Japanese Gardens

Photo from: https://www.gankofood.co.jp/shop/detail/ya-nijyoen/

Japanese Gardens

project the great nature into limited space

Photo from: https://www.gankofood.co.jp/shop/detail/ya-nijyoen/

≃ numrical simulations

Try to represent the **real world** in a **limited space**.

Various own techniques in numerical simulations

— Luscher's finite-volume method
— finite-size scaling for phase transitions
This study ➡ — Lee-Yang zeros
— …

HHIQCD2024, YITP, Kyoto, Oct. 30, 2024

Lee-Yang Zeros for locating a critical point

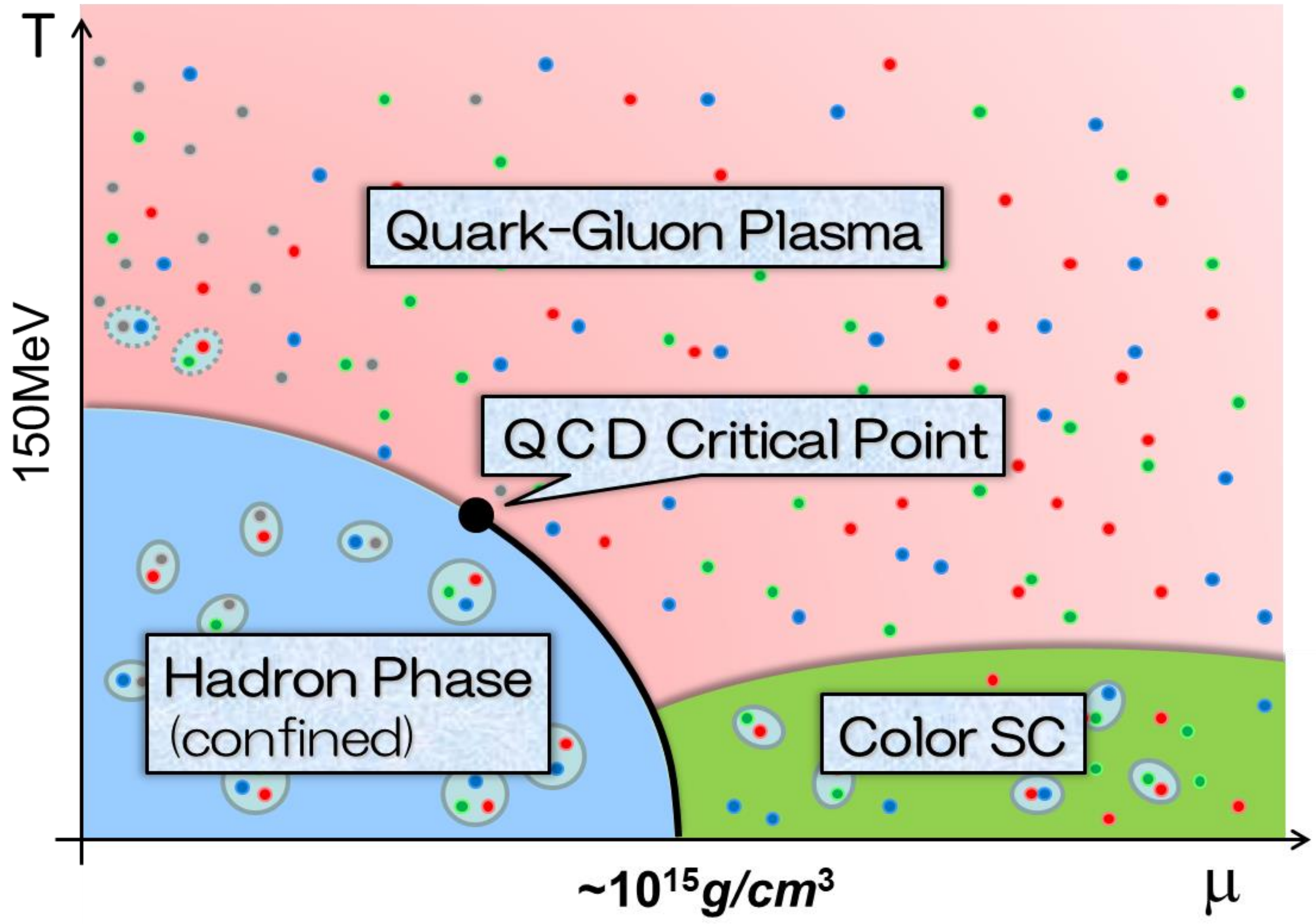
Masakiyo Kitazawa
(YITP, Kyoto)

In collab. with **Tatsuya Wada**, Kazuyuki Kanaya

Special thanks to S. Ejiri

Wada, MK, Kanaya, arXiv:2410.19345 [hep-lat]

QCD Phase Diagram



Rich phase structure in QCD

- QCD critical point(s)
- color superconductivity

Sign problem

- difficulty in lattice QCD Monte-Carlo simulations at $\mu \neq 0$

Various approaches

- Taylor expansion method
- Imaginary chem. pot.
- Complex Langevin
- Lifschetz thimble
- Lee-Yang edge singularity
- ...

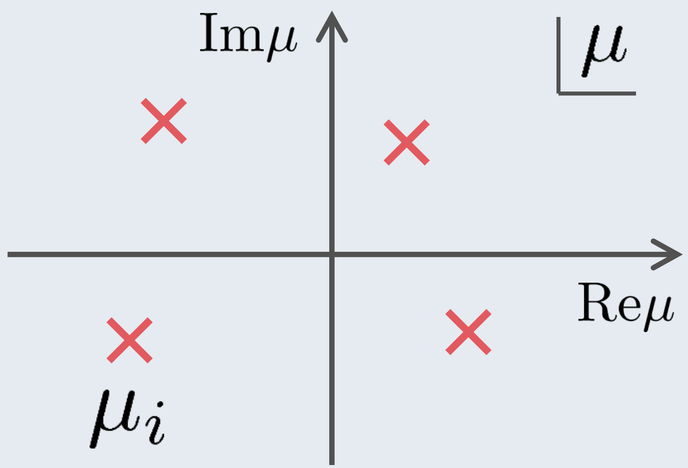
Lee-Yang Zeros

Yang, Lee; Lee, Yang ('52)

Partition Function $Z(T,\mu)$

Finite V ➤ Polynomial of μ (or T)

$$Z(T,\mu) = \prod_i (\mu - \mu_i)$$



➤ **zeros** on the complex plane

=Lee-Yang Zeros

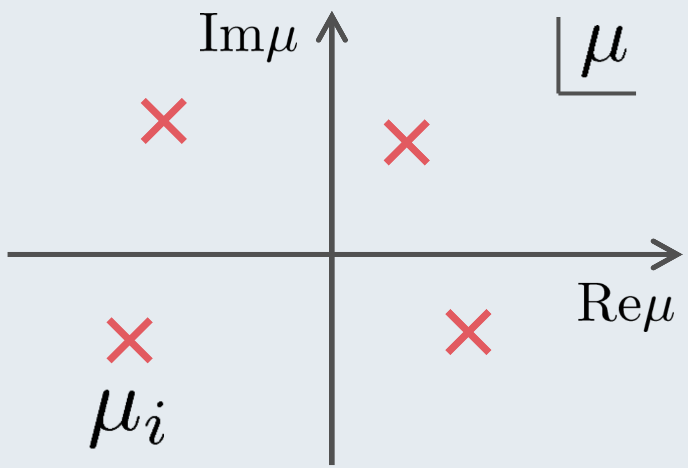
Lee-Yang Zeros

Yang, Lee; Lee, Yang ('52)

Partition Function $Z(T,\mu)$

Finite V ➤ Polynomial of μ (or T)

