HHIQCD 2024

Modified homotopy approach for diffractive production in the saturation region

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Thanks to Fondecyt (Chile) grant 1231829
and to UTFSM for support.

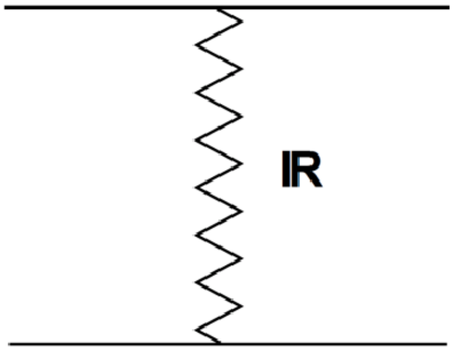
10.1103/PhysRevD.110.054045

Outline

- Yesterday and Today
- Diffraction at high energy
- Conclusions

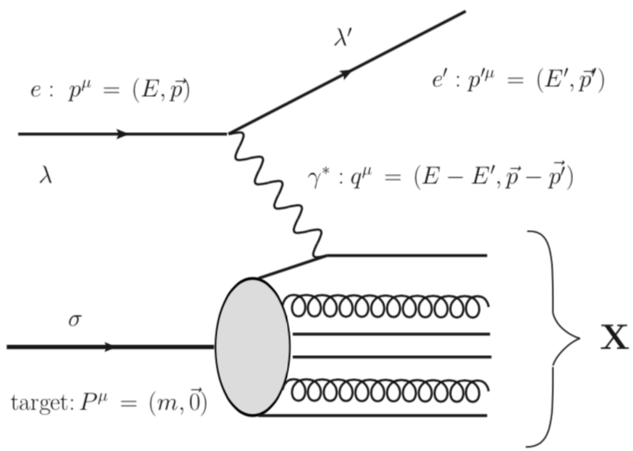
Yesterday

- The theoretical approach to high energy scattering in the **pre-QCD** era was established by V. Gribov and is known as **Gribov‘s Reggeon Calculus (1968).**
- The brick from which we wanted to do this was the **Pomeron** (**Reggeon** with the intercept close to 1).
- Of course one defect of the approach has been seen from the beginning, namely **the absence of a theoretical idea** how to select the interaction between Pomerons.
- The **death** of the Reggeon Approach was in **1974.**



Yesterday

- The **microscopic theory of QCD** was established by **Fritzsch, Gell-Mann and Leutwyler (1973), Gross and Wilczek (1973), and Weinberg (1973).**
- One of the simplest scattering processes that occur at short distances is the reaction $e + p \longrightarrow e' + X$



$$\text{incoming proton momentum}: \; P$$

$$\text{photon's virtuality}: \; Q^2 = -q^2 = (k - k')^2$$

$$\text{fraction of electron energy transferred to the proton}: \; y = \frac{P \cdot q}{P \cdot k} = \frac{Q^2}{sx}$$

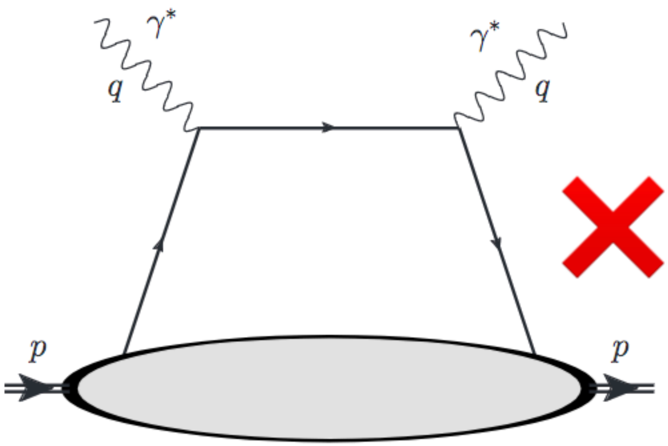
$$\text{photon} - \text{proton system energy}: \; s = (P + q)^2 = W^2$$

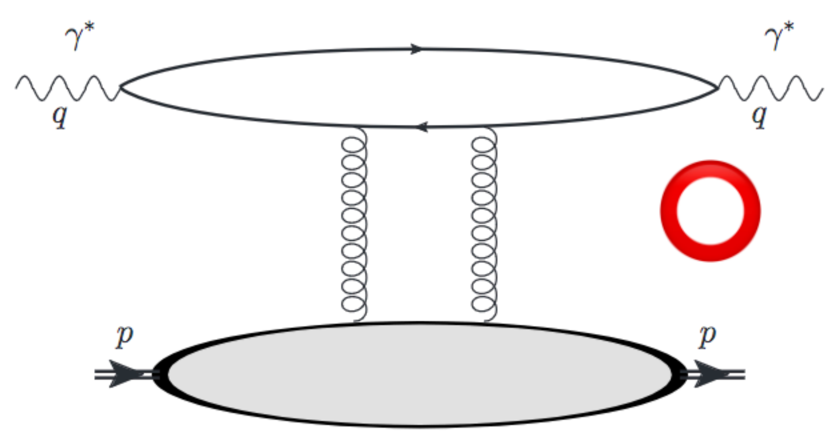
$$\text{Bjorken} - x: \; x = \frac{Q^2}{2P \cdot q} = \frac{Q^2}{Q^2 + W^2 - m_p^2}$$

- The **Deep Inelastic Scattering (DIS)** experiment allows us to investigate the structure of the hadron at short distances by observing the recoil electron e′

Yesterday

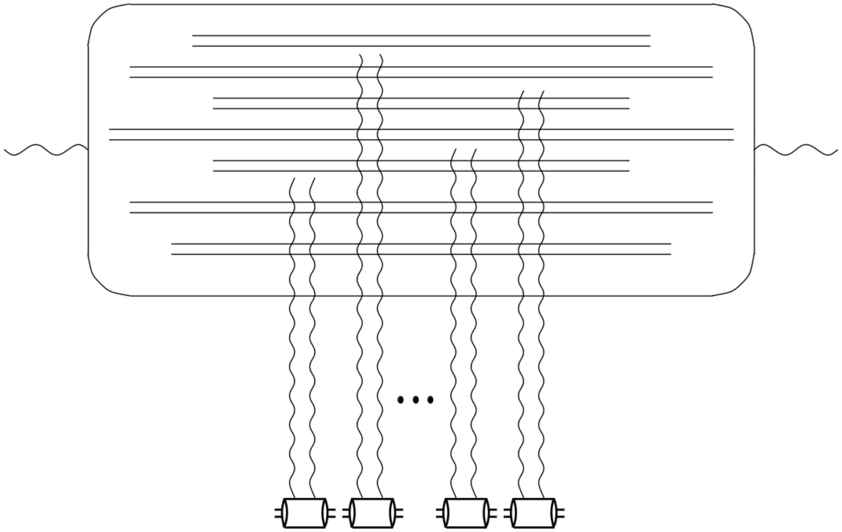
- Previous figure was in the infinite momentum frame (IMF).
- However, since our goal is to study the high energy behavior of QCD, it's better to use the **dipole picture of DIS (Gribov (1970); Bjorken and Kogut (1973); Bertsch et al. (1981); Frankfurt and Strikman (1988); Kopeliovich et al. (1981); Mueller (1990); Nikolaev and Zakharov(1991))**

