9th October 2024

Neutron Stars in Covariant $f(Q)$ Gravity

Based on Alwan, et.al. *JCAP* 09 (2024) 011

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@ CSQCD2024, YITP

Outline

01 Background

02 Covariant formulation of $f(Q)$ and TOV Equations

03 Neutron Star Properties and the Observational Constraint

04 Slowly Rotating Neutron Stars and $\bar{I} - C$ Universal Relations

05 Summary and Future Works

Background

Neutron Stars Characteristics

- Neutron Stars has become natural laboratories for studying the behavior of high-density nuclear matter.

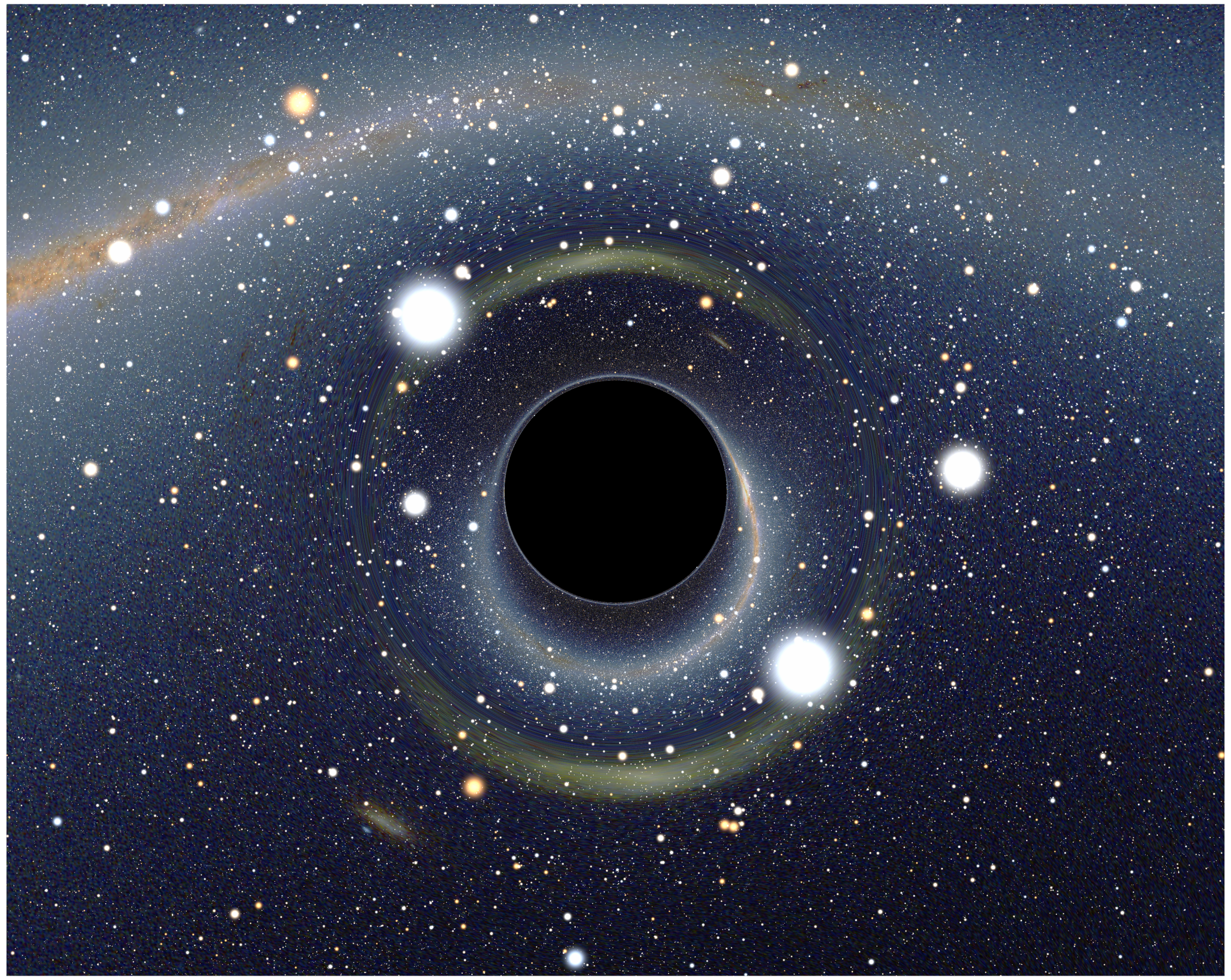

Figure 1. Compact Objects in Astrophysics

Source : https://www.h-its.org/

Background

Neutron Stars Characteristics

- Neutron Stars has become natural laboratories for studying the behavior of high-density nuclear matter.
- Using approriate EoS, we can get macroscopic properties such as Mass-Radius, tidal deformability, and the stellar momentum of inertia

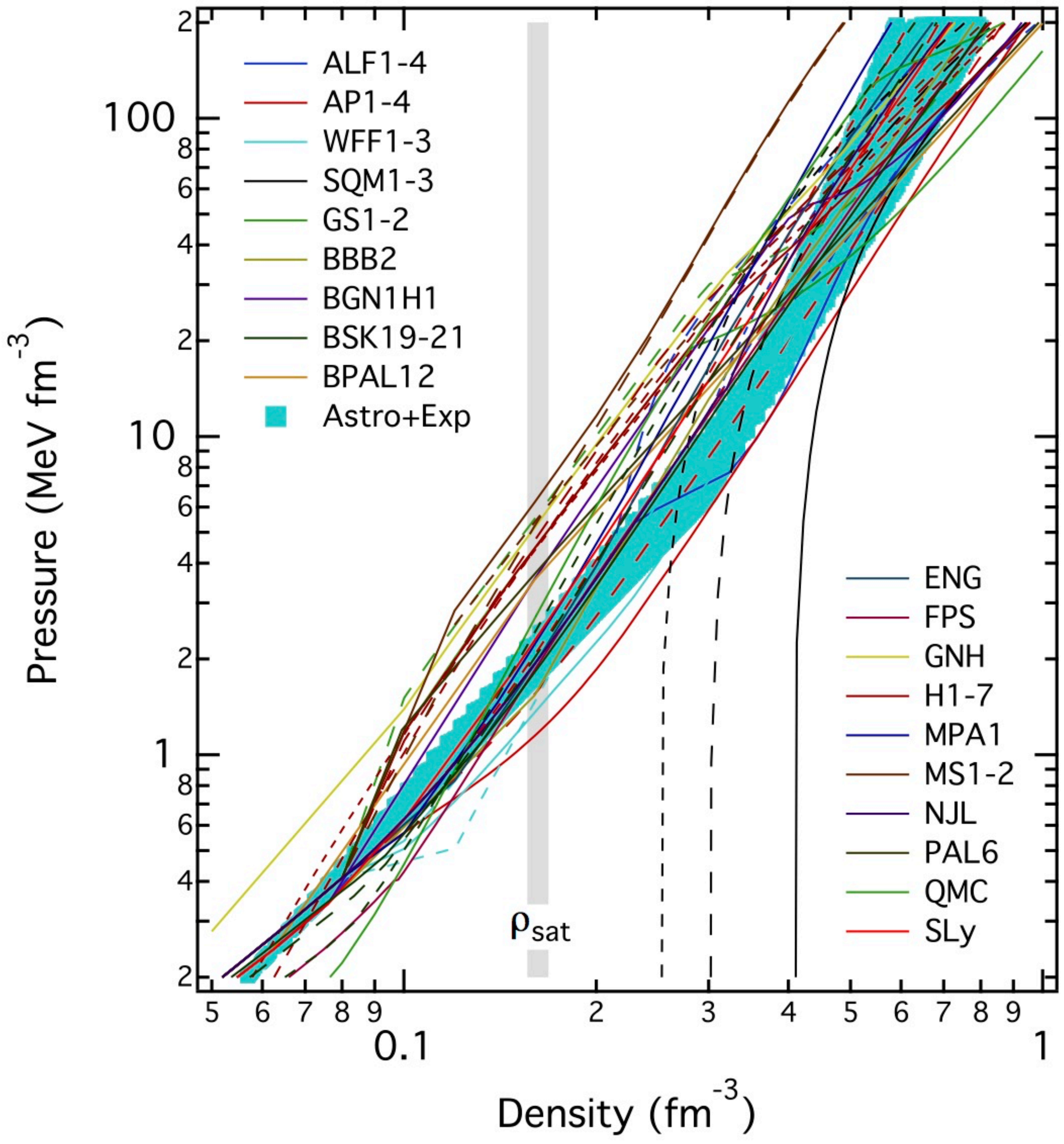

Figure 2. Various EoS in Neutron Star

Source : Ozel, Feryal and Paulo C. C. Freire. "Masses, Radii, and the Equation of State of Neutron Stars." Annual Review of Astronomy and Astrophysics 54 (2016): 401-440.

Background

Neutron Stars Characteristics

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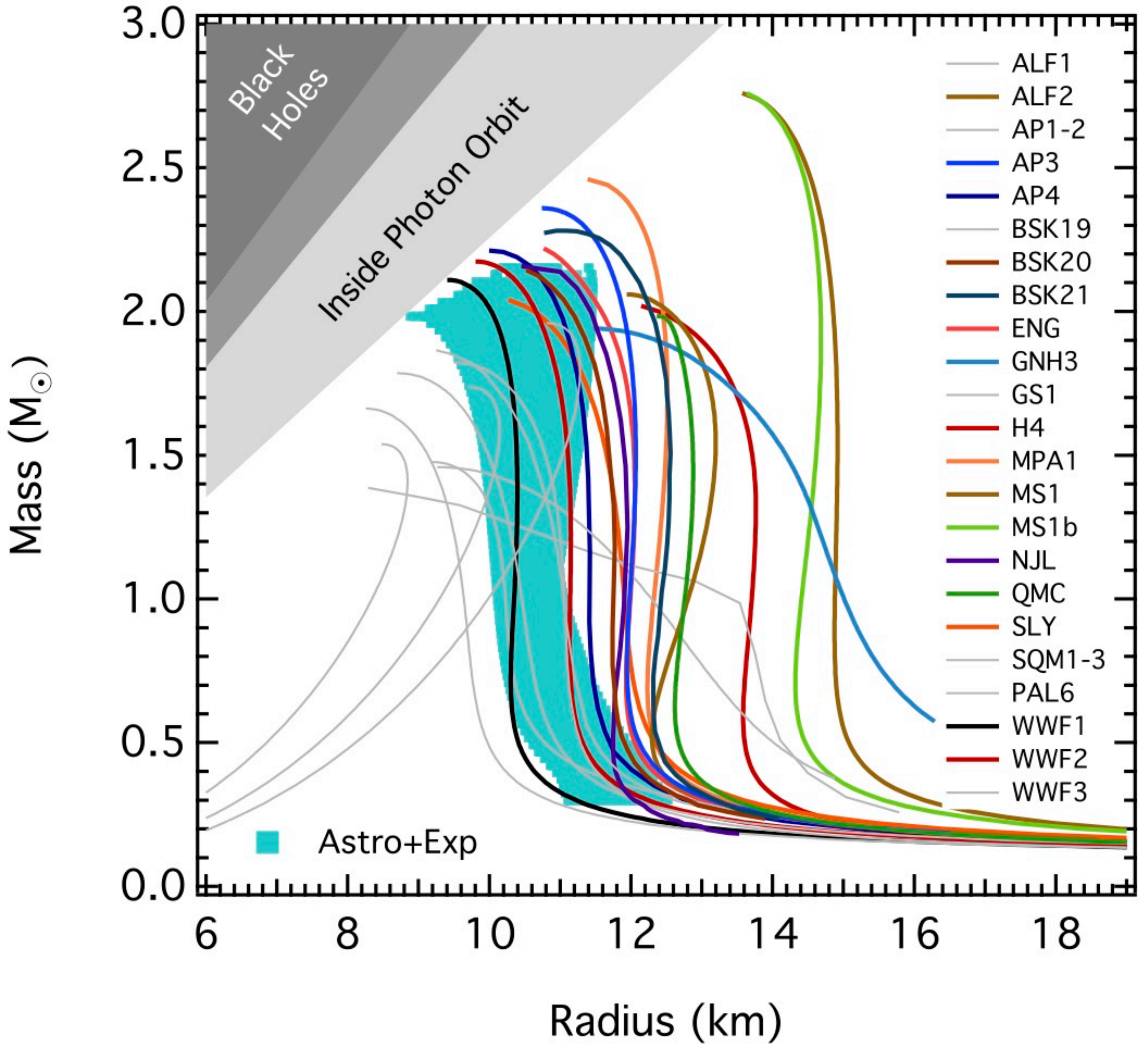

