Update on α_s from the decoupling method

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based on [**Eur.Phys.J.C 82 (2022) 12, 1092** (2209.14204 [hep-lat])]

& work in collaboration with:

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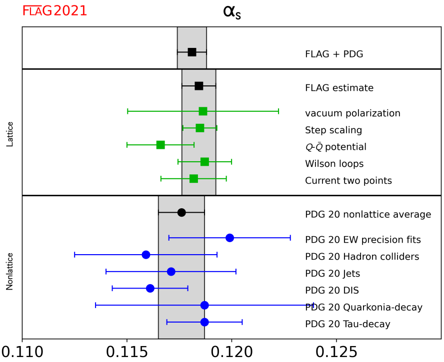
HHIQCD2024
Yukawa Institute for Theoretical Physics
Kyoto University, Japan
25 October 2024

Outline

- Current status of $\alpha_s(m_Z)$
- The ALPHA collaboration project
- Reformulation in terms of the Λ-parameter
- Large scale differences and step-scaling function (SSF)
- Decoupling of heavy quarks as a renormalization tool
- Numerical set-up, continuum and decoupling limits
- Error budget
- Recent improvements
- Conclusions

Current status of $\alpha_s(m_Z)$



- Best result from lattice QCD:

$$\alpha_s(m_Z) = 0.1184(8)$$

- Step-scaling method: error is statistics dominated!
- Most other results (on and off the lattice) systematics dominated (hadronization models, quark hadron duality violations, non-perturbative effects at low energies,...)

Reducing the current error (0.7%) to below half a percent is a challenge!

ALPHA collaboration project (origins in the 1990's)

Build on CLS effort [Bruno et al, JHEP 1502 (2015) 043]:

- $N_\mathrm{f} = 2 + 1$ state of the art lattice QCD simulations
- nonperturbatively O(a) improved Wilson quarks & Lüscher-Weisz gauge action;
- open boundary conditions (avoids topology freezing)

Use 3 input parameters from experiment, e.g.

$$F_{\pi,K}, m_\pi, m_K \qquad \Rightarrow \qquad m_u = m_d, m_s, g_0$$

$\Rightarrow$ everything else becomes a prediction, for instance

$$\alpha_s^{(N_\mathrm{f}=3)}(1000 \times F_{\pi,K}) \quad \text{(in any renormalization scheme)}$$

Final goal: $\alpha_s^{(N_\mathrm{f}=5)}(m_Z)$ in the $\overline{\mathrm{MS}}$-scheme

- Requires matching to $N_\mathrm{f} = 5$ across the charm and bottom thresholds;
- Perturbation theory to 4-loop order satisfactory [Athenodorou et al. '18]

The QCD Λ-parameter vs. $\alpha_s(\mu) = \bar{g}^2(\mu)/4\pi$

The coupling $\alpha_s(\mu)$ can be traded for its associated Λ-parameter:

$$\Lambda = \mu\varphi(\bar{g}(\mu)) = \mu\left[b_0\bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} dg\left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

- exact solution of Callan-Symanzik equation: $\left(\mu\frac{\partial}{\partial\mu} + \beta(\bar{g})\frac{\partial}{\partial\bar{g}}\right)\Lambda = 0$
- Number N_f of massless quarks is fixed.
- If the coupling $\bar{g}(\mu)$ non-perturbatively defined so is its β-function!
- $\beta(g)$ has asymptotic expansion $\beta(g) = -b_0g^3 - b_1g^5 - b_2g^7..$

  $$b_0 = (11 - \tfrac{2}{3}N_\mathrm{f})/(4\pi)^2, \qquad b_1 = (102 - \tfrac{38}{3}N_\mathrm{f})/(4\pi)^4, \quad \ldots$$

  $b_{0,1}$ are universal, scheme-dependence starts with 3-loop coefficient b_2.
- Scheme dependence of Λ almost trivial:

  $$g_\mathrm{X}^2(\mu) = g_\mathrm{Y}^2(\mu) + c_\mathrm{XY}g_\mathrm{Y}^4(\mu) + \ldots \quad \Rightarrow \quad \frac{\Lambda_\mathrm{X}}{\Lambda_\mathrm{Y}} = \mathrm{e}^{c_\mathrm{XY}/2b_0}$$

$\Rightarrow$ can use $\Lambda_{\overline{\mathrm{MS}}}$ as reference (even though the $\overline{\mathrm{MS}}$-scheme is purely perturbative!)