$$Z(T,\mu) = \prod_i (\mu - \mu_i)$$



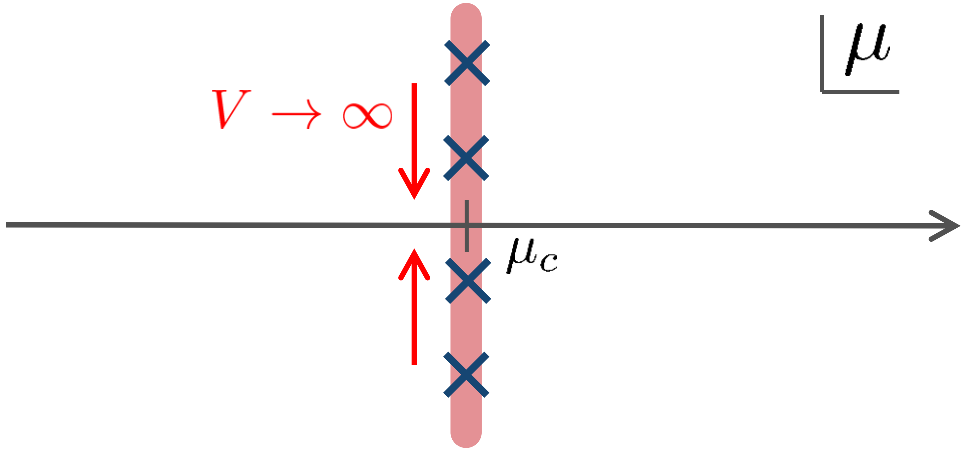
➤ **zeros** on the complex plane

=Lee-Yang Zeros

Phase Transition & LYZ

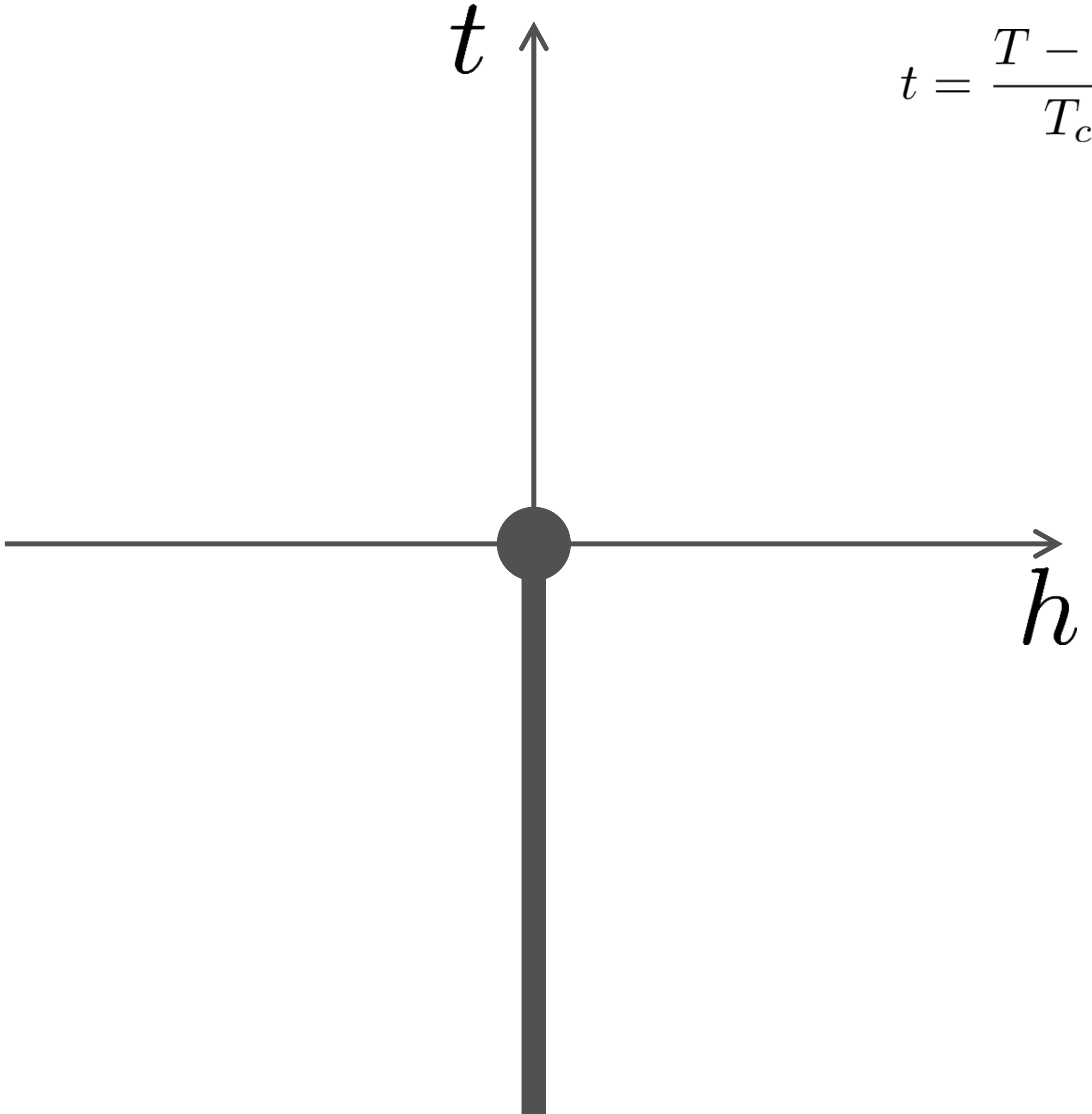
First-order transition

at $\mu = \mu_c$



— For $V \to \infty$, LYZs are accumulated on the line crossing the real axis at $\mu = \mu_c$.

LYZ around a Critical Point in Ising Model



$$t = \frac{T - T_c}{T_c}$$

1st-transition

singularity on the real h axis

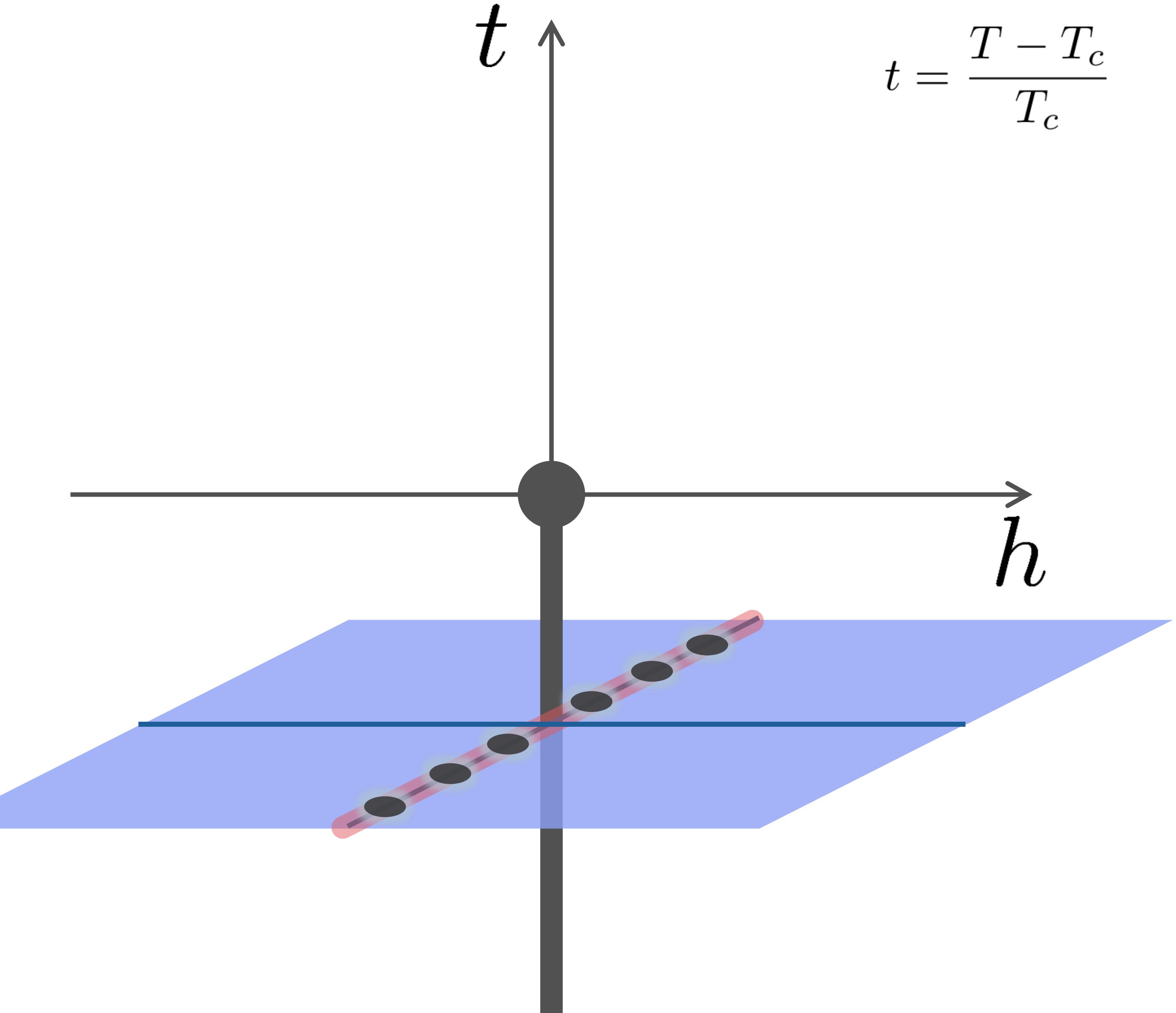
Crossover

no singularity on the real axis

Note:
LYZ in complex-h plane are purely imaginary.

Lee-Yang, 1952

LYZ around a Critical Point in Ising Model



$$t = \frac{T - T_c}{T_c}$$

1st-transition

singularity on the real h axis

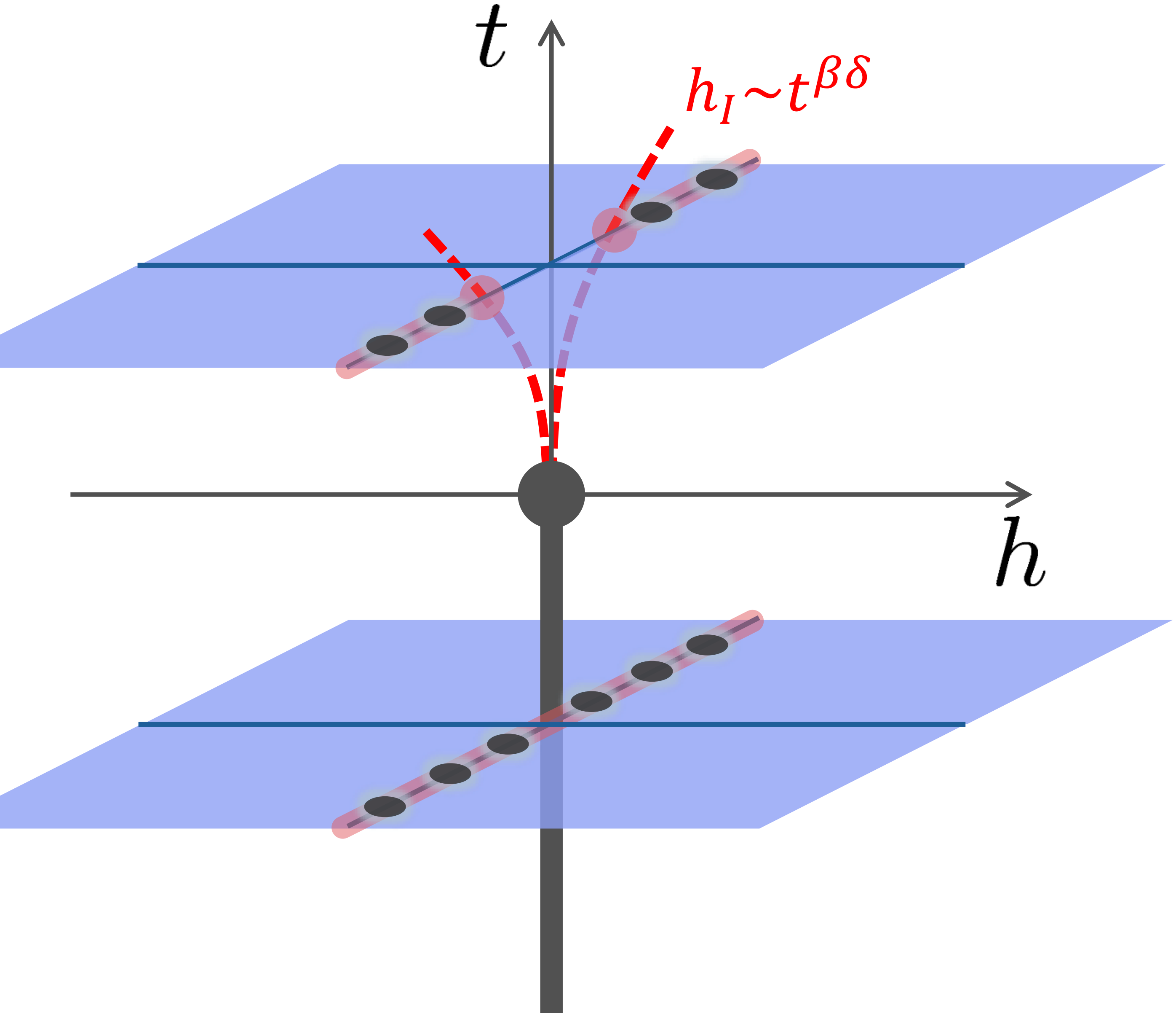
Crossover

no singularity on the real axis

Note:
LYZ in complex-h plane are purely imaginary.