Figure 3. Mass-Radius Relation under various EoS

Source : Ozel, Feryal and Paulo C. C. Freire. "Masses, Radii, and the Equation of State of Neutron Stars." Annual Review of Astronomy and Astrophysics 54 (2016): 401-440.

Background

Neutron Stars as High-Densed Laboratory

- Due to its **high compactness, high-energy behavior, and abundance of observational data**, signatures of modified gravity theories may emerge, providing insights beyond the predictions of GR, which makes it **an ideal environment to test the limits of General Relativity**.

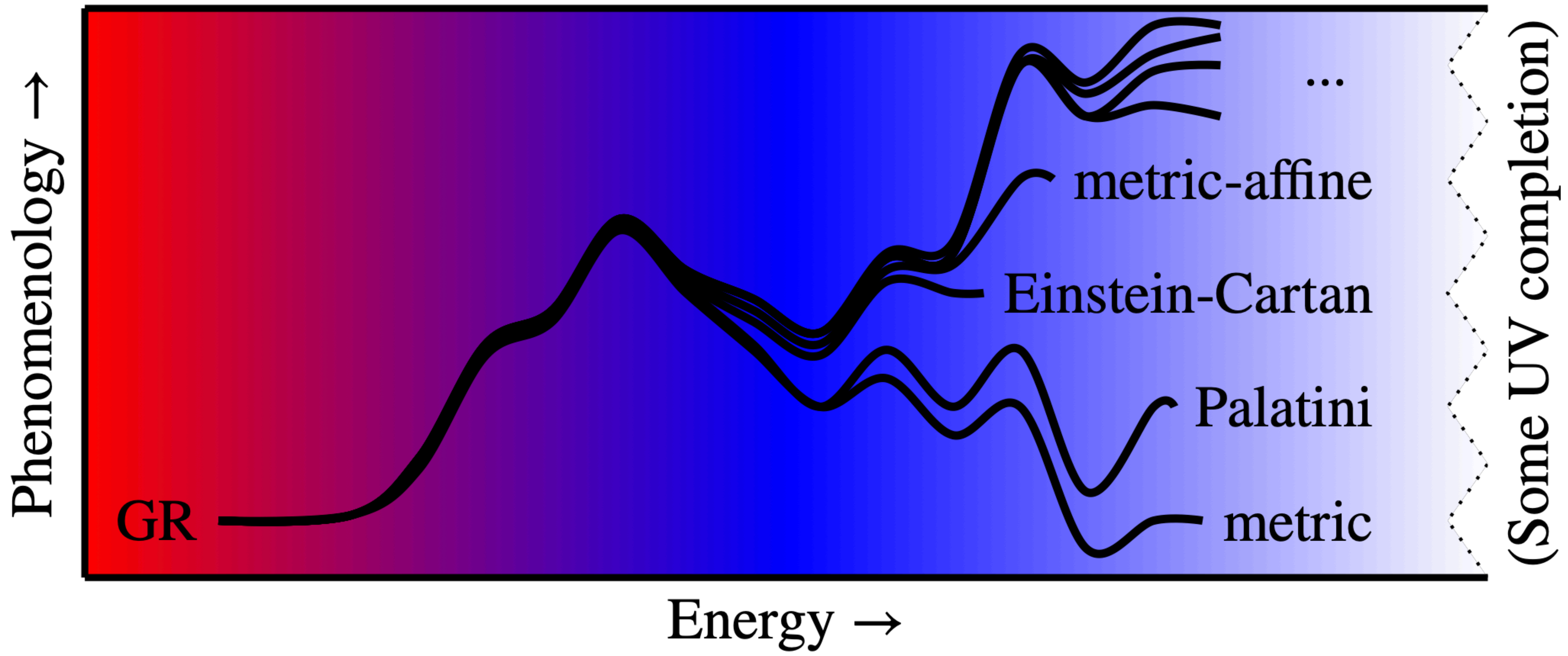

Figure 4. As a low-energy effective theory, Einstein's GR has many equivalent geometrical formulations that will only be distinguishable by experimental bounds on their high-energy phenomenology

Source : Barker, Will and Sebastian Zell. "Consistent particle physics in metric-affine gravity from extended projective symmetry." (2024). arXiv:2402.14917

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- Due to its **high compactness, high-energy behavior, and abundance of observational data**, signatures of modified gravity theories may emerge, providing insights beyond the predictions of GR, which makes it **an ideal environment to test the limits of General Relativity**.
- The **presence of exotic matter** in NS could help address some of the problems faced by GR and the Standard Model across different energy scales.

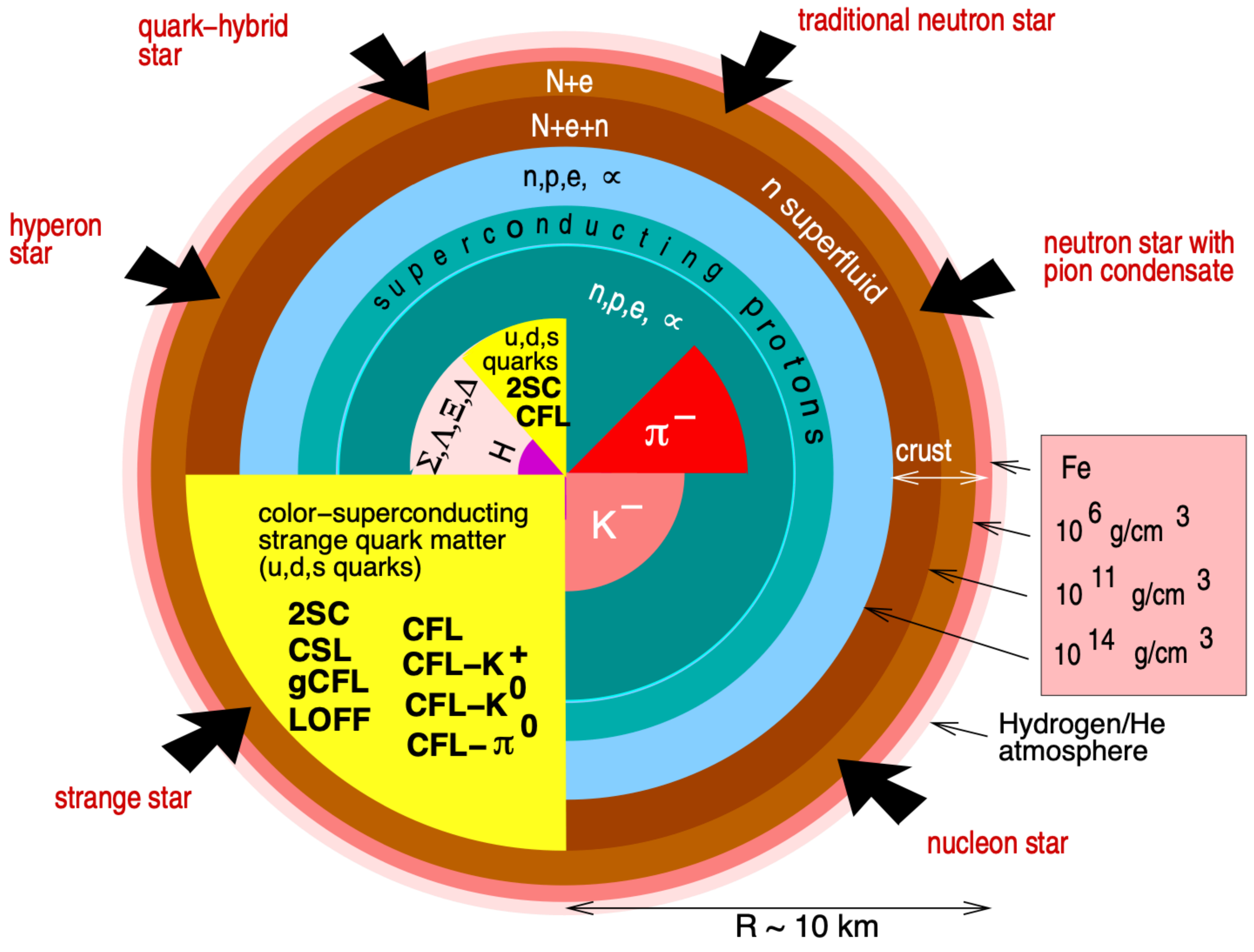

Figure 5. Neutron Star Structure

Source : Weber, Fridolin. “Strange quark matter and compact stars.” Progress in Particle and Nuclear Physics 54 (2004): 193-288.

Background

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- The **presence of exotic matter** in neutron stars could help address some of the problems faced by GR and the Standard Model (SM) across different energy scales.

- Modified gravity theories providing alternative options to describe the interiors and changing the macroscopic and microscopic properties of neutron stars.