The QCD Λ-parameter and $\alpha_s(\mu) = \bar{g}^2(\mu)/4\pi$

$$\Lambda = \mu\varphi(\bar{g}(\mu)) = \mu\left[b_0\bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} dg \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

- Continuum relation, exact at any scale μ:
  - require large μ to evaluate integral perturbatively
  - require small μ to match hadronic scale

$\Rightarrow$ problem of large scale differences:

- The scale μ must reach the perturbative regime: $\mu \gg \Lambda$
- lattice cutoff must still be larger: $\mu \ll a^{-1}$
- spatial volume must be large enough to contain pions: $L \gg 1/m_\pi$
- Taken together a naive estimate gives

$$L/a \gg \mu L \gg m_\pi L \gg 1 \quad \Rightarrow L/a \simeq \mathrm{O}(10^3)$$

$\Rightarrow$ widely different scales cannot be resolved simultaneously on a single lattice!

The step scaling solution

- Widely different scales cannot be resolved simultaneously on a *single* lattice

$\Rightarrow$ break calculation up in steps [Lüscher, Weisz, Wolff '91; Jansen et al. '95]:

1. define $\bar{g}^2(L)$ that runs with the space-time volume, i.e. $\mu = 1/L$
2. construct the step-scaling function

$$\sigma(u) = \bar{g}^2(2L)\Big|_{u=\bar{g}^2(L)}$$

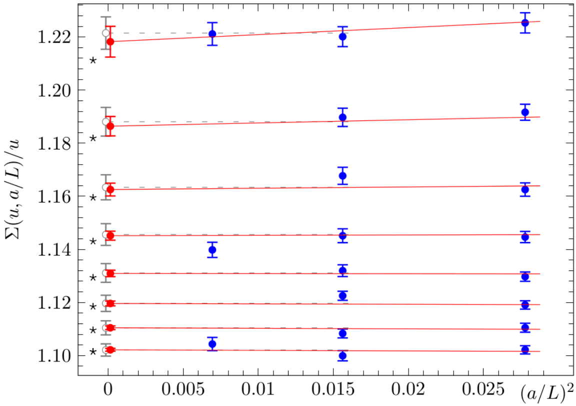
   for a range of values $u \in [u_{\text{min}}, u_{\text{max}}]$

3. iteratively step up/down in scale by factors of 2:

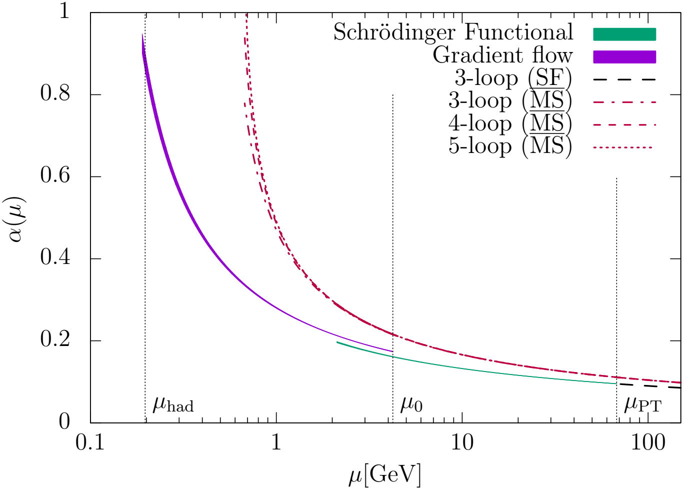
$$\bar{g}^2(L_{\text{max}}) = u_{\text{max}} \equiv u_0, \quad u_k = \sigma(u_{k+1}) = \bar{g}^2(2^{-k}L_{\text{max}}), \quad k = 0, 1, \ldots$$

4. match to hadronic input at a hadronic scale L_{max}, i.e. $F_K L_{\max} = \mathrm{O}(1)$
5. once arrived in the perturbative regime $L_{\text{pert}} = 2^{-n} L_{\text{max}}$ one now knows $u_n = \bar{g}^2(L_{\text{pert}})$; determine $L_{\text{pert}}\Lambda$ and combine to obtain Λ/F_K.

