Threshold cusp structures in multi-channel scattering

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arXiv:2405.08436 [hep-ph]

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1. Introduction
2. N-channel cusp
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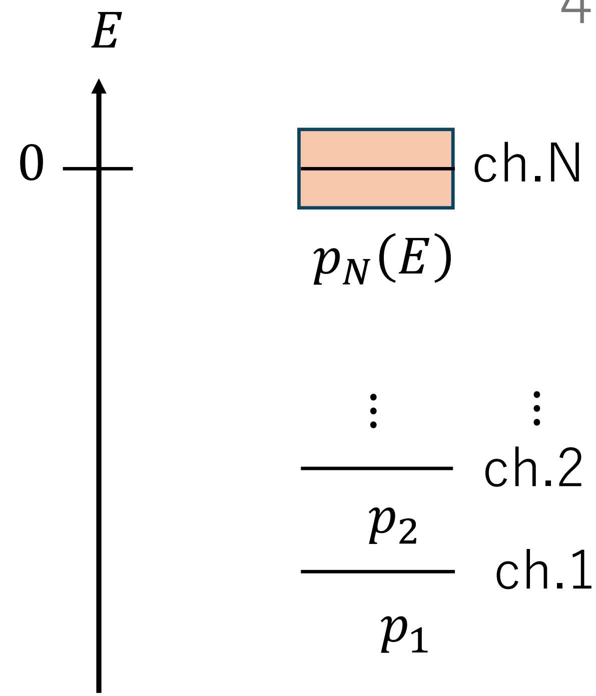
Background

Exotic candidates $X(3872), f_0(980)$

For the analysis of the near-threshold states

⇨ **Threshold cusp**
Channel couplings

Problem：**the constraint** in 2ch. scattering[1]



This work： **considering the general case（N channel scattering）**

： **studying the relation of the number of channels and cusp**

[1] K. Sone, T. Hyodo, arXiv:2405.08436 [hep-ph]

Threshold cusp

The scattering amplitude has a term $-ip_N$ from the unitarity:

S-wave : $f_{ij} \propto \dfrac{1}{-\dfrac{1}{X_{ij}} - ip_N + O(p_N{}^2)}$

cross section : $\sigma_{ij} \sim 4\pi \dfrac{p_j}{p_i} |f_{ij}|^2$

$p_N \propto \sqrt{E}$: relative momentum

X_{ij} : complex constant

:not ERE

Expansion around the Nth threshold $(E = 0)$

$\sigma_{ij}^{above}(p_N) \propto 1 + \underline{2Im(X_{ij})p_N}$ $(E > 0)$: **above** the threshold

$\sigma_{ij}^{blow}(\kappa_N) \propto 1 + \underline{2Re(X_{ij})\kappa_N}$ $p_N = i\kappa_N \propto i\sqrt{|E|}$ $(E < 0)$: **below** the threshold

