Dispersive approach to nonperturbative QCD

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Conventional QCD approaches

- QCD observables involve nonpert dynamics. How to handle it?
- Factorization theorem: absorb nonpert dynamics into universal PDFs; break down at high powers eventually
- QCD sum rules: simple but hard to control uncertainties from assumption of quark-hadron duality, determination of stability window in Borel mass; less predictive power
- Lattice QCD, 1st principle but tedious numerics; inapplicable to complicated processes (D -> pi pi,…)
- Effective theories (chiral pert theory,…), models (chiral quark model,…)

Our proposal---dispersive approach

- Adopt dispersion relation like sum rules---based only on analyticity of physical observables (rigorousness)
- OPE in Euclidean region calculable to high orders and powers with universal condensates (no breakdown, systematical improvement of precision)
- Handle dispersion relation as inverse problem---solve integral equation directly to get unknown spectral functions with OPE inputs (mature mathematical tools available)
- Have shown uniqueness of solution
- Higher predictive power without strong assumptions

Formalism

Contour integration

- Two-current correlator

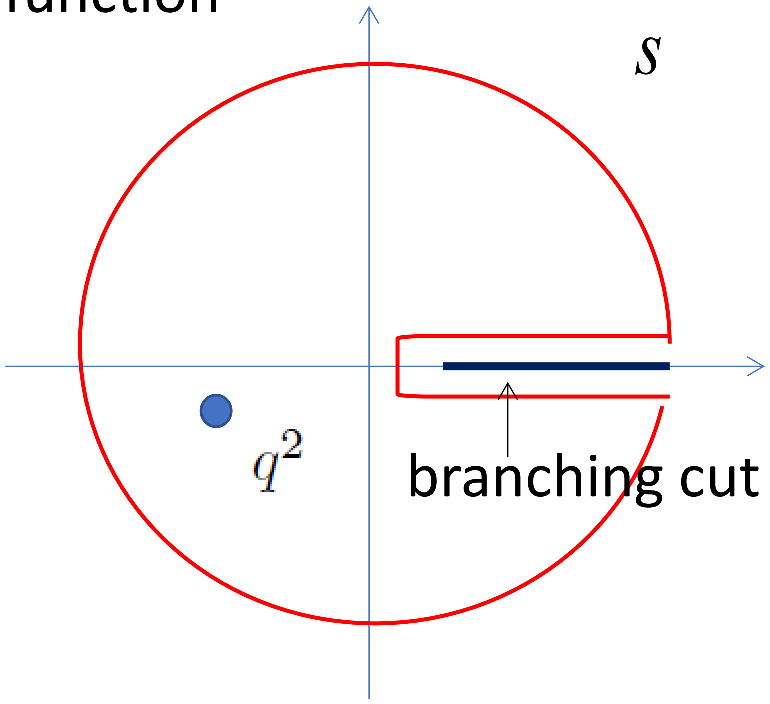
$$J_\mu = (\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)/\sqrt{2}.$$

$$\begin{aligned}\Pi_{\mu\nu}(q^2) &= i\int d^4x e^{iq\cdot x}\langle 0|T[J_\mu(x)J_\nu(0)]|0\rangle \\ &= (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2)\end{aligned}$$

← vacuum polarization function

- Identity from contour integration

$$\Pi(q^2) = \frac{1}{2\pi i}\oint ds\frac{\Pi(s)}{s-q^2}$$



Quark side

- Correlator at large q^2 (deep Euclidean region)
- Operator product expansion (OPE) reliable

parameter characterizing factorization breakdown

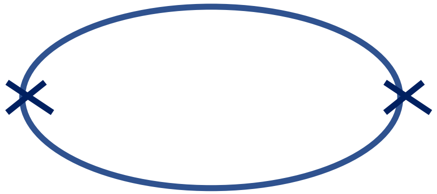
higher order

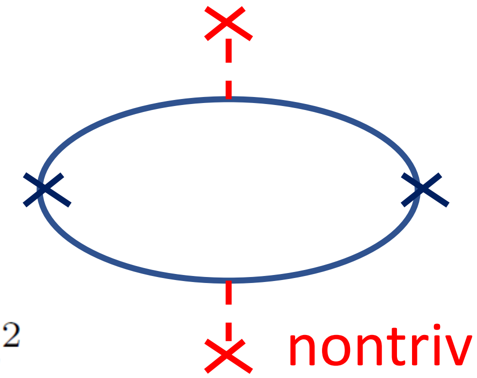
higher powers

$$\Pi^{\text{OPE}}(q^2) = \frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln\frac{\mu^2}{-q^2} + \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q\bar{q}q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{(q^2)^3}$$

perturbative piece

4-quark condensate factorized into product of 2-quark condensates





nontrivial vacuum

$$\frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln\frac{\mu^2}{-q^2} \equiv c\ln\frac{\mu^2}{-q^2}$$