Continuum limit $\sigma(u) = \lim_{a/L \to 0} \Sigma(u, a/L)$

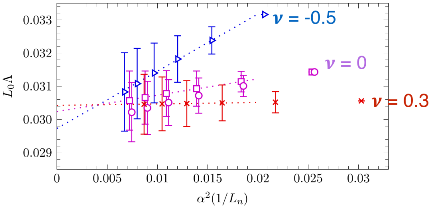


Non-perturbative running of α_s in $N_f = 3$ QCD



Result for Λ [ALPHA '17]

- The Λ-parameter can now be evaluated at a very high perturbative scale $\mu_{\rm PT}$ with $\mu_{\rm had}/\mu_{\rm PT}$ known (a power of 2 if related by step-scaling).
- The remaining uncertainty is parametrically $\propto \alpha^2(\mu_{\rm PT})$ if the β-function is



known to 3-loop order

- Note: observation of this dependence requires data over a large range of scales, so that α^2 varies significantly!
- The result

$$\Lambda^{(3)}_{\overline{\rm MS}} = 341(12)\text{MeV}$$

translates to $\alpha_s(m_Z) = 0.11852(84)$ (with 4-loop matching across charm and bottom thresholds)

Decoupling of heavy quarks as a renormalization tool

- Perturbative matching across the charm and bottom quark thresholds seems to be very accurate in $\overline{\mathrm{MS}}$ scheme;

  available up to 4-loop order [Bernreuther and Wetzel '82; Grozin et al. '11; Chetyrkin et al.'05; Schröder and Steinhauser '05; Kniehl et al. '06; Gerlach, Herren and Steinhauser '18]

- [ALPHA '18]: non-perturbative test of decoupling, comparison with perturbative 4-loop description in the $\overline{\mathrm{MS}}$-scheme.

$\Rightarrow$ suggests that perturbative decoupling yields very high precision for quarks around the charm mass and above!

- Idea: match QCD with $N_\mathrm{f} = 3$ heavy quarks non-perturbatively to QCD with $N_\mathrm{f} = 0$ by taking the infinite mass limit.

- Potential gain: Non-perturbative running could be done in the $N_\mathrm{f} = 0$ theory (computationally cheaper and much more precise).

Decoupling of heavy quarks

- Consider QCD with $N_\mathrm{f} = 3$ heavy quarks of RGI mass M

$$M = \overline{m}_s(\mu)\,\left[2b_0\bar{g}_s^2(\mu)\right]^{-\frac{d_0}{2b_0}} \exp\left\{-\int_0^{\bar{g}_s(\mu)}\left[\frac{\tau_s(x)}{\beta_s(x)} - \frac{d_0}{b_0 x}\right]\,dx\right\}\,,$$

  with $\overline{m}_s(\mu)$ the running mass in scheme s.

- At scales $\mu \ll M$, the fundamental theory ($N_\mathrm{f} = 3$-flavour QCD) can be described by an effective theory, $N_\mathrm{f} = 0$ QCD (i.e. pure Yang-Mills theory):

$$\bar{g}_s^{(3)}(\mu/\Lambda_s^{(3)}, M) = \bar{g}_s^{(0)}(\mu/\Lambda^{(0)}) + \mathrm{O}(\mu^2/M^2)\,, \qquad (1)$$

- in PT this leads to

$$[\bar{g}_{\overline{\mathrm{MS}}}^{(0)}(\mu)]^2 = C\left(\bar{g}_{\overline{\mathrm{MS}}}^{(3)}(m_\star)\right)[\bar{g}_{\overline{\mathrm{MS}}}^{(3)}(m_\star)]^2, \qquad m_\star = \overline{m}_{\overline{\mathrm{MS}}}(m_\star),$$

  and for $\mu = m_\star$ one finds $C(x) = 1 + c_2x^4 + c_3x^6 + c_4x^8 + \ldots$.