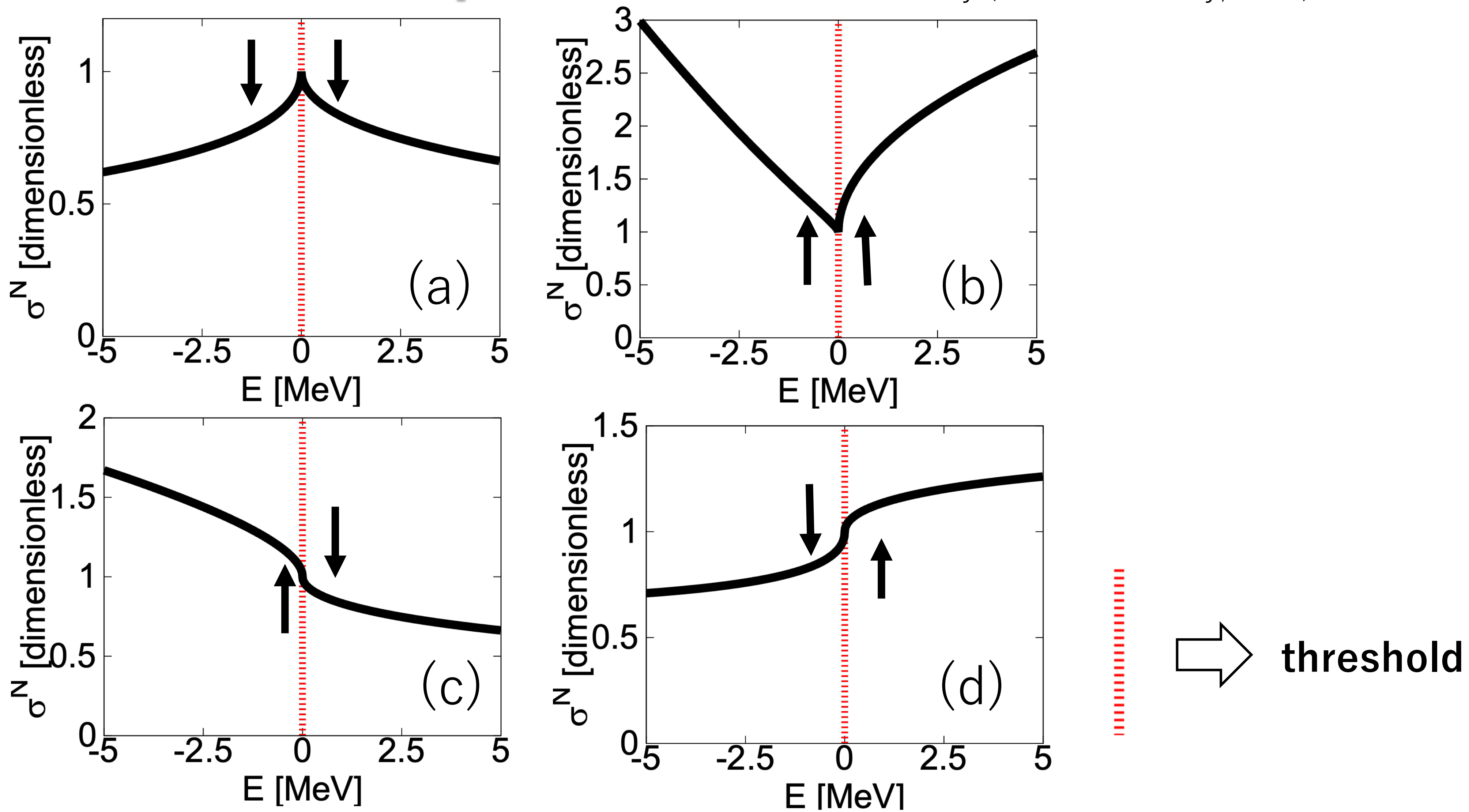
⇨ Discontinuity in derivative at the threshold : **threshold cusp**

Slopes are determined by the coefficients of p_N

$d\sigma/dE\big|_{E=0} \rightarrow \pm\infty$ ⇨ **4 kinds of cusps**(next slide)

4 kinds of cusps

[2] L. D. Landau and E. M. Lifshitz, Quantum Mechanics: Non-relativistic Theory (Addison-Wesley, 1965)