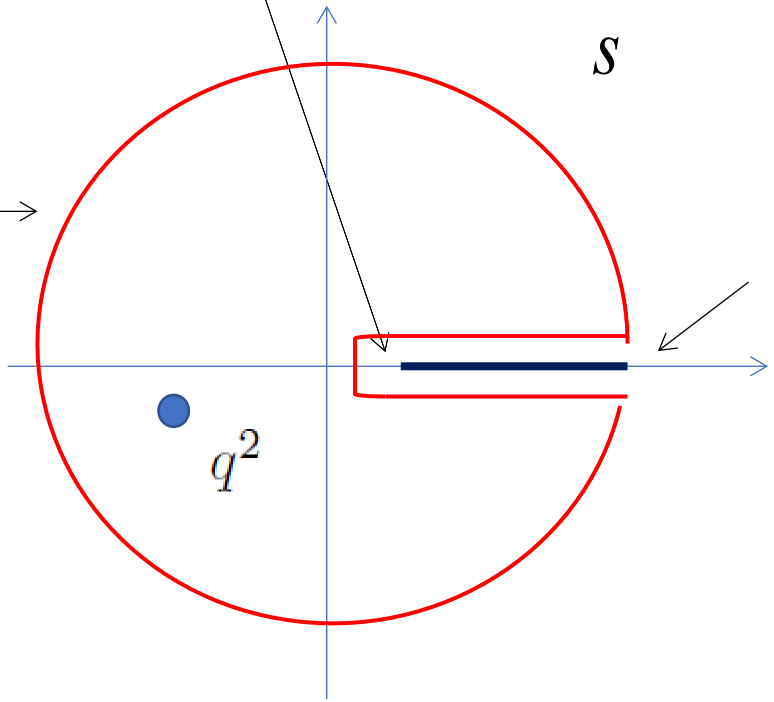
Hadron side

Dispersive integral

nonperturbative spectral function

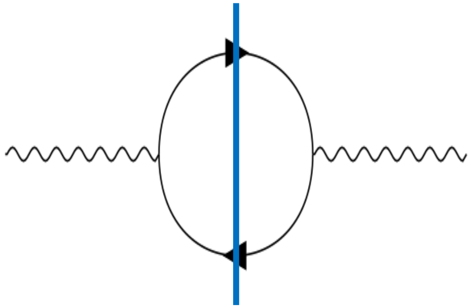
perturbative result

$$\frac{1}{2\pi i}\oint ds\frac{\Pi(s)}{s-q^2}=\frac{1}{\pi}\int_0^R ds\frac{\mathrm{Im}\Pi(s)}{s-q^2}+\frac{1}{2\pi i}\int_C ds\frac{\Pi^{\mathrm{pert}}(s)}{s-q^2}$$



contribution from large circle C of radius R will be cancelled

branch cut caused by real intermediate states due to time-like $q^2 > 0$ (log term)



Dispersion relation

- Rewrite pert piece as contour integral

$$\Pi^{\mathrm{OPE}}(q^2) = \frac{1}{2\pi i}\oint ds\,\frac{\Pi^{\mathrm{pert}}(s)}{s-q^2} + \frac{1}{12\pi}\frac{\langle \alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q \bar{q} q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle \bar{q} q\rangle^2}{(q^2)^3}$$

due to analyticity of perturbation theory

- Equality of two sides gives dispersion relation
- Contributions from big circles cancel, and unknown spectral function from branch cuts remains

arbitrary radius

$$\frac{1}{\pi}\int_0^R ds\,\frac{\mathrm{Im}\Pi(s)}{s-q^2} = \frac{1}{\pi}\int_0^R ds\,\frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s-q^2} + \frac{1}{12\pi}\frac{\langle \alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q \bar{q} q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle \bar{q} q\rangle^2}{(q^2)^3}$$

UV subtraction

- Subtracted spectral function

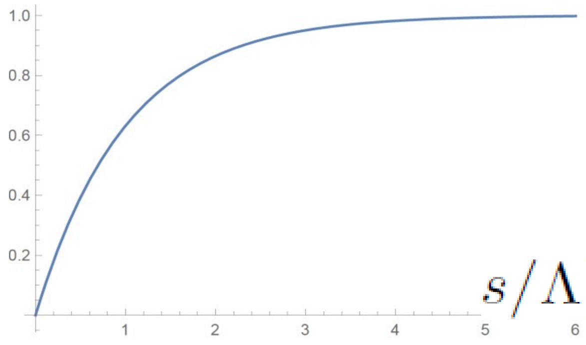
arbitrary R turned into arbitrary scale

$$\Delta\rho(s,\Lambda) = \rho(s) - \frac{1}{\pi}\mathrm{Im}\Pi^{\mathrm{pert}}(s)[1-\exp(-s/\Lambda)]$$

- Maintain low-energy behavior $\rho(s) \sim s$ at $s \to 0$

Kwon et al 2008

- Bear resonance structure the same as $\rho(s)$
- Circle radius R can be pushed to infinity



$$\int_0^\infty ds\frac{\Delta\rho(s,\Lambda)}{s-q^2} = \int_0^\infty ds\frac{ce^{-s/\Lambda}}{s-q^2} + \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q\bar{q}q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{(q^2)^3}$$

- No duality assumed at any finite s

Fredholm equation of the 1st kind

Weakness of sum rules

- Presume existence of ground state, parametrized as pole
- How to handle excited-state contribution?
- Rely on parametrization, quark-hadron duality

$$\mathrm{Im}\Pi(q^2) = \pi f_V^2 \delta(q^2 - m_V^2) + \mathrm{Im}\Pi^{\mathrm{pert}}(q^2)\theta(q^2 - s_0)$$

observables: decay constant, mass

continuum threshold

- Duality may fail

equivalent to q, related via Borel transform

- Stability in unphysical Borel mass?
- Usually not; rely on discretionary prescription; tune s0 to make 70% (30%) perturbative (nonperturbative) contribution