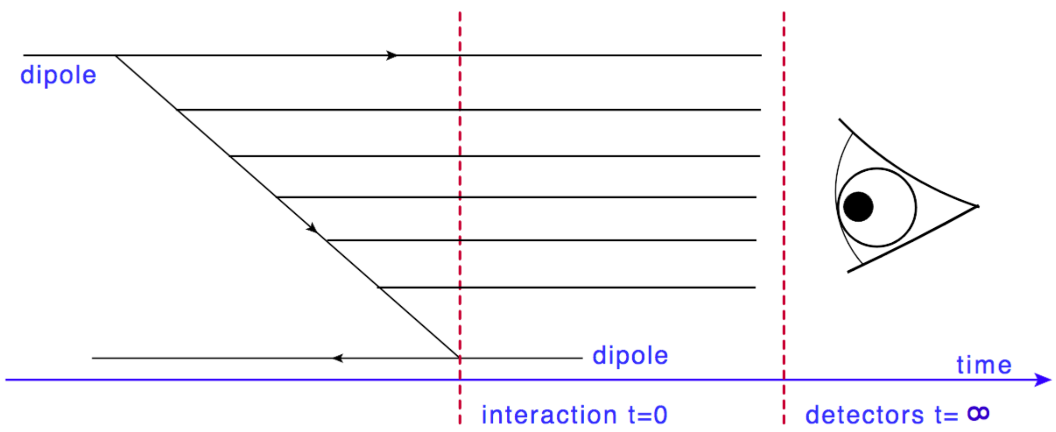




Yesterday

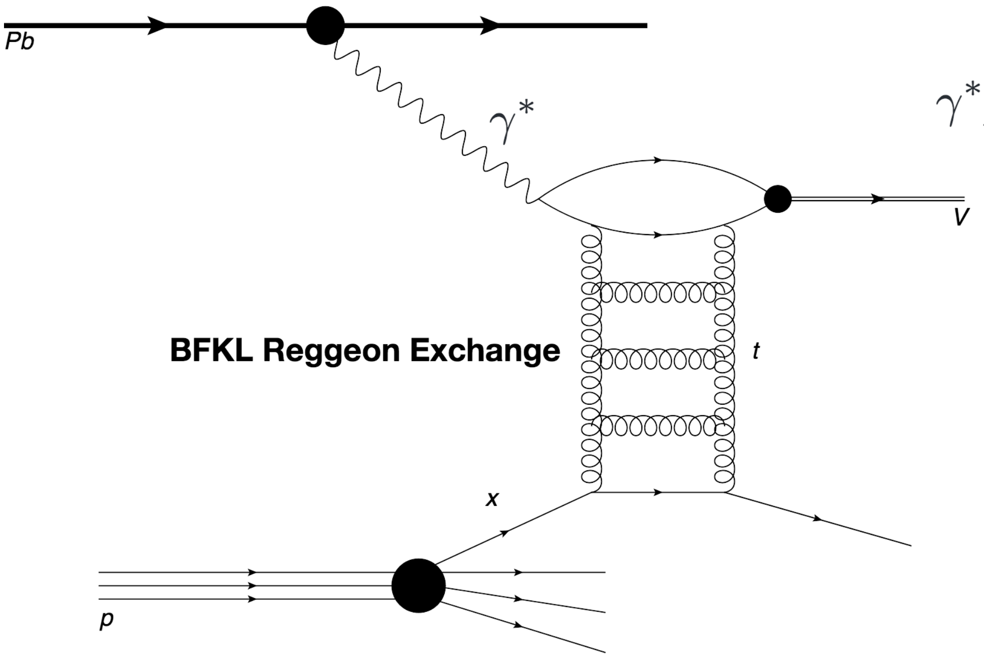
- **Colour dipoles are the correct degrees of freedom at high energy QCD.**
- The physical picture of DIS presented is the following: virtual photon splits into a **quark–antiquark pair**, which, by the time it reaches the target develops a cascade of dipoles, each of which independently interacts with the target.

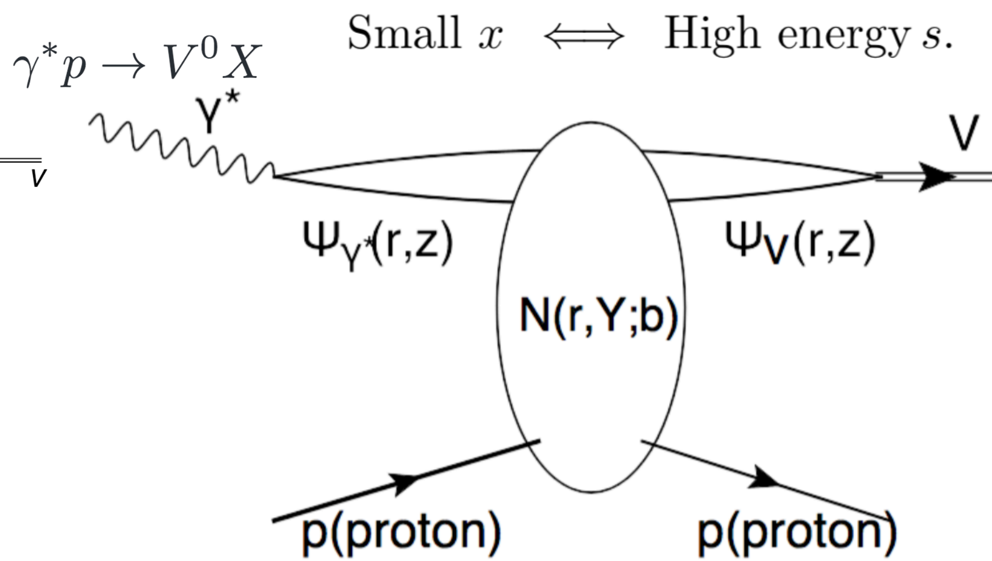




Yesterday

- First attempt to the study of high energy limit in QCD began with the derivation of the **Balitsky-Fadin-Kuraev-Lipatov (BFKL) Pomeron (1975).**





- The BFKL equation represents an important step toward understanding of high energy asymptotics of QCD. But also raised some important questions.

Yesterday

- The BFKL evolution equation for the dipole-target scattering amplitude N(x10, b, Y; R) was derived

$$\frac{\partial}{\partial Y} N\left(\boldsymbol{x}_{10}, \boldsymbol{b}, Y; R\right) =$$

(with $K^{\mathrm{LO}}\left(\boldsymbol{x}_{02}, \boldsymbol{x}_{12}; \boldsymbol{x}_{10}\right) = \frac{x_{10}^2}{x_{02}^2\, x_{12}^2}$)

$$\bar{\alpha}_S \int \frac{d^2 \boldsymbol{x}_2}{2\,\pi} K\left(\boldsymbol{x}_{02}, \boldsymbol{x}_{12}; \boldsymbol{x}_{10}\right) \left(N\left(\boldsymbol{x}_{12}, \boldsymbol{b} - \frac{1}{2}\boldsymbol{x}_{20}, Y; R\right) + N\left(\boldsymbol{x}_{20}, \boldsymbol{b} - \frac{1}{2}\boldsymbol{x}_{12}, Y; R\right) - N\left(\boldsymbol{x}_{10}, \boldsymbol{b}, Y; R\right)\right)$$

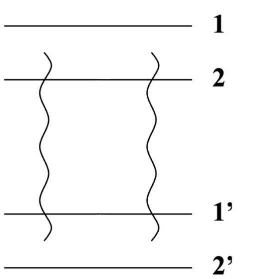
- Using a Mellin transform with respect to Y, and then expanding the dipole distribution on a conformal basis, we find (**Lipatov, 1986**)

$$N^{BFKL}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \boldsymbol{\rho}_{1'}, \boldsymbol{\rho}_{2'}; Y) = 8\,\bar{\alpha}_S^2 \sum_{\text{even } n}^{\infty} \int_{-\infty}^{\infty} d\nu \int e^{\bar{\alpha}_S \chi(n,\nu) Y} \frac{\nu^2 + \frac{n^2}{4}}{[\nu^2 + \frac{(n+1)^2}{4}][\nu^2 + \frac{(n-1)^2}{4}]} G^{n,\nu}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \boldsymbol{\rho}_1', \boldsymbol{\rho}_2')$$

With $\omega(n,\nu) = \bar{\alpha}_S \chi(n,\nu) = \bar{\alpha}_S (2\psi(1) - \psi(\frac{1+|n|}{2} + i\nu) - \psi(\frac{1+|n|}{2} - i\nu))$, $h = \frac{1}{2} + i\nu + \frac{n}{2}$, $\tilde{h} = \frac{1}{2} + i\nu - \frac{n}{2}$

$$G^{n,\nu}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \boldsymbol{\rho}_1', \boldsymbol{\rho}_2') = b_{n,-\nu} w^h w^{*\tilde{h}} {}_2F_1(h, h, 2h; w) {}_2F_1(\tilde{h}, \tilde{h}; 2\tilde{h}; w^*)$$
$$+ b_{n,\nu} w^{1-h} w^{*1-\tilde{h}} {}_2F_1(1-h, 1-h, 2-2h; w) {}_2F_1(1-\tilde{h}, 1-\tilde{h}; 2-2\tilde{h}; w^*),$$

$$b_{n,\nu} = \pi^3\, 2^{4i\nu} \frac{\Gamma\left(-i\nu + \frac{1+|n|}{2}\right)}{\Gamma\left(i\nu + \frac{1+|n|}{2}\right)} \frac{\Gamma\left(i\nu + \frac{|n|}{2}\right)}{\Gamma\left(1 - i\nu + \frac{|n|}{2}\right)}; \quad w = \frac{\rho_{12}\rho_{1'2'}}{\rho_{11'}\rho_{22'}}, \quad \rho_i = x_{i,1} + i\, x_{i,2}; \quad \rho_{ik} = \rho_i - \rho_k$$



Yesterday

- At large values of Y=ln(1/x) the main contribution stems from the first term with n = 0. We have

$$H^{\gamma}(w,w^*) = 8\bar{\alpha}_S^2 \frac{(\gamma-\frac{1}{2})^2}{(\gamma(1-\gamma))^2}\{b_\gamma w^\gamma w^{*\gamma}{}_2F_1(\gamma,\gamma,2\gamma;w){}_2F_1(\gamma,\gamma;2\gamma;w^*)$$
$$+ b_{1-\gamma} w^{1-\gamma} w^{*1-\gamma}{}_2F_1(1-\gamma,1-\gamma,2-2\gamma;w){}_2F_1(1-\gamma,1-\gamma;2-2\gamma;w^*)\}$$

With $\gamma = \frac{1}{2} + i\nu$, $b_\gamma = \pi^3\, 2^{4(1/2-\gamma)} \frac{\Gamma(\gamma)}{\Gamma(1/2-\gamma)} \frac{\Gamma(1-\gamma)}{\Gamma(1/2+\gamma)}$, $\bar{\alpha}_S = \frac{\alpha_S N_c}{\pi}$