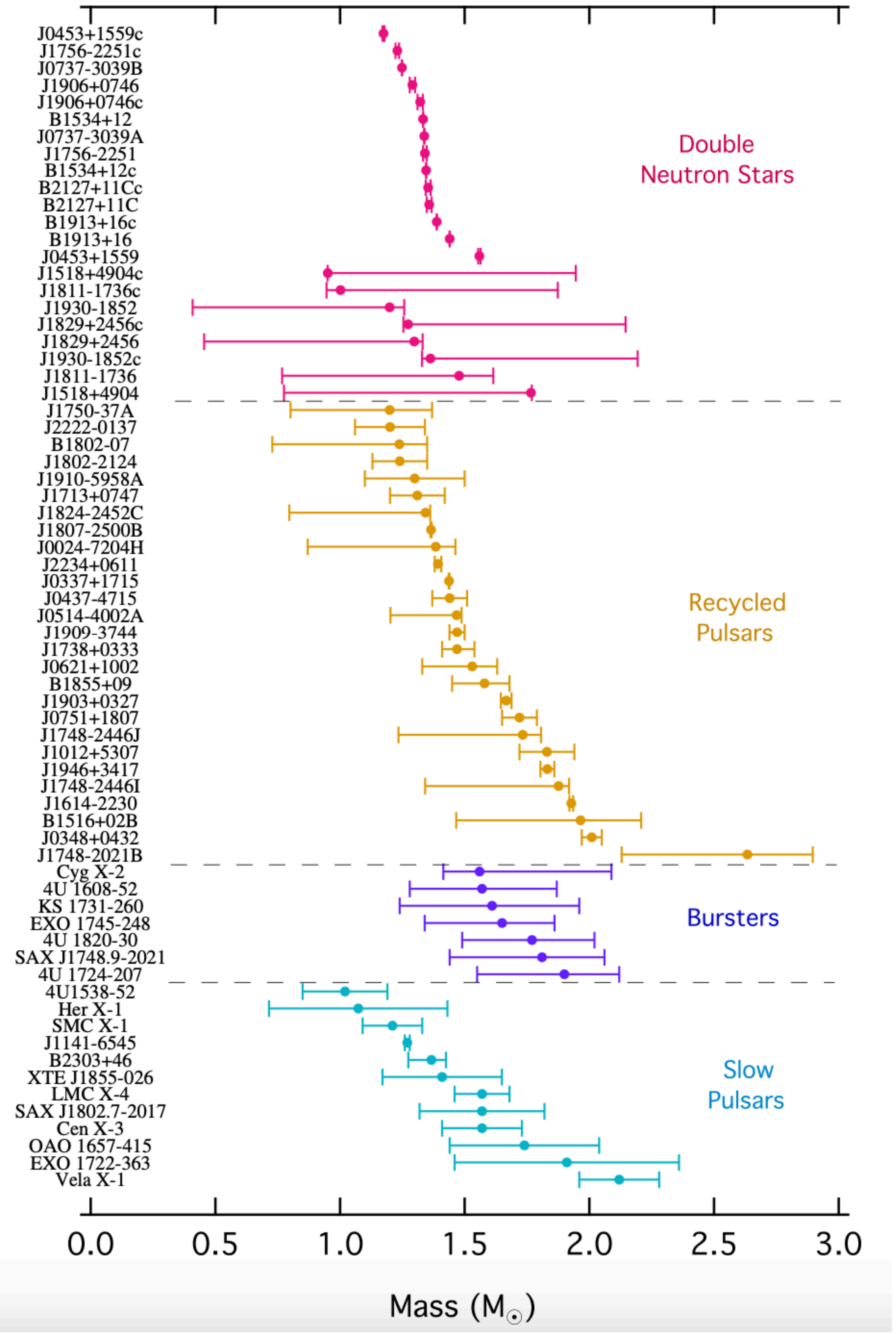

Figure 6. Mass-Radius Relation from Observational Data

Source : Ozel, Feryal and Paulo C. C. Freire. "Masses, Radii, and the Equation of State of Neutron Stars." Annual Review of Astronomy and Astrophysics 54 (2016): 401–440

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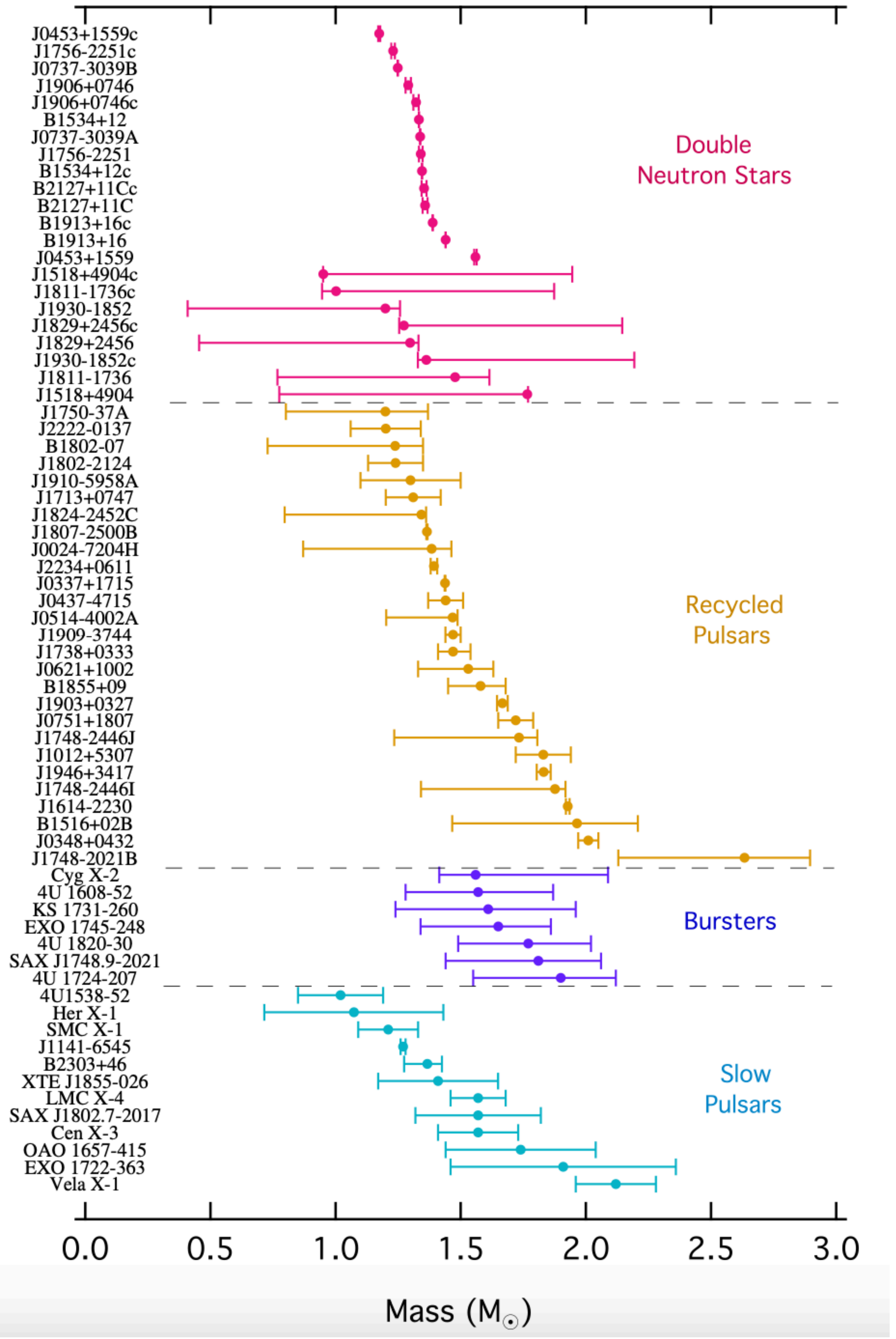
For example:

- The observation from GW190814 observations with mass 2.59 ± 0.08 $M_{\odot}$.
- From pulsar observations, PSR J0952–0607 with mass 2.35 ± 0.17 $M_{\odot}$.



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Modified Gravity

Metric-Affine Gravity

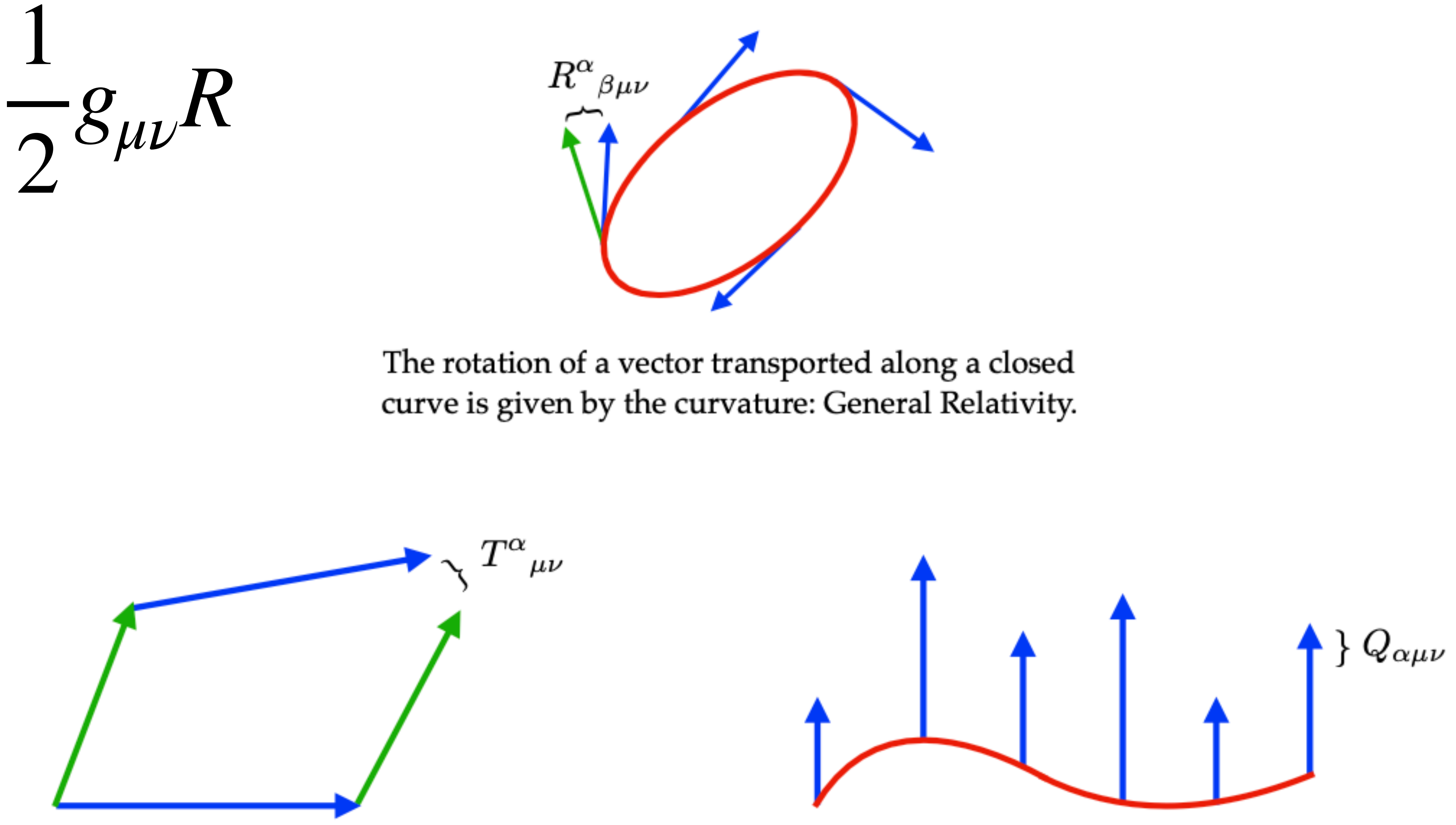
- In Rienmanian geometry $\Gamma^{\alpha}_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\}.$ → $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$
- But, can decompose the connection become more general as

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\} + K^{\alpha}_{\mu\nu} + L^{\alpha}_{\mu\nu}, \text{ where}$$

$$K^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} (T_{\lambda\mu\nu} + T_{\nu\mu\lambda} + T_{\mu\nu\lambda}),$$

$$L^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} (Q_{\lambda\mu\nu} - Q_{\mu\lambda\nu} - Q_{\nu\lambda\mu}),$$

where $K^{\alpha}_{\mu\nu}$ is contorsion tensor and $L^{\alpha}_{\mu\nu}$ is disformation tensor.