Phenomenological applications

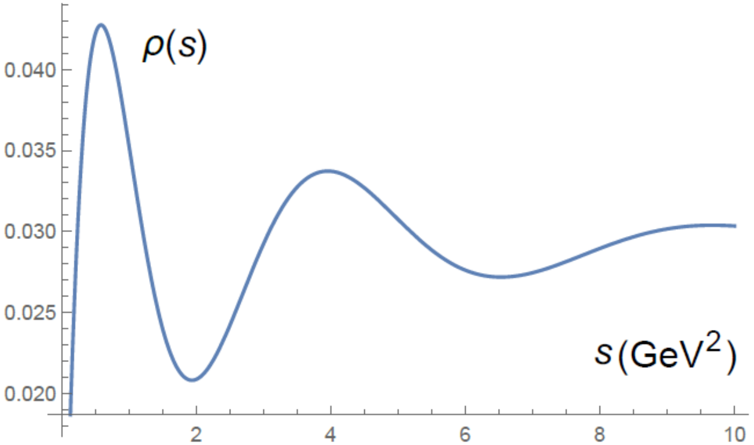
Set aside technical detail of solving the integral equation $\int_0^\infty dy \frac{\rho(y)}{x-y} = \omega(x)$

rho meson spectral function

- OPE input known in the literature

$$\langle m_q \bar{q}q \rangle = 0.007 \times (-0.246)^3 \ \mathrm{GeV}^4, \ \ \langle \alpha_s G^2 \rangle = 0.08 \ \mathrm{GeV}^4$$
$$\alpha_s \langle \bar{q}q \rangle^2 = 1.49 \times 10^{-4} \ \mathrm{GeV}^6, \ \ \alpha_s = 0.5, \ \ \kappa = 2.5.$$

solution of
spectral function



local duality violation

excited states

rho meson peak emerges !

rho meson mass

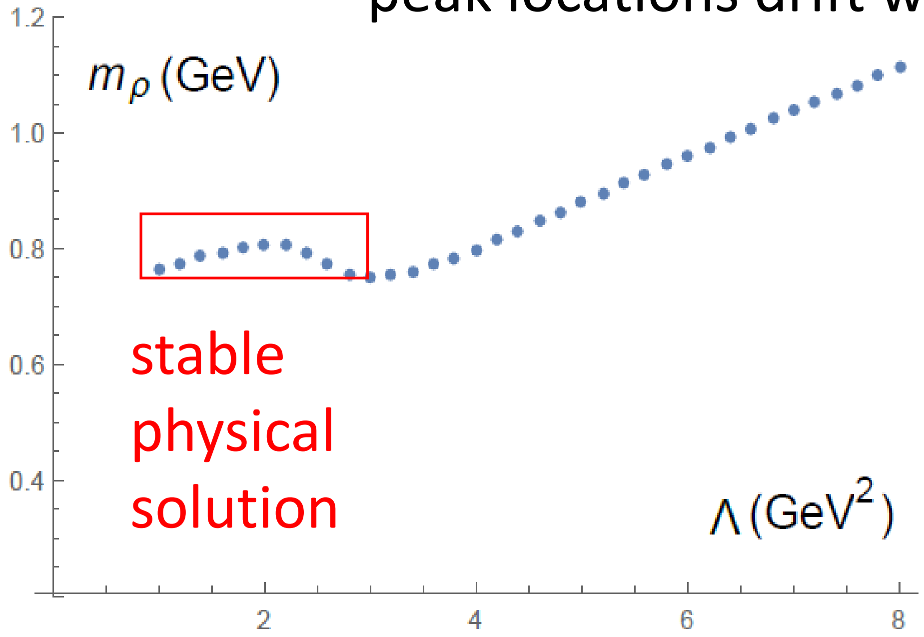
- Vary Λ , find peak location
- Physical solution insensitive to Λ
- Tiny error, stable solution

$$m_\rho = (0.77 \pm 0.02)\ \mathrm{GeV}$$

- Including condensate variation

$$m_\rho = (0.77 \pm 0.04)\ \mathrm{GeV}$$

scaling (unphysical) region
peak locations drift with Λ



stable
physical
solution

- Variable changes $x = q^2/\Lambda \quad y = s/\Lambda$

$$\int_0^\infty dy \frac{\Delta\rho(y)}{x-y} = \int_0^\infty dy \frac{ce^{-y}}{x-y} - \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{x^2\Lambda^2} - 2\frac{\langle m_q\bar{q}q\rangle}{x^2\Lambda^2} - \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{x^3\Lambda^3}$$

