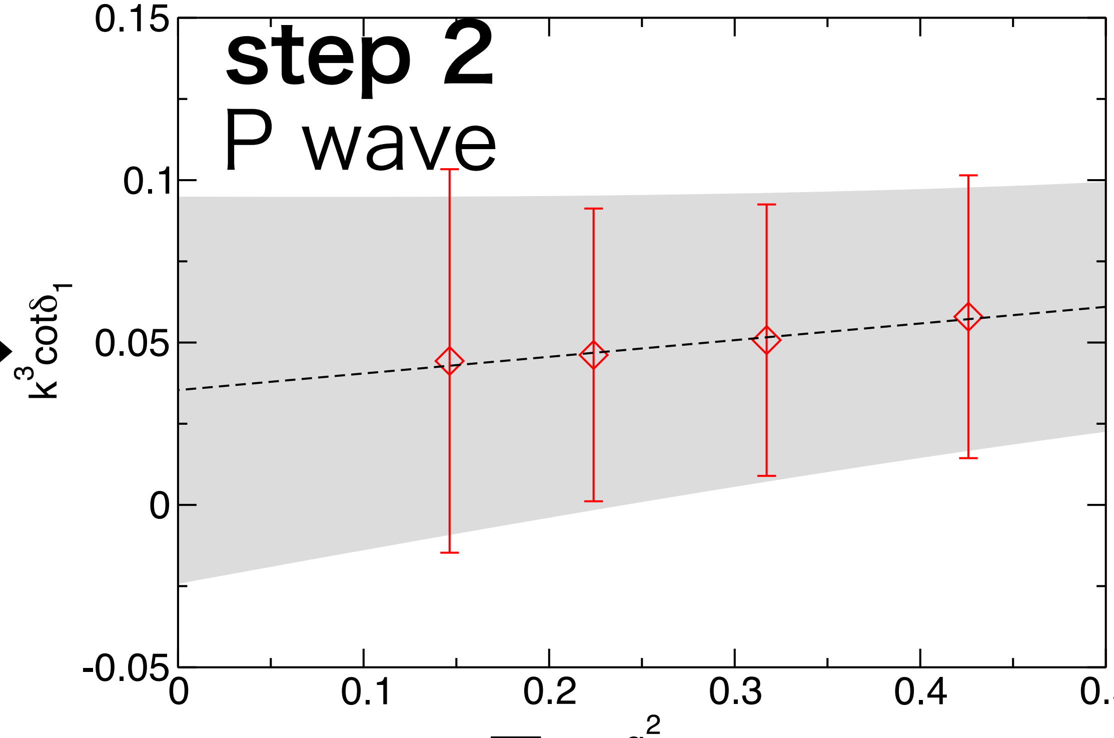
- 2+1 flavor $32^3 \times 64$ PACS-CS gauge configuration with $m_\pi = 295,\ 411$ MeV and $L = 2.9$ fm.
- **Clover** action for **up** and **down** quarks.
- **Relativistic heavy quark (RHQ)** action for **charm** and **bottom** quarks.
- Charm and Bottom quarks obey to twisted BCs.
- Using operators