Lee-Yang, 1952

LYZ around a Critical Point in Ising Model

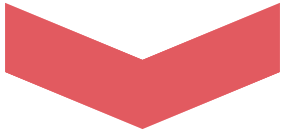


1st-transition

singularity on the real h axis

Crossover

no singularity on the real axis



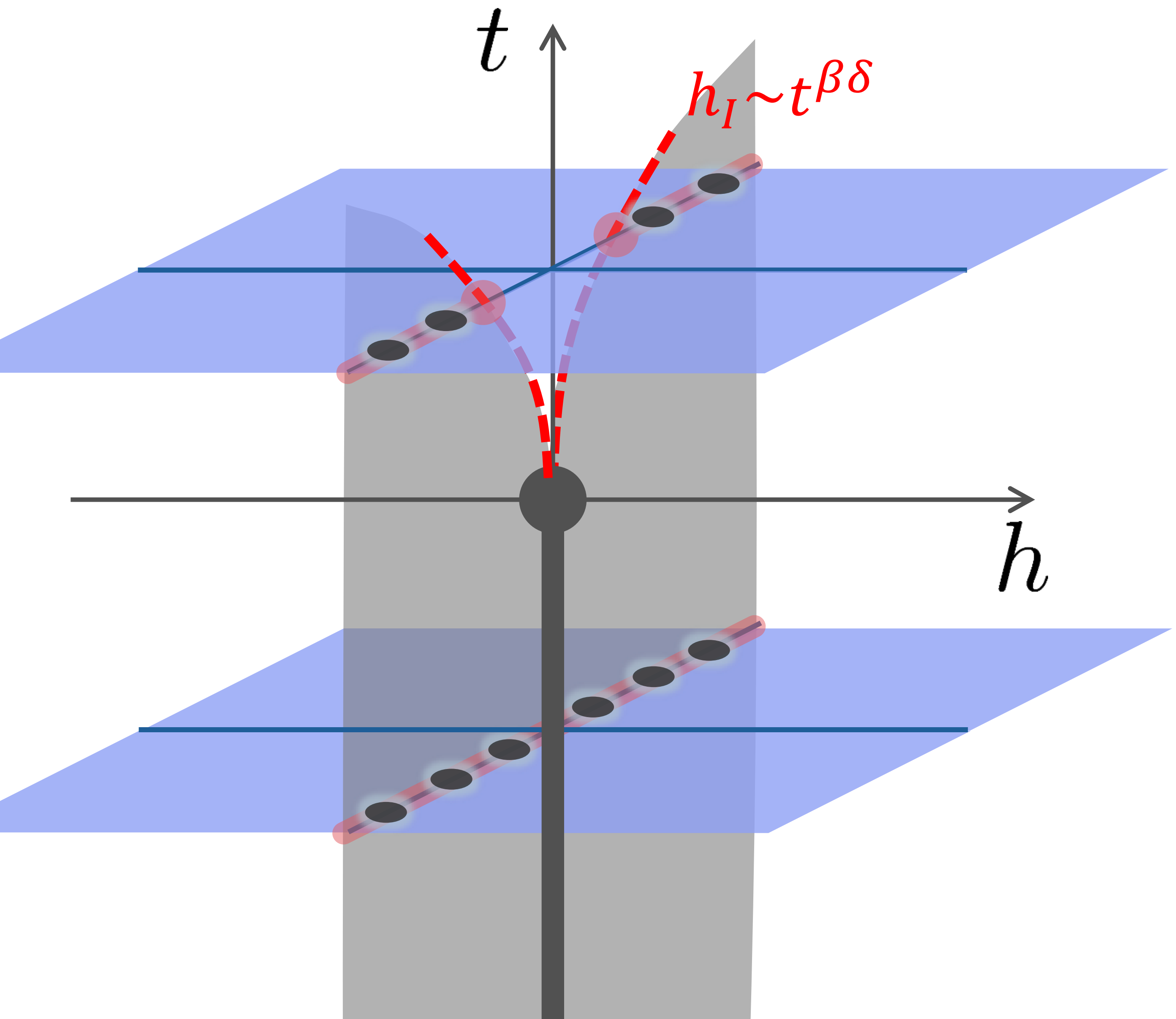
LY edge singularity

Starting from the CP

Its behavior is governed by the the scaling function.

$h_I \sim t^{\beta\delta}$

LYZ around a Critical Point in Ising Model



1st-transition

singularity on the real h axis

Crossover

no singularity on the real axis

LY edge singularity

Starting from the CP

Its behavior is governed by the the scaling function.

$h_I \sim t^{\beta\delta}$

Recent Progress in LYZ/LYES and Lattice

Analytic Structure

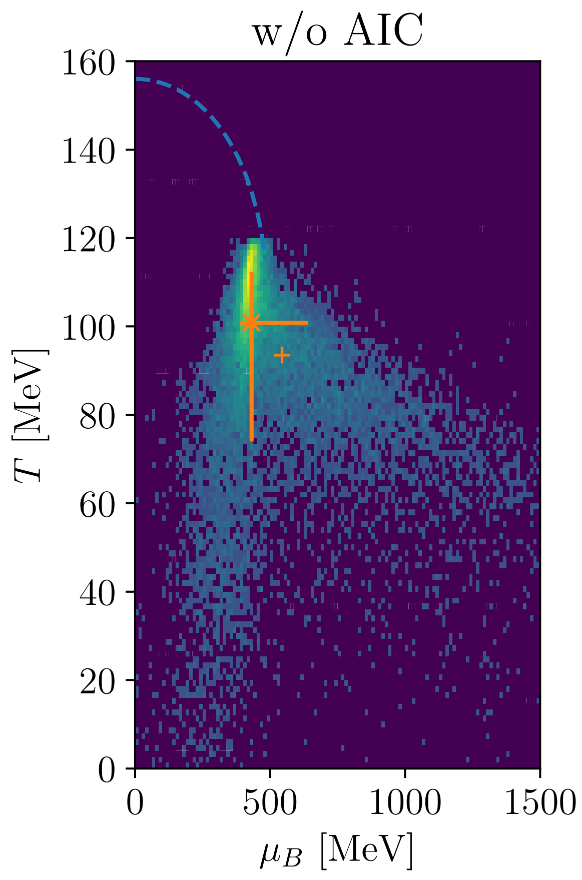
— Scaling functions, FRG, …

An, Mesterhazy, Stephanov ('16)
Johnson, Rennecke, Skokov ('23)
Karsch, Schmidt, Singh ('23)
…

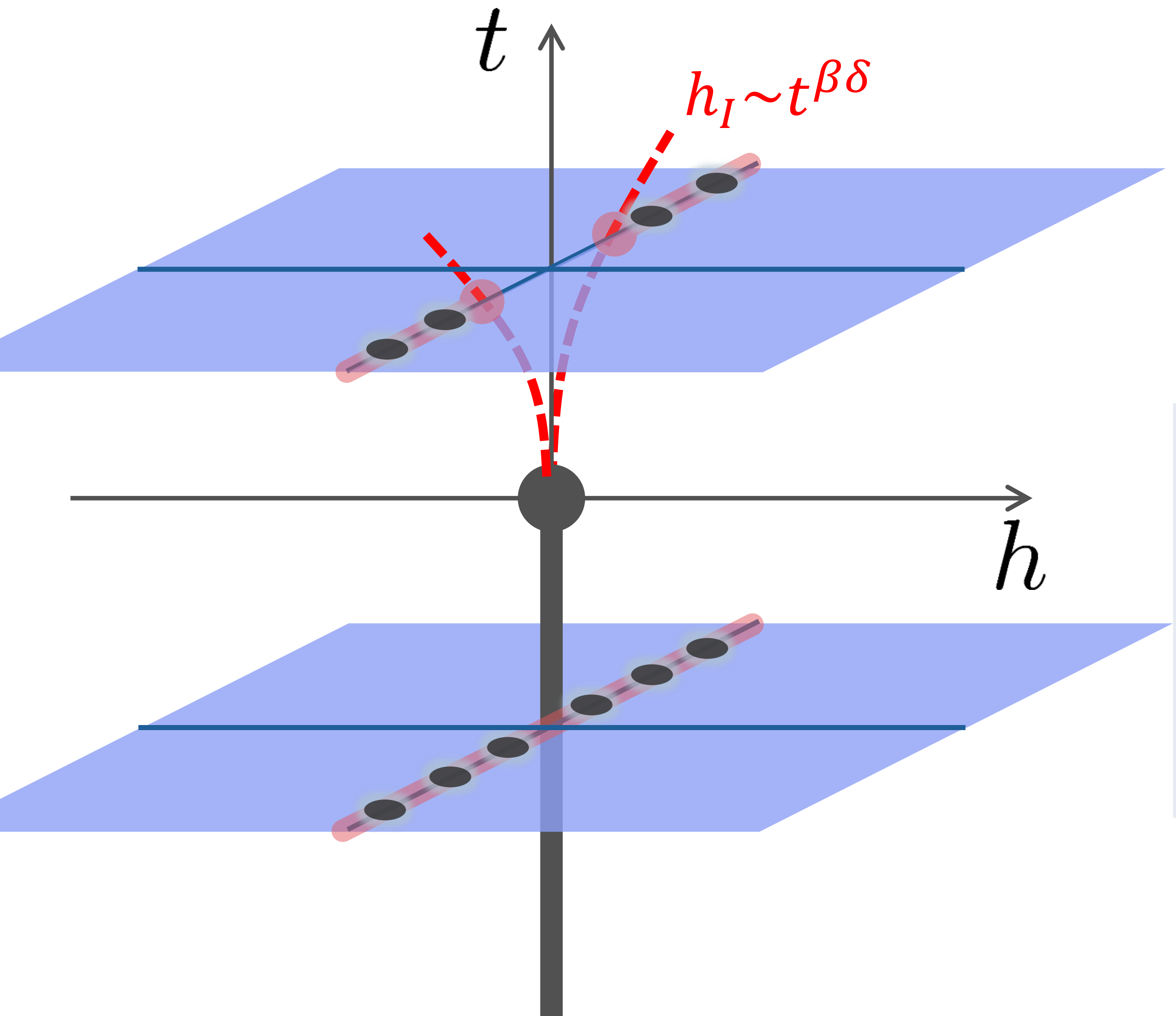
Locating QCD-CP at $\mu \neq 0$ on the lattice?

Clarke+, arXiv:2405.10196
Alexander+ Lattice2024

— Taylor exp. + Imaginary μ + Pade approx.
— Identify the 1st LYZ to be LYES



Purpose of This Study



On finite volume,

1st LYZ ≠ LYES

Purpose of This Study

—Understand finite-volume effects on LYZ
—Exploit them for the CP searches

Finite-Size Scaling

Scaling Hypothesis

$$F_{\text{sing}}(t, h, L^{-1}) = \tilde{F}_{\text{sing}}(L^{y_t} t, L^{y_h} h)$$

$$Z_{\text{sing}}(t, h, L^{-1}) = \tilde{Z}_{\text{sing}}(L^{y_t} t, L^{y_h} h)$$