- Scaling behavior due to disappearance of power corrections at high Λ

Excited states

- To access excited state, ground-state contribution must be deducted from correlator, i.e., from spectral function to suppress interference
- Parametrize rho(770) contribution as delta-function $F_0\delta(s-m_\rho^2)$

$$F_0 = \int_0^\infty ds\Delta\rho_0(s,\Lambda) = 0.22\ \mathrm{GeV}^2$$

- Subtract it from two sides of dispersion relation

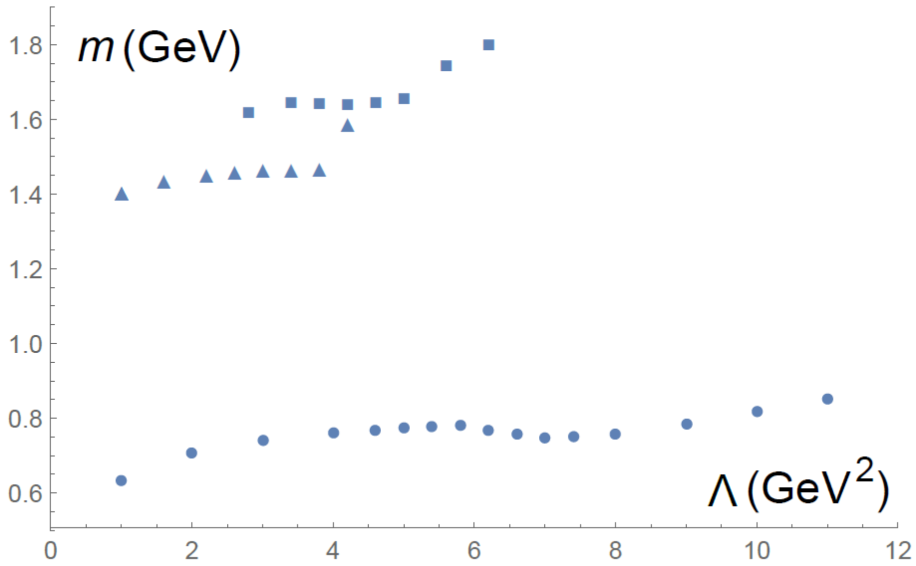
$$\overset{\text{unknown}}{\int_0^\infty dy\frac{\Delta\rho(y)}{x-y}} = \int_0^\infty dy\frac{ce^{-y}-f_0\delta(y-r_0)}{x-y} - \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{x^2\Lambda^2} - 2\frac{\langle m_q\bar{q}q\rangle}{x^2\Lambda^2} - \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{x^3\Lambda^3}$$

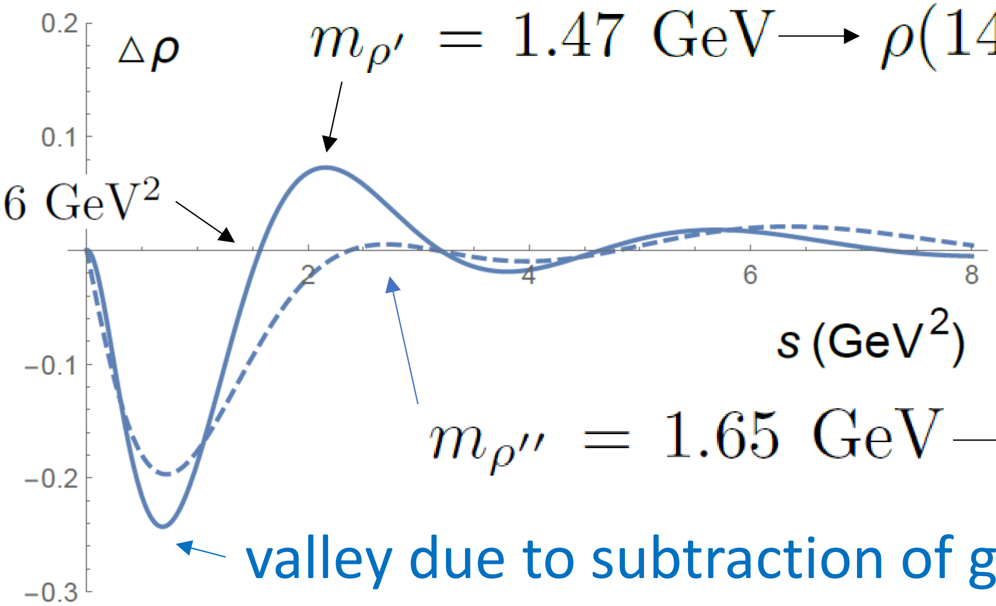
$$x = q^2/\Lambda \qquad y = s/\Lambda \qquad f_0 = F_0/\Lambda \qquad r_0 = m_\rho^2/\Lambda$$

rho resonances

- To get 2nd excited state, further subtract

$$F_1\delta(s-m^2_{\rho'}) \quad F_1=\int_{t_1}^{\infty} ds\Delta\rho_1(s)=0.11\ \mathrm{GeV}^2$$





$m_{\rho'} = 1.47\ \mathrm{GeV} \longrightarrow \rho(1450)$

$t_1 = 1.56\ \mathrm{GeV}^2$

$m_{\rho''} = 1.65\ \mathrm{GeV} \longrightarrow \rho(1700)\ \ 1720 \pm 20\ \mathrm{MeV}$

valley due to subtraction of ground state

uncertainties involved in lower states propagated to higher states, enlarged through sequential subtractions

- Adopting BW form, instead of delta-function, $m_{\rho'}$ increases by 5%

Scalar glueballs

- After checking our formalism, apply it to scalar glueballs

subtracted spectral function, cannot resolve fine structure with finite-power inputs