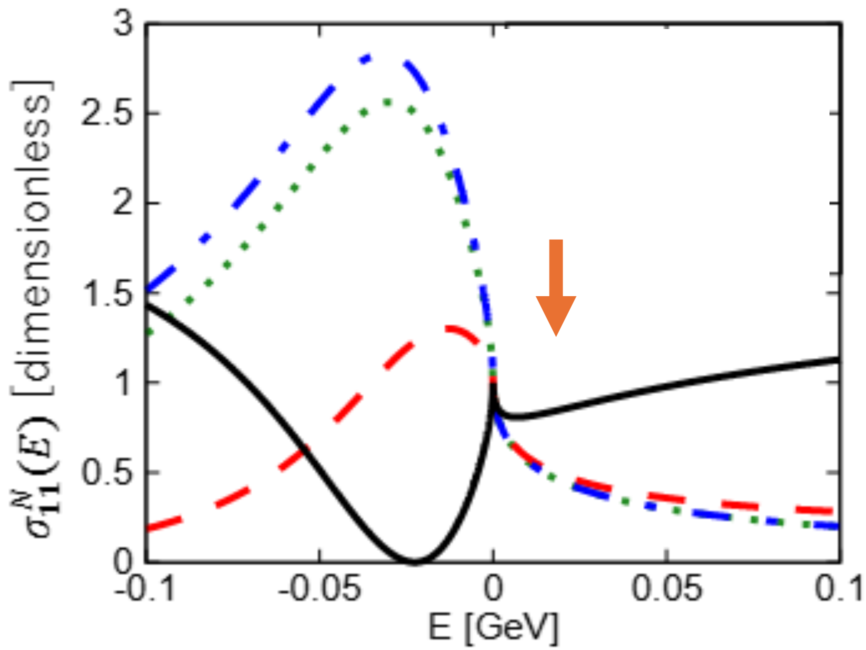


2 channel constraint

Slopes of the cross section in general case:

$$\sigma_{11}^{above}(p_N) \propto 1 + 2Im(X_{11})p_2$$

$$\sigma_{11}^{blow}(\kappa_N) \propto 1 + 2Re(X_{11})\kappa_2$$



2 channel case:

$$Im(X_{11}) = Im(a_2)$$ **constraint**

⇩

$$\sigma_{11}^{above}(p_N) \propto 1 + 2Im(a_2)p_2$$

The slope **above** the threshold must be **negative**[1] ↓

[1] K. Sone, T. Hyodo, arXiv:2405.08436 [hep-ph]

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Unitarity

A general form of the scattering amplitude derived from the unitarity：

$$f^{-1}(E) = \begin{pmatrix} K_{11}^{-1} - ip_1 & K_{12}^{-1} & \cdots & K_{1i}^{-1} & \cdots & K_{1N}^{-1} \\ K_{12}^{-1} & K_{22}^{-1} - ip_2 & \cdots & K_{2i}^{-1} & \cdots & K_{2N}^{-1} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ K_{1i}^{-1} & K_{2i}^{-1} & \cdots & K_{ii}^{-1} - ip_i & \cdots & K_{iN}^{-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{1N}^{-1} & K_{2N}^{-1} & \cdots & K_{iN}^{-1} & \cdots & K_{NN}^{-1} - i\underline{p_N(E)} \end{pmatrix}$$

K : K-matrix

p_i : relative momentum of ch. i

- The imaginary part $-ip_i$ in **diagonal components**
- up to first order of $\underline{p_N}$
- **Focusing on Nth threshold** ⇨ Applicable near the threshold

Near-threshold amplitude

Calculating the amplitude from inverse from

$$f_{ij}(E) = \frac{n_{ij}(E)}{D(E)}$$

$D(E)$: common to all components

$n_{ij}(E)$: different for each component

$D(E)$: determinant of f^{-1}

$$D(E) = det\begin{pmatrix} K_{11}^{-1} - ip_1 & K_{12}^{-1} & \cdots & K_{1i}^{-1} & \cdots & K_{1N}^{-1} \\ K_{12}^{-1} & K_{22}^{-1} - ip_2 & \cdots & K_{2i}^{-1} & \cdots & K_{2N}^{-1} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ K_{1i}^{-1} & K_{2i}^{-1} & \cdots & K_{ii}^{-1} - ip_i & \cdots & K_{iN}^{-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{1N}^{-1} & K_{2N}^{-1} & \cdots & K_{iN}^{-1} & \cdots & K_{NN}^{-1} - ip_N(E) \end{pmatrix}$$