- Taking the integral over γ using the method of steepest decent, we see that

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \approx \chi(\gamma_{cr}) + \frac{1}{2}\chi''(\gamma_{cr})(\gamma-\gamma_{cr})^2$$

$$N(r,R;Y,b) = \underbrace{\sqrt{\frac{1}{2\pi\bar{\alpha}_S\chi''(\gamma_{cr})}}\, 8\bar{\alpha}_S^2 \frac{(\gamma_{cr}-\frac{1}{2})^2}{(\gamma_{cr}(1-\gamma_{cr}))^2} b_{\gamma_{cr}}}_{=N_0} \left(\left(\frac{r^2R^2}{(\mathbf{b}^2-\frac{1}{2}(\mathbf{r}-\mathbf{R}))^2(\mathbf{b}^2+\frac{1}{2}(\mathbf{r}-\mathbf{R}))^2}\right) e^{\bar{\alpha}_S \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}} Y}\right)^{1-\gamma_{cr}}$$

Yesterday

- Defining

$$\xi = \ln\left(\frac{r^2 R^2}{(\mathbf{b}^2 - \frac{1}{2}(\mathbf{r}-\mathbf{R}))^2(\mathbf{b}^2 + \frac{1}{2}(\mathbf{r}-\mathbf{R}))^2}\right) = \ln(r^2 Q_s^2(Y=0,b)),$$

$$\kappa = \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}, \qquad Q_s^2(Y,b) = Q_s^2(Y=0,b)\, e^{\bar{\alpha}_S\,\kappa Y},$$

$$z = \ln\left(x_{01}^2\, Q_s^2(Y,b)\right) = \bar{\alpha}_S\,\kappa\, Y + \xi, \qquad 1-\gamma_{cr} = \bar{\gamma},$$

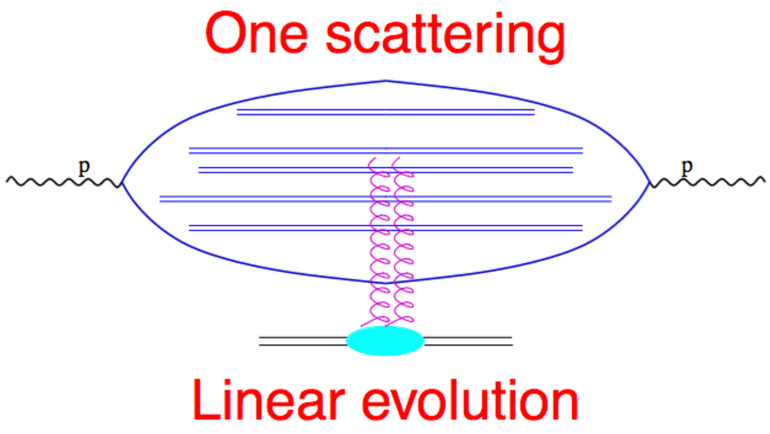
- We obtain

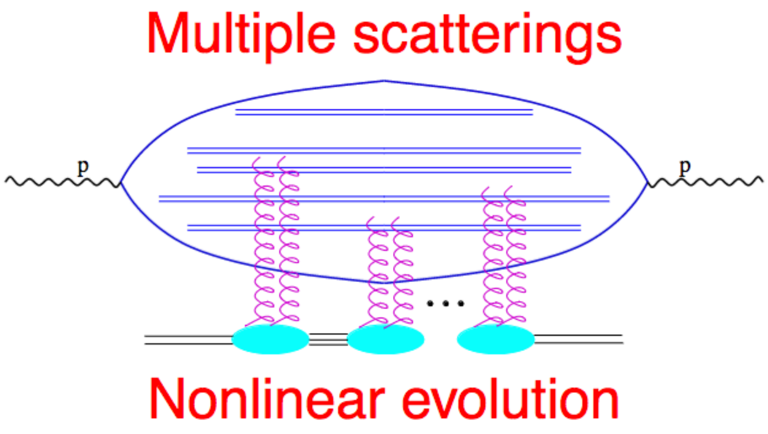
$$N(z) = N_0 e^{z\bar{\gamma}}$$

- This shows the so called **geometric scaling.**
- Value of γ_{cr} is found by solving $\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}} = -\frac{d\chi(\gamma_{cr})}{d\gamma_{cr}}$, we find $\gamma_{cr} = 0.37$ and for $\bar{\alpha}_S = 0.2$ we get $N_0 = 0.25$

Today

- Seminal paper of **Gribov, Levin and Ryskin (GLR, 1983)** put forward the idea that nonlinear effects in QCD evolution lead to saturation of gluonic density at high enough energy.





- Nonlinear term slows down the small-x evolution, leading to **parton saturation** and to total cross sections adhering to the **black disk limit** $\sigma_{tot} \leq 2\pi R^2$.

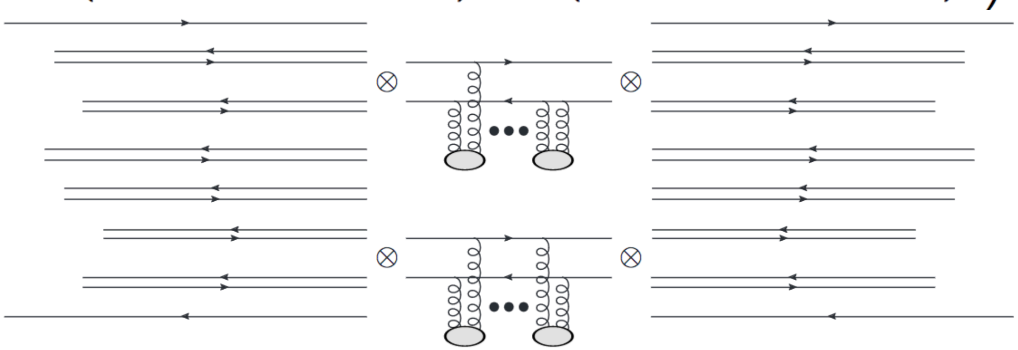
Today

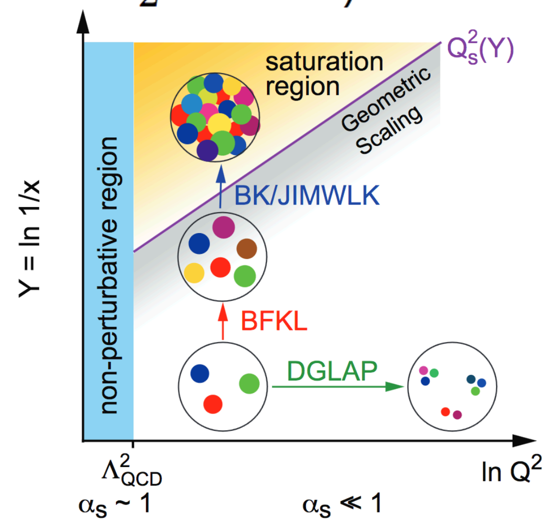
- This BFKL Parton cascade leads to **Balitsky-Kovchegov (BK) equation** for the amplitude and gives the theoretical description of the DIS **(Mueller (1994), Balistky (1999), Kovchegov (1999), Levin and Lublinsky (2003)).**

$$\frac{\partial}{\partial Y} N\left(\boldsymbol{x}_{10}, \boldsymbol{b}, Y; R\right) =$$

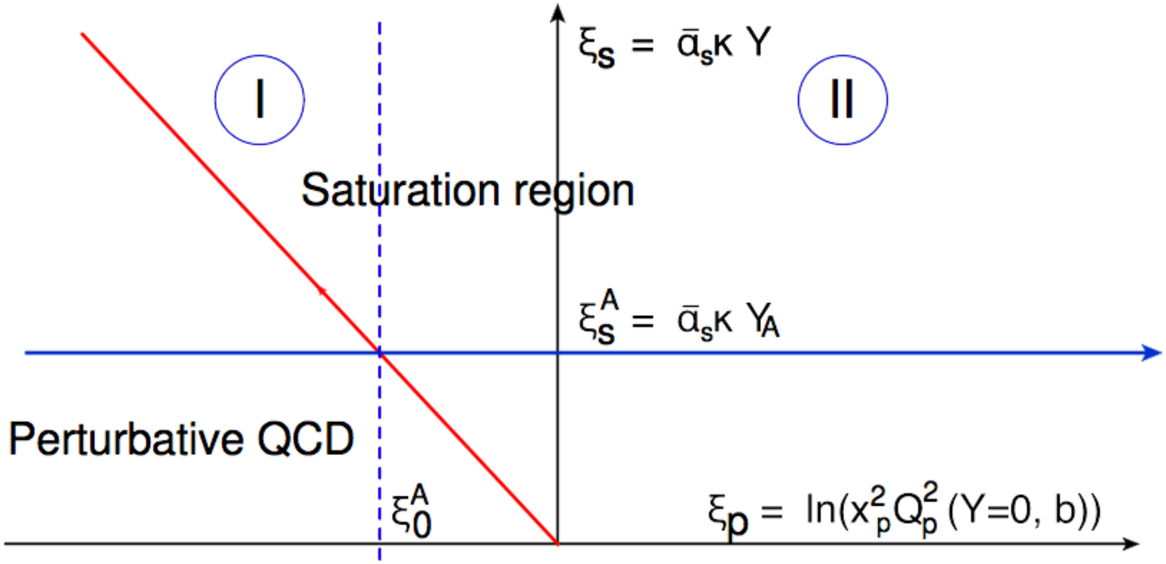
$$\bar{\alpha}_S \int \frac{d^2 \boldsymbol{x}_2}{2\,\pi} K\left(\boldsymbol{x}_{02}, \boldsymbol{x}_{12}; \boldsymbol{x}_{10}\right) \left(N\left(\boldsymbol{x}_{12}, \boldsymbol{b} - \frac{1}{2}\boldsymbol{x}_{20}, Y; R\right) + N\left(\boldsymbol{x}_{20}, \boldsymbol{b} - \frac{1}{2}\boldsymbol{x}_{12}, Y; R\right) - N\left(\boldsymbol{x}_{10}, \boldsymbol{b}, Y; R\right) \right.$$

$$\left. - N\left(\boldsymbol{x}_{12}, \boldsymbol{b} - \frac{1}{2}\boldsymbol{x}_{20}, Y; R\right) N\left(\boldsymbol{x}_{20}, \boldsymbol{b} - \frac{1}{2}\boldsymbol{x}_{12}, Y; R\right) \right)$$