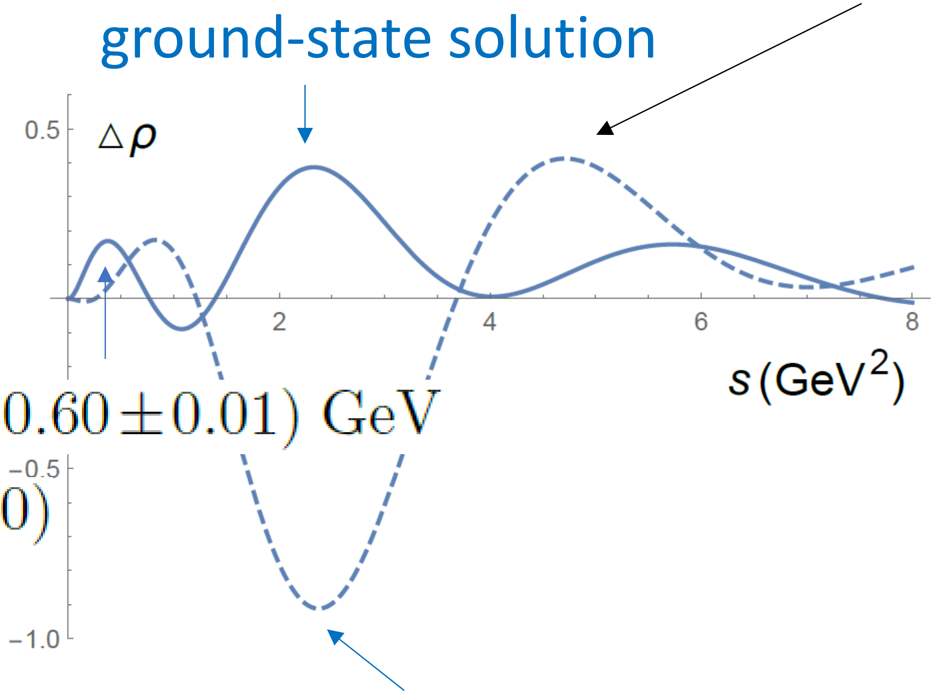
$$m_S = (1.53\pm0.02)\ \mathrm{GeV}$$

$$\longrightarrow f_0(1370),\ f_0(1500)\ \ f_0(1710)$$

$$2187 \pm 14\ \mathrm{MeV}$$

$$m_{S'} = 2.17 \pm 0.01\ \mathrm{GeV} \longrightarrow f_0(2200)$$

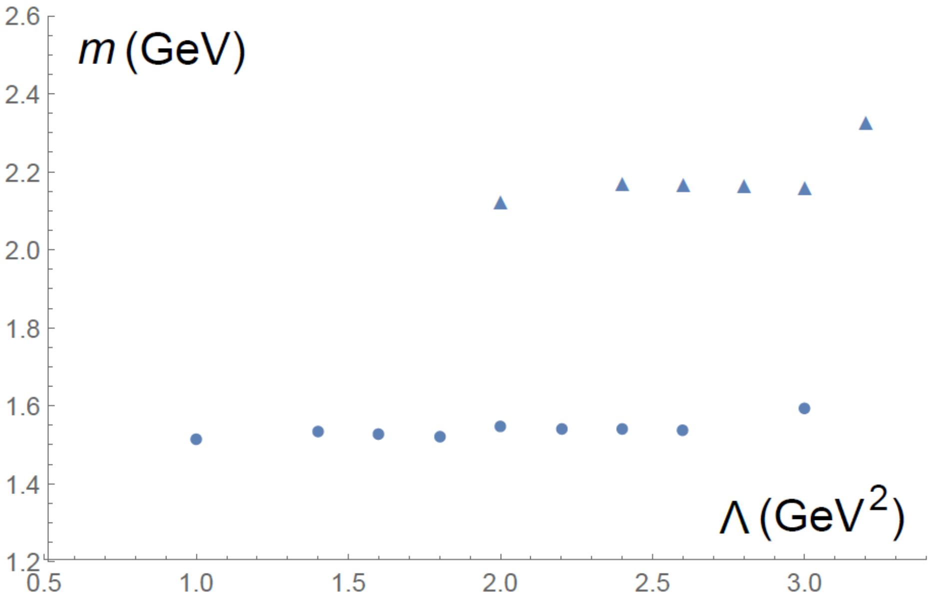
ground-state solution



$$m_{S_1} = (0.60\pm0.01)\ \mathrm{GeV}$$

$$\longrightarrow f_0(500)$$

valley due to subtraction of ground state