The rotation of a vector transported along a closed curve is given by the curvature: General Relativity.

The non-closure of parallelograms formed when two vectors are transported along each other is given by the torsion: Teleparallel Equivalent of General Relativity.

The variation of the length of a vector as it is transported is given by the non-metricity: Symmetric Teleparallel Equivalent of General Relativity.

Figure 7. Illustration geometrical meaning for Curvature, Torsion, and Non-Metricity

Source : Beltrán Jiménez, Jose, Lavinia Heisenberg and Tomi S. Koivisto. "The Geometrical Trinity of Gravity." Universe (2019)

Modified Gravity

The Geometrical Trinity of Gravity

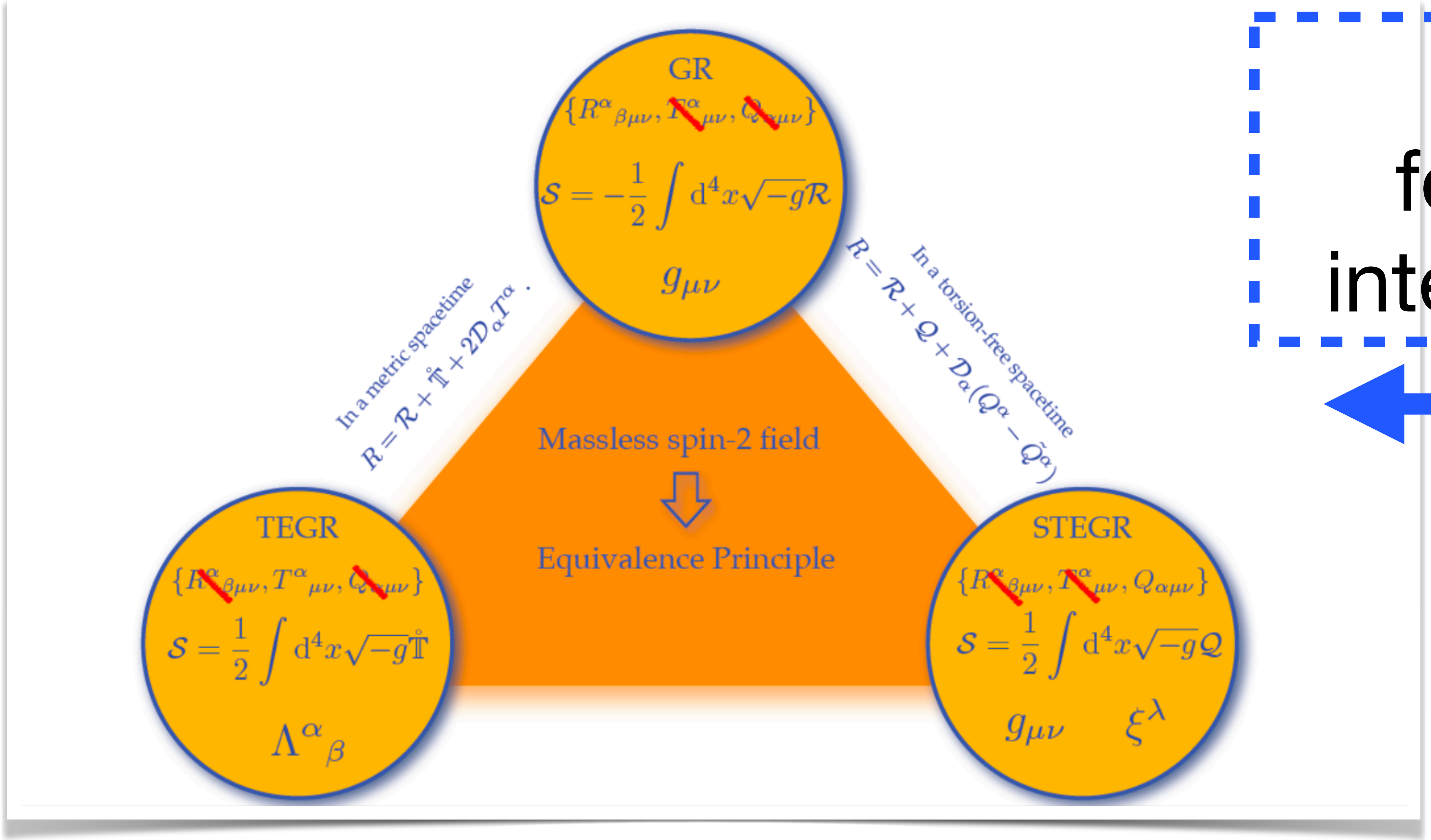


Figure 8. The Geometrical Trinity of Gravity

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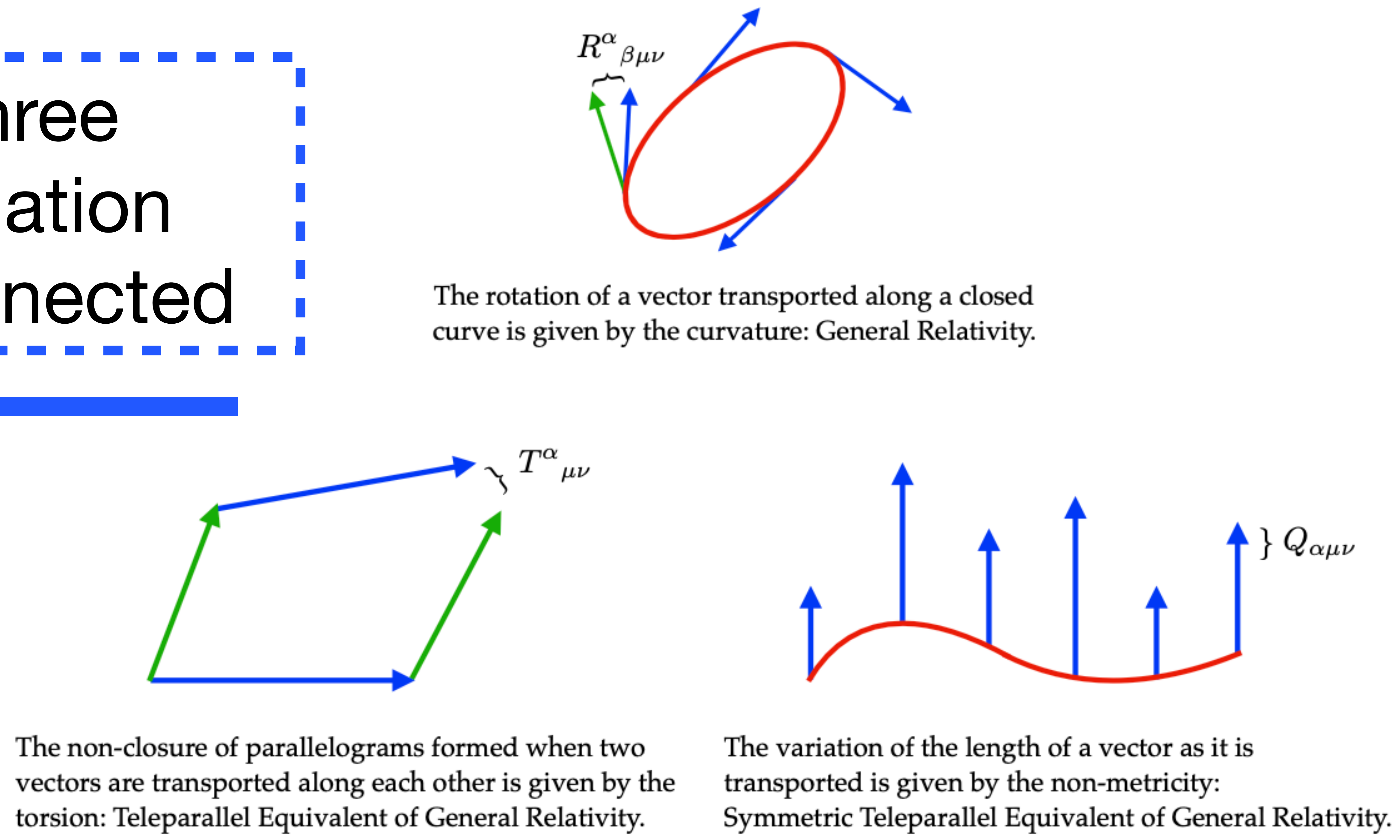
All three formulation interconnected