- Reformulation with $P = \varphi_{\overline{\mathrm{MS}}}^{(0)}\left(g_\star\sqrt{C(g_\star)}\right)/\varphi_{\overline{\mathrm{MS}}}^{(3)}(g_\star)$, $g_\star = g_{\overline{\mathrm{MS}}}^{(3)}(m_\star)$:

$$\frac{\Lambda_{\overline{\mathrm{MS}}}^{(3)}}{\mu_\mathrm{dec}} = \frac{\Lambda_{\overline{\mathrm{MS}}}^{(0)}}{\Lambda_s^{(0)}} \times \lim_{M/\mu_\mathrm{dec}\to\infty}\left[\frac{\varphi_s^{(0)}\left(\bar{g}_s^{(3)}(\mu_\mathrm{dec}, M)\right)}{P\left(\frac{M}{\mu_\mathrm{dec}}\Big/\frac{\Lambda_{\overline{\mathrm{MS}}}^{(3)}}{\mu_\mathrm{dec}}\right)}\right]$$

Coupling definitions

Set-up such that it benefits from various previous projects: running quark mass [Campos et al '18], $N_\mathrm{f} = 3$ coupling [ALPHA'17]

- Definition of massless renormalized couplings: use gradient flow GF scheme in finite volume

$$\bar{g}^2_{\mathsf{GF}}(\mu) = \mathcal{N}^{-1} \sum_{k,l=1}^{3} \frac{t^2 \langle \mathrm{tr}\,\{G_{kl}(t,x) G_{kl}(t,x)\}\, \delta_{Q,0} \rangle}{\langle \delta_{Q,0} \rangle} \Bigg|^{x_0 = T/2,\; c = \sqrt{8t}/L}_{\mu = 1/L, T = L,\; M = 0}$$

- use both $T = L$ (GF scheme) and $T = 2L$ (GFT scheme) with projection to topological charge $Q = 0$ sector (part of scheme definition)
- 1-parameter families of schemes, parameter $c = \sqrt{8t}/L$
- $T = 2L$ chosen to suppress both cutoff effects linear in a and large mass effects linear in $1/M$ from Euclidean time boundaries.

Numerical set-up

Lines of constant physics:

- $\bar{g}_{\mathsf{GF}}(\mu_{\mathsf{dec}}) = 3.949 \quad \Rightarrow \mu_{\mathsf{dec}} = 789(15)\mathrm{MeV}$

  Varying $L/a = 1/(a\mu_{\mathsf{dec}})$ between $L/a = 12 - 48$ defines a sequence of values $\beta = 6/g_0^2 \in [4.3, 5.2]$
- Define range of values $z = M/\mu_{\mathsf{dec}} \in \{1.972, 4, 6, 8, 10, 12\}$ up to $\mathrm{O}(a^2)$ effects (non-trivial!) and find corresponding bare mass values.
- At these values of the bare parameters choose $T = 2L$ and compute the couplings in a massive scheme

$$\bar{g}^{(3)}_{\mathsf{GFT},c}(\mu_{\mathsf{dec}}, z)$$

- require aM to be small and $z = ML = M/\mu_{\mathsf{dec}} \gg 1$

$\Rightarrow$ potentially a difficult multiscale problem; using $\mu = 1/L$ alleviates part of it.

O(a) improvement, rôle of b_g

- Lattice QCD with Wilson quarks is affected by lattice artefacts of O(a), due to explicit chiral symmetry breaking.

$\Rightarrow$ can be restored by tuning the corresponding counterterms in the action and composite fields

- Not all counterterms are known non-perturbatively! With $m_q = m_0 - m_{cr}(g_0^2)$, the O($a$) improved bare coupling is

$$\tilde{g}_0^2 = g_0^2(1 + b_g(g_0^2)am_q),$$

and the O(a) improved RGI mass can be written as

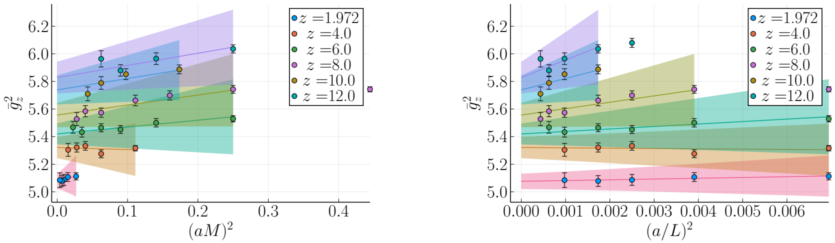
$$M = Z_M(\tilde{g}_0^2, a\mu)m_q(1 + b_m(g_0^2)am_q), \qquad Z_M = \underbrace{\frac{M}{\overline{m}(\mu)}}_{\text{RG running}} \times Z_m(\tilde{g}_0^2, a\mu)$$

- If one wants to vary the quark mass at fixed lattice spacing one needs to fix $\tilde{g}_0^2$. This requires b_g, given to 1-loop order by,

$$b_g(g_0^2) = 0.01200 \times N_f g_0^2 + O(g_0^4)$$

$\Rightarrow$ in ALPHA '22 we assumed an uncertainty of b_g of the same size as the one-loop term.

Continuum extrapolations



Data for $z = M/\mu_{\rm dec} \in \{1.972, 4, 6, 8, 10, 12\}$ (here with $c = 0.36$), extrapolated to $a = 0$ using

- individual fits for each z-value ($b_{\rm g}$-uncertainty not included in error bars):

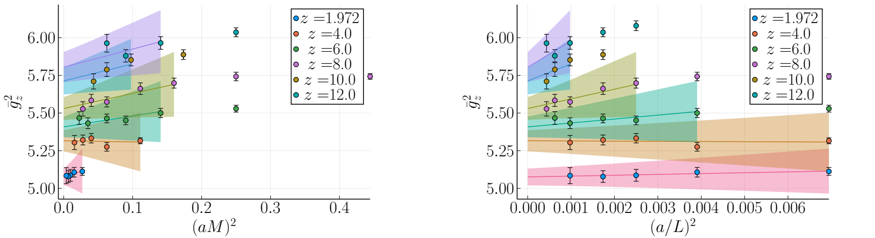
$$\bar{g}^2(z_i, a) = C_i + p_i \left[\alpha_{\overline{\rm MS}}(a^{-1})\right]^{\hat{\Gamma}} (a\mu_{\rm dec})^2$$

- fit form motivated by Symanzik expansion with RG improvement

  [Balog et al. '09; Husung et al.'19]

- 2 cuts in $(aM)^2 < 0.16, 0.25$, fits are carried out for various $\hat{\Gamma} \in [-1, 1]$ (lines in plot for $\hat{\Gamma} = 0$.

Continuum extrapolations



Data for $z = M/\mu_{\rm dec} \in \{1.972, 4, 6, 8, 10, 12\}$ (here with $c = 0.36$), extrapolated to $a = 0$ using

- global fits (bands in plot, contain $b_{\rm g}$-uncertainty);

$$\bar{g}^2(z_i, a) = C_i + p_1[\alpha_{\overline{\rm MS}}(a^{-1})]^{\hat{\Gamma}}(a\mu_{\rm dec})^2 + p_2[\alpha_{\overline{\rm MS}}(a^{-1})]^{\hat{\Gamma}'}(aM_i)^2 \,.$$

- fit form motivated by Symanzik and large mass expansions.
- 2 cuts in $(aM)^2 < 0.16, 0.25$, with fixed $\hat{\Gamma} \in [-1, 1]$ and $\hat{\Gamma}' \in [-1/9, 1]$
- $z = 1.972$ seems to be at the edge of large mass regime; precautioniary measure: cut $z > 2$ and include $z = 1.972$ with different slope parameter

Combining with pure gauge theory results [Dalla Brida & Ramos '19]

- The continuum values of the massive couplings $g^{(3)}_{\mathrm{GFT}}(\mu_{\mathrm{dec}}, M)$ can now be matched with the corresponding $g^{(0)}_{\mathrm{GFT}}(\mu_{\mathrm{dec}})$, up to power corrections $1/M$.
- The step-scaling procedure in pure gauge theory gives us the function $\varphi^{(0)}_{\mathrm{GF}}(g)$

$$\frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{dec}}} = \frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}}{\Lambda^{(0)}_{\mathrm{GF}}} \times \varphi^{(0)}_{\mathrm{GF}}\left(\bar{g}^{(0)}_{\mathrm{GF}}(\mu_{\mathrm{dec}})\right) .$$

$\Rightarrow$ requires matching GFT,c with the $T = L, c = 0.3$ scheme GF

$$\bar{g}^{(0)}_{\mathrm{GF}}(\mu) = \chi_{\mathrm{c}}\left(\bar{g}^{(0)}_{\mathrm{GFT,c}}(\mu)\right)$$

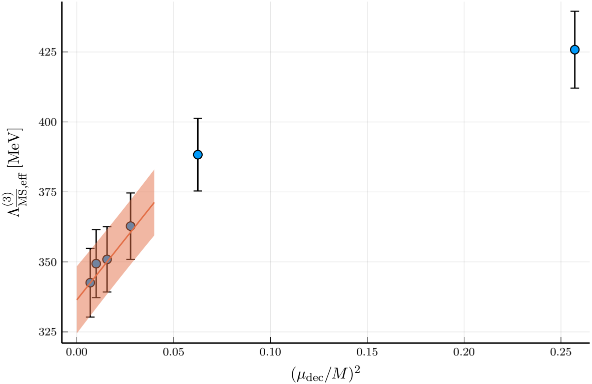
$\Rightarrow$ define $g = \chi_{\mathrm{c}}\left(\bar{g}^{(3)}_{\mathrm{GFT,c}}(\mu, M)\right)$ as input to $\varphi^{(0)}_{\mathrm{GF}}(g)$

- Combining all this at $\mu = \mu_{\mathrm{dec}}$, solve equation for target ρ,

$$\rho \times \underbrace{P\left(z/\rho\right)}_{\mathrm{PT} + \mathrm{O}\left(\alpha^4_{\overline{\mathrm{MS}}}(m_\star)\right)} = \frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{dec}}}, \qquad \rho = \frac{\Lambda^{(3)}_{\overline{\mathrm{MS}},\mathrm{eff}}}{\mu_{\mathrm{dec}}} = \frac{\Lambda^{(3)}_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{dec}}} + \mathrm{O}(1/z^2)$$

- Corrections $\mathrm{O}(1/z)$ from boundaries strongly suppressed due to $T = 2L$.

Decoupling limit extrapolation



Extrapolate continuum results for $z \to \infty$:

$$\Lambda^{3}_{\overline{\mathrm{MS}},\ \mathrm{eff}} = \Lambda^{3}_{\overline{\mathrm{MS}}} + \frac{B}{z^2}\left[\alpha_{\overline{\mathrm{MS}}}(m_\star)\right]^{\hat{\Gamma}_m}$$

Extrapolation at

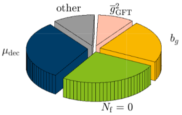
- fixed $c \in [0.3, 0.42]$ (here $c = 0.36$)
- fixed $\hat{\Gamma}_m \in [0, 1]$ (here $\hat{\Gamma}_m = 0$, variation with $\hat{\Gamma}_m$ is used as error estimate)

Result from decoupling, ALPHA '22

Our best estimate:

$$\Lambda_{\overline{\rm MS}}^{(3)} = 336(10)(6)_{b_{\rm g}}(3)_{\hat{\Gamma}_m}\,{\rm MeV} = 336(12){\rm MeV} \quad \Rightarrow \quad \alpha_s(m_Z) = 0.11823(84)$$

- Total error is of the same size as in ALPHA '17 $(341(12)\mathrm{MeV})$



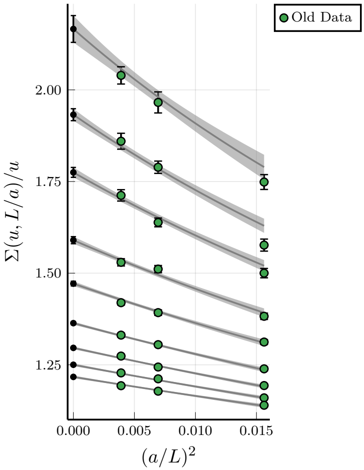
contributions to error^2

- only 28% common (squared) error with ALPHA '17 ⇒ combine:
  $\Lambda_{\overline{\rm MS}}^{(3)} = 339.5(9.6) \quad \Rightarrow \quad \alpha_s(m_Z) = 0.1184(7)$ (statistics dominated!)
- Clear path to further error reduction:
  - Improve determination of $\Lambda_{\overline{\rm MS}}^{(0)}/\mu_{\rm dec}$
  - Improve physical scale setting from CLS ensembles
  - Improve continuum extrapolation of SSF at low energies
  - Non-perturbative determination of $b_{\rm g}$

⇒ all these improvements are underway.

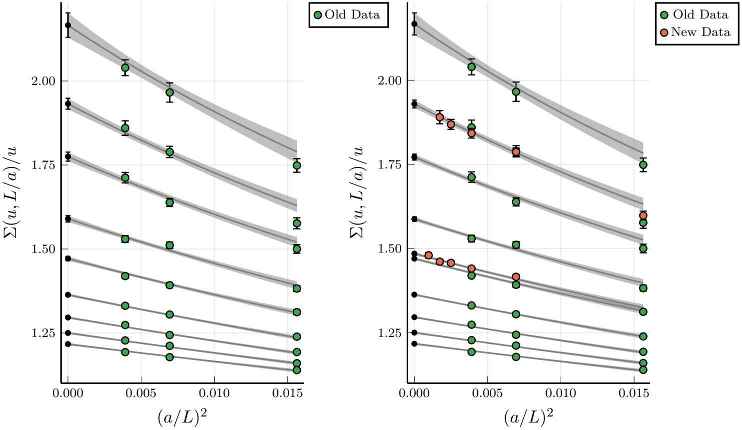
Improvement 1: ALPHA 17 SSF for GF coupling

old data: significant cutoff effects, small lattices ($L/a = 8$) given smaller weight:

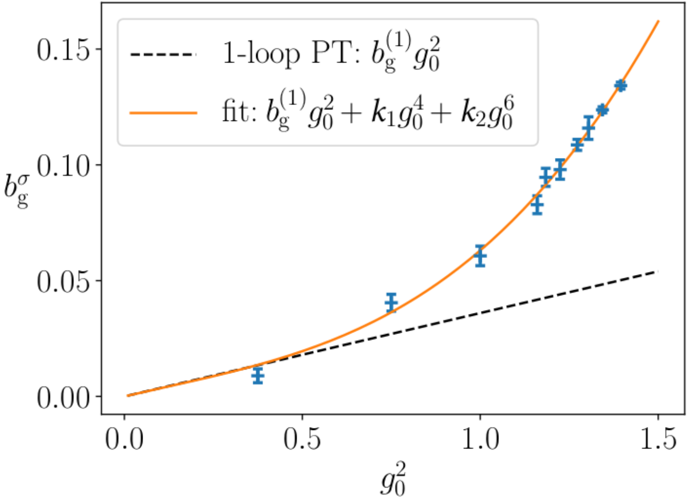


ALPHA 17 SSF for GF coupling

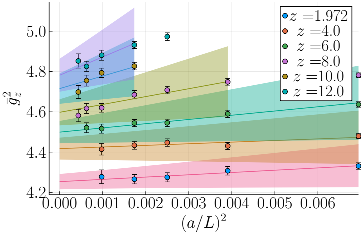
with new data from ALPHA coll. HQET project (courtesy Fritzsch, Kuberski, Heitger): very nice confirmation & improvement of old continuum extrapolations!



Nonperturbative result for b_g, ALPHA '24

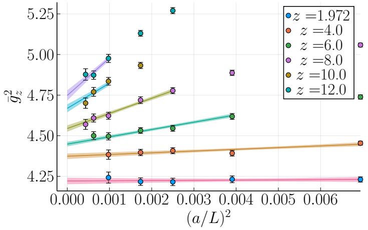


Improvement 2. The continuum extrapolation of massive couplings



Previous determination, ALPHA '22

- Most of error from estimate of $b_g - b_g^{1-\text{loop}}$
- This is a systematic!
- But error in $\bar{g}^2(\mu, M)$ subdominant (assuming 100 percent error on 1-loop b_g)



NP determination of b_g (ALPHA '24)

- Much more precise continuum values
- Completely removes largest systematic effect in α_s

Conclusions

- Current re-analysis with improvements by the ALPHA collaboration will achieve a 0.5 percent **statistics dominated** error on $\alpha_s(m_Z)$.
- Low energy scale setting with $\sqrt{t_0}$ shows inconsistencies at the level of quoted errors, however, these are not relevant for α_s!
- The decoupling result rests on $N_\mathrm{f} = 0$ step-scaling study by Dalla Brida & Ramos '19.
- Change of perspective: Rather than a test-bed for QCD methods, $N_\mathrm{f} = 0$ results can contribute to physics results via decoupling!

$\Rightarrow$ Several new $N_\mathrm{f} = 0$ results, assessed & discussed in FLAG 24 report (publication imminent)

- Use the precise lattice result for α_s as input for phenomenological analyses at colliders!

STAY TUNED!