Semi-Universal Analytic Inversion of the TOV Equation

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Symmetry Parameter Constraints from a Lower Bound on Neutron-matter Energy

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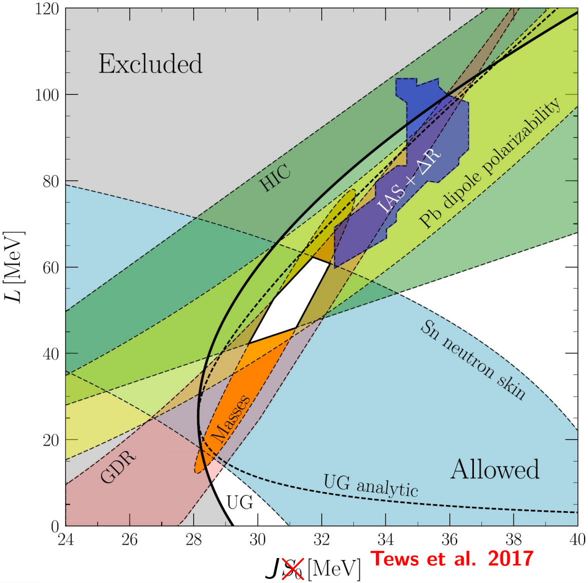
Abstract

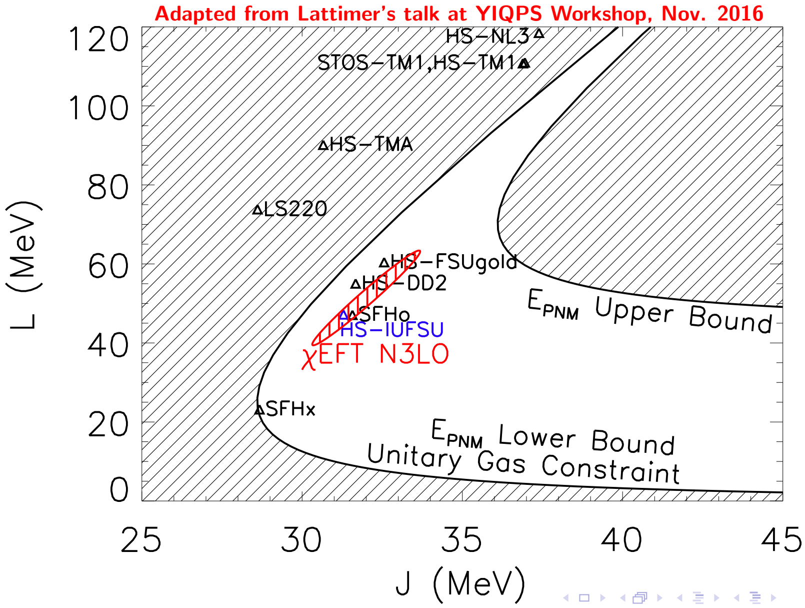
We propose the existence of a lower bound on the energy of pure neutron matter (PNM) on the basis of unitary-gas considerations. We discuss its justification from experimental studies of cold atoms as well as from theoretical