Today



- Saturation region of QCD for the elastic scattering amplitude. The critical line (z=0) is shown in red. The initial condition for scattering with the dilute system of partons (with proton) is given at ξs = 0. For heavy nuclei the initial conditions are placed at YA = (1/3) ln A ≫ 1, where A is the number of nucleon in a nucleus.

Today

- Performing a "Fourier transform"

$$N\left(x_{01}^2, b, Y\right) = x_{01}^2 \int \frac{d^2 k_\perp}{2\pi} e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_{01}} \widetilde{N}\left(k_\perp, b, Y\right) = x_{01}^2 \int_0^\infty k_\perp dk_\perp J_0\left(x_{01} k_\perp\right) \widetilde{N}\left(k_\perp, b, Y\right)$$

- We write **(Kovchegov (2000)).**

$$\frac{\partial \widetilde{N}\left(k_\perp, b, Y\right)}{\partial Y} = \bar{\alpha}_S \left\{ \chi\left(-\frac{\partial}{\partial \tilde{\xi}}\right) \widetilde{N}\left(k_\perp, b, Y\right) - \widetilde{N}^2\left(k_\perp, b, Y\right) \right\}$$

Where $\tilde{\xi} = \ln\left(Q_s^2\left(Y = Y_A, b\right)/k_\perp^2\right)$ and $\tilde{z} = \bar{\alpha}_S \kappa\left(Y - Y_A\right) + \tilde{\xi} = \ln\left(Q_s^2\left(Y, b\right)/k_\perp^2\right)$

- Differentiating this equation over $\tilde{\xi}$ we get

$$\frac{\partial^2 \widetilde{N}\left(k_\perp, b, Y\right)}{\partial Y\, \partial \tilde{\xi}} =$$

$$\bar{\alpha}_S \left\{ \chi_0\left(-\frac{\partial}{\partial \tilde{\xi}}\right) \frac{\partial \widetilde{N}\left(k_\perp, b, Y\right)}{\partial\, \tilde{\xi}} + \widetilde{N}\left(k_\perp, b, Y\right) - 2 \frac{\partial \widetilde{N}\left(k_\perp, b, Y\right)}{\partial\, \tilde{\xi}} \widetilde{N}\left(k_\perp, b, Y\right) \right\}$$

Today

Where $\chi_0(\gamma) = \chi(\gamma) - \frac{1}{\gamma}$, $\gamma = -\frac{\partial}{\partial\tilde{\xi}}$.

- Introducing the variable $\tilde{z}$ instead of $\tilde{\xi}$ and the new function M as

$$\frac{\partial \tilde{N}\left(k_\perp, b, \delta\tilde{Y}\right)}{\partial\,\tilde{z}} = \frac{1}{2} + M\left(\tilde{z}, b, \delta\tilde{Y}\right) \quad \text{or} \quad \tilde{N}\left(\tilde{z}, b, \delta\tilde{Y}\right) = \frac{1}{2}\tilde{z} + \int_0^{\tilde{z}} d\tilde{z}' M\left(\tilde{z}', b, \delta\tilde{Y}\right) = \frac{\tilde{z}+\zeta}{2} + \int_{\tilde{z}}^{\infty} d\tilde{z}' M\left(\tilde{z}', b, \delta\tilde{Y}\right)$$

We can re-write the previous equation in the form

$$\frac{\partial M\left(\tilde{z}, b, \delta\tilde{Y}\right)}{\partial\delta\tilde{Y}} + \kappa\frac{\partial M\left(\tilde{z}, b, \delta\tilde{Y}\right)}{\partial\tilde{z}} = \chi_0\left(\frac{\partial}{\partial\tilde{z}}\right) M\left(\tilde{z}, b, \delta\tilde{Y}\right) - \left(\tilde{z} + \zeta\right) M\left(\tilde{z}, b, \delta\tilde{Y}\right) + M\left(\tilde{z}, b, \delta\tilde{Y}\right)\int_{\tilde{z}}^{\infty} d\tilde{z}' M\left(\tilde{z}', b, \delta\tilde{Y}\right)$$

with $\delta\tilde{Y} = \overline{\alpha}_S\left(Y - Y_A\right)$ and $\zeta = \int_0^{\infty} dz' M\left(\tilde{z}', b, \delta\tilde{Y}\right)$.

- At large $\tilde{z}$, M is small so we can neglect the last term. This gives a “linear equation”, but with the term $\left(\tilde{z} + \zeta\right) M\left(\tilde{z}, b, \delta\tilde{Y}\right)$ actually coming from the $\tilde{N}^2$ term

Today

- For region I, we have

$$\kappa \frac{dM_0^I(\tilde{z},b)}{d\tilde{z}} = \chi_0\left(\frac{d}{d\tilde{z}}\right) M_0^I(\tilde{z},b) - (\tilde{z}+\zeta)\, M_0^I(\tilde{z},b)$$

- We solve this equation using the Mellin transform

$$M_0^I(\tilde{z},b) = \int\limits_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\gamma}{2\pi i} e^{\gamma(\tilde{z}+\zeta)} m_0^I(\gamma,b)$$

- We find M0 and then we back to coordinates. The result is

$$N_0^I(z) = 1 - C\exp\left(-\frac{(z-\tilde{z})^2}{2\kappa}\right)$$

With $\tilde{z} = 2(\ln 2 + \psi(1)) - \zeta$

Today

- For region II, we have to solve

$$\frac{\partial M_0^{II}}{\partial\,\delta\tilde{Y}} + \kappa\frac{\partial M_0^{II}}{\partial\tilde{z}} = \chi_0\left(\frac{\partial}{\partial\tilde{z}}\right)M_0^{II} - (\tilde{z}+\zeta)M_0^{II}$$

- We solve this equation using the Mellin transform

$$M_0^{II}\left(\tilde{z},\delta\tilde{Y}\right) = \int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\gamma}{2\pi i}e^{(\tilde{z}+\zeta)\,\gamma}m_0^{II}\left(\gamma,\delta\tilde{Y}\right)$$

- Final result for the elastic amplitude in region II is

$$N_0^{II}(\xi,z) = 1 - G(\xi)\exp\left(-\frac{(z-\tilde{z})^2}{2\kappa}\right)$$

with $G(\xi) = \exp\left(\frac{(\xi-\tilde{z})^2}{2\kappa} - \frac{e^{\xi}}{4}\right)$

Today

- From the matching condition on the line ξ_0^A

$$N_0^I\left(z=\kappa\delta\tilde{Y}+\xi_0^A\right) \;=\; N_0^{II}(\xi=\xi_0^A, z=\kappa\delta\tilde{Y}+\xi_0^A)$$

- We find constant C

$$C=\exp\left(\frac{(\xi_0^A-\tilde{z})^2}{2\kappa}-\frac{e^{\xi_0^A}}{4}\right)$$