$$F = F_{\text{sing}} + F_{\text{reg}}$$

$$Z = Z_{\text{sing}} \times Z_{\text{reg}}$$

LYZ in the scaling region on finite volume

$$Z(t, h, L^{-1}) \sim \tilde{Z}_{\text{sing}}(L^{y_t} t, L^{y_h} h) = 0$$

$$L^{y_h} h^{(i)} = \tilde{h}^{(i)}_{\text{LY}}(L^{y_t} t)$$

LYZ in 3d-Ising Model

$$H = -\sum_{\langle i,j\rangle} s_i s_j - h\sum_i s_i$$

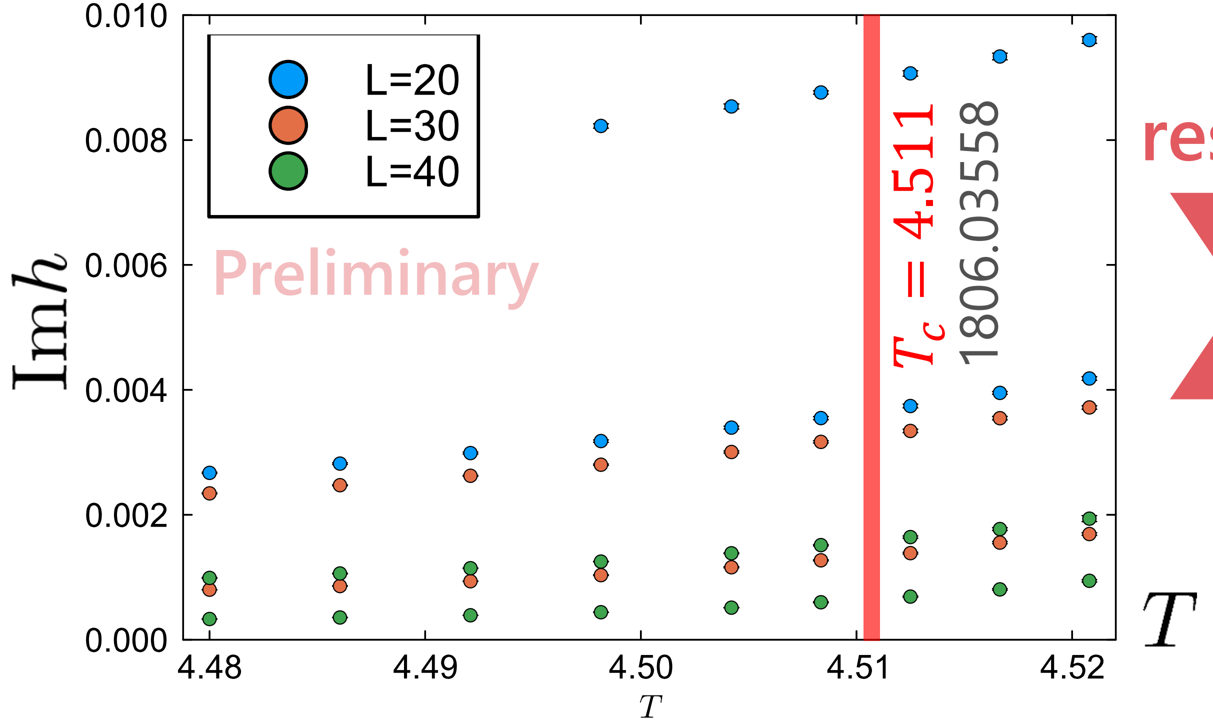
Monte-Carlo + reweighting

LYZ

$$\frac{Z(t,\tilde{h})}{Z(t,h)} = 0$$

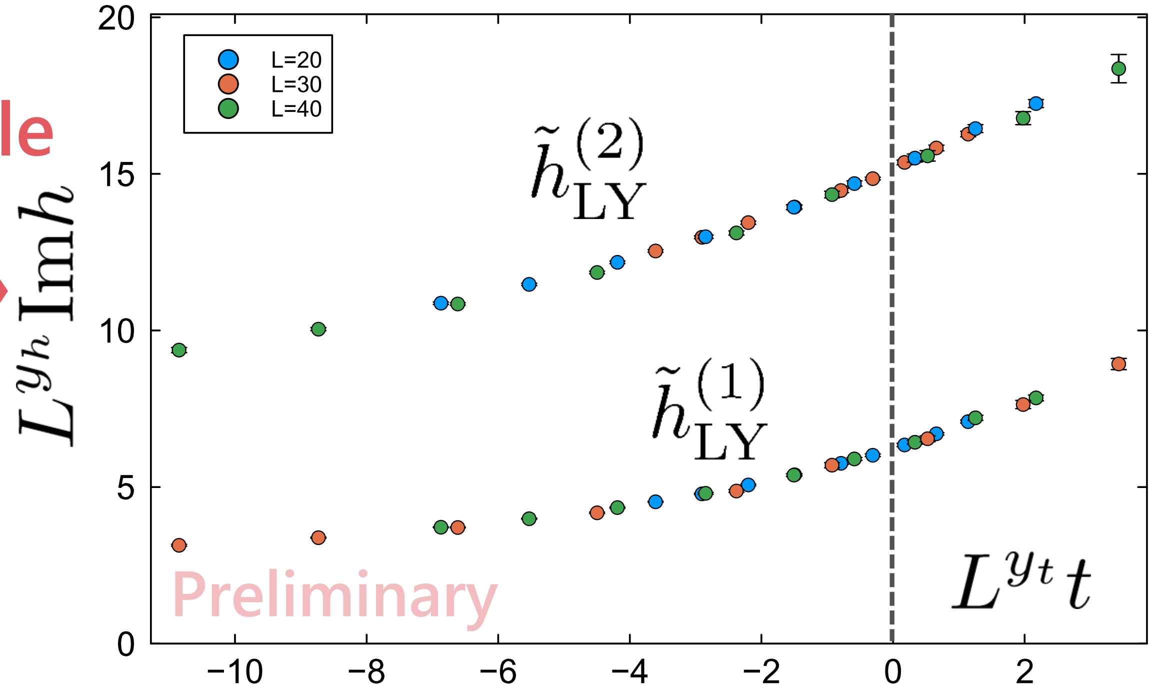
$$L^{y_h} h^{(i)} = \tilde{h}^{(i)}_{\mathrm{LY}}(L^{y_t} t)$$

1st & 2nd LYZ



rescale

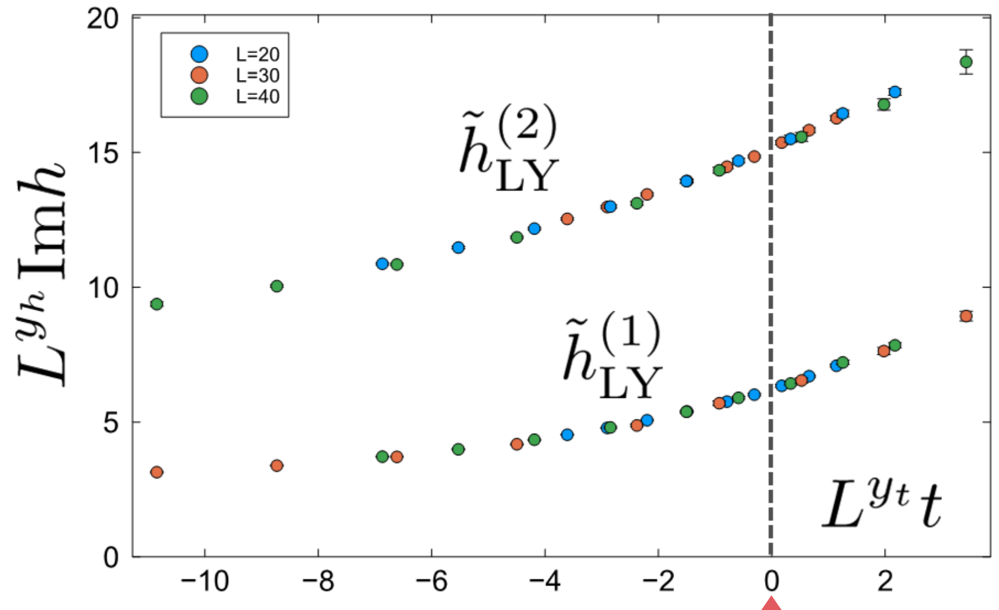
Rescaled



LYZ is away from the real axis at the CP on finite L.

Where is QCD Critical Point?

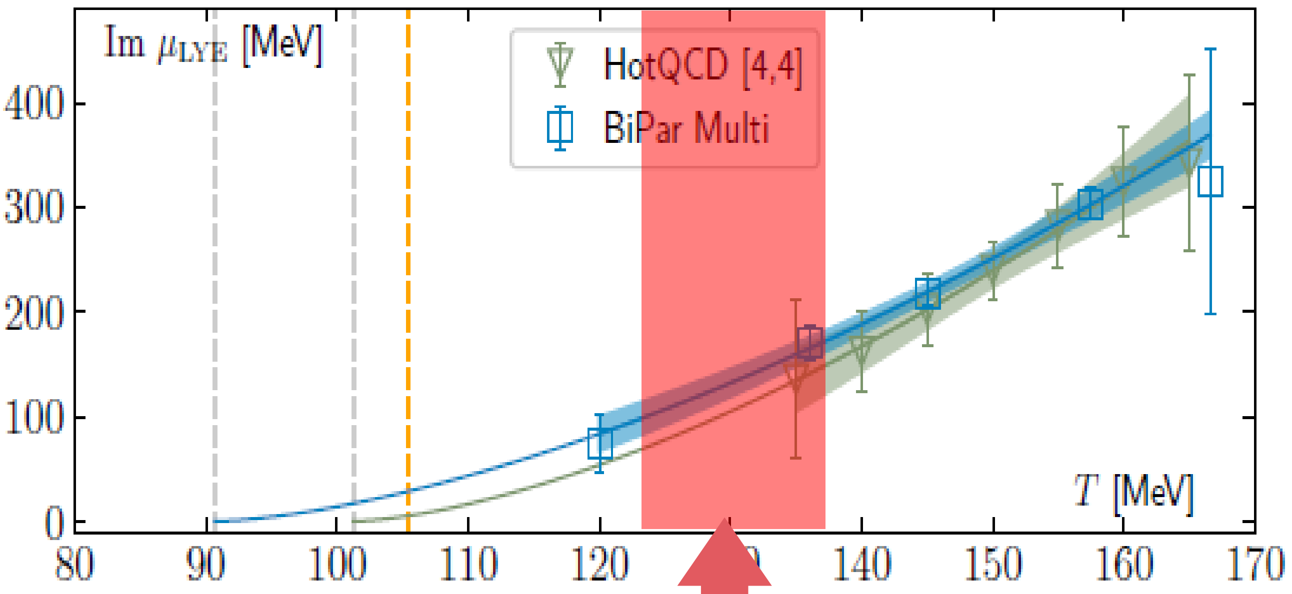
Ising model



$T = T_c$

LYZ in QCD

Clarke+, arXiv:2405.10196

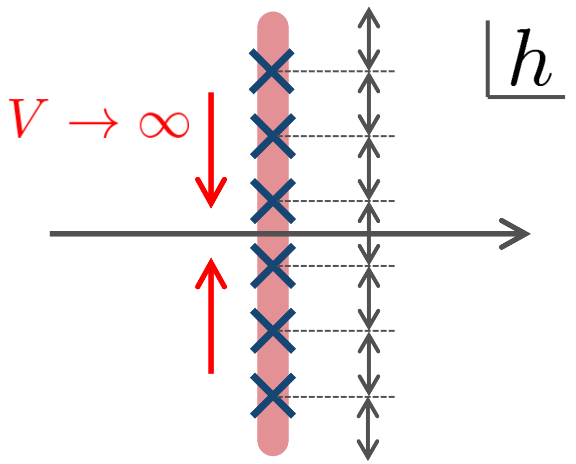


CP around here?