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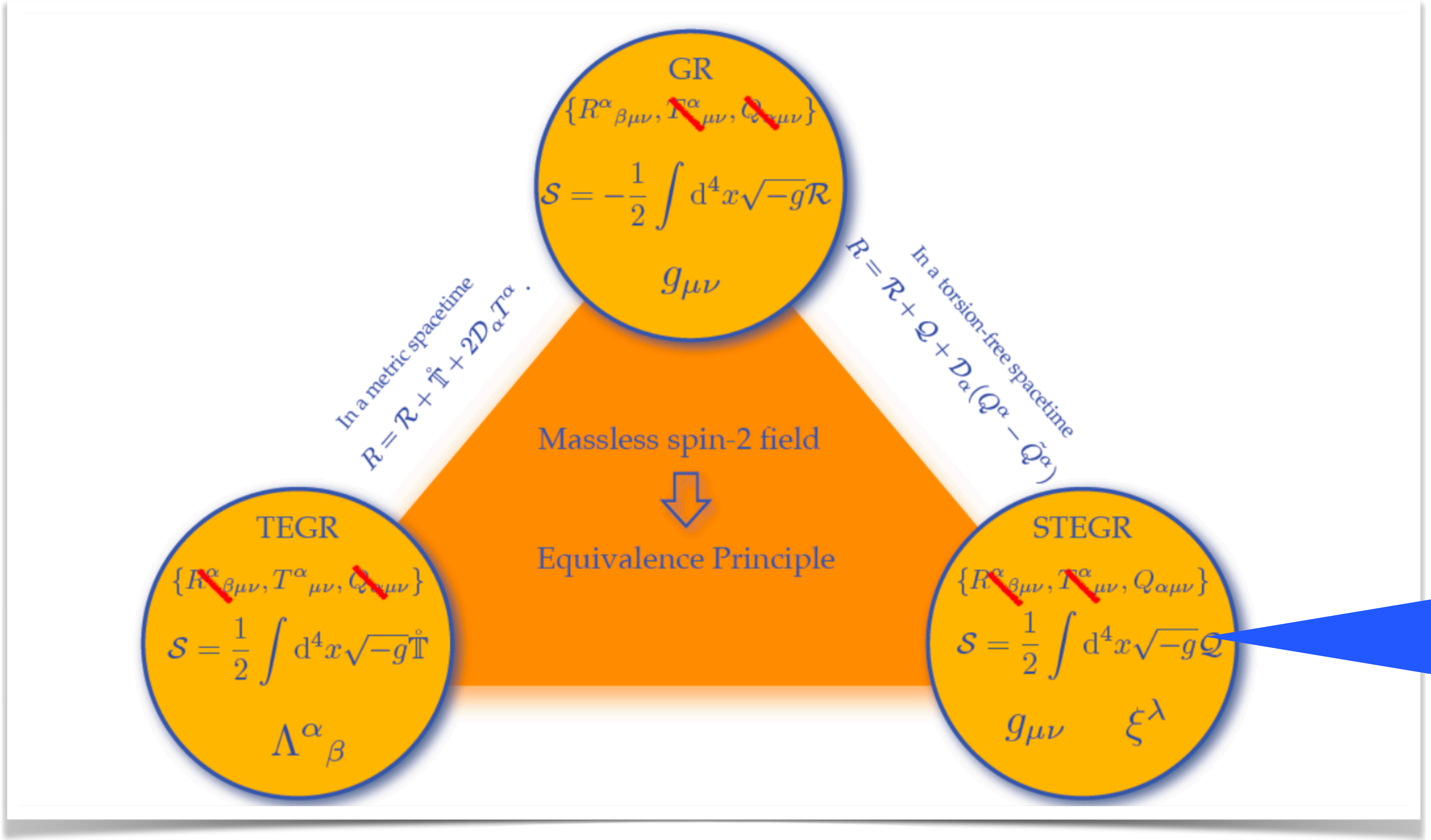

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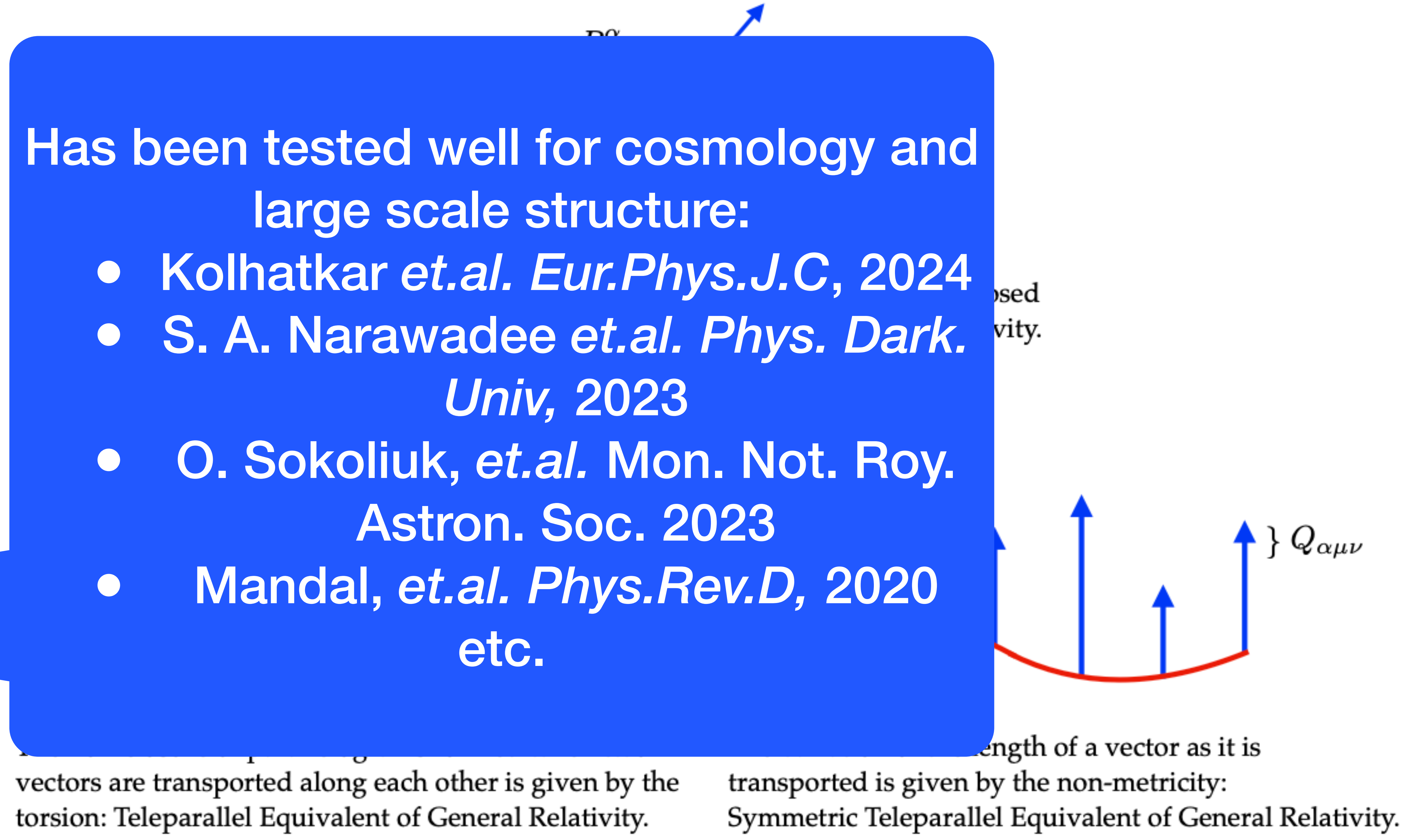

Has been tested well for cosmology and large scale structure:

- Kolhatkar *et.al. Eur.Phys.J.C*, 2024
- S. A. Narawadee *et.al. Phys. Dark. Univ*, 2023
- O. Sokoliuk, *et.al.* Mon. Not. Roy. Astron. Soc. 2023
- Mandal, *et.al. Phys.Rev.D*, 2020

etc.

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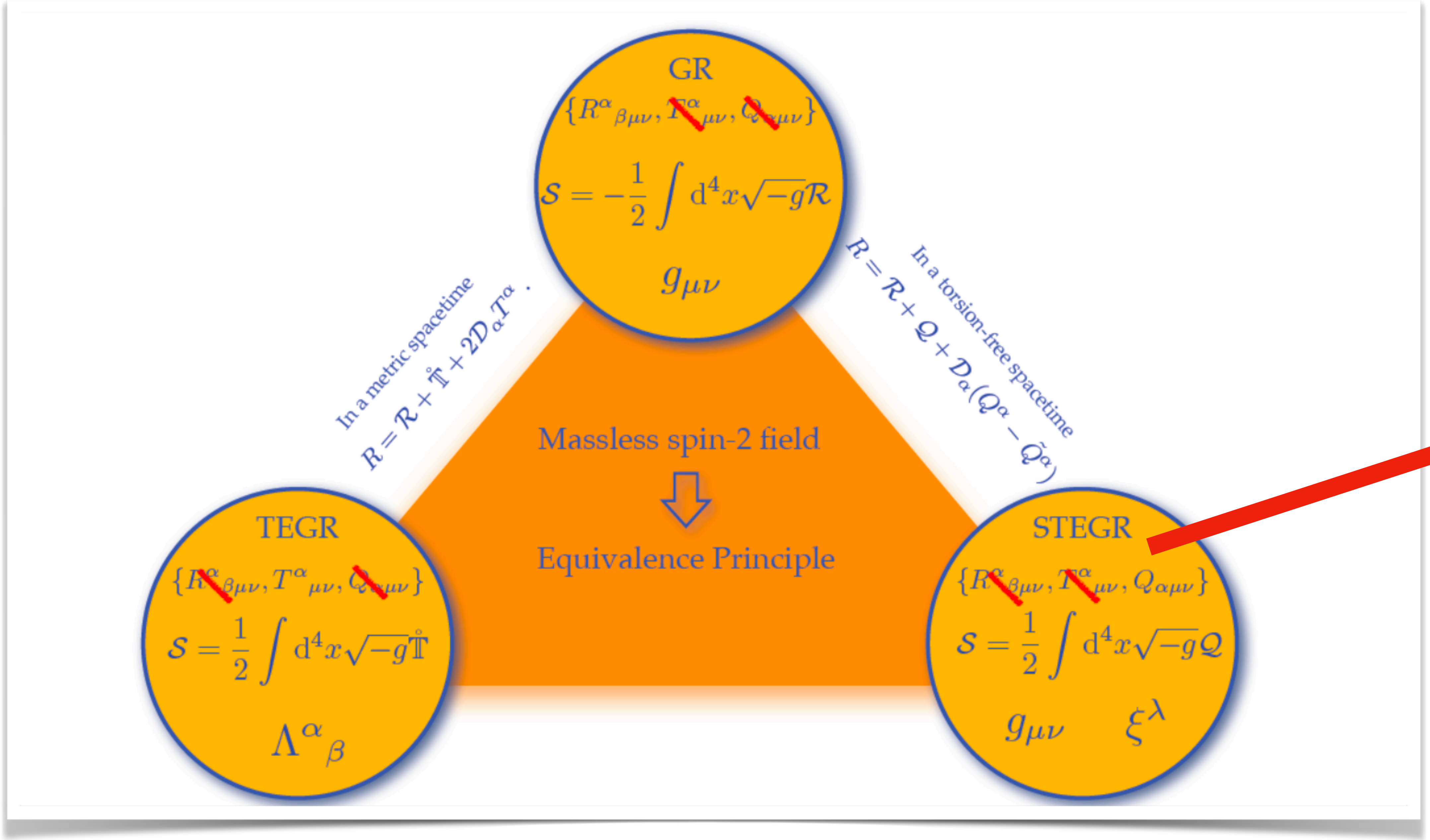
Motivations



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- Can $f(Q)$ remain consistent when tested on astrophysical objects like NS?
- Are neutron stars sufficient to reveal the signature of $f(Q)$ gravity?
- How does non-metricity affect the structure of NS?
- Can $f(Q)$ provide an alternative explanation for neutron stars reaching higher masses?