K_{ij}^{-1} : real number

$p_i \ (i \neq N)$: constant

$$n_{ij}(E) = det\begin{pmatrix} K_{11}^{-1} - ip_1 & \cdots & K_{1j}^{-1} & \cdots & K_{1N}^{-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ K_{1i}^{-1} & \cdots & K_{ij}^{-1} & \cdots & K_{iN}^{-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{N1}^{-1} & \cdots & K_{Nj}^{-1} & \cdots & K_{NN}^{-1} - ip_N(E) \end{pmatrix}$$: cofactor of f^{-1}

Denominator

Decomposing the denominator using the property of determinant

$$D(E) = det\begin{pmatrix} K_{11}^{-1} - ip_1 & K_{12}^{-1} & \cdots & K_{1N}^{-1} \\ K_{12}^{-1} & K_{22}^{-1} - ip_2 & \cdots & K_{2N}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ K_{1N}^{-1} & K_{2N}^{-1} & \cdots & K_{NN}^{-1} \end{pmatrix} + det\begin{pmatrix} K_{11}^{-1} - ip_1 & K_{12}^{-1} & \cdots & 0 \\ K_{12}^{-1} & K_{22}^{-1} - ip_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K_{1N}^{-1} & K_{2N}^{-1} & \cdots & -ip_N \end{pmatrix}$$

$= D_1$ $\qquad$ $= -iD_2 p_N(E)$

D_1, D_2 : complex constants

$$D(E) = D_2(D_1/D_2 - ip_N(E)) \iff \text{ERE} : f \propto \left(-\frac{1}{a_0} + \frac{1}{2}r_e p^2 - ip + O(p^4)\right)^{-1}$$

the scattering length of channel N : $a_N = -D_2/D_1$

From the optical theorem, $Im(a_N) < 0$

Numerator(diagonal components)

n_{ii} : Same expansion as $D(E)$ ⇨ $(N-1)\times(N-1)$ determinant(**cofactor** of f^{-1})

$$n_{ii}(E) = det\begin{pmatrix} K_{11}^{-1}-ip_1 & \cdots & K_{1i-1}^{-1} & K_{1i+1}^{-1} & \cdots & K_{1N}^{-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ K_{1i-1}^{-1} & \cdots & K_{i-1i-1}^{-1}-ip_{i-1} & K_{i-1i+1}^{-1} & \cdots & K_{i-1N}^{-1} \\ K_{1i+1}^{-1} & \cdots & K_{i-1i+1}^{-1} & K_{i+1i+1}^{-1}-ip_{i+1} & \cdots & K_{i+1N}^{-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{1N}^{-1} & \cdots & K_{i-1N}^{-1} & K_{i+1N}^{-1} & \cdots & K_{NN}^{-1}-ip_N(E) \end{pmatrix}$$

Imaginary part are only in diagonal components

⇩

n_1^d, n_2^d : complex constants

$$n_{ii}(E) = n_2^d(n_1^d/n_2^d - ip_N(E))$$

Defining b_{ii} : $b_{ii} \equiv -n_1^d/n_2^d$

constraint : $Im(b_{ii}) < 0$ ⇨ **Same condition as a_N**

Numerator (off-diagonal components)

In case of n_{ij} $(i \neq j \neq N)$,

$$n_{ij}(E) = det\begin{pmatrix} K_{11}^{-1} - ip_1 & \cdots & K_{1i-1}^{-1} & K_{1i+1}^{-1} & \cdots & K_{1N}^{-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ K_{1j-1}^{-1} & \cdots & K_{j-1i-1}^{-1} & K_{j-1i+1}^{-1} & \cdots & K_{j-1N}^{-1} \\ K_{1j+1}^{-1} & \cdots & K_{i-1j+1}^{-1} & K_{i+1j+1}^{-1} & \cdots & K_{j+1N}^{-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{1N}^{-1} & \cdots & K_{i-1N}^{-1} & K_{i+1N}^{-1} & \cdots & K_{NN}^{-1} - ip_N(E) \end{pmatrix}$$

n_{ij} : cofactors of $[f^{-1}]_{ij}$ have imaginary parts in **off-diagonal components**