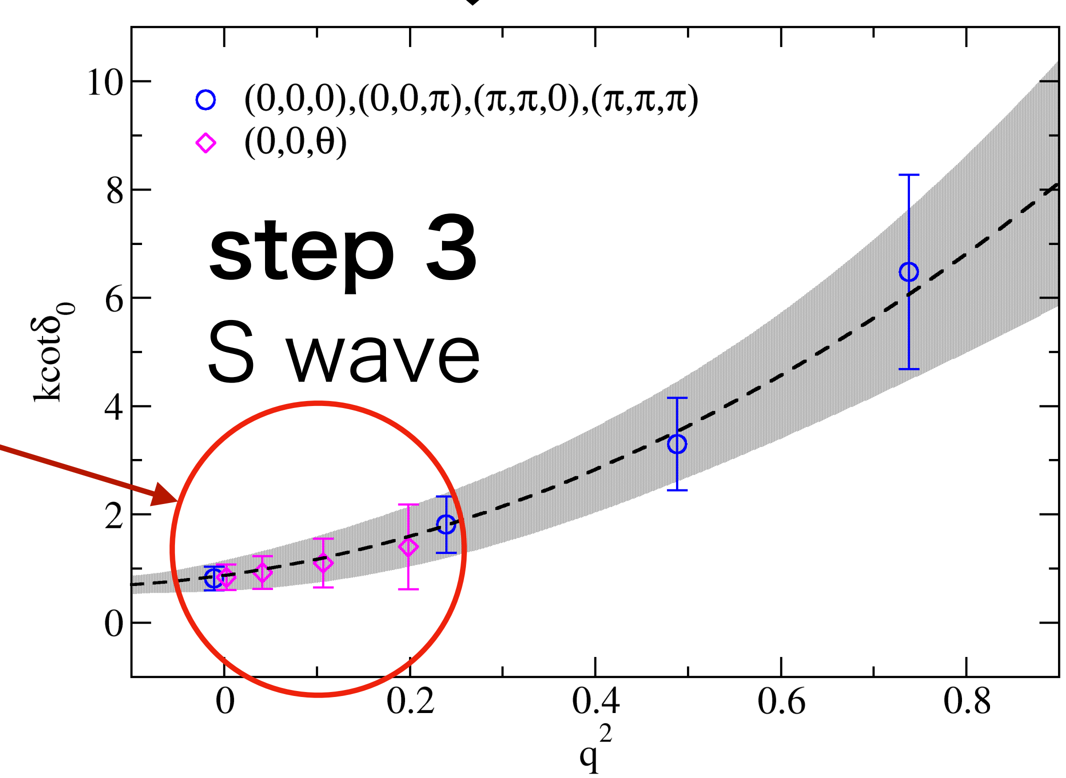
$$O_j = \frac{(\bar{u}\gamma_5 c)\cdot(\bar{d}\gamma_j c) - (\bar{d}\gamma_5 c)\cdot(\bar{u}\gamma_j c)}{\sqrt{2}}$$