Lee-Yang Zero Ratios

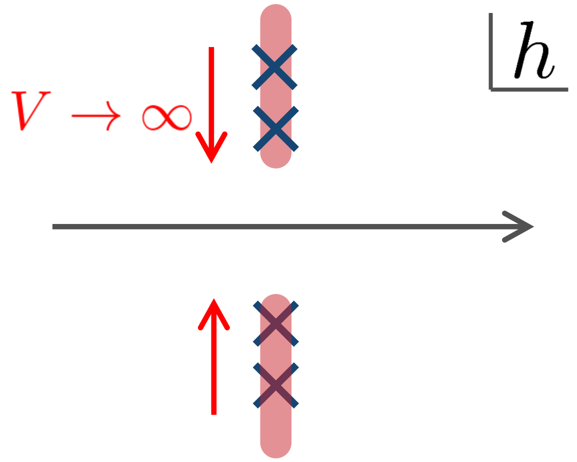
$$R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$$

First-Order Side $(t < 0)$



$$R_{nm}(t) \xrightarrow[V\to\infty]{} \frac{2n-1}{2m-1}$$

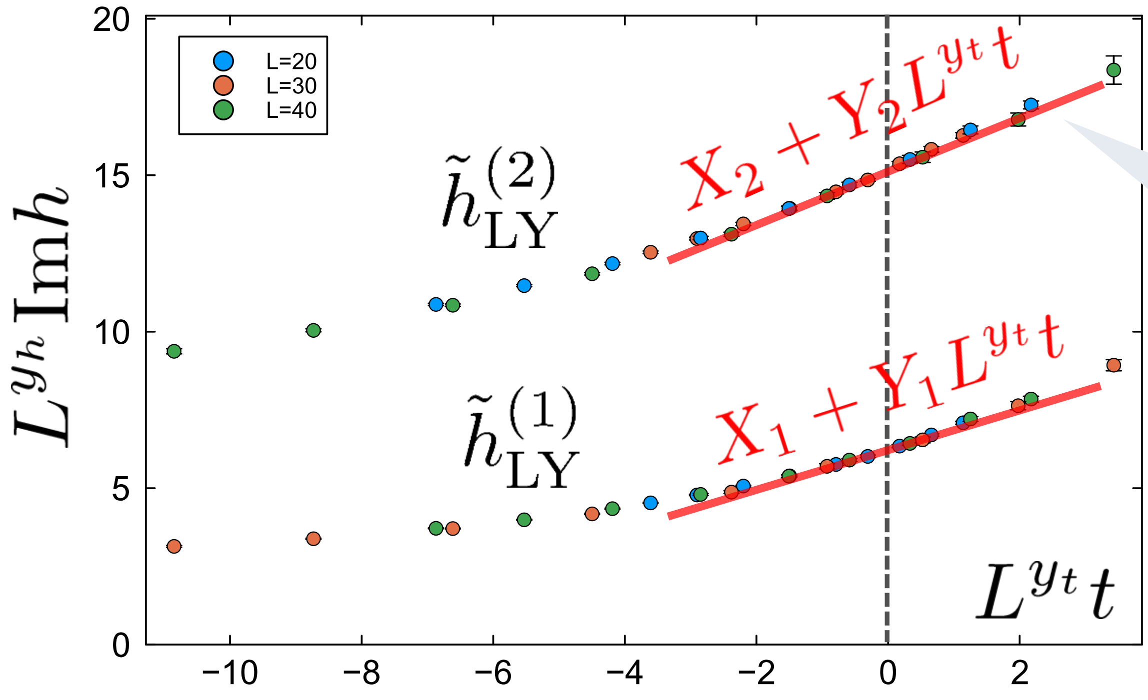
Crossover Side $(t > 0)$



$$R_{nm}(t) \xrightarrow[V\to\infty]{} 1$$

Lee-Yang Zero Ratios

$$R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$$



$$C_{nm} = \frac{Y_n}{X_n} - \frac{Y_m}{X_m}$$

Linear Approx. at t = 0

$$L^{y_h} h = \tilde{h}^{(i)}_{\mathrm{LY}}(L^{y_t} t)$$
$$= X_i + Y_i L^{y_t} t + \mathcal{O}(t^2)$$

$$R_{nm}(t) = \frac{X_n}{X_m}\left(1 + C_{nm} t L^{y_t} + \mathcal{O}(t^2)\right)$$

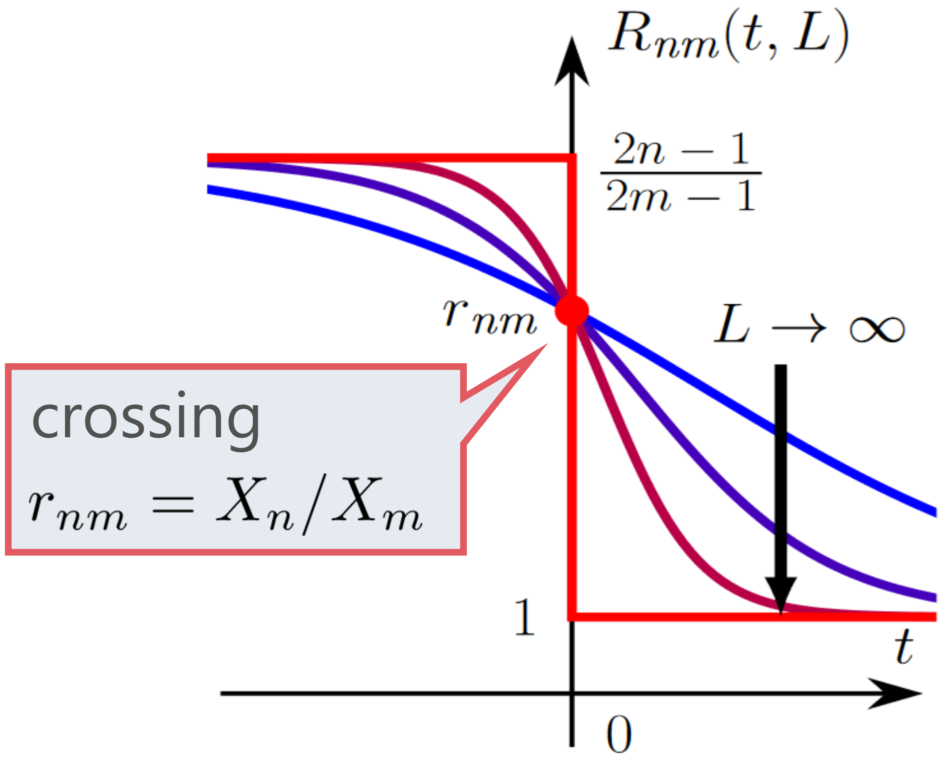
LYZ Ratios $R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$

Wada, MK, Kanaya, 2410.19345

$$R_{n1}(t) \xrightarrow[V\to\infty]{} \begin{cases} 2n-1 & t<0 \text{ (1st order)} \\ 1 & t>0 \text{ (crossover)} \end{cases}$$

$$R_{n1}(t) = \frac{X_n}{X_1}\left(1 + C_{n1} t L^{y_t} + \mathcal{O}(t^2)\right)$$

near $t = 0$



— $R(0)$ is L independent, the universal value.

—Crossing point of various L gives the CP.

—Reminiscent of Binder-cumulant analysis

LYZ Ratios $R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$

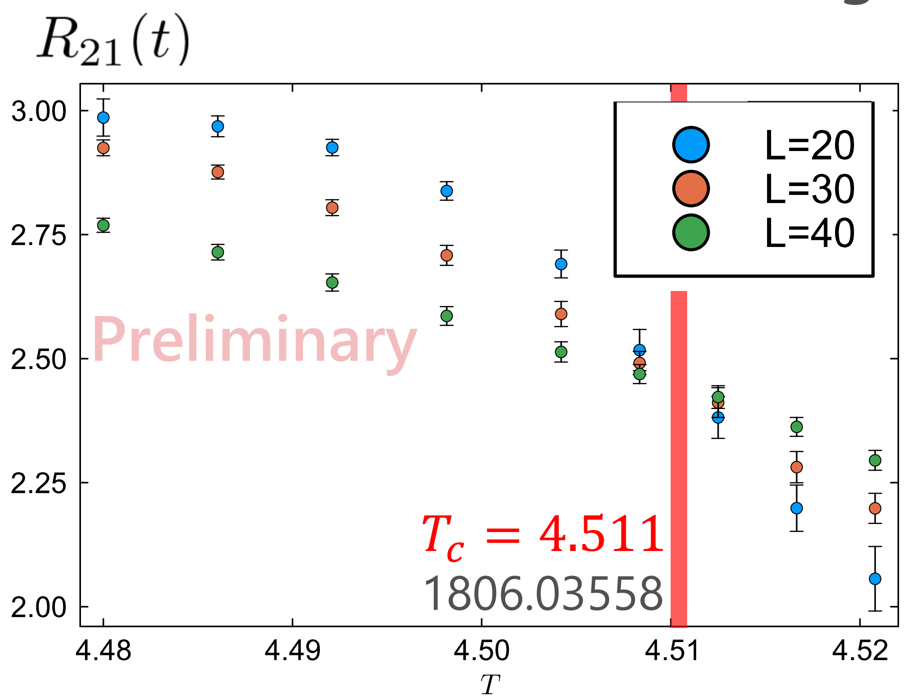
Wada, MK, Kanaya, 2410.19345

$$R_{n1}(t) \xrightarrow[V\to\infty]{} \begin{cases} 2n-1 & t<0 \text{ (1st order)} \\ 1 & t>0 \text{ (crossover)} \end{cases}$$

$$R_{n1}(t) = \frac{X_n}{X_1}\left(1 + C_{n1} t L^{y_t} + \mathcal{O}(t^2)\right)$$

near $t = 0$

Numerical Result in 3d-Ising



$R_{21}(0) \simeq 2.42$

— $R(0)$ is L independent, the universal value.

— Crossing point of various L gives the CP.

— Reminiscent of Binder-cumulant analysis

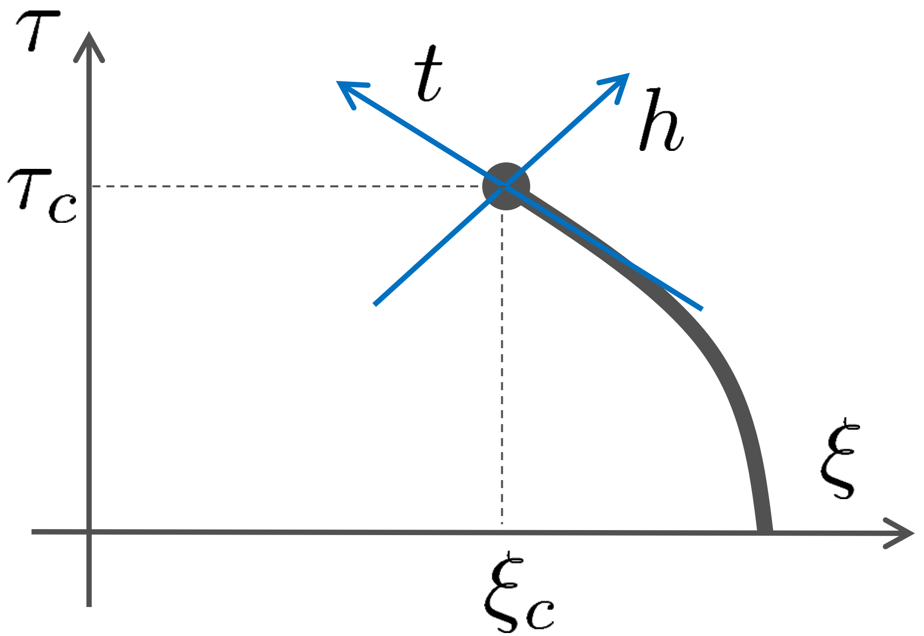
$\bar{y} = y_t - y_h = -0.894$

CP in a General System

- CP on a $\tau - \xi$ plane
- LYZ on the complex ξ plane

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$

$$L^{y_h} h^{(n)} \simeq X_i + Y_i L^{y_t} t$$



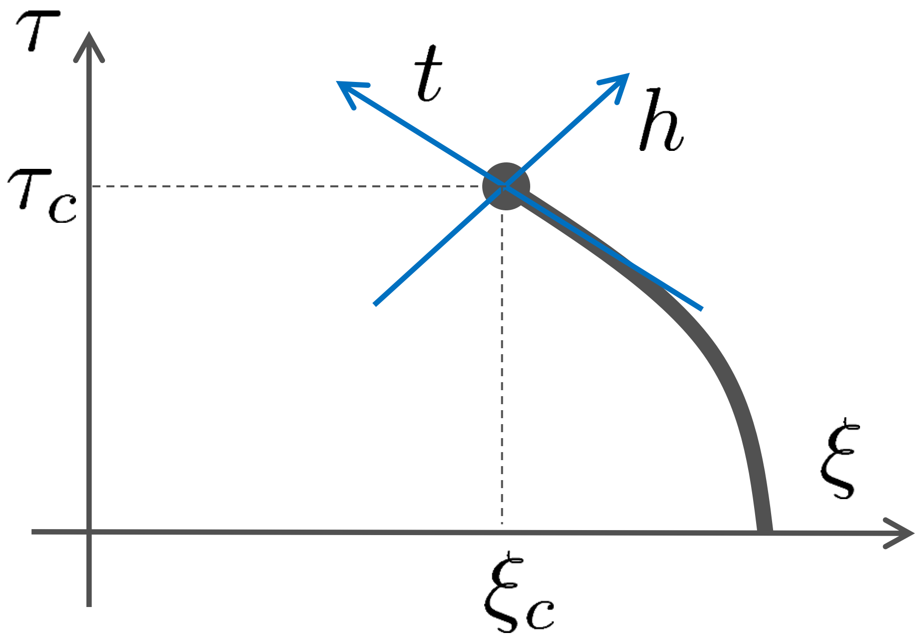
CP in a General System

$\bar{y} = y_t - y_h = -0.894$

- CP on a $\tau - \xi$ plane
- **LYZ on the complex ξ plane**

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$

$$L^{y_h} h^{(n)} \simeq X_i + Y_i L^{y_t} t$$



$$\begin{cases} \xi_R^{(n)} = \xi_c - \dfrac{a_{21}}{a_{22}} \delta\tau + \mathcal{O}(L^{2\bar{y}}) \\ \xi_I^{(n)} = \dfrac{X_n}{a_{22}} L^{-y_h} + \dfrac{\det AY_n}{a_{22}^2} \delta\tau L^{\bar{y}} + \mathcal{O}(L^{2\bar{y}}) \end{cases}$$