Pseudoscalar glueballs

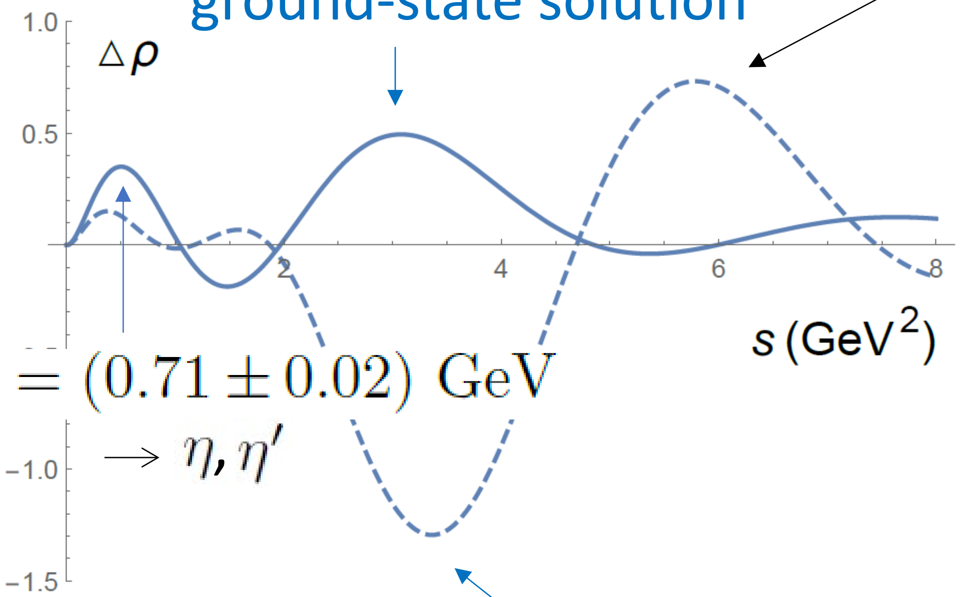
- For pseudoscalar glueballs

glueball-like X(2370)
measured by BESIII
should be 1st excited state

$$2395 \pm 11(\mathrm{stat})^{+26}_{-94}(\mathrm{syst})\ \mathrm{MeV}$$

$$m_P = (1.75 \pm 0.02)\ \mathrm{GeV} \rightarrow \eta(1760)$$

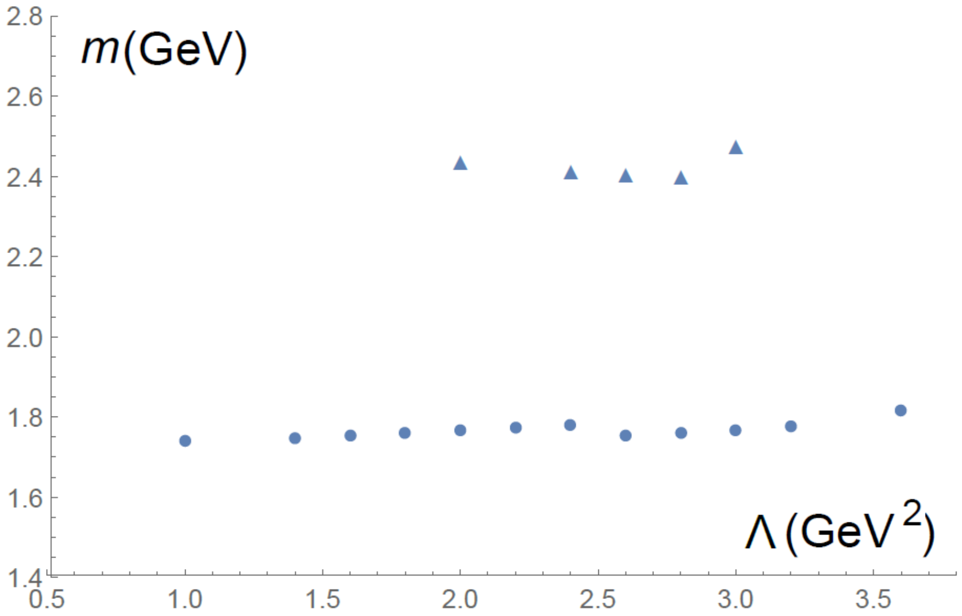
$$m_{P'} = 2.41 \pm 0.04\ \mathrm{GeV} \longrightarrow X(2370)$$