Calculation according to the strategy

For DD^* system with $m_\pi = 411$ MeV



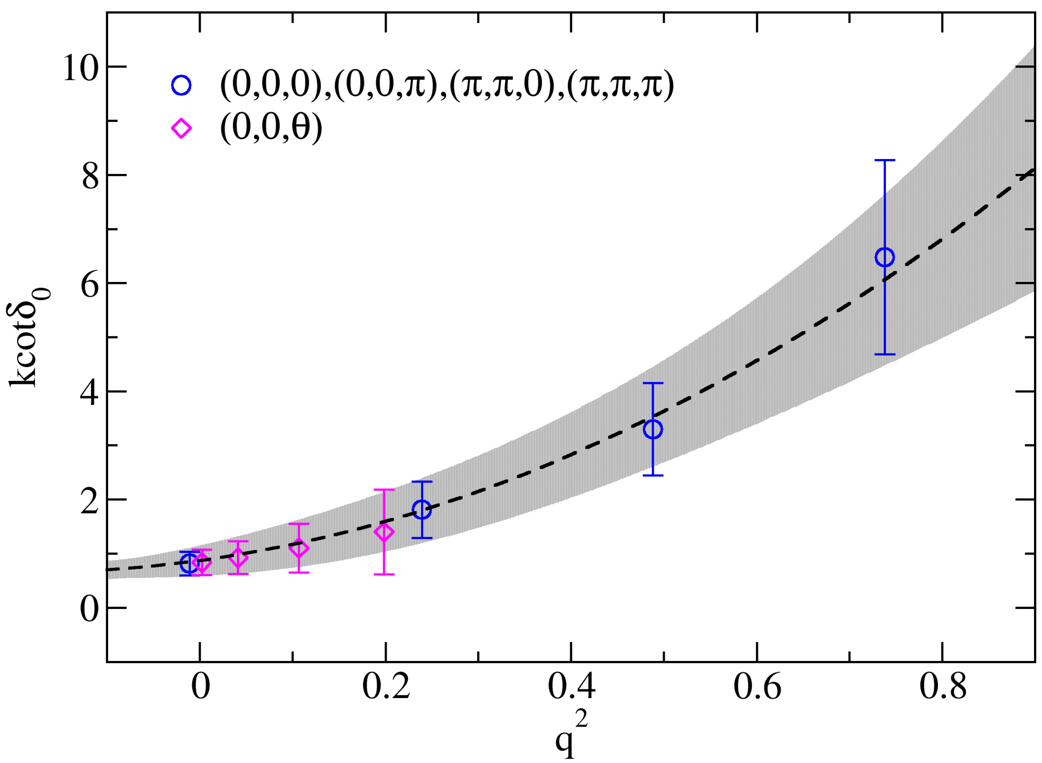


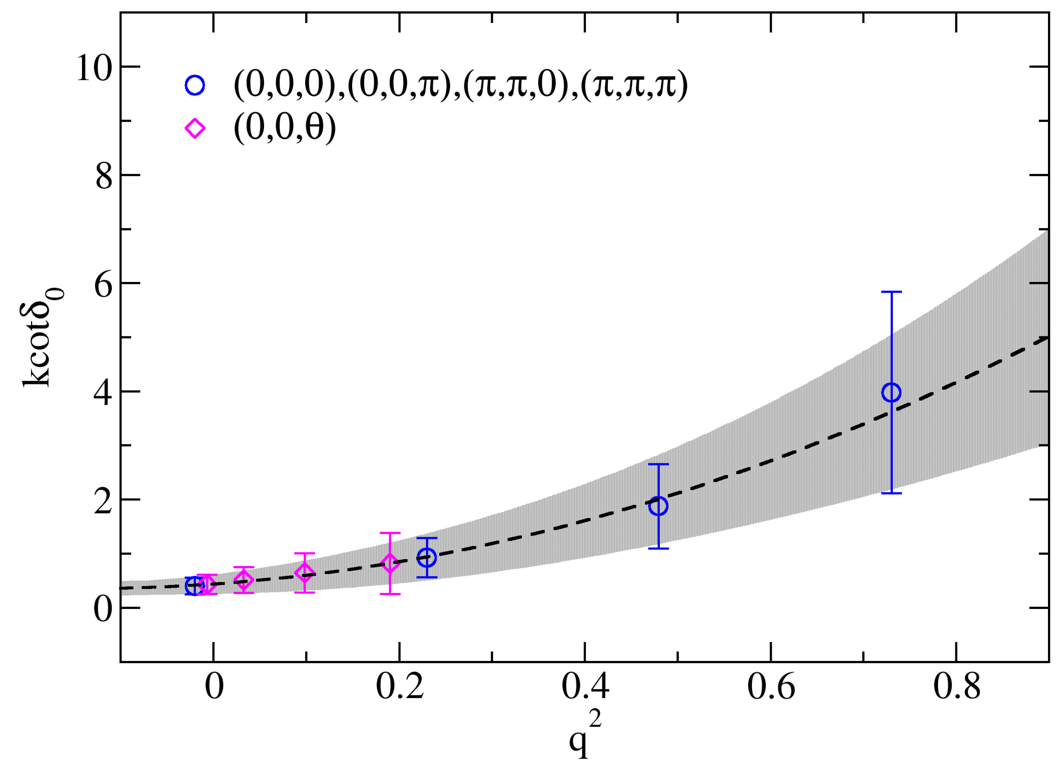


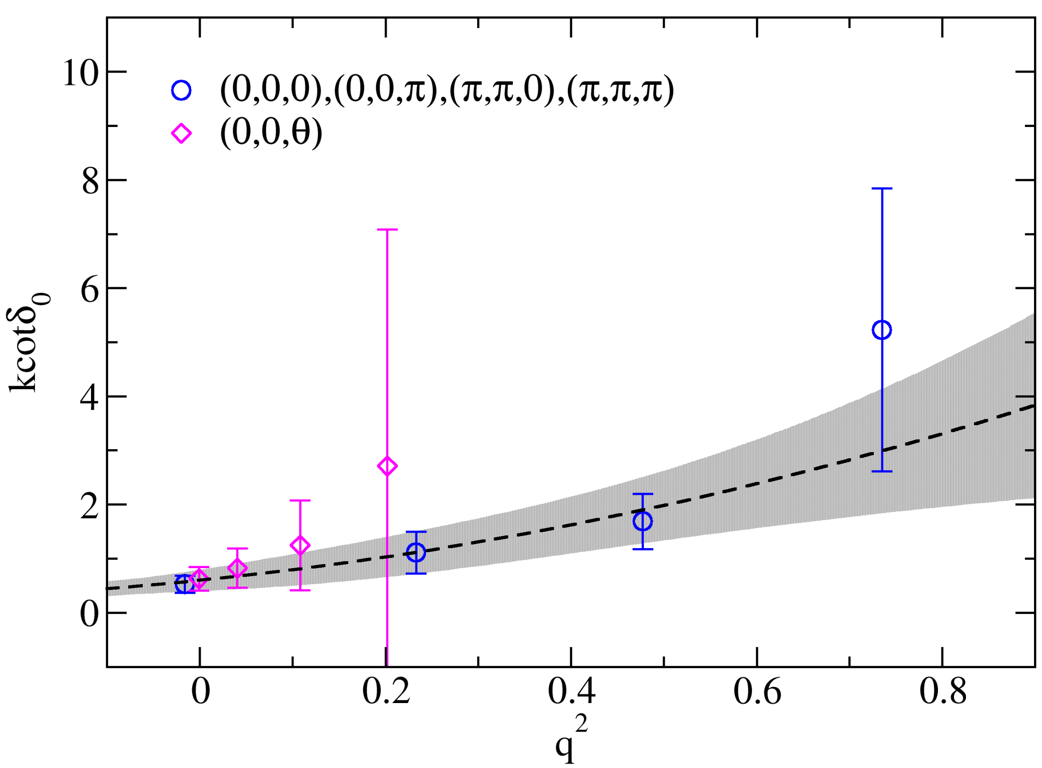