The Unitay Gas Conjecture



Tews et al. 2017

Imposing An Upper Neutron-Matter Energy Bound

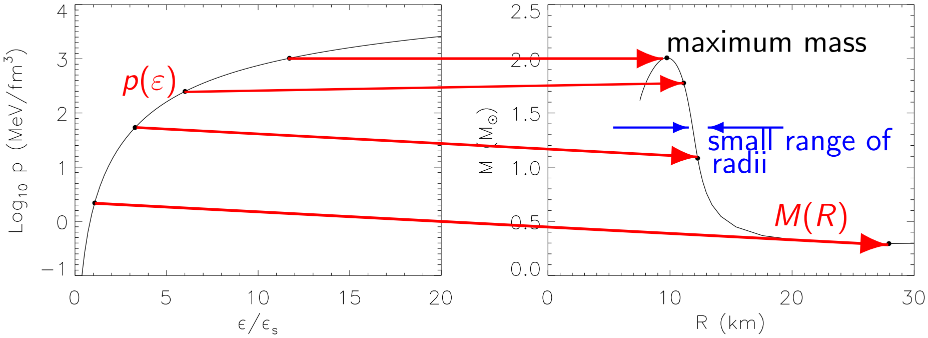


Neutron Star Structure

Tolman-Oppenheimer-Volkov equations

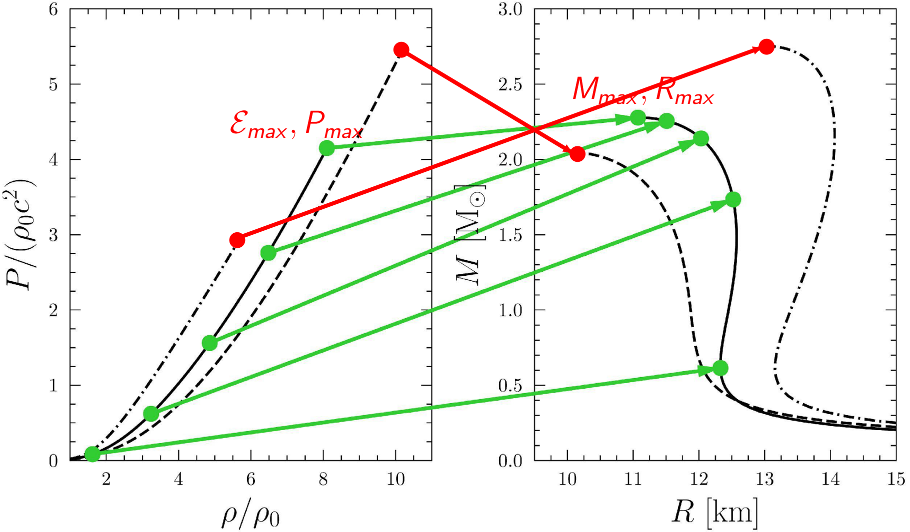
$$\frac{dp}{dr} = -\frac{G}{c^4}\frac{(mc^2+4\pi pr^3)(\varepsilon+p)}{r(r-2Gm/c^2)}$$

$$\frac{dm}{dr} = 4\pi\frac{\varepsilon}{c^2}r^2$$



Equation of State ← Observations

Maximum Mass As a Unique Scaling Point



$M_{\max}$, $R_{\max}$, $\mathcal{E}_{\max}$, $P_{\max}$ Correlation

Ofengeim et al's finding suggest the power-law correlations

$$\mathcal{E}_{\rm c,max} = (1.809 \pm 0.36)\left(\frac{R_{\max}}{10{\rm km}}\right)^{-1.98}\left(\frac{M_{\max}}{M_\odot}\right)^{-0.171} {\rm GeV\ fm}^{-3},$$

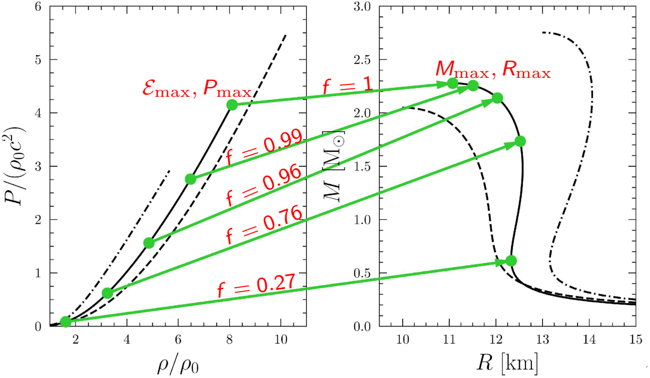
$$P_{\rm c,max} = (118.5 \pm 6.2)\left(\frac{R_{\max}}{10{\rm km}}\right)^{-5.24}\left(\frac{M_{\max}}{M_\odot}\right)^{2.73} {\rm MeV\ fm}^{-3},$$

which are accurate to about 5% in fitting $\mathcal{E}_{c,max}$ and $P_{c,max}$.