$L \to \infty$

generalization

LY Edge Singularity

$$\begin{cases} \mathrm{Re}\xi_{\mathrm{LYES}} \simeq c_1 \tau \\ \mathrm{Im}\xi_{\mathrm{LYES}} \simeq c_2 \tau^{\beta\delta} \end{cases}$$

Stephanov, 2006

LYZ Ratios for General CP

$$\bar{y} = y_t - y_h = -0.894$$

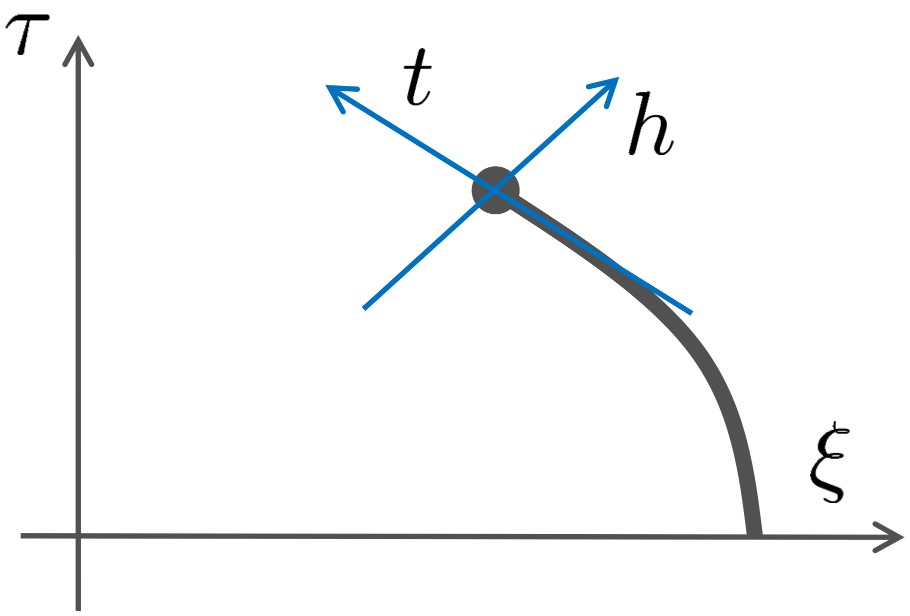
LYZ Ratio

$$R_{nm}(t) = \frac{\xi_{\rm I}^{(n)}(\tau)}{\xi_{\rm I}^{(m)}(\tau)} = \frac{X_n}{X_m}\left(1 + C\tau L^{y_t} + \mathcal{O}(\tau^2)\right)\left(1 + DL^{2\bar{y}} + \mathcal{O}(L^{4\bar{y}})\right)$$

nonzero for $a_{12} \neq 0$

$$C = \frac{\det A}{a_{22}}\left(\frac{Y_2}{X_2} - \frac{Y_1}{X_1}\right), \quad D = \frac{a_{12}^2}{a_{22}^2}(Y_1^2 - Y_2^2)$$

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$



LYZ Ratios vs Binder Cumulant

$\bar{y} = y_t - y_h = -0.894$

LYZ Ratio

$$R_{nm}(t) = \frac{\xi_{\rm I}^{(n)}(\tau)}{\xi_{\rm I}^{(m)}(\tau)} = \frac{X_n}{X_m}\left(1 + C\tau L^{y_t} + \mathcal{O}(\tau^2)\right)\left(1 + DL^{2\bar{y}} + \mathcal{O}(L^{4\bar{y}})\right)$$

nonzero for $a_{12} \neq 0$

$$C = \frac{\det A}{a_{22}}\left(\frac{Y_2}{X_2} - \frac{Y_1}{X_1}\right), \quad D = \frac{a_{12}^2}{a_{22}^2}(Y_1^2 - Y_2^2)$$

Binder cumulant

Jin+, PRD86, 2017

$$B_4(t) = b_4\left(1 + c\tau L^{y_t} + \mathcal{O}(t^2)\right)\left(1 + dL^{\bar{y}} + \mathcal{O}(L^{2\bar{y}})\right)$$

nonzero for $a_{12} \neq 0$

Deviation at $t = 0$ due to $a_{12} \neq 0$ converges faster in LYZ ratio.

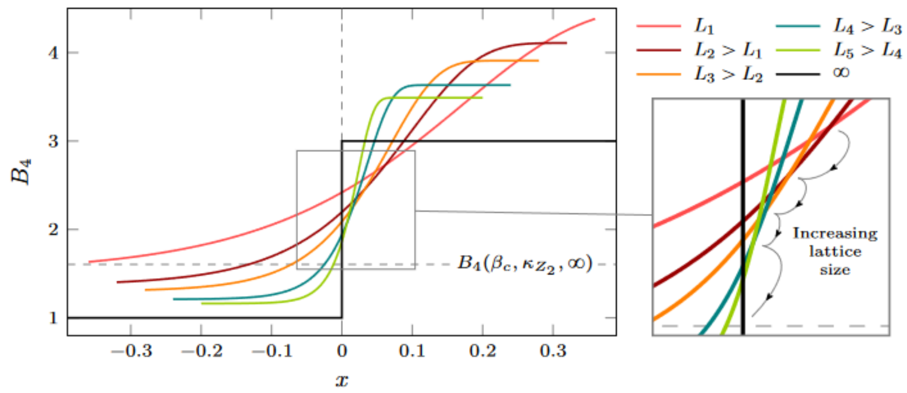
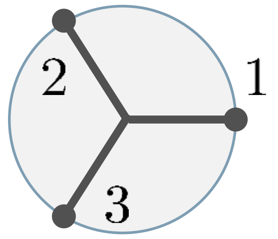


fig from Cuteri+, PRD ('21)

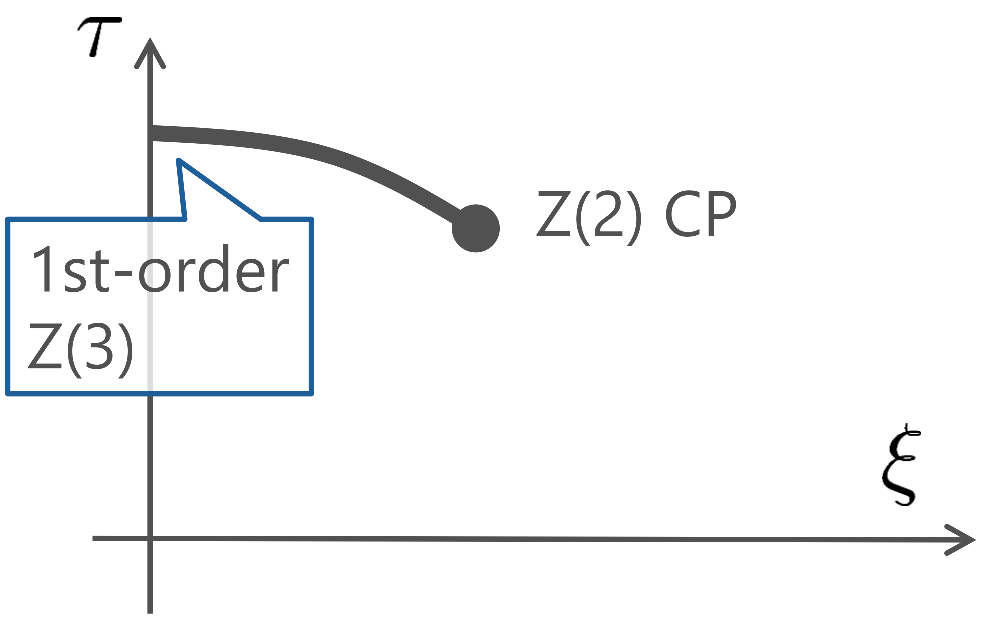
Numerical Analysis: 3d 3-State Potts Model

$$H = -\tau \sum_{\langle i,j \rangle} \delta_{\sigma_i,\sigma_j} - \xi \sum_i \delta_{\sigma_i,1} \quad \sigma_i = 1,2,3$$