⇨ different property from n_{ii}

n_1^o, n_2^o : complex

$$n_{ij}(E) = n_2^o(n_1^o/n_2^o - ip_N(E)) \qquad (i \neq j \neq N)$$

Defining z_{ij} : $z_{ij} = -n_1^o/n_2^o$

No constraint is imposed on z_{ij} ⇨ Different from a_N, b_{ii}

Summary so far

$$f(E) = \begin{pmatrix} f_{11}(E) & f_{12}(E) & \cdots & f_{1i}(E) & \cdots & f_{1N}(E) \\ f_{12}(E) & f_{22}(E) & \cdots & f_{2i}(E) & \cdots & f_{2N}(E) \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ f_{1i}(E) & f_{2i}(E) & \cdots & f_{ii}(E) & \cdots & f_{iN}(E) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{1N}(E) & f_{2N}(E) & \cdots & f_{iN}(E) & \cdots & f_{NN}(E) \end{pmatrix}$$

Diagonal: $f_{ii}(E) = \frac{n_2^d}{D_2} \frac{-\frac{1}{b_{ii}} - ip_N}{-\frac{1}{a_N} - ip_N} = f_{ii}(0) \frac{1 + ib_{ii}p_N(E)}{1 + ia_N p_N(E)} \quad Im(b_{ii}), Im(a_N) < 0$

Off-diagonal: $f_{ij}(E) = f_{ij}(0) \frac{1 + iz_{ij}p_N(E)}{1 + ia_N p_N(E)} \quad (i \neq j \neq N)$

Slopes of the cross section

Slopes of the cross section determine the **shape of cusp**

s-wave cross section ：$\sigma_{ij} = \frac{p_j}{p_i} \int f f^* d\Omega = 4\pi \frac{p_j}{p_i} \left| f_{ij} \right|^2$

Expansion of f_{ii}(E) around the threshold $(p_N = 0)$:

$$f_{ii}(E) \cong f_{ii}(0)(1 + i b_{ii} p_N)(1 - i a_N p_N)$$

Cross section :

$\sigma_{ii}^{above}(p_N) \cong |f_{ii}(0)|^2 [1 + \underline{2Im(a_N - b_{ii}) p_N}]$ $(E > 0)$: **above** the threshold

$\sigma_{ii}^{blow}(\kappa_N) \cong |f_{ii}(0)|^2 [1 + \underline{2Re(a_N - b_{ii}) \kappa_N}]$ $(E < 0)$: **below** the threshold

$\kappa_N = \sqrt{2\mu_N |E|}$

Slopes are detrmined by $\underline{a_N - b_{ii}}$

$\sigma_{ii}(p_N)$ **can reproduce the 4 kinds of cusps** (**no constraint**)

Slopes of the cross section

Expansion of $f_{ij}(E)$ around the threshold $(p_N = 0)$:

$$f_{ij}(E) \cong f_{ij}(0)(1 + iz_{ij}p_N)(1 - ia_N p_N)$$

Off-diagonal components$(i \neq j \neq N)$:

$$\sigma_{ij}^{above}(p_N) \cong |f_{ij}(0)|^2[1 + \underline{2Im(a_N - z_{ij})p_N}]$$

$$\sigma_{ij}^{blow}(\kappa_N) \cong |f_{ij}(0)|^2[1 + \underline{2Re(a_N - z_{ij})\kappa_N}]$$

Slopes are determined by $a_N - z_{ij}$

$\sigma_{ij}(p_N)$ can reproduce the 4 kinds of cusps (however, $N > 3$[latter])

⇨ General N channel case : **No constraint** (4 kinds of cusps)