Covariant $f(Q)$ formulation

- Here we use the lagrangian action as:

$$S = \int \frac{1}{2\kappa} f(Q)\sqrt{-g}\ d^4x + \int \mathscr{L}_m \sqrt{-g}\ d^4x,$$

- Where non-metricity scalar is defined as $Q = Q_{\lambda\mu\nu}P^{\lambda\mu\nu}$. $Q_{\lambda\mu\nu}$ and $P^{\lambda}{}_{\mu\nu}$ are called as non-metricity tensor and conjugate and given as

$$Q_{\lambda\mu\nu} := \nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^{\alpha}{}_{\lambda\mu} g_{\alpha\nu} - \Gamma^{\alpha}{}_{\lambda\nu} g_{\alpha\mu}\ ;$$

$$P^{\lambda}{}_{\mu\nu} = -\frac{1}{4} Q^{\lambda}{}_{\mu\nu} + \frac{1}{4}\left(Q_{\mu}{}^{\lambda}{}_{\nu} + Q_{\nu}{}^{\lambda}{}_{\mu}\right) + \frac{1}{4} Q^{\lambda} g_{\mu\nu} - \frac{1}{8}\left(2\tilde{Q}^{\lambda} g_{\mu\nu} + \delta^{\lambda}_{\mu} Q_{\nu} + \delta^{\lambda}_{\nu} Q_{\mu}\right).$$

- Using least action principle, we can get the filed equation as (D. Zhao, *Eur. Phys. J. C.,* 2022).

$$f_Q \mathring{G}_{\mu\nu} + \frac{1}{2} g_{\mu\nu}(Qf_Q - f) + 2f_{QQ} P^{\lambda}{}_{\mu\nu} \mathring{\nabla}_\lambda Q = \kappa T_{\mu\nu}$$

where, f_Q is derivative of f with respect to Q and $\mathring{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, with $R_{\mu\nu}$ and R are the Riemannian Ricci tensor and scalar respectively which are constructed by the Levi-Civita connection.

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$$f_Q \mathring{G}_{\mu\nu} + \frac{1}{2} g_{\mu\nu}(Q f_Q - f) + 2 f_{QQ} P^{\lambda}{}_{\mu\nu} \mathring{\nabla}_\lambda Q = \kappa T_{\mu\nu} \xrightarrow{f(Q)=Q} \mathring{G}_{\mu\nu} = \kappa T_{\mu\nu}$$

where, f_Q is derivative of f with respect to Q and $\mathring{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, with $R_{\mu\nu}$ and R are the Riemannian Ricci tensor and scalar respectively which are constructed by the Levi-Civita connection.

Covariant $f(Q)$ formulation

Field Equations under Spherical Symmetric Metric

- Assuming perfect fluid matter and using static and spherically symmetric metric as

$$ds^2 = -e^{A(r)}dt^2 + e^{B(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

Geometry Profile

- The field equations become

$$\kappa\mathcal{T}_{tt} = \frac{e^{A-B}}{2r^2}\left\{r^2e^Bf + 2f'_Qr(e^B-1) + f_Q\left[(e^B-1)(2+rA') + (1+e^B)rB'\right]\right\},$$

$$\kappa\mathcal{T}_{rr} = \frac{-1}{2r^2}\left\{r^2e^Bf + 2f'_Qr(e^B-1) + f_Q\left[(e^B-1)(2+rA'+rB') - 2rA'\right]\right\},$$

$$\kappa\mathcal{T}_{\theta\theta} = -\frac{r}{4e^B}\left\{f_Q\left[-4A' - r(A')^2 - 2rA'' + rA'B' + 2e^B(A'+B')\right] + 2e^Brf - 2f'_QrA'\right\}.$$

where $Q = \frac{(e^{-B}-1)(A'+B')}{r}$ and $f'_Q = f_{QQ}\frac{dQ}{dr}$.

TOV Equations in Covariant $f(Q)$ Gravity

- From the field equations, we can extract the TOV equations as

$$A'' = \frac{-((1+e^B)f_Q r(A')^2) + 2e^B(-1+e^B)(r(f+2p\kappa)+f_Q B') + A'(2(-1+e^B)^2 f_Q}{2(-1+e^B)f_Q r}$$

$$+\frac{e^B r^2(f+\kappa(p-\rho+e^B(p+\rho))) + (-1+e^B)f_Q r B')}{2(-1+e^B)f_Q r}$$

$$B' = \frac{-\kappa e^B(p+\rho)r + f_Q A'}{f_Q}$$

$$p' = -\frac{(p+\rho)}{2}A'$$

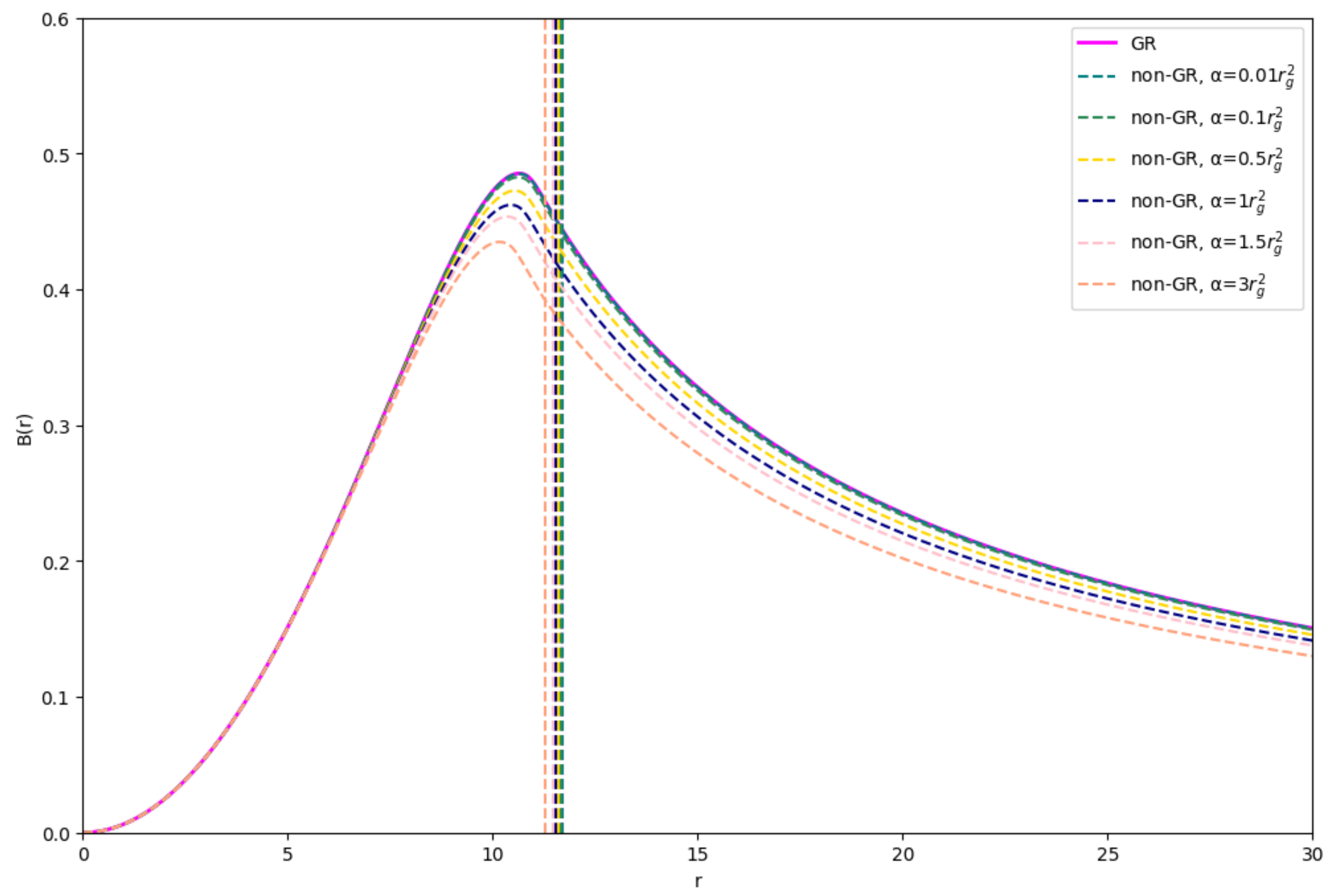
→

Solved using realistic EoS SLy and APR4
—> Previous works using Polytropic EoS
R.-H. Lin, *et. al., Phys. Rev. D, 2022*

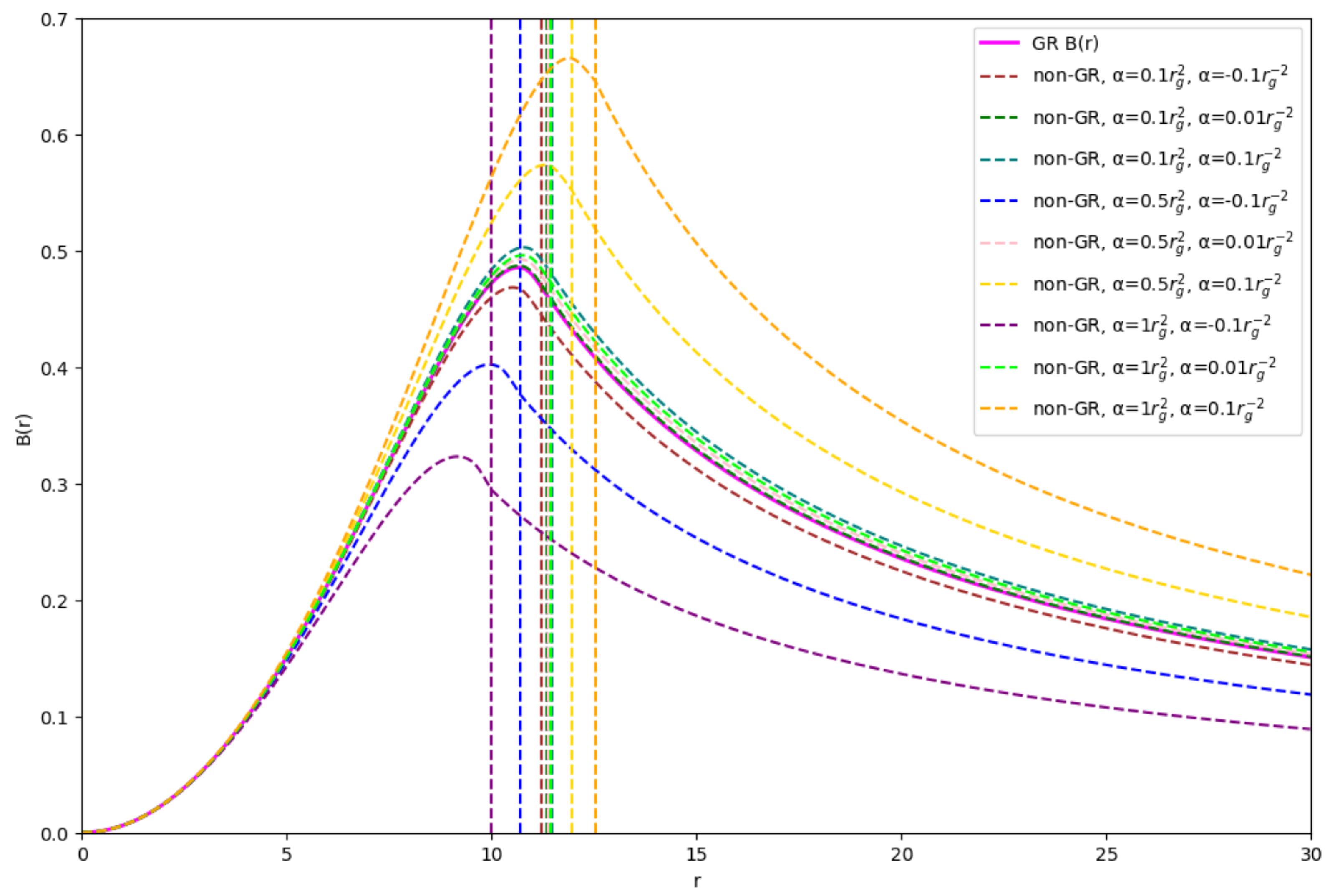
TOV Equations in Covariant $f(Q)$ Gravity