Points along $M - R$ curves, at $M = fM_{\max}$, have similarly accurate correlations:

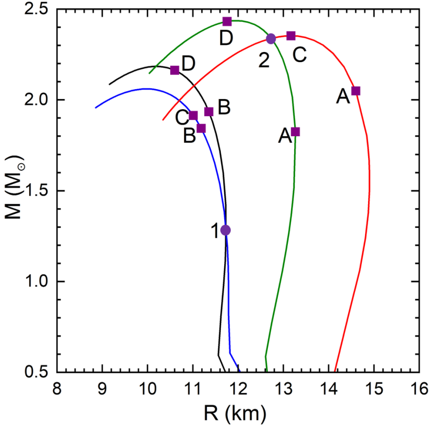
$$\mathcal{E}_{c,f} = a_{\mathcal{E},f} R_{fM_{max}}^{b_{\mathcal{E},f}} M_{max}^{c_{\mathcal{E},f}}$$

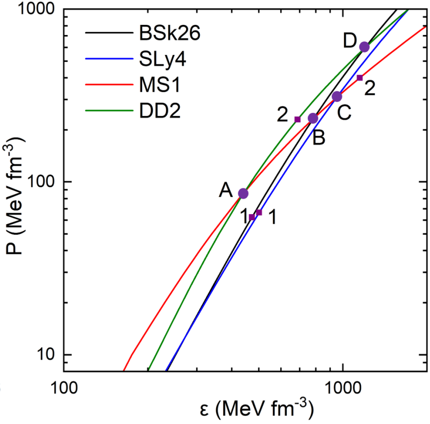
$$P_{c,f} = a_{P,f} R_{fM_{max}}^{b_{P,f}} M_{max}^{c_{P,f}}$$



(M, R) Is Not Equivalent To $(\mathcal{E}_c, P_c)$

The maximum mass point (M_{max}, R_{max}) predicts $(\mathcal{E}_{c,max}, P_{c,max})$ to about 5-10%, and similarly for a given fractional maximum mass fM_{max}.
The inversion is not unique as two different EOSs predicting the same (M, R) (numbers in figure) arrive at those values from integration via different paths in $(\mathcal{E}, P)$ space. Similarly, two EOSs with identical values of $(\mathcal{E}_c, P_c)$ (letters) do not have the same (M, R) values.





Correlations at $M = fM_{\rm max}$

Thus, more information than (M, R) needed. We find precision is greatly improved using a 2nd radius from a grid of fractional M_{max} points, *e.g.*, $f \in [1, 0.95, 0.9, 0.85, 4/5, 3/4, 2/3, 0.6, 0.5, 0.4, 1/3]$.

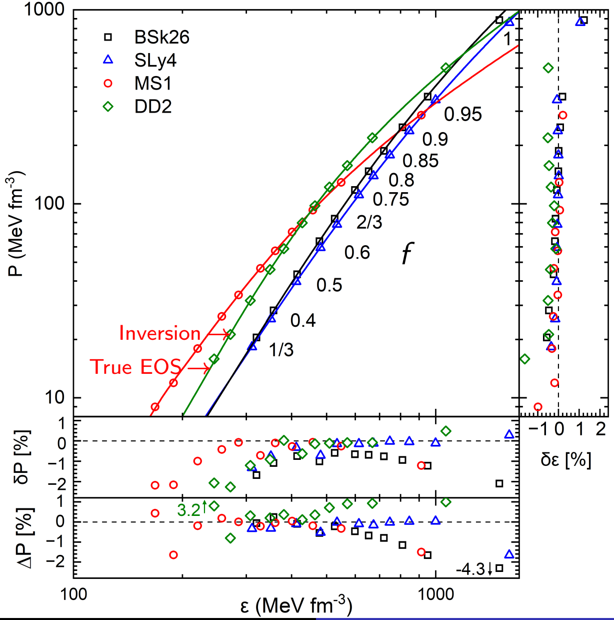
$$\mathcal{E}_f = a_{\mathcal{E},f}\left(\frac{R_{f_1}}{10{\rm km}}\right)^{b_{\mathcal{E},f_1}}\left(\frac{R_{f_2}}{10{\rm km}}\right)^{c_{\mathcal{E},f_2}}\left(\frac{M_{max}}{M_\odot}\right)^{d_{\mathcal{E},f}},$$