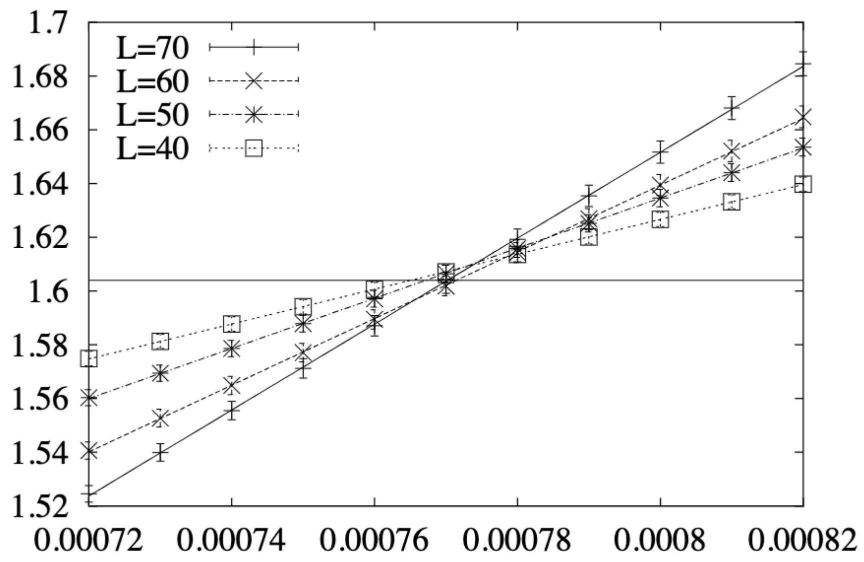
Monte-Carlo + reweighting

$\sigma_i =$ 

Phase Diagram



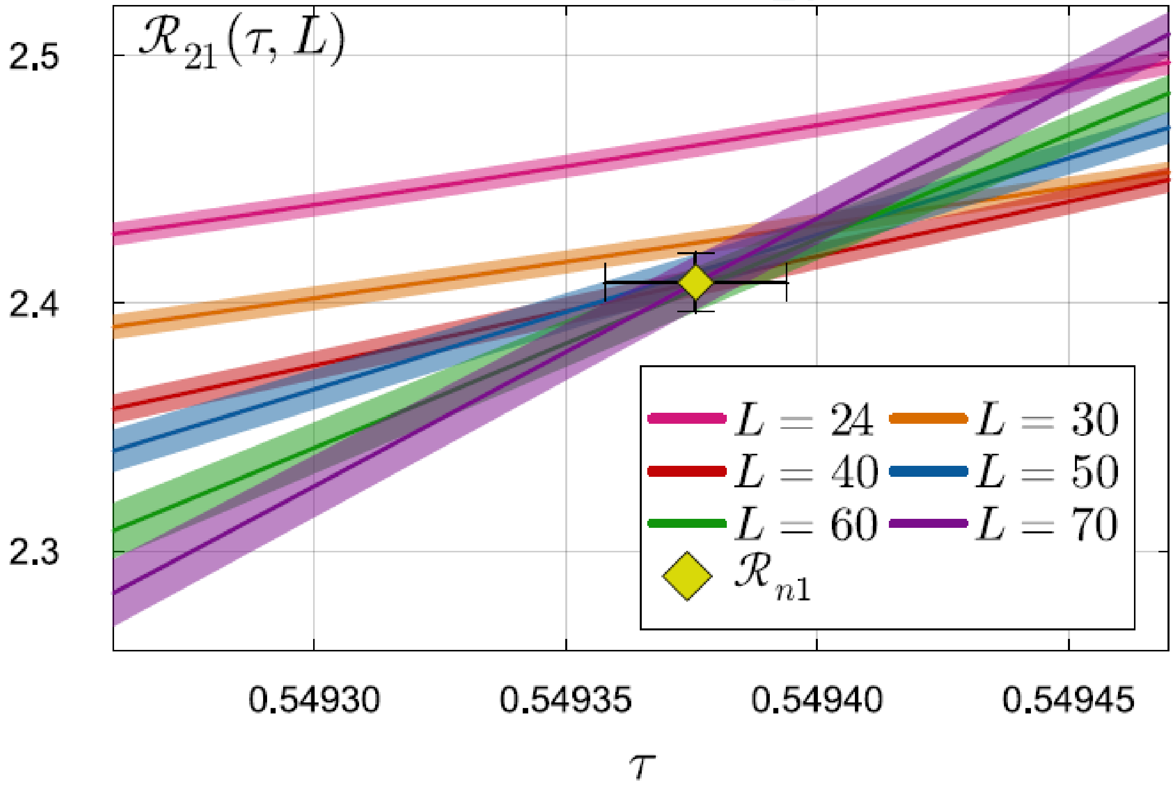
Binder-Cumulant Analysis



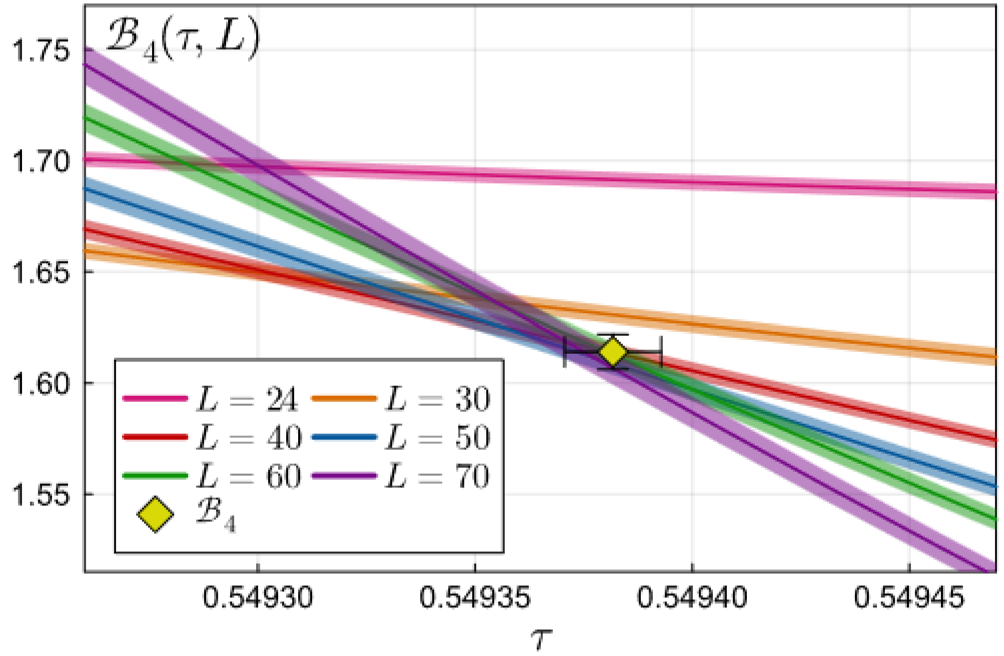
Karsch, Stickan, 2000

3d 3-State Potts Model: LYZ Ratio

LYZ Ratio (R_{21})



Binder Cumulant

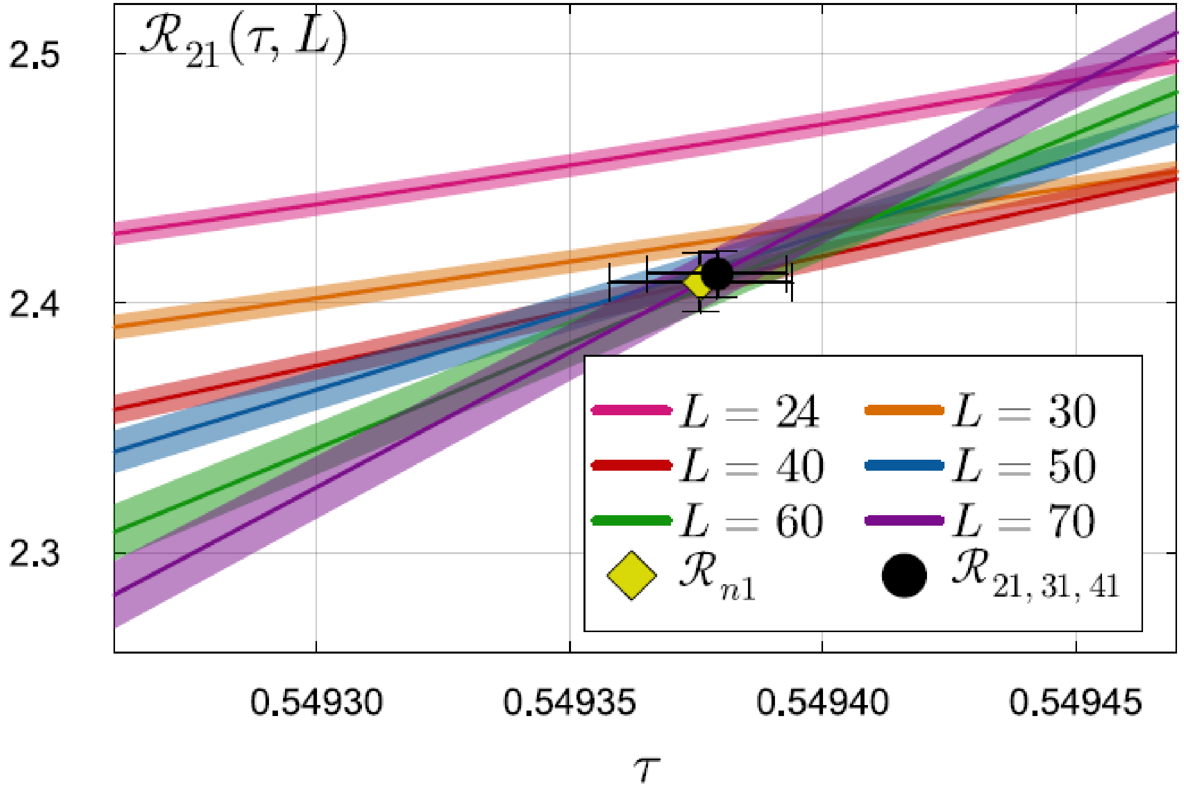


fit data	τ_c	y_t	r_{n1} or b_4	χ^2/dof
$\mathcal{R}_{21}$	0.549375(18)	1.53(19)	2.408(12)	0.38
$\mathcal{B}_4$	0.549382(11)	1.63(13)	1.614(8)	0.69

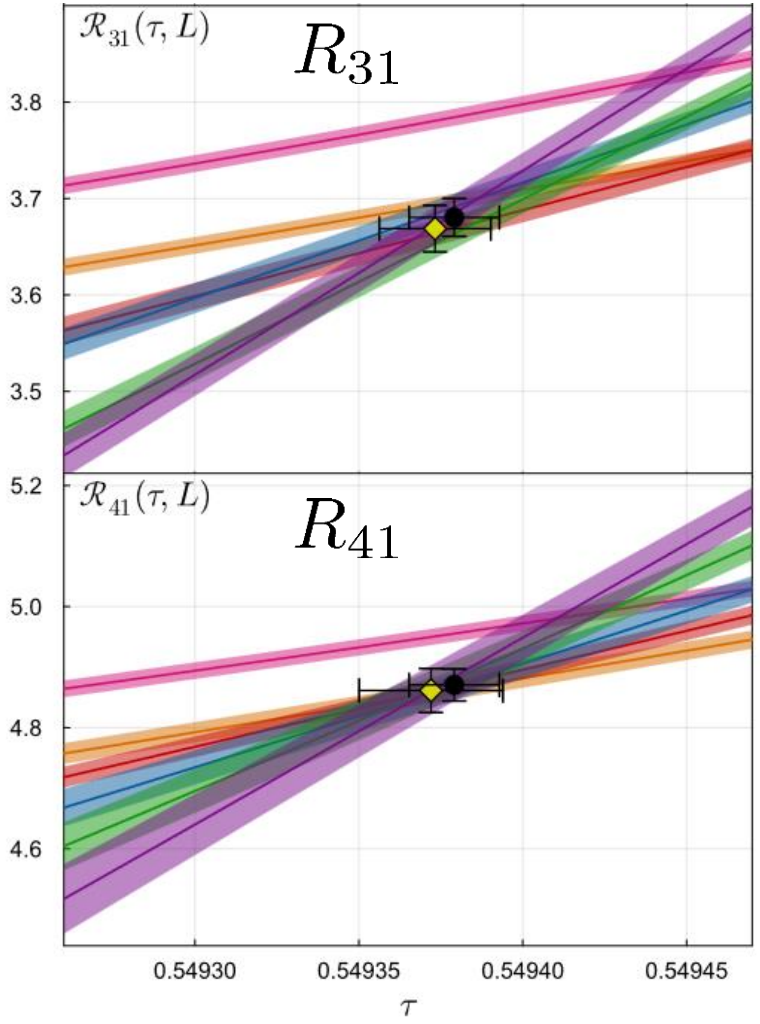
R_{n1} and B_4 give the same value of τ_c within statistics.