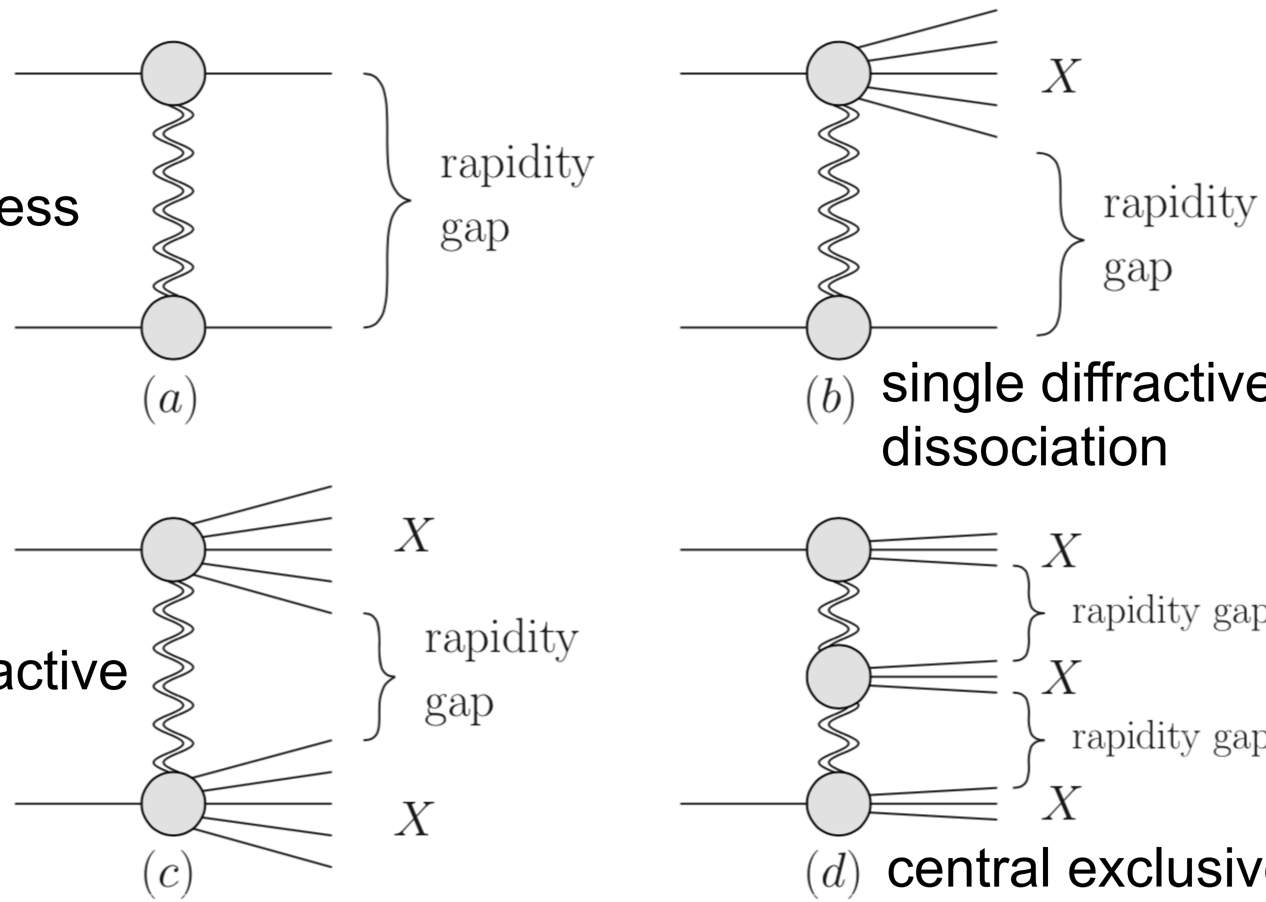
- And from matching conditions at z=0 (red line)

$$N_0^I(z=0)=N_0 \qquad \left.\frac{d\ln N_0^I(z)}{dz}\right|_{z=0}=\bar{\gamma}$$

we find $\tilde{z}=-\frac{\kappa\bar{\gamma}N_0}{1-N_0}=-1.02$, $C=0.83$, $\xi_0^A=0.56$.

Diffraction at high energy

- An event is considered diffractive if it contains a rapidity gap (interval over which no particles are produced).



elastic process

single diffractive dissociation

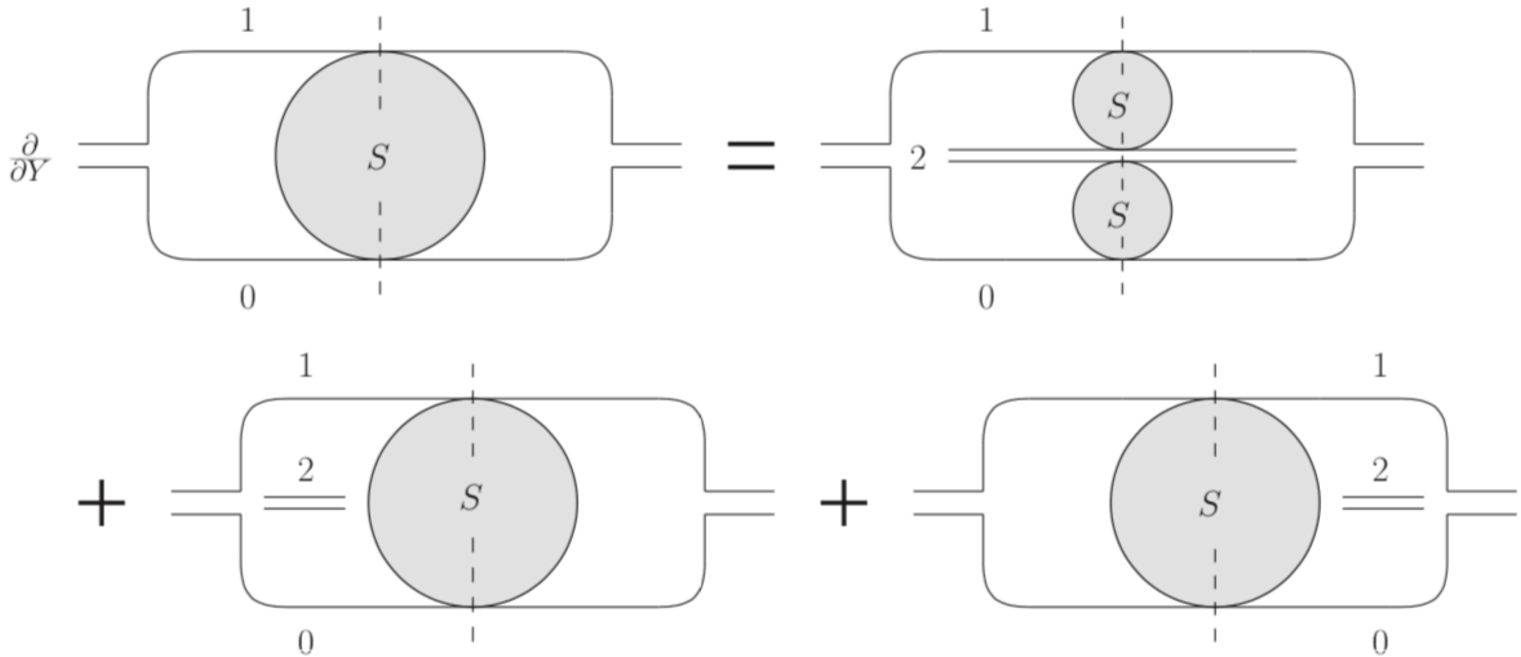
double diffractive dissociation

central exclusive diffraction

Diffraction at high energy

- The equation for the S-matrix in the BFKL cascade has the following form in the dipole approach to QCD (Mueller 1994)

$$\frac{d}{dY}S\left(Y,r;\boldsymbol{b}\right) = \bar{\alpha}_S \int d^2r \underbrace{\frac{r^2}{r'^2\left(\boldsymbol{r}-\boldsymbol{r}'\right)^2}}_{K\left(r,r'\right)} \left\{ S\left(Y,\boldsymbol{r}';\boldsymbol{b}-\frac{1}{2}(\boldsymbol{r}-\boldsymbol{r})\right) S\left(Y,\boldsymbol{r}-\boldsymbol{r}';\boldsymbol{b}-\frac{1}{2}\boldsymbol{r}\right) - S\left(Y,r;\boldsymbol{b}\right) \right\}$$



Diffraction at high energy

- Writing

$$S^D(Y, r; \boldsymbol{b}) = 1 - 2N(Y, r; \boldsymbol{b}) + N^D(Y, Y_0, \mathbf{r}, b)$$

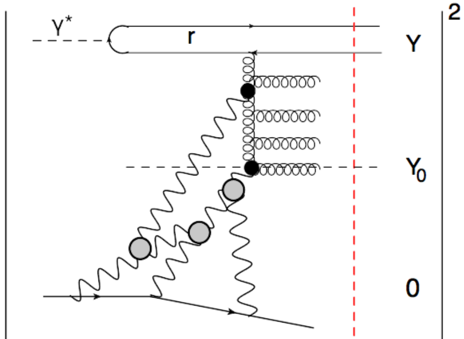
- And plugging this in the S-matrix equation we obtain the **Kovchegov-Levin equation (2000)**

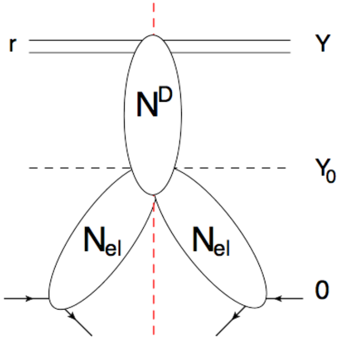
$$\frac{\partial N^D(Y, Y_0, r_{10}; b)}{\partial Y_M} =$$

$$\frac{\bar{\alpha}_S}{2\pi} \int d^2 r_2\, K(r_{10}|r_{12}, r_{02}) \{ N^D(Y, Y_0, r_{12}; b) + N^D(Y, Y_0, r_{20}; b) - N^D(Y, Y_0, r_{10}; b)$$

$$+ N^D(Y; Y_0, r_{12}; b) N^D(Y; Y_0, r_{20}; b) - 2\, N^D(Y; Y_0, r_{12}; b)\, N(Y; r_{20}; b)$$