The low energy behavior can be obtained through P wave calculations.

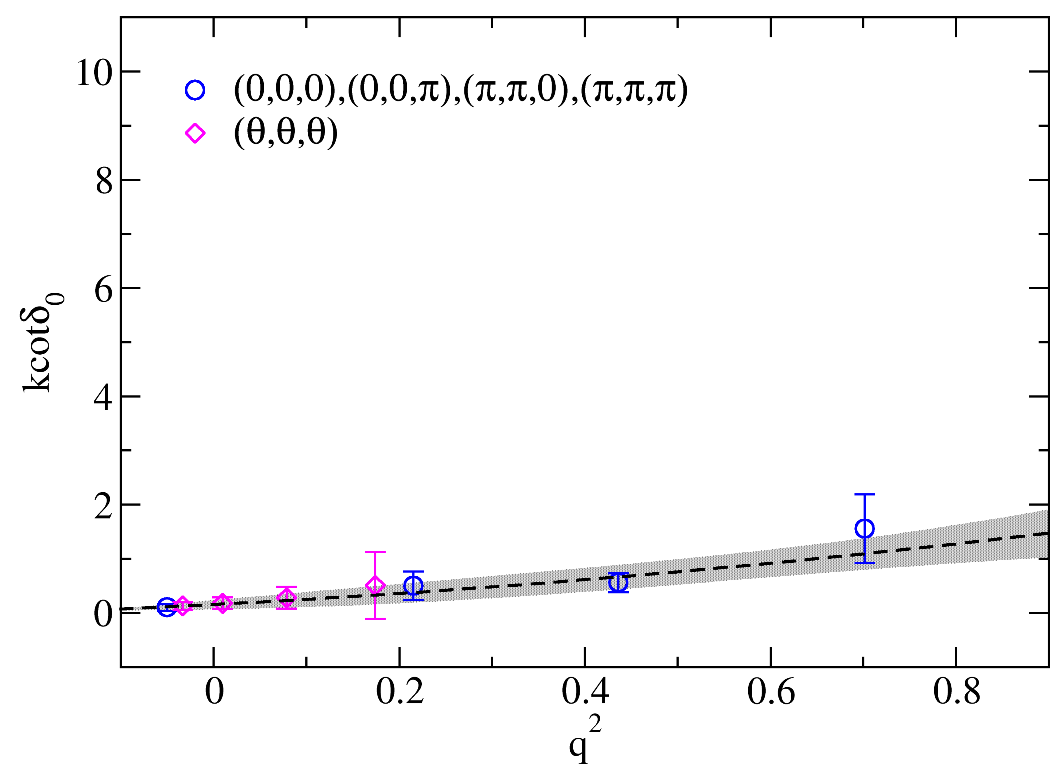
Result of $k\cot\delta_0(k)$

m_π	Charm	Bottom
411 MeV	(see below)	(see below)
295 MeV	(see below)	(see below)