3d 3-State Potts Model: LYZ Ratio

LYZ Ratio (R_{21})

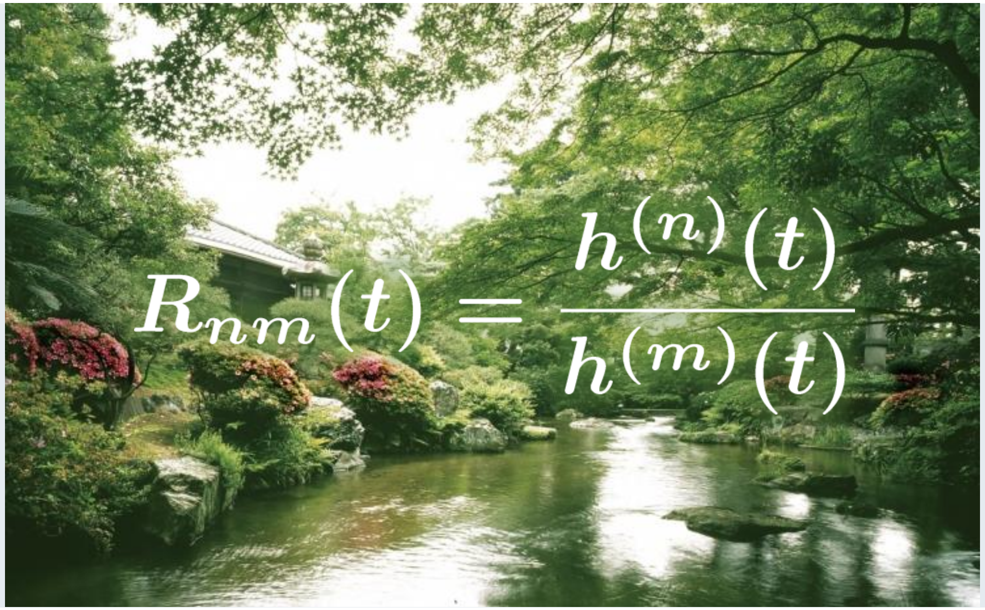


fit data	τ_c	y_t	r_{n1} or b_4	χ^2/dof
$\mathcal{R}_{21}$	0.549375(18)	1.53(19)	2.408(12)	0.38
$\mathcal{R}_{21,31,41}$	0.549379(14)	1.70(16)	–	0.56
$\mathcal{B}_4$	0.549382(11)	1.63(13)	1.614(8)	0.69



Combined use of LYZ can improve statistics!

Summary



Lee-Yang-zero ratios

A new method to utilize the **finite-size effects** of **Lee-Yang zeros** for the **CP searches**.

Outlook

- Determination of r_{nm} in each universality class
- More sophisticated utilization of $R_{nm}(t, L)$
- Other quantities: ξ_c, mixing matrix A, etc.
- Application to **QCD-CP** and other CPs

Question: Can lattice QCD find the 2nd LYZ??

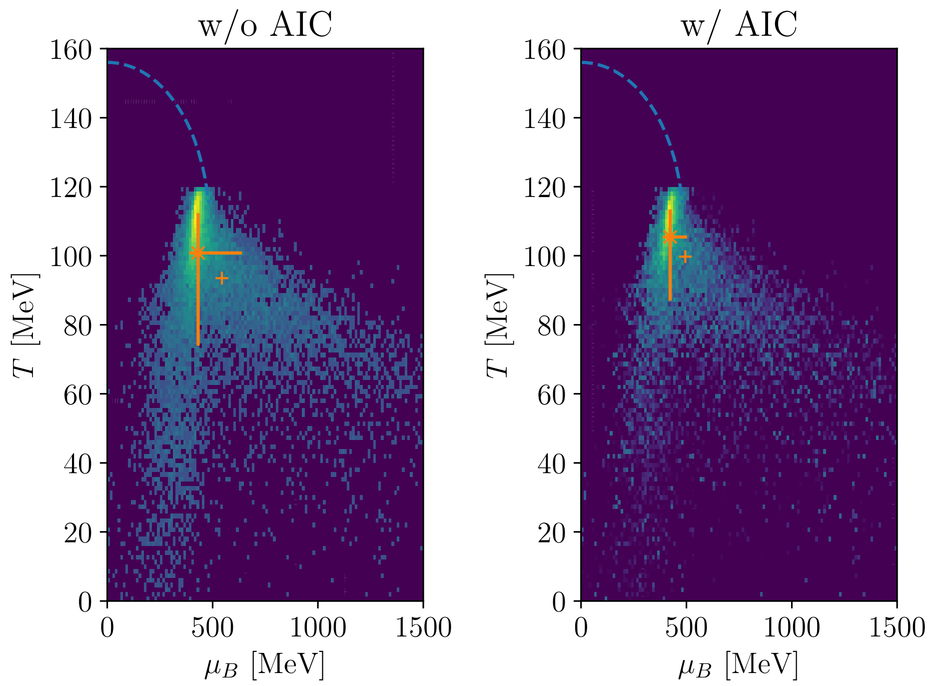
Enjoy tomorrow’s banquet!!

Photo from: https://www.gankofood.co.jp/shop/detail/ya-nijyoen/

backup

QCD-CP Search with Lee-Yang Zeros

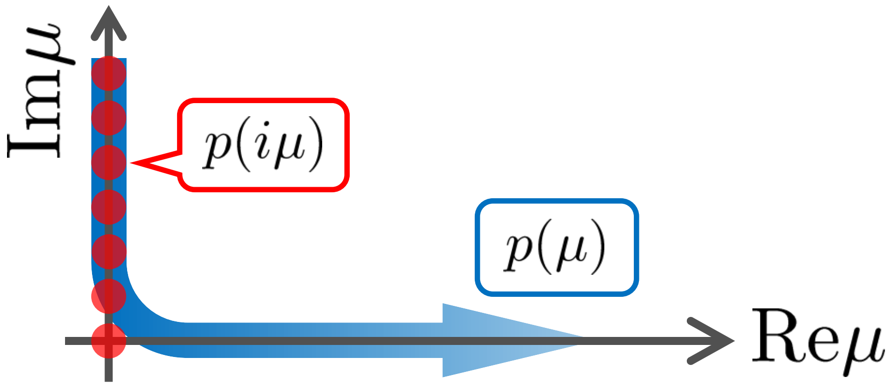
D. Clarke+ 2405.10196 [hep-lat]



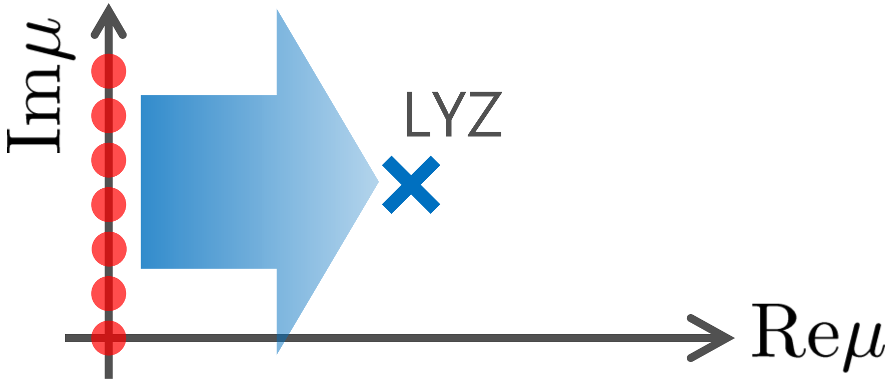
$$\begin{cases} \mu^{\mathrm{CEP}} = 422^{+80}_{-35}\mathrm{MeV} \\ T^{\mathrm{CEP}} = 105^{+8}_{-18}\mathrm{MeV} \end{cases}$$

See also, Alexander+ Lattice2024

Imaginary chem. pot.



Lee-Yang zero



analytic continuation via Pade approx.

QCD-CP and LYZ

arXiv:2405.10196v1 [hep-lat] 16 May 2024

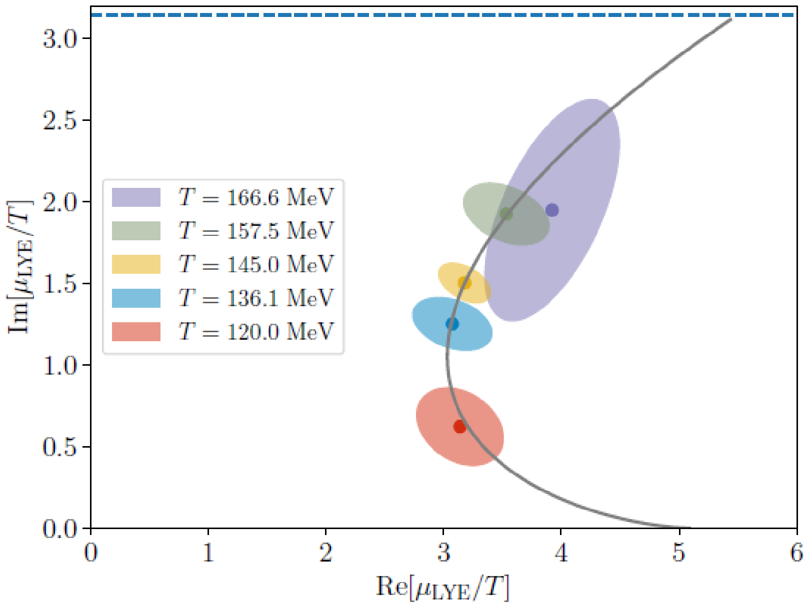


FIG. 3. Singularities at $T = 166.6, 157.5, 145.0, 136.1$ and 120.0 MeV. The dashed line lies at $\hat{\mu}_B = i\pi$.

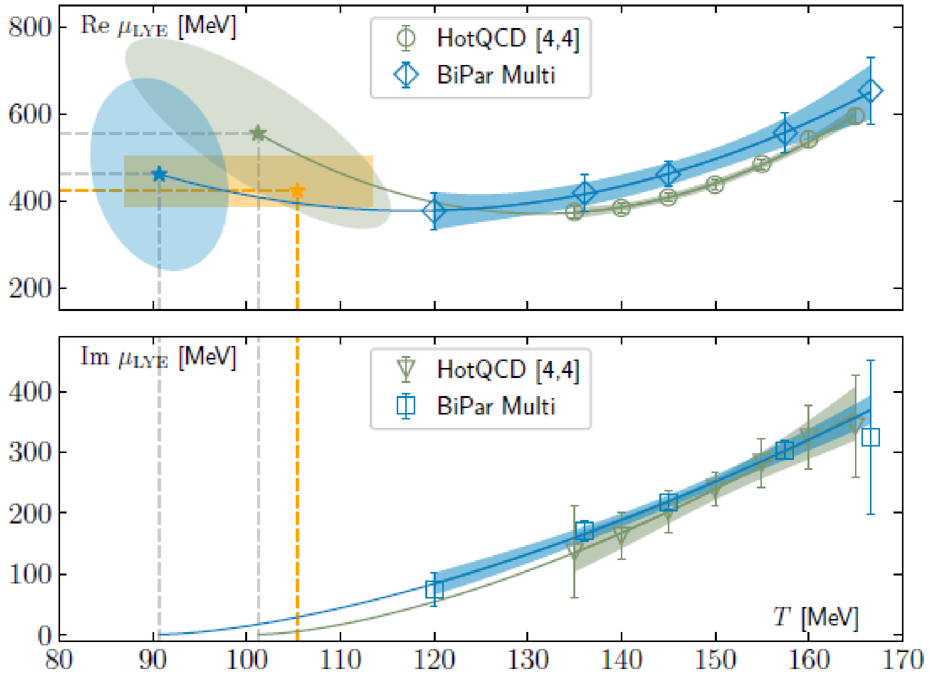


FIG. 4. Scaling fits for the LYE singularities related to the CEP. Green data come from a [4,4] Padé from Ref. [7]. Blue data come from the multi-point Padé. *Top*: Scaling of the real part. *Bottom*: Scaling of the imaginary part. The ellipses shown in the top panel represent the 68% confidence region deduced from the covariance matrix of the fit. The orange box indicates the AIC weighted estimate (6).