- Here we will consider three models of $f(Q)$:
  - $f(Q) = Q + \alpha Q^2$ (Lin, *et. al., Phys. Rev. D, 2022)*
  - $f(Q) = Q + \alpha e^{\beta Q}$ (Anagnostopoulos, *et al., Phys. Lett. B*, 2021)
  - $f(Q) = Q - \alpha \ln(1 - \beta Q)$ (N´ajera, *et.al. Mon. Not. Roy. Astron. Soc., 2023*)
- α will act as a coarse tuning parameter, determining the effects of the additional term on the total function.
- β will act as a fine tuning parameter, controls the growth of exponential and logarithmic term.

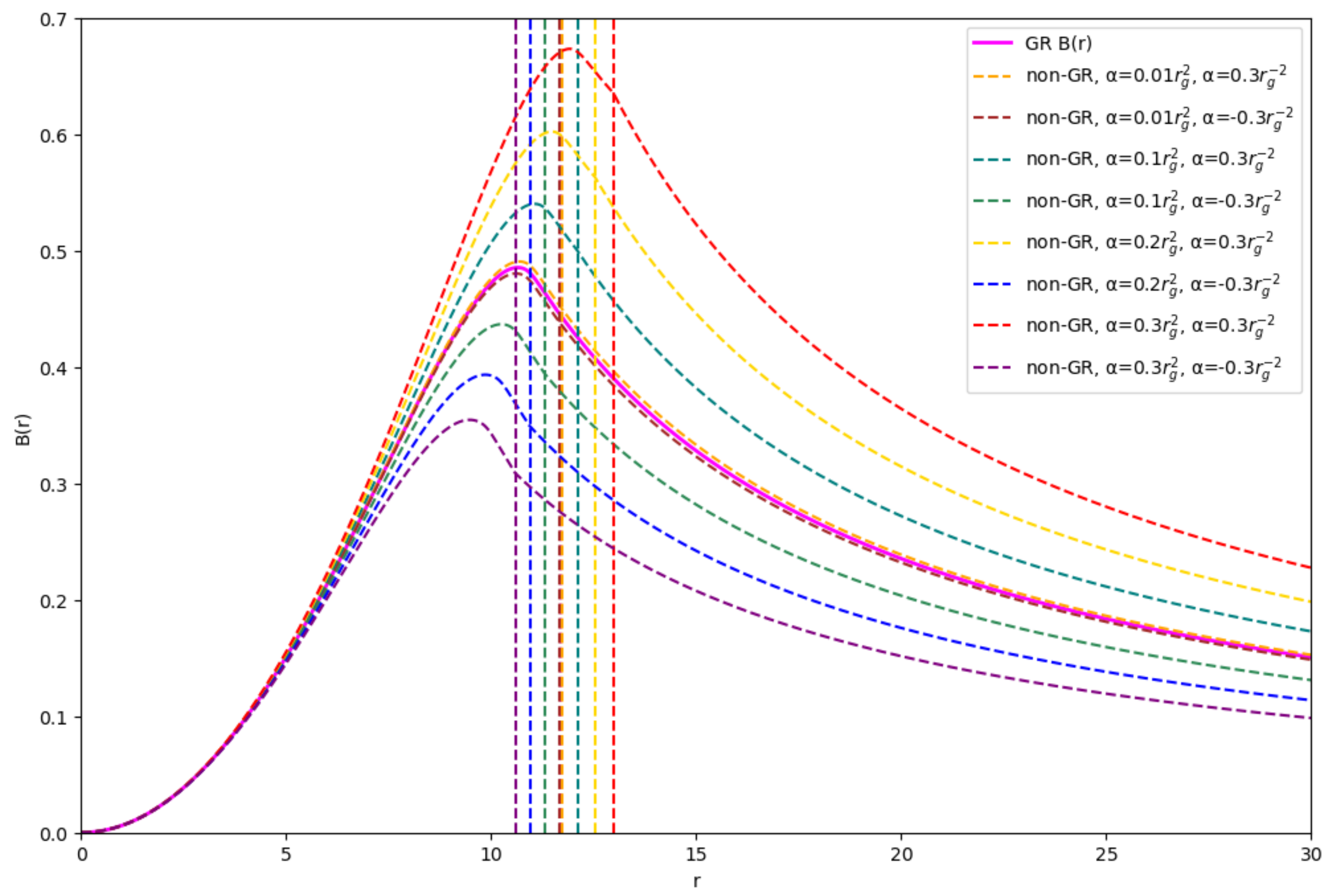
$B(r)$ and $Q(r)$ profile



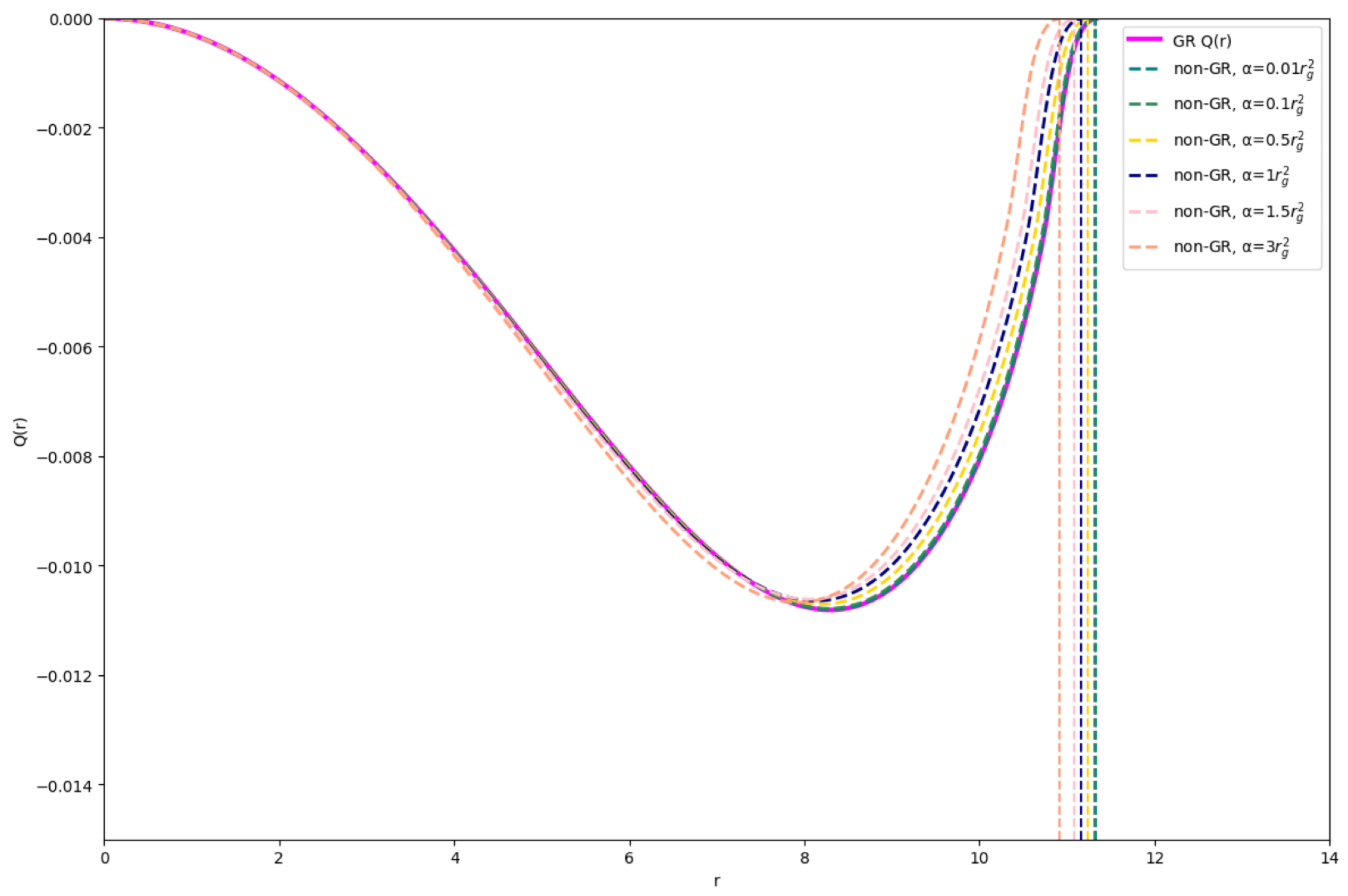
B(r) profile using $f(Q) = Q + \alpha Q^2$