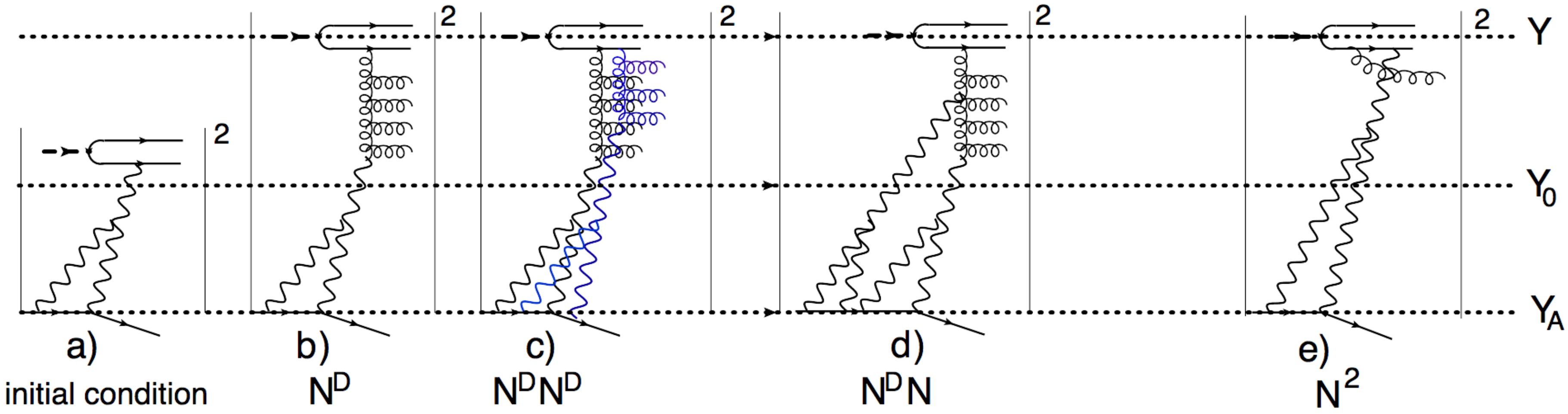
$$- 2\, N(Y; r_{12}; b) N^D(Y; Y_0, r_{20}; b) + 2N(Y; r_{12}; b) N(Y; r_{20}; b)\},$$ with $Y_M = Y - Y_0$





Diffraction at high energy

- The graphic representation of the terms of of the KL equation for diffraction production



- Introducing a new function:

$$\mathcal{N}(z, \delta\tilde{Y}, \delta Y_0) = 2\,N(z, \delta\tilde{Y}) - N^D(z, \delta\tilde{Y}, \delta Y_0)$$

where $\delta\tilde{Y} = \bar{\alpha}_S\,(Y - Y_A)$, $\delta Y_0 = \bar{\alpha}_S\,(Y_0 - Y_A)$, we rewrite

$$\frac{\partial \mathcal{N}_{01}}{\partial Y} = \bar{\alpha}_S \int \frac{d^2 x_{02}}{2\pi} \frac{x_{01}^2}{x_{02}^2\, x_{12}^2} \left\{ \mathcal{N}_{02} + \mathcal{N}_{12} - \mathcal{N}_{02}\mathcal{N}_{12} - \mathcal{N}_{01} \right\}$$

Diffraction at high energy

- With initial conditions

$$\mathcal{N}(z \to z_0, \delta\tilde{Y} = \delta Y_0, \delta Y_0) = 2\,N(z_0, \delta Y_0) - N^2(z_0, \delta Y_0)$$

- Replacing $\mathcal{N}(z, \delta Y_0) = 1 - \Delta^D(z, \delta Y_0)$, it takes the form

$$\frac{\partial \Delta^D_{01}}{\partial Y} = \bar{\alpha}_S \int \frac{d^2 x_{02}}{2\pi} \frac{x^2_{01}}{x^2_{02}\, x^2_{12}} \left\{ \Delta^D_{02} \Delta^D_{12} - \Delta^D_{01} \right\}$$

- with the initial conditions for Δ^D as

$$\text{Region I}: \quad \Delta^D_{01}(z \to z_0, \delta Y_0) = C^2 \exp\left(- \frac{(z_0 - \tilde{z})^2}{\kappa} \right)$$

$$\text{Region II}: \quad \Delta^D_{01}\left(z \to z_0, \delta\tilde{Y} \to \delta Y_0, \delta Y_0\right) = 1 - N_{in} = G^2(\xi) \exp\left(- \frac{(z_0 - \tilde{z})^2}{\kappa} \right)$$

$$\text{with} \quad G(\xi) = \exp\left(\frac{(\xi - \tilde{z})^2}{2\,\kappa} - \frac{1}{4} e^{\xi} \right)$$

Diffraction at high energy

- The homotopy method we can use is as follows

$$\mathscr{H}(p,u) = \mathscr{L}[u_p] + p\,\mathscr{N}_{\mathscr{L}}[u_p] = 0$$

With $u_p(Y,\boldsymbol{x}_{10},\boldsymbol{b}) = u_0(Y,\boldsymbol{x}_{10},\boldsymbol{b}) + p\,u_1(Y,\boldsymbol{x}_{10},\boldsymbol{b}) + p^2\,u_2(Y,\boldsymbol{x}_{10},\boldsymbol{b}) + \ldots$

- In this work we include in $\mathscr{L}[u_p]$ part of the non-linear corrections. First, we simplify the non-linear term as

$$\bar{\alpha}_S \int \frac{d^2 x_{02}}{2\pi} \frac{x_{01}^2}{x_{02}^2\, x_{12}^2} \Delta^D_{02}\Delta^D_{12} \to \Delta^D_{01} \int_0^z dz'\,\Delta^D_{02} = \Delta^D_{01}\left(\zeta_\Delta - \int_z^\infty dz'\,\Delta^D_{02}\right) \text{ with } \zeta_\Delta = \int_0^\infty dz'\,\Delta^D_{02}$$

So we take

$$\mathscr{L}(\Delta^D_0) = \left(\frac{\partial}{\partial \delta\tilde{Y}} + z - \zeta_\Delta\right)\Delta^D_0 + \Delta^D_0\left(z,\delta\tilde{Y},z_0\right)\int_z^\infty dz'\,\Delta^D_0\left(z',\delta\tilde{Y},z_0\right)$$

$$\mathscr{N}_{\mathscr{L}}[\Delta^D] = \bar{\alpha}_S \int \frac{d^2 x_{02}}{2\pi} \frac{x_{01}^2}{x_{02}^2\, x_{12}^2} \Delta^D_0(x_{02})\,\Delta^D_0(x_{12}) - \Delta^D_0 \int^{x_{01}^2} \frac{dx_{02}^2}{x_{02}^2} \Delta^D_{02}$$

Diffraction at high energy

- The first iteration (p=0) gives

$$\mathscr{L}\left(\Delta_0^D\right) = 0; \quad \left(\frac{\partial}{\partial \delta \tilde{Y}} + z - \zeta_\Delta\right)\Delta_0^D + \Delta_0^D\left(z, \delta\tilde{Y}, z_0\right)\int_z^\infty dz' \Delta_0^D\left(z', \delta\tilde{Y}, \boldsymbol{z}_0\right) = 0;$$

- Introducing $\Delta^{(0)}\left(z,\xi_s\right) = 1 - \mathscr{N}_{01}\left(z,\xi_s\right) = \exp\left(-\Omega^{(0)}\left(z,\xi_s\right)\right)$

  with $\xi_s = \kappa\,\delta\tilde{Y}$, we obtain

$$\kappa\frac{\partial^2\Omega^{(0)}\left(z,\xi_s\right)}{\partial\xi_s\,\partial z} = 1 - e^{-\Omega^{(0)}\left(z,\xi_s\right)}$$

- For region I, it reduces to

$$\kappa\frac{d^2\Omega^{(0)}\left(z,z_0\right)}{dz^2} = 1 - e^{-\Omega^{(0)}\left(z,z_0\right)}$$

Diffraction at high energy

- Solution is found by defining $\frac{d\Omega^{(0)}(z)}{dz} = p\left(\Omega^{(0)}\right)$ so that

$$\frac{1}{2}\kappa \frac{dp^2}{d\Omega^{(0)}} = 1 - e^{-\Omega^{(0)}(z)}$$

- With the solution

$$p = \frac{d\Omega^{(0)}}{dz} = \sqrt{\frac{2}{\kappa}\left(\Omega^{(0)} + \exp\left(-\Omega^{(0)}\right) - 1\right) + C_1}$$

- Integrating and applying initial conditions, we get

$$\int_a^{\Omega^{(0)}} \frac{d\,\Omega'}{\sqrt{\Omega' + \exp\left(-\Omega'\right) - a}} = \sqrt{\frac{2}{\kappa}}\left(z - \tilde{z}\right)$$

with $a = \left(z_0 - \tilde{z}\right)^2 / (2\kappa) - 2\ln C$

Diffraction at high energy

- To solve the integral, we assume $\Omega^{(0,I)}$ is large, so that we can use the expansion $\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \frac{(2n-1)!!x^n}{2^n n!}$

$$\frac{1}{\sqrt{\Omega' + \exp\left(-\Omega'\right) - a}} = \frac{1}{\sqrt{\Omega' - a}} \left(1 + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{k!} \left(-\frac{e^{-\Omega'}}{2\left(\Omega' - a\right)}\right)^k\right)$$