CDD zero

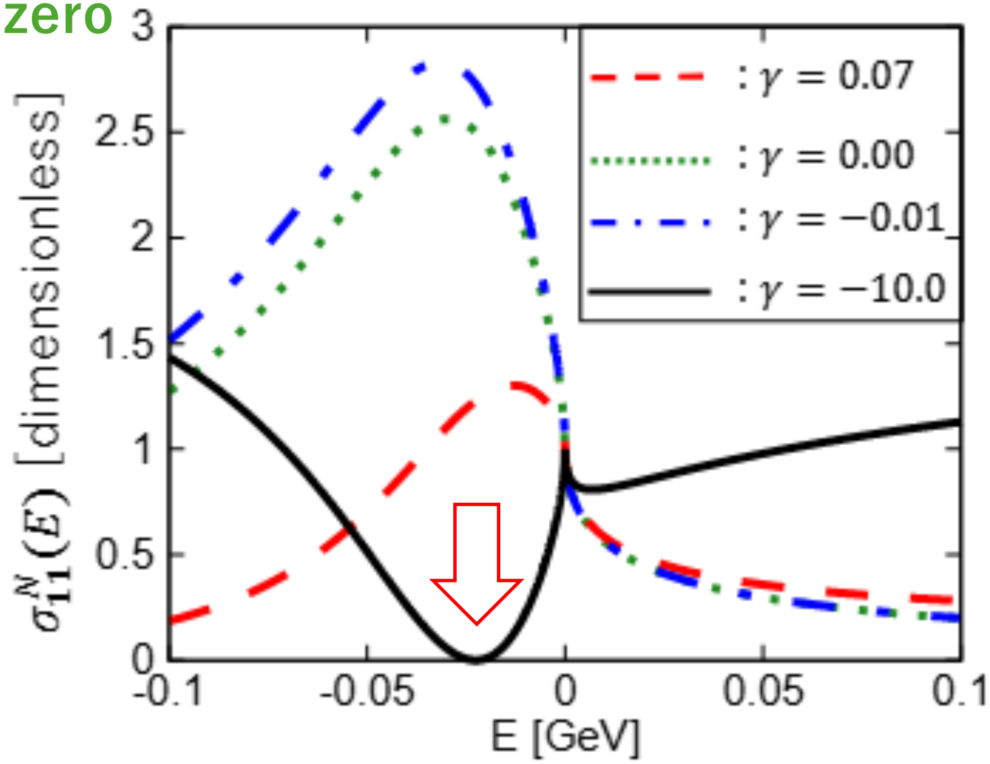
Roles of the parameters b_{ii} and z_{ij}

One of the reason for dip structures：CDD zero

In case with near-threshold CDD zero :

Diagonal : Zero point：$p_{ii}^{zero} \cong i/b_{ii}$

Off-diagonal : Zero point：$p_{ij}^{zero} \cong i/z_{ij}$



CDD zero can make dip structures

Summary so far

We discussed the slopes of the cross section in N channel case.

Results：

Slopes of the cross sections are determined by a_N, b_{ii} and z_{ij} $\quad (Im(a_N), Im(b_{ii}) < 0)$

General N channel case : **No constraint** (4 kinds of cusps)

Roles of b_{ii} and z_{ij} ：

In this model, the CDD zero is determined by b_{ii} and z_{ij}

⇩ **Near-threshold CDD zero**

Slopes of the cross sections are determined by **a_N and the position of CDD zero**

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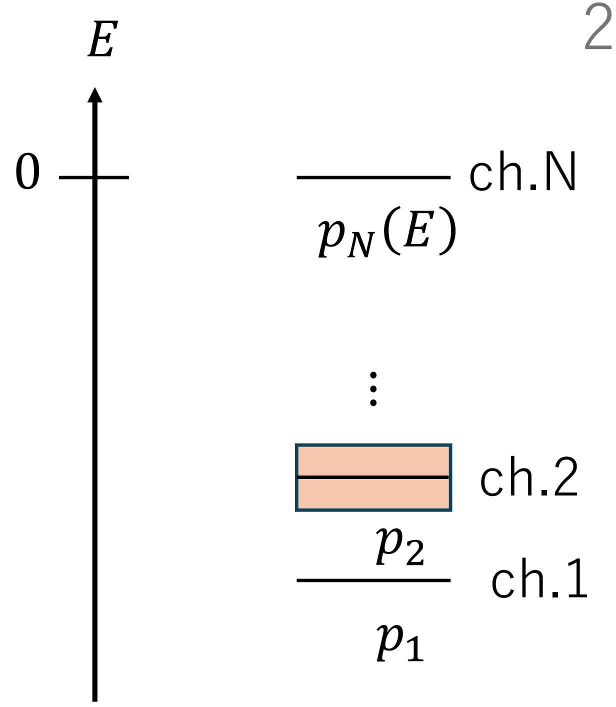
Near the second threshold