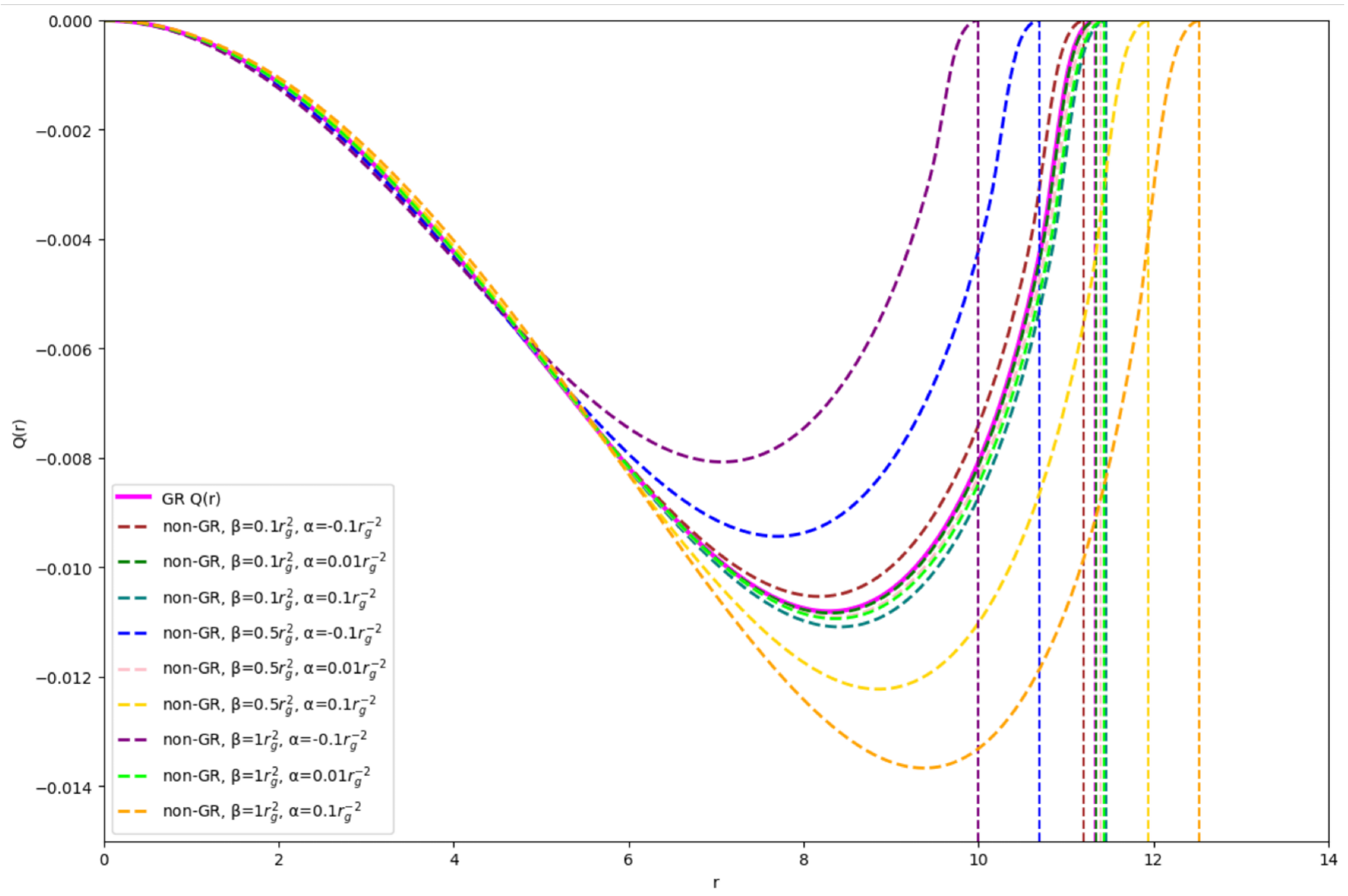
B(r) profile using $f(Q) = Q + \alpha e^{\beta Q}$



B(r) profile using $f(Q) = Q - \alpha \ln(1 - \beta Q)$



Non-metricity profile using $f(Q) = Q + \alpha Q^2$



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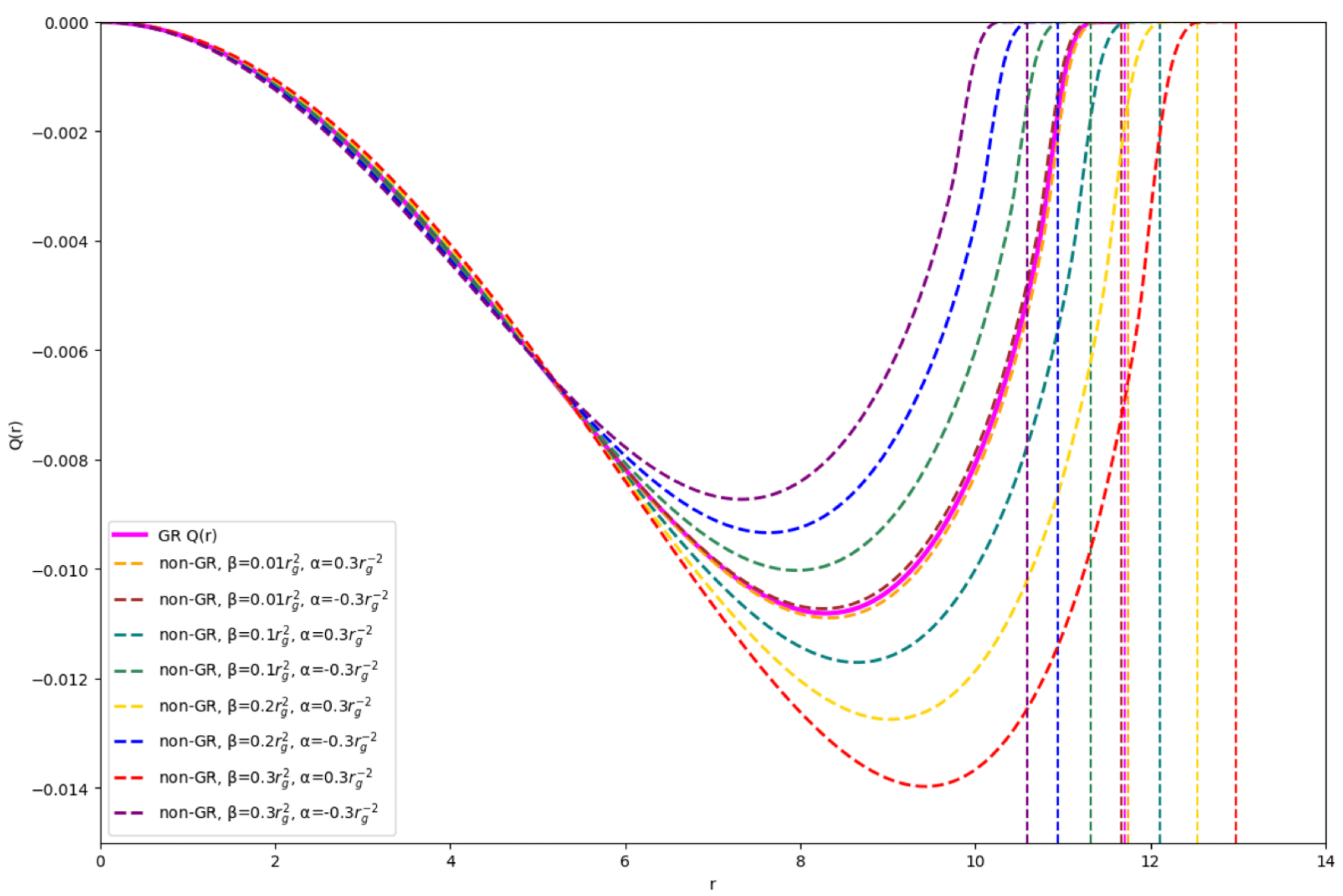

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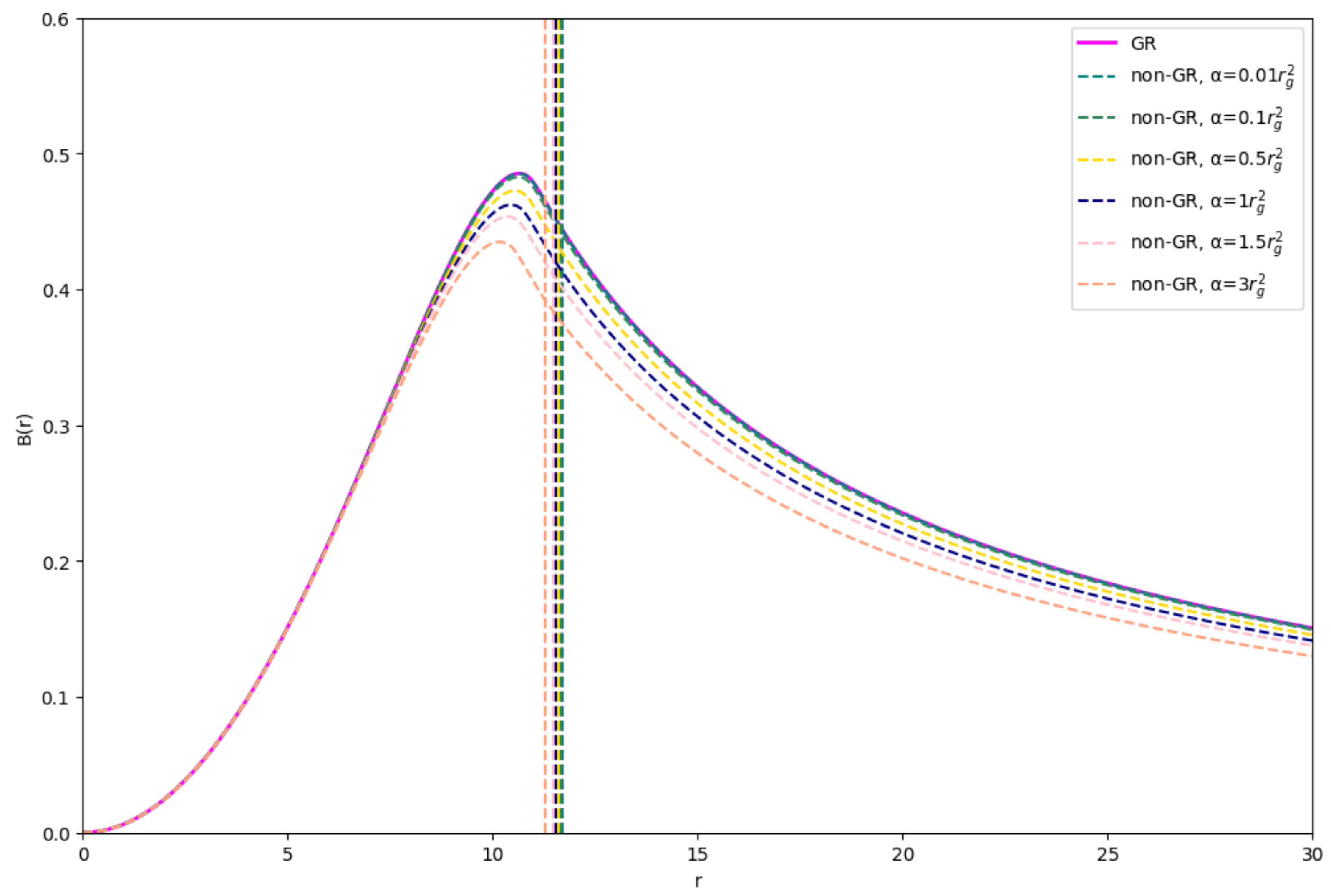

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