$$P_f = a_{P,f}\left(\frac{R_{f_1}}{10{\rm km}}\right)^{b_{P,f_1}}\left(\frac{R_{f_2}}{10{\rm km}}\right)^{c_{P,f_2}}\left(\frac{M_{max}}{M_\odot}\right)^{d_{P,f}},$$

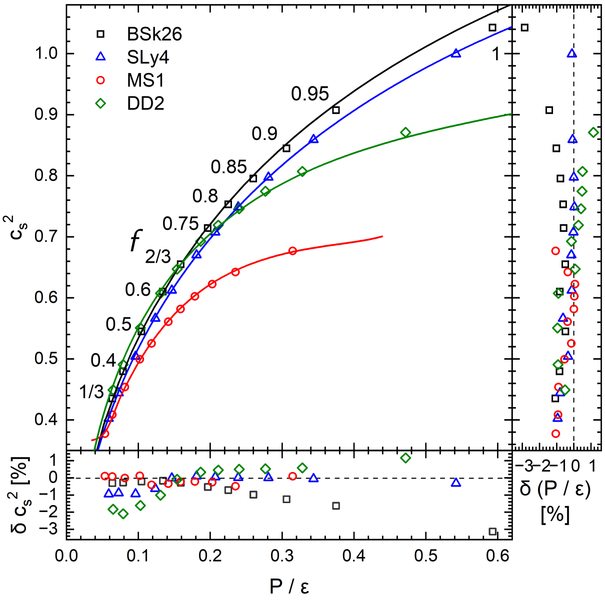
$f = M/M_{\rm max}$	f_1	f_2	$\Delta(\ln \mathcal{E}_f)$	f_1	f_2	$\Delta(\ln P_f)$
1	0.95	0.9	0.00469	1	3/5	0.0123
0.95	0.95	4/5	0.00275	0.95	3/5	0.00722
0.90	0.95	2/3	0.00227	0.95	2/5	0.00517
0.85	0.95	1/2	0.00237	0.9	2/5	0.00491
4/5	0.9	1/2	0.00230	0.85	2/5	0.00463
3/4	0.85	1/2	0.00239	4/5	2/5	0.00539
2/3	3/4	1/2	0.00277	2/3	2/5	0.00513
3/5	3/4	2/5	0.00339	2/3	1/3	0.0172
1/2	2/3	1/3	0.00477	1/2	2/5	0.00996
2/5	1/2	1/3	0.00706	1/2	1/3	0.0187
1/3	1/2	1/3	0.0122	2/5	1/3	0.0259

greatly reduced uncertaintites!