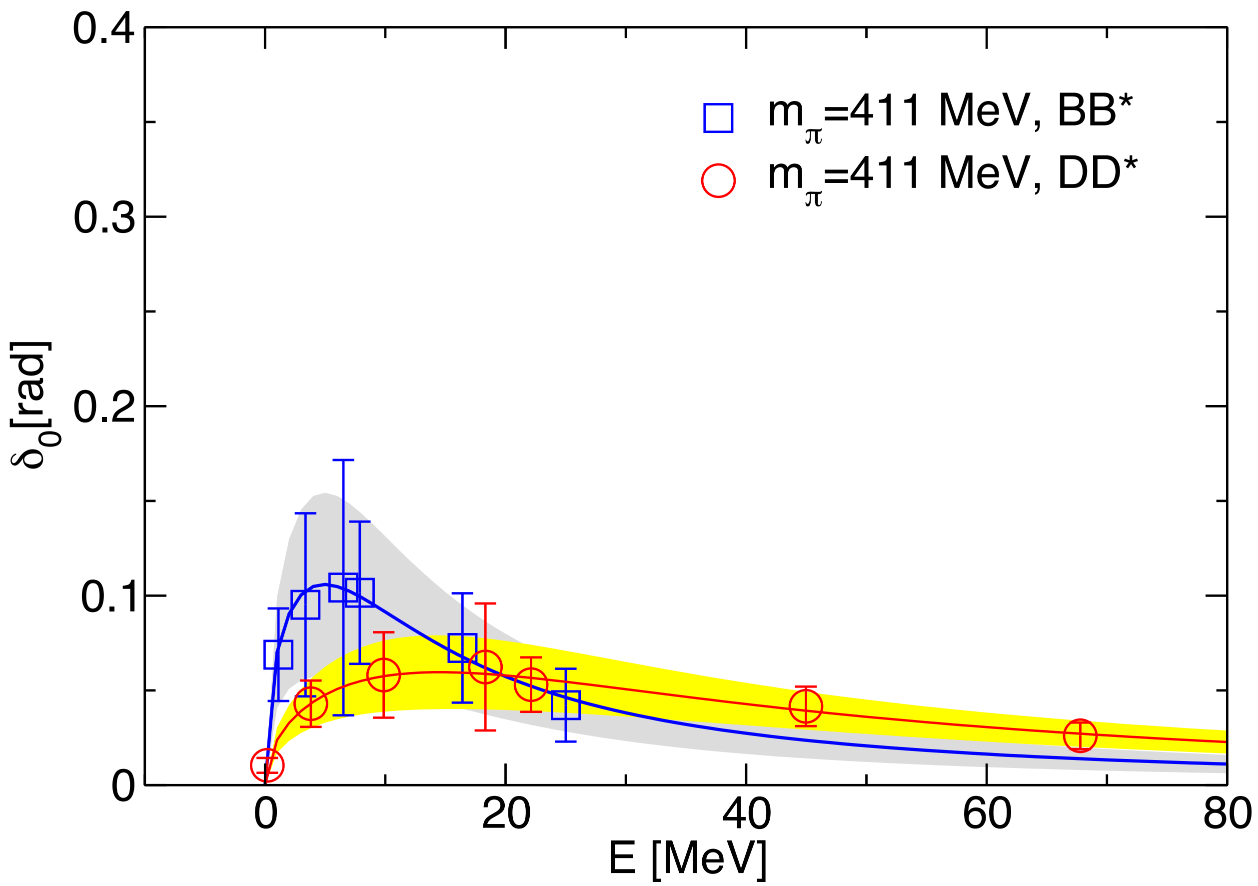




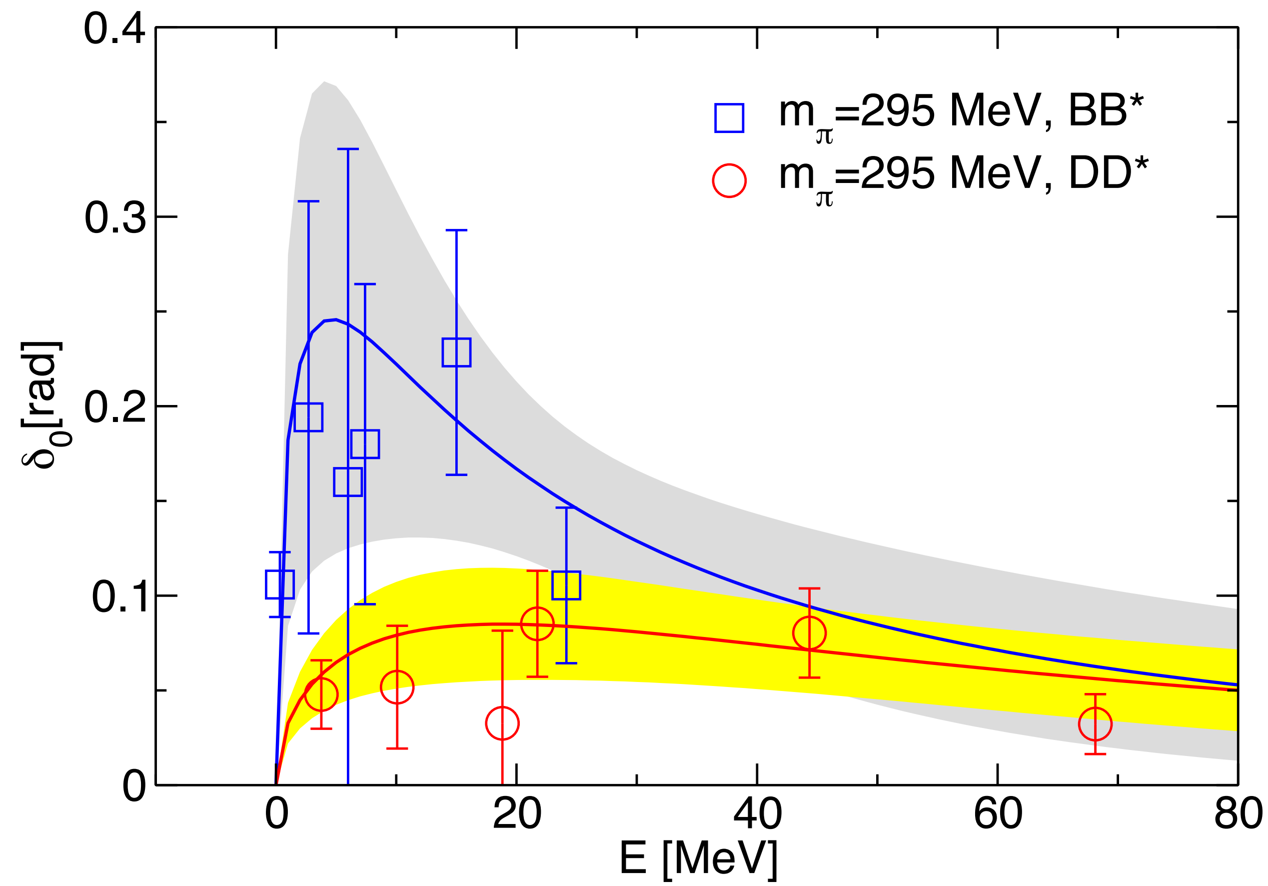


S wave scattering phase shifts

For $m_\pi = 411$ MeV



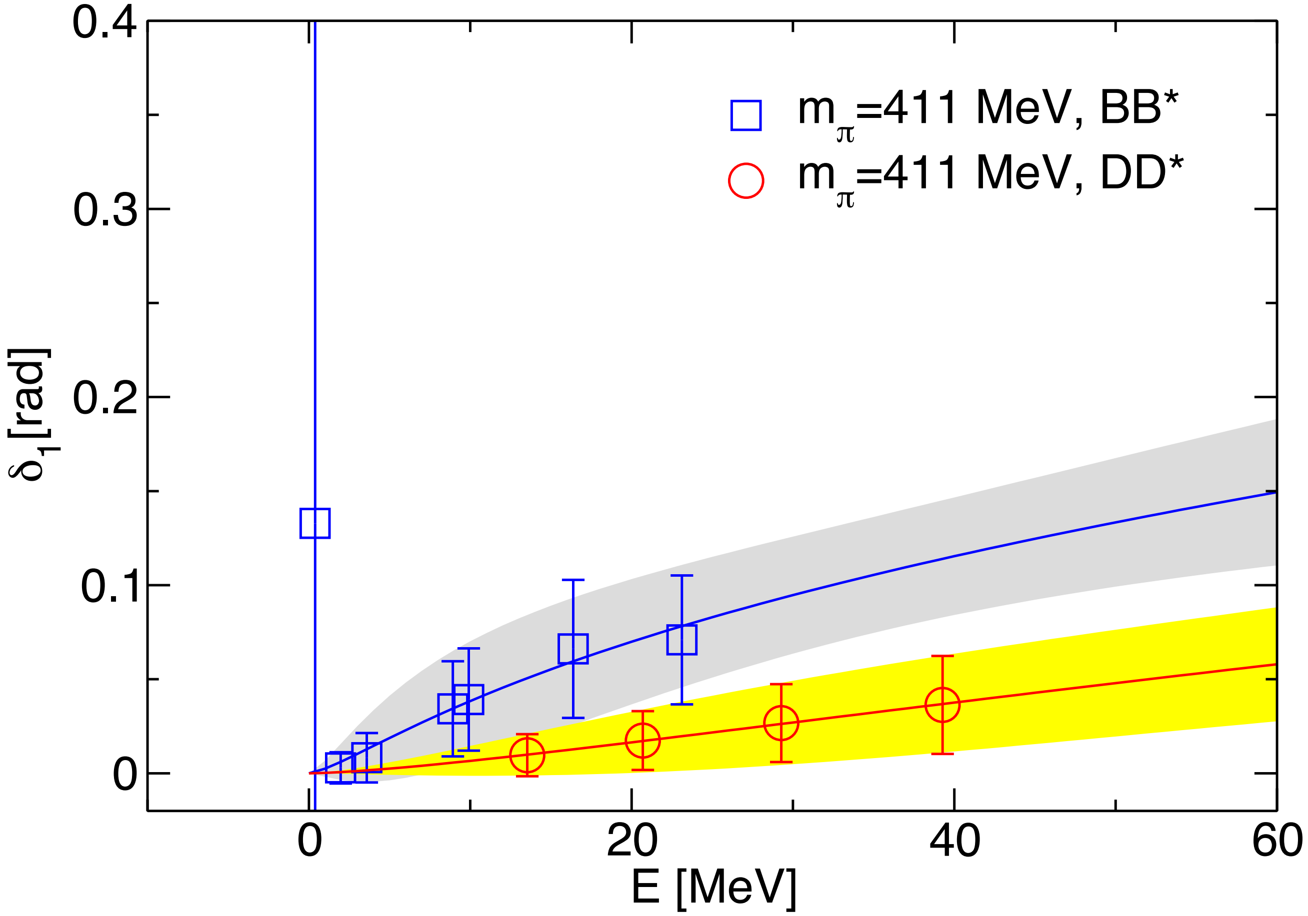
For $m_\pi = 295$ MeV



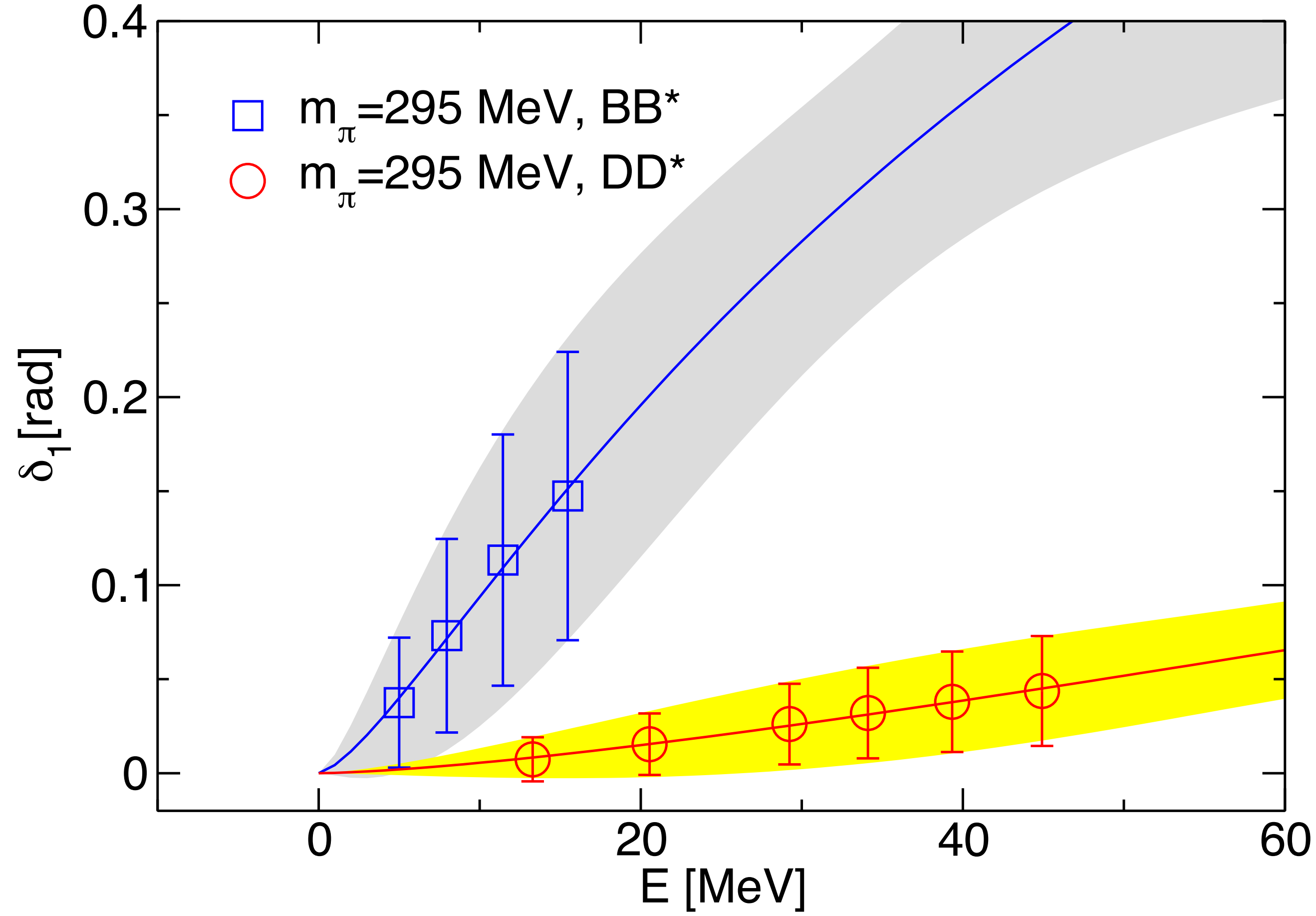
Scattering phase shifts get close to the behavior of a bound state as m_π gets smaller and heavy quark mass get heavier.

P wave scattering phase shifts

For $m_\pi = 411$ MeV



For $m_\pi = 295$ MeV



BB^* system has large scattering phase shift compared to the DD^*.

Summary

- **S** and **P** wave scattering phase shift are extracted through Lüscher's method under twisted BCs with $m_\pi = 295, 411$ MeV for DD^* and BB^* systems.
  - We properly separated the S and P wave effects in Lüscher's formula.
  - $qq\bar{u}\bar{d}$ system seems to get close to **a bound state** as $m_\pi \to m_{\rm phys}$ and heavy quark mass gets heavier.

Prospects

- Calculation at lighter masses, more statistics.
- Specification of the bound state.