Here, we show the constraint in 2 channel case

⇨ Focusing on second lowest threshold



We should consider **2 × 2 matrix** in channel space

$$f^{-1} = \begin{pmatrix} K_{11}^{-1} - ip_1 & K_{12}^{-1} \\ K_{12}^{-1} & K_{22}^{-1} - ip_2 \end{pmatrix}$$

⇨
- Higher channels than the ch.2 are absorbed to K_{ij}
- It doesn't matter haw many channels the system has

We discuss the shape of threshold cusp at the second threshold

Near the second threshold

The inverse of amplitude： $f^{-1} = \begin{pmatrix} K_{11}^{-1} - ip_1 & K_{12}^{-1} \\ K_{12}^{-1} & K_{22}^{-1} - ip_2 \end{pmatrix}$

・No off-diagonal components satisfying $i \neq j \neq 2$ ⇨ We consider f_{11}

Diagonal component

Numerator：$n_{11} = det \begin{pmatrix} K_{11}^{-1} - ip_1 & K_{12}^{-1} \\ K_{12}^{-1} & K_{22}^{-1} - ip_2 \end{pmatrix}$ ⇨ $f_{11} = f_{11}(0)\dfrac{1 + ib_{11}p_2}{1 + ia_2p_2}$

a_2：scattering length

b_{11}：**real number** ⇨ $Im(b_{11}) = 0$：**A special constraint of 2 channel**

Slope： $\sigma_{11}^{above}(p_2) \cong |f_{11}(0)|^2[1 + 2Im(a_2 - b_{22})p_2]$ $(E > 0)$：**above**

$Im(a_2) < 0$ ⇨ The slope of σ_{11}^{above} **must be negative**

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The third threshold

- Diagonal components : **same as N channel case** ⇨ ip_1 or ip_2 remains in n_{ii}

  Off-diagonal ($i \neq j \neq 3$) ⇨ only $f_{12}(p_3)$

Numerator : $n_{12} = det\begin{pmatrix} K_{11}^{-1} - ip_1 & K_{12}^{-1} & K_{13}^{-1} \\ K_{12}^{-1} & K_{22}^{-1} - ip_2 & K_{23}^{-1} \\ K_{13}^{-1} & K_{23}^{-1} & K_{33}^{-1} - ip_3 \end{pmatrix}$ ⇨ $f_{12} = f_{12}(0)\dfrac{1 + iz_{12}p_3}{1 + ia_3p_3}$

ip_1, ip_2 **vanish**

a_3 : scattering length

z_{12} : **real number** ⇨ $Im(z_{12}) = 0$: the special constraint of three channel

(1,2) slope : $\sigma_{12}^{above}(p_N) \cong |f_{ij}(0)|^2[1 + 2Im(a_2 - z_{12})p_N]$