ground-state solution

$$m_{P_1} = (0.71 \pm 0.02)\ \mathrm{GeV}$$

$$\rightarrow \eta, \eta'$$

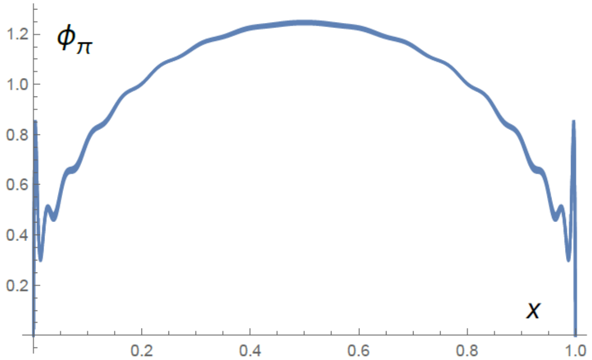


valley due to subtraction of ground state

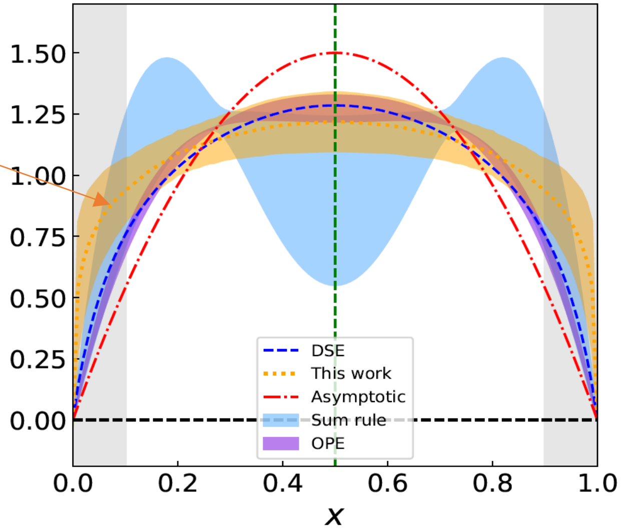
Conclusion

- Dispersive approach, compared to conventional QSR, is free of arbitrary parameters, gives definite predictions with controllable uncertainties
- Applied to spectroscopies of rho and glueballs as test
- Predicted lightest scalar (pseudoscalar) glueball to be admixture of f0(1370), f0(1500) and f0(1710) (eta(1760)
- f0(2200) and X(2370) are 1st excited glueball states
- Can also derive x dependence of light-cone distribution amplitude (LCDA)

pion LCDA



quasi-LCDA lattice



Back-up slides

Fredholm integral equation

- Goal is to solve ill-posed integral equation

unknown spectral density
to be solved

$$\int_0^\infty dy \frac{\rho(y)}{x-y} = \omega(x)$$

OPE input

1st kind of Fredholm integral equation

- How to solve it? Notoriously difficult
- Discretization does not work

ill-posedness

- Discretizing integral equation fails

$$\sum_i M_{ij}\rho_j = \omega_i \qquad M_{ij} = \begin{cases} 1/(i-j), & i \neq j \\ 0, & i = j \end{cases}$$

unknowns input

- Rows Mij and M(i+1)j become almost identical for fine meshes, det(M) ~ 0
- Matrix M becomes singular; M^{-1} diverges quickly
- Solution diverges and sensitive to variation of inputs

Strategy

- Suppose $\rho(y)$ decreases quickly enough
- Expansion into powers of 1/x justified

$$\frac{1}{x-y} = \sum_{m=1}^{N} \frac{y^{m-1}}{x^m} \qquad \omega(x) = \sum_{n=1}^{N} \frac{b_n}{x^n}$$

true for OPE

- Suppose $\omega(x)$ can be expanded
- Decompose

$$\rho(y) = \sum_{n=1}^{N} a_n y^\alpha e^{-y} L_{n-1}^{(\alpha)}(y)$$

generalized Laguerre polynomials

depend on $\rho(y)$ at $y \to 0$

- Orthogonality

$$\int_0^\infty \underline{y^\alpha e^{-y}} L_m^{(\alpha)}(y) L_n^{(\alpha)}(y) dy = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{mn}$$

Solution

- Equating coefficients of $1/x^n$

$$Ma = b \qquad M_{mn} = \int_0^\infty dy y^{m-1+\alpha} e^{-y} L_{n-1}^{(\alpha)}(y)$$

matrix

input $b = (b_1, b_2, \cdots, b_N)$

unknown $a = (a_1, a_2, \cdots, a_N)$

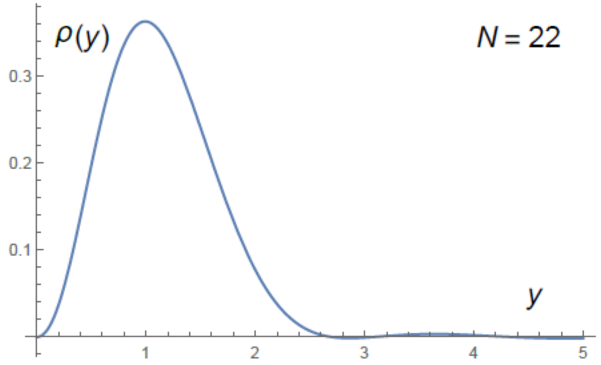
- Solution $a = M^{-1}b$
- True solution can be approached by increasing N, but M^{-1} diverges with N
- Additional polynomial gives $1/x^{N+1}$ correction due to orthogonality, beyond considered precision

Test examples

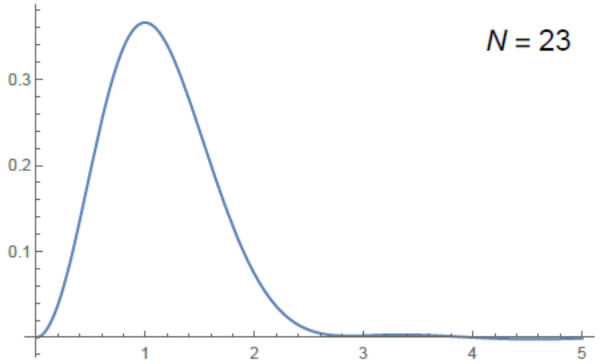
- Generate mock data from $\rho(y) = y^2 e^{-y^2}$

$$b_n = \int_0^\infty dy y^{n-1} y^2 e^{-y^2} \Longleftarrow \int_0^\infty dy \frac{\rho(y)}{x-y} = \omega(x)$$