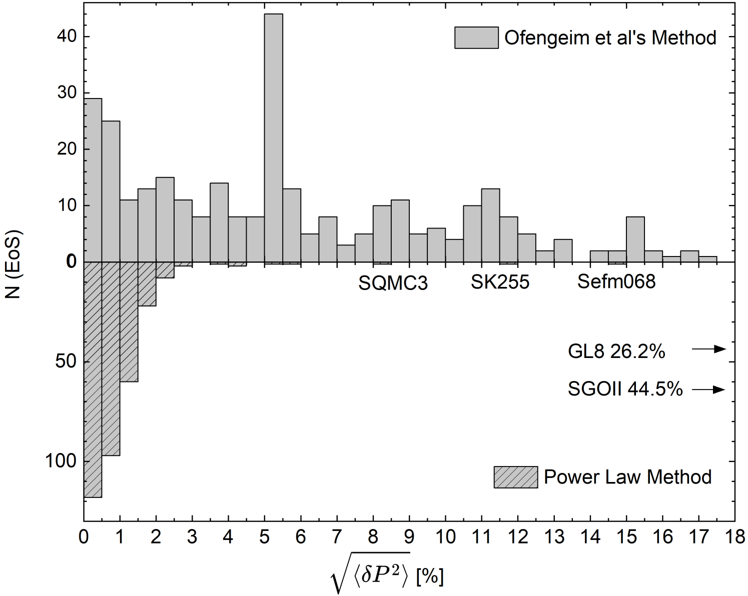
Testing the Inversion



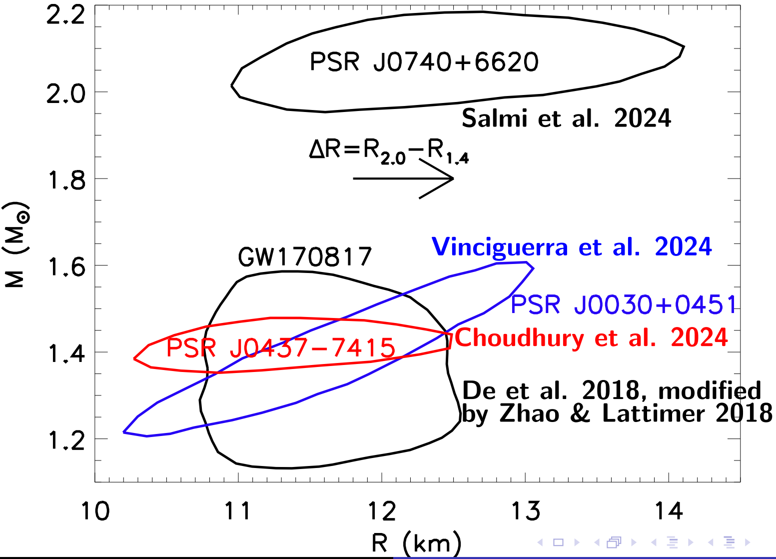
Testing the Inversion for $c_s^2 - P/\mathcal{E}$



Comparing Inversions



Summary of Astrophysical Observations



Inversion of $M - R$ Data

Instead of inverting an $M - R$ curve one may wish to infer the EOS from $M - R$ data. Traditional Bayesian inversions begin with $M - R$ priors generated by sampling millions of trials using a specific EOS parameterization with uniform distributions of parameters within selected ranges.

One problem with our approach is that M_{max} and R_{max} are not known. One can form analytical correlations between (M, R) and $(\mathcal{E}_c, P_c)$, but these have only moderate accuracy since this inversion is not unique. More information than the $M - R$ point itself is necessary to improve the inversion.

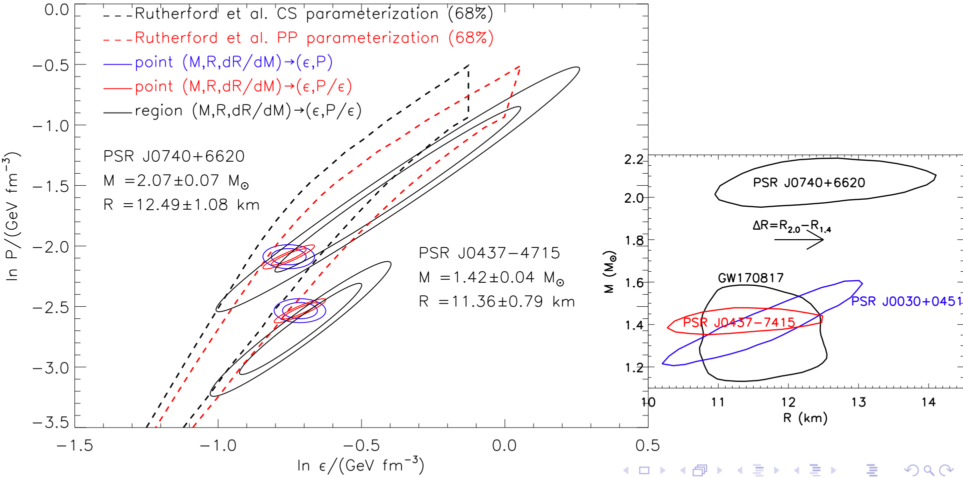
One possibility is to include the inverse slope dR/dM at the (M, R) point. Generally, one can determine a correlation between a quantity $G \in [\mathcal{E}_c, P_c, \mathrm{etc.}]$ and $(M, R, dR/dM)$ in the form