- So defining function $\mathscr{U}\left(\Omega^{(0,I)}\right)$ as

$$\mathscr{U}\left(\Omega^{(0,I)}\right) = \int_a^{\Omega^{(0,I)}} \frac{d\Omega'}{\sqrt{\Omega' + \exp\left(-\Omega'\right) - a}}$$

- We obtain

$$\mathscr{U}\left(\Omega^{(0,I)}\right) = 2\sqrt{\Omega^{(0,I)} - a} + \sum_{k=1}^{\infty} \frac{(-1)^k k^{k-\frac{1}{2}} (2k-1)!!}{2^k\, k!} e^{-a\,k} \left(\frac{(-1)^k 2^k \sqrt{\pi}}{(2k-1)!!} - \Gamma\left(\frac{1}{2} - k, (k(-a + \Omega^{(0,I)})\right)\right)$$

Diffraction at high energy

- Now using the asymptotic expansion $\Gamma(a,z) \sim z^{a-1}e^{-z}$ we obtain

$$2\sqrt{\Omega^{(0,I)} - a} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-1)!!}{2^k k\, k!} \left(\frac{1}{\Omega^{(0,I)} - a}\right)^{k+\frac{1}{2}} \exp\left(-k\,\Omega^{(0,I)}\right) = \sqrt{\frac{2}{\kappa}}\,(z-\tilde{z})$$

- And for large z leads to

$$\Omega^{(0,I)}(z) - a = \frac{(z-\tilde{z})^2}{2\,\kappa} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-1)!!}{2^k k\, k!} \left(\frac{1}{\frac{(z-\tilde{z})^2}{2\,\kappa}}\right)^{k+\frac{1}{2}} \exp\left(-k\,\frac{(z-\tilde{z})^2}{2\,\kappa}\right)$$

- Finally $\Delta_0^{(0,I)} = \exp(-\Omega^{(0,I)})$ can be rewritten as follows

$$\Delta_0^{(0,I)}(z) = \Delta_{LT}(z) \exp\left(-a + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-1)!!}{2^k k\, k!} \left(\frac{2\kappa}{(z-\tilde{z})^2}\right)^{k+\frac{1}{2}} \Delta_{LT}^{k}(z)\right)$$

where $\Delta_{LT}(z) = \exp\left(-\frac{(z-\tilde{z})^2}{2\,\kappa}\right)$

Diffraction at high energy

- For region II, we have

$$\frac{\partial^2 \Omega^{(0)}(z,\xi_s)}{\partial z^2} - \frac{\partial^2 \Omega^{(0)}(z,\xi_s)}{\partial t^2} = \frac{1}{\kappa}\left(1 - e^{-\Omega^{(0)}(z',\xi_s)}\right)$$

where $z = \xi_s + \xi$ and $t = \xi_s - \xi$.

- This equation has the traveling wave solution

$$\int_{\Omega_0^{(0)}(z_0,\xi_{0,s})}^{\Omega^{(0)}(z,\xi_s)} \frac{d\Omega'}{\sqrt{C_1 + \frac{2}{\kappa(\mu^2-\nu^2)}\left(\Omega' + \exp(-\Omega')\right)}} = \mu\, z + \nu\, t + C_2$$

- We rewrite as follows for satisfy initial conditions

$$\int_{\Omega_0}^{\Omega^{(0)}(z,\xi_s)} \frac{d\Omega'}{\sqrt{-\Omega_0 + \Omega' + \exp(-\Omega')}} = \sqrt{\frac{2}{\kappa}}\left((1+\nu)\, z + \nu\, t - \tilde{z} - 2\nu\, \xi_{0,s}\right)$$

Diffraction at high energy

- Therefore $\Omega^{(0,II)}$ is the solution to the equation

$$\mathscr{U}\left(\Omega^{(0,II)},\Omega_0\right) = \sqrt{\frac{2}{\kappa}}\left(\left(1+\nu\right)\,z + \nu\,t - \hat{z}\right)$$

where $\hat{z} = \tilde{z} + 2\nu\xi_{0,s}$

- Following the same steps, we obtain

$$\Delta_0^{(0,II)}\left(z,\xi_s\right) = \tilde{\Delta}_{LT}\left(z,\xi_s\right)\exp\left(-\Omega_0 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(2k-1)!!}{2^k k\,k!}\left(\frac{2\kappa}{\left((1+\nu)z+\nu t-\hat{z}\right)^2}\right)^{k+\frac{1}{2}}\tilde{\Delta}_{LT}^k\left(z,\xi_s\right)\right)$$

where $\tilde{\Delta}_{LT}\left(z,\xi_s\right) = \exp\left(-\frac{\left((1+\nu)z+\nu t-\hat{z}\right)^2}{2\,\kappa}\right)$

- By doing the matching on the line $\xi = \xi_0^A$, we find $\nu = 0$

Diffraction at high energy

- Let's rewrite the KL equation as follows

$$n^D_{01} \;=\; \frac{\bar{\alpha}_S}{2\pi}\int d^2 r_2\, K\left(r_{10}|r_{12},r_{02}\right)\left\{N^D_{12}+N^D_{02}\;-\;N^D_{01}+N^D_{12}N^D_{02}\;-\;2\,N^D_{12}\,N_{02}-\;2\,N_{12}N^D_{02}\;+\;2N_{12}N_{02}\right\}$$

- We see that

$$n^D\left(z,Y_M,z_0,\delta Y_0;b\right)\;=\;\frac{\partial N^D(z,\delta\tilde{Y},\delta Y_0)}{\partial\delta\tilde{Y}}\;=\;-\frac{\partial\mathcal{N}(z,\delta\tilde{Y},\delta Y_0)}{\partial\delta\tilde{Y}}\;+\;2\frac{\partial N(z,\delta\tilde{Y},\delta Y_0)}{\partial\delta\tilde{Y}}$$

- Applying chain rule on function $\mathscr{U}$

$$\frac{d\mathscr{U}\left(\Omega^{(0,I)}\right)}{dz}=\frac{d\mathscr{U}\left(\Omega^{(0,I)}\right)}{d\Omega^{(0,I)}}\frac{d\Omega^{(0,I)}}{dz}=\frac{d\Omega^{(0,I)}}{dz}\frac{1}{\sqrt{\Omega^{(0,I)}+\exp\left(-\Omega^{(0,I)}\right)-a}}=\sqrt{\frac{2}{\kappa}}$$

- We obtain

$$\frac{\partial\mathcal{N}(z,\delta\tilde{Y},\delta Y_0)}{\partial\delta\tilde{Y}}=\kappa\frac{d\Omega^{(0,I)}}{dz}\exp\left(-\Omega^{(0,I)}\right)=\sqrt{2\,\kappa}\sqrt{\Omega^{(0,I)}+\exp\left(-\Omega^{(0,I)}\right)-a}\;\exp\left(-\Omega^{(0,I)}\right)$$

Diffraction at high energy

- Therefore

$$n^{(D,I)} = -\sqrt{2\kappa\,\left(\Omega^{(0,I)} + \Delta_0^{(0,I)} - a^I\right)\,\Delta_0^{(0,I)}} + 2\sqrt{2\kappa\left(\Omega^{(0,I)} + \Delta_0^{(0,I)} - a^I/2\right)\,\Delta_0^{(0,I)}}$$

$$n^{(D,II)} = -\sqrt{2\kappa\,\left(\Omega^{(0,II)} + \Delta_0^{(0,II)} - a^{II}\right)\,\Delta_0^{(0,II)}} + 2\sqrt{2\kappa\left(\Omega^{(0,II)} + \Delta_0^{(0,II)} - a^{II}/2\right)\,\Delta_0^{(0,II)}}$$

where $a^I = (z_0 - \tilde{z})^2/(2\kappa) - 2\ln C$ and $a^{II} = \frac{(z_0-\tilde{z})^2}{2\,\kappa} - \frac{(z_0-\xi_{0,s}-\tilde{z})^2}{\kappa} + \frac{1}{2}\exp\left(z_0 - \xi_{0,s}\right)$

- Finally, amplitude $N^D(Y;Y_0,r_{01};b)$ is given by

$$N^D(Y;Y_0,r_{01};b) = N^2_{el}(Y_0) + \int_{Y_0}^{Y} dY'\sigma_{diff}(Y',Y_0,b) \equiv N^2_{el}(Y_0) + \int_0^{Y_M} dY' n^D(Y'_M,Y_0,r_{01};b)$$

- For relate with experiment, we should use formulas

$$\sigma^{\rm diff}(Y,Y_0,Q^2) = \int d^2r_\perp \int dz\,|\Psi^{\gamma^*}(Q^2;r_\perp,z)|^2\,\sigma^{diff}_{\rm dipole}(r_\perp,Y,Y_0)$$

where $\sigma^{diff}_{\rm dipole}(r_\perp,Y,Y_0) = \int d^2b\,d^2b'\,N^D(r_\perp,Y,Y_0;\mathbf{b})$