$Im(a_2) < 0$ ⇨ The slope of σ_{12}^{above} **must be negative**

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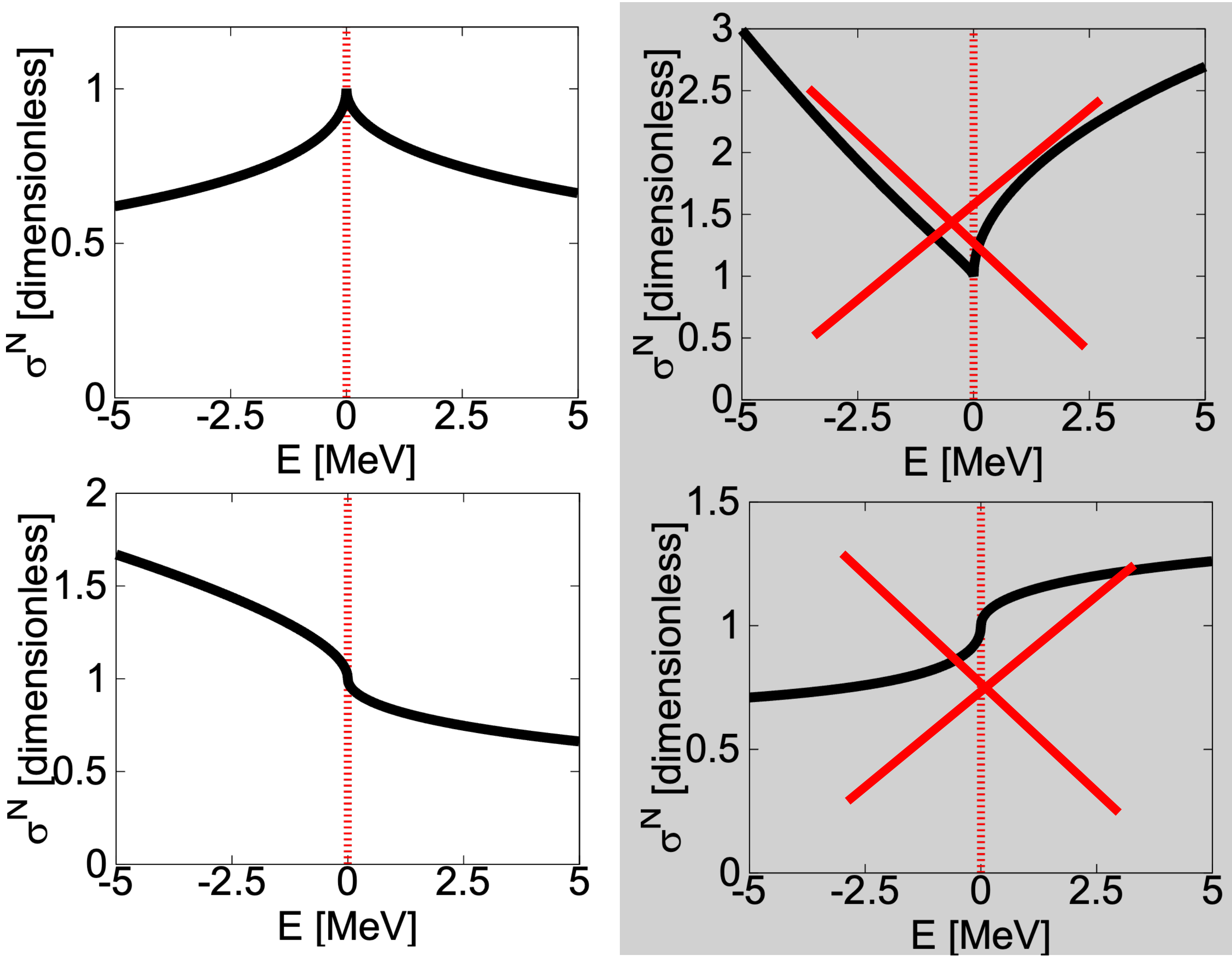
Summary

2 channel σ_{11}

3 channel σ_{12}



Summary

This work：we discuss the shape of threshold cusp with channel couplings

Results：

- General case(N channel case) : 4 kinds of cusps are possible
- There are constraints in 2ch. (1,1) and 3ch. (1,2) components

⇨ Only 2 kinds of cusps are possible[1]

We should consider the conditions in few channel scattering.

We should use a scattering amplitude with correct numbers of the channels.

[1] K. Sone, T. Hyodo, arXiv:2405.08436 [hep-ph]