$$\ln G = \ln a_G + b_G \ln M + c_G \ln R + d_G(dR/dM).$$

Including dR/dM information improves correlations by factors of about 2. It is also found that inferred values of $\mathcal{E}_c$ and P_c are highly correlated; fits to $P_c/\mathcal{E}_c$ have much smaller uncertainty than fits to $\mathcal{E}_c$ or P_c.

Comparison to Traditional Bayesian Inference

From two $M - R$ regions obtained from observations select random pairs of points and determine dR/dM. Then, using the above correlation formulae, infer two $\mathcal{E}_c - P_c$ uncertainty regions (after rejecting pairs that violate the conditions $0 \le dP_c/d\mathcal{E}_c \le 1$ and $dP_c/dM > 0$).

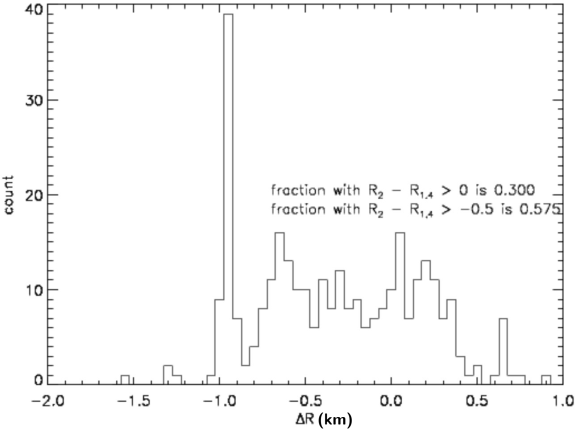


Importance of $\Delta R = R_{2.0} - R_{1.4}$

- J0437-4715:
$R = 11.36^{+0.95}_{-0.63}$ km
- J0740+6620:
$R = 12.49^{+1.28}_{-0.88}$ km

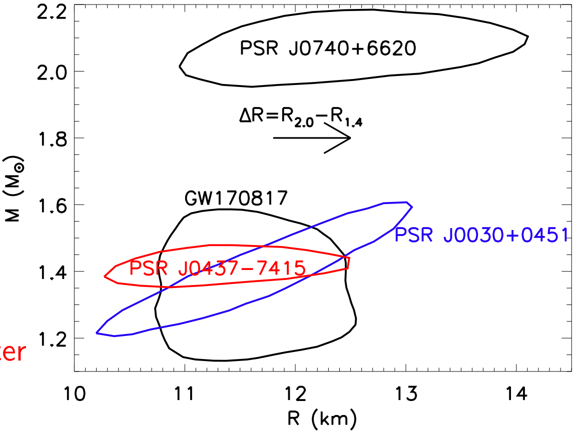
$\Delta R = +1.13^{+1.59}_{-1.08}$ km

313 Skyrme + RMF forces with $M_{max} \geq 2.0 M_\odot$



Applications

- Analytic inversion of TOV equations with arbitrarily high accuracies (depends on number of R_f values).
- Existing techniques use parameterized EOS models in probabilistic (Bayesian) approaches having unquantified systematic uncertainties stemming from the model and parameter choices (prior distributions).



• Since M and R can't uniquely determine $\mathcal{E}_c$ and P_c, we use the value of (dR/dM) to improve accuracies.

• Correlations of c_s with M, R and dR/dM can be used to further improve the fidelity of inversions and also for interpolating within the $\mathcal{E}_f - P_f$ grid. They could also allow probing the composition of the neutron star interior (phase transitions, etc.).

• Correlations of $\tilde{\Lambda}, \bar{I}$ and BE$/M$ with M and R also exist and aren't sensitive to dR/dM.

• This inversion technique might have wider applicability to other physics or engineering problems.