Diffraction at high energy

- Now, the equation for the second iteration is

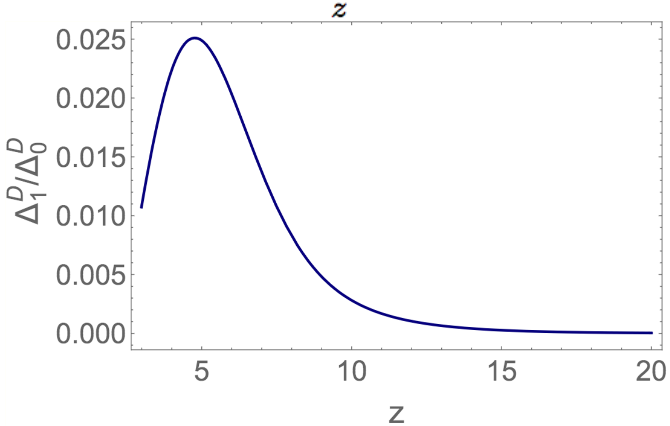
$$\left(\kappa\frac{\partial}{\partial z} + z - \zeta\right)\Delta_1^D(z,z_0) + \underbrace{\Delta_0^D(z,z_0)\int_z^\infty dz'\Delta_1^D(z',\boldsymbol{z}_0) + \Delta_1^D(z,z_0)\int_z^\infty dz'\Delta_0^D(z',\boldsymbol{z}_0)}_{\sim(\Delta^0)^3} = -\underbrace{\mathcal{N}_{\mathcal{L}}[\Delta_0^D(z)]}_{\sim(\Delta^0)^2}$$

- Taking into account only terms of the order of $(\Delta^0)^2$

$$\Delta_1^D(z,z_0) = -\Delta_0^D(z,z_0)\int_z^\infty dz'\frac{1}{\Delta_0^D(z',z_0)}\mathcal{N}_{\mathcal{L}}[\Delta_0^D(z')]$$

- The particular solution can be written as follows

$$\Delta_1^D(z,z_0) = -\Delta_0^D(z,z_0)\int_z^\infty dz'\frac{1}{\Delta_0^D(z',z_0)}\mathcal{N}_{\mathcal{L}}[\Delta_0^D(z')]$$



Diffraction at high energy

- This was numerical calculation, but you can also do some simplifications for obtain an analytical solution

$$\left(\kappa \frac{\partial}{\partial z} + z\right)\Delta^D_1(z,\xi,z_0) = -\mathcal{N}_{\mathcal{L}}[\Delta^D_0(z)] = -\exp\left(-\frac{(z-2\ln 2)^2}{\kappa} - \tilde{\phi}(\xi,z_0)\right)$$

$$\kappa\,\Delta^D_1(z,\xi,z_0) = \Delta^D_0(z,\xi,z_0)\int_{z_0}^{z} dz' \frac{\mathcal{N}_{\mathcal{L}}[\Delta^D_0(z')]}{\Delta^D_0(z,\xi,z_0)}$$
$$= \Delta^D_0(z,\xi,z_0)\sqrt{\frac{\pi\kappa}{2}}e^{\frac{4\ln^2 2}{\kappa}}\left(-\text{erf}\left(\frac{z_0-4\ln 2}{\sqrt{2\kappa}}\right)+\text{erf}\left(\frac{z-4\ln 2}{\sqrt{2\kappa}}\right)\right)$$

- Finally, let me briefly talk you about $\Delta^{pQCD}_0(z_D,\xi,z_0)$, region where $z_0 \leq 0$ and initial conditions are

$$\mathcal{N}^D(z\to z_0,\delta\tilde{Y}=\delta Y_0) = 2\,N^{\text{BFKL}}(z_0,\delta Y_0) - \left(N^{\text{BFKL}}(z_0,\delta Y_0)\right)^2 = 2\,N_0\left(r^2\,Q^2_s(\delta Y_0)\right)^{\bar{\gamma}} - N^2_0\left(r^2\,Q^2_s(\delta Y_0)\right)^{2\bar{\gamma}}$$

or

$$\Omega^{0,pQCD}_0 = \mathcal{N}^D(z\to z_0,\delta\tilde{Y}=\delta Y_0,\delta Y_0) = 2\,N_0\left(r^2\,Q^2_s(\delta Y_0)\right)^{\bar{\gamma}} - N^2_0\left(r^2\,Q^2_s(\delta Y_0)\right)^{2\bar{\gamma}} = 2N_0\,e^{\bar{\gamma}\,z_0} - N^2_0 e^{2\,\bar{\gamma}\,z_0}$$

Diffraction at high energy

- We have

$$\int_{\Omega_0^{0,pQCD}}^{\Omega^{0,pQCD}} \frac{d\,\Omega'}{\sqrt{\Omega' + e^{-\Omega'} - 1 + \frac{1}{2}\left(\bar{\gamma}^2\,\kappa\,-\,1\right)\left(\Omega_0^{0,pQCD}\right)^2}} = \sqrt{\frac{2}{\kappa}}\,(z\,-\,z_0)$$

- and we rewrite this as

$$\underbrace{\int_{1}^{\Omega^{0,pQCD}} \frac{d\,\Omega'}{\sqrt{\Omega' + e^{-\Omega'} - 1}}}_{\mathscr{U}_1\left(\Omega^{0,pQCD}\right)} + \underbrace{\int_{\Omega_0^{0,pQCD}}^{1} \frac{d\,\Omega'}{\sqrt{\Omega' + e^{-\Omega'} - 1 + \frac{1}{2}\left(\bar{\gamma}^2\,\kappa\,-\,1\right)\left(\Omega_0^{0,pQCD}\right)^2}}}_{\mathscr{U}_2\left(\Omega_0^{0,pQCD}\right)} = \sqrt{\frac{2}{\kappa}}\,(z\,-\,z_0)$$

- Considering $\Omega^{0,pQCD}$ large for $\mathscr{U}_1\left(\Omega^{0,pQCD}\right)$ and $\Omega^{0,pQCD}$ small for $\mathscr{U}_2\left(\Omega_0^{(pQCD,0)}\right)$, we obtain

Diffraction at high energy

$$\Omega^{0,pQCD}\left(\tilde{\zeta},\Omega_0^{0,pQCD}\right) > 1 \quad \Delta_0^{pQCD}\left(z_D,\xi,z_0\right) = \exp\left(-\Omega^{0,pQCD}(\tilde{\zeta})\right)$$

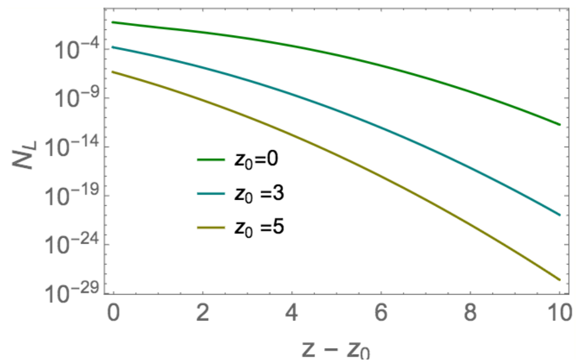
$$\text{with} \quad \tilde{\zeta} = \sqrt{\frac{2}{\kappa}}\left(z - z_0\right) - \mathscr{U}_2\left(\Omega_0^{(pQCD,0)}\right)$$

$$\Omega^{0,pQCD}\left(\tilde{\zeta},\Omega_0^{0,pQCD}\right) < 1 \quad \Delta_0^{pQCD}\left(z_D,\xi,z_0\right) = \Omega_0^{0,pQCD}\left(\cosh\left(\frac{z-z_0}{\sqrt{\kappa}}\right) + \bar{\gamma}\sqrt{\kappa}\sinh\left(\frac{z-z_0}{\sqrt{\kappa}}\right)\right)$$

where $\mathscr{U}_2(\Omega_0^{(pQCD,0)}) = -\sqrt{2}\left(\frac{1}{2}\ln(\Omega_0^{(pQCD,0)})^2 + \frac{1}{2}\ln\frac{(\bar{\gamma}^2\kappa - 1)}{4}\frac{1+\frac{1}{\bar{\gamma}\sqrt{\kappa}}}{1-\frac{1}{\bar{\gamma}\sqrt{\kappa}}}\right)$

- Finally, for the second iteration we have

$$\Delta_1^D\left(z_D,\xi,z_0\right) = \Delta_0^{pQCD}\left(z_D,\xi,z_0\right)\int_{z_0}^{z_D} dz'\frac{1}{\Delta_0^{pQCD}\left(z',\xi,z_0\right)}\mathscr{N}_{\mathscr{L}} \quad ,$$



Conclusions

- **It has been shown** that the **zero order solution (first iteration)** is a **good approximation** for solving non-linear equation that appear in QCD, and that the iteration procedure, which is being partly numerical, leads to **small corrections**.
- However, there are still some **open questions.** What about for a running coupling? (actually, in development!), what about for other nonlinear equations? (in development too), how to confront these results with experimental data?, etc.

Thank you for your attention!

ありがとう!