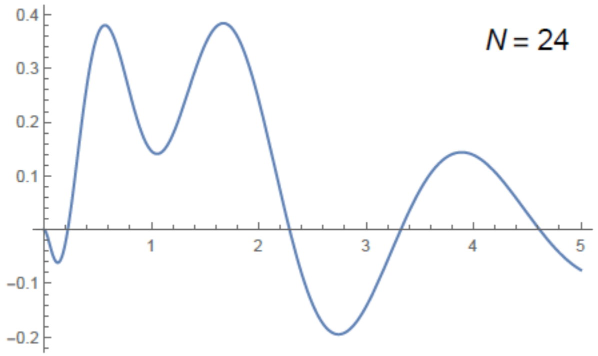
- Compute matrix M with $\alpha = 2$
- Solution stable for N > 20, becomes oscillatory as N=24 due to divergent M^{-1}



$$a_{22}/a_{21} \approx 1$$



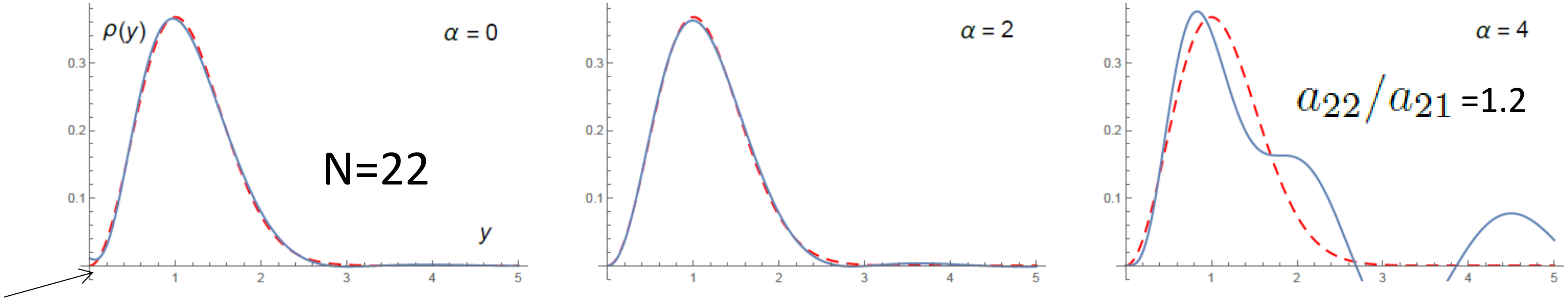
$$a_{23}/a_{22} \approx 2$$



$$a_{24}/a_{23} \approx 58$$

Boundary conditions

- Test choices of α (red: true solution)



deviation

almost perfect

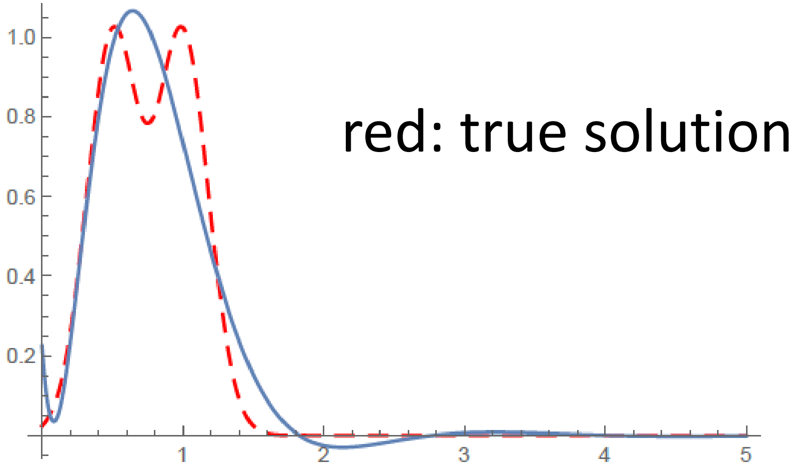
completely different

- Parameter α determined by boundary conditions of solution
- Boundary conditions help getting correct solutions

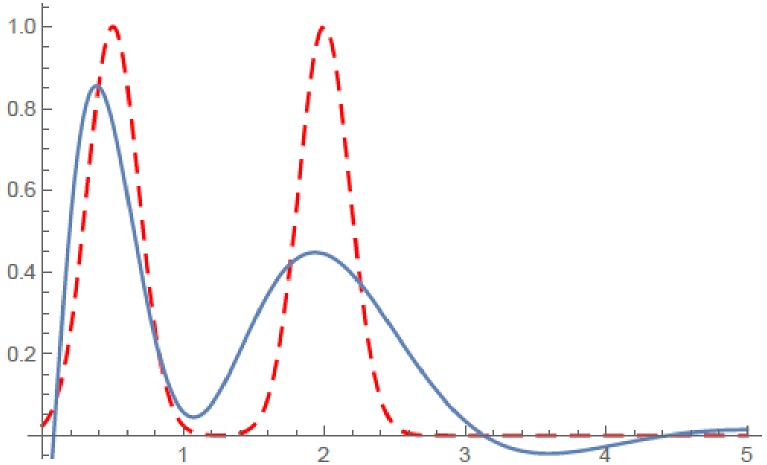
Resolution

- e^{-y} implies resolution power $\Delta y \sim 1$
- Test double peak functions

$$\begin{aligned} \rho_1(y) &= e^{-20(y-0.5)^2} + e^{-20(y-1.0)^2} & \Delta y &\sim 0.5 \\ \rho_2(y) &= e^{-20(y-0.5)^2} + e^{-20(y-2.0)^2} & \Delta y &\sim 1.5 \end{aligned}$$



red: true solution



- Fine structure cannot be resolved (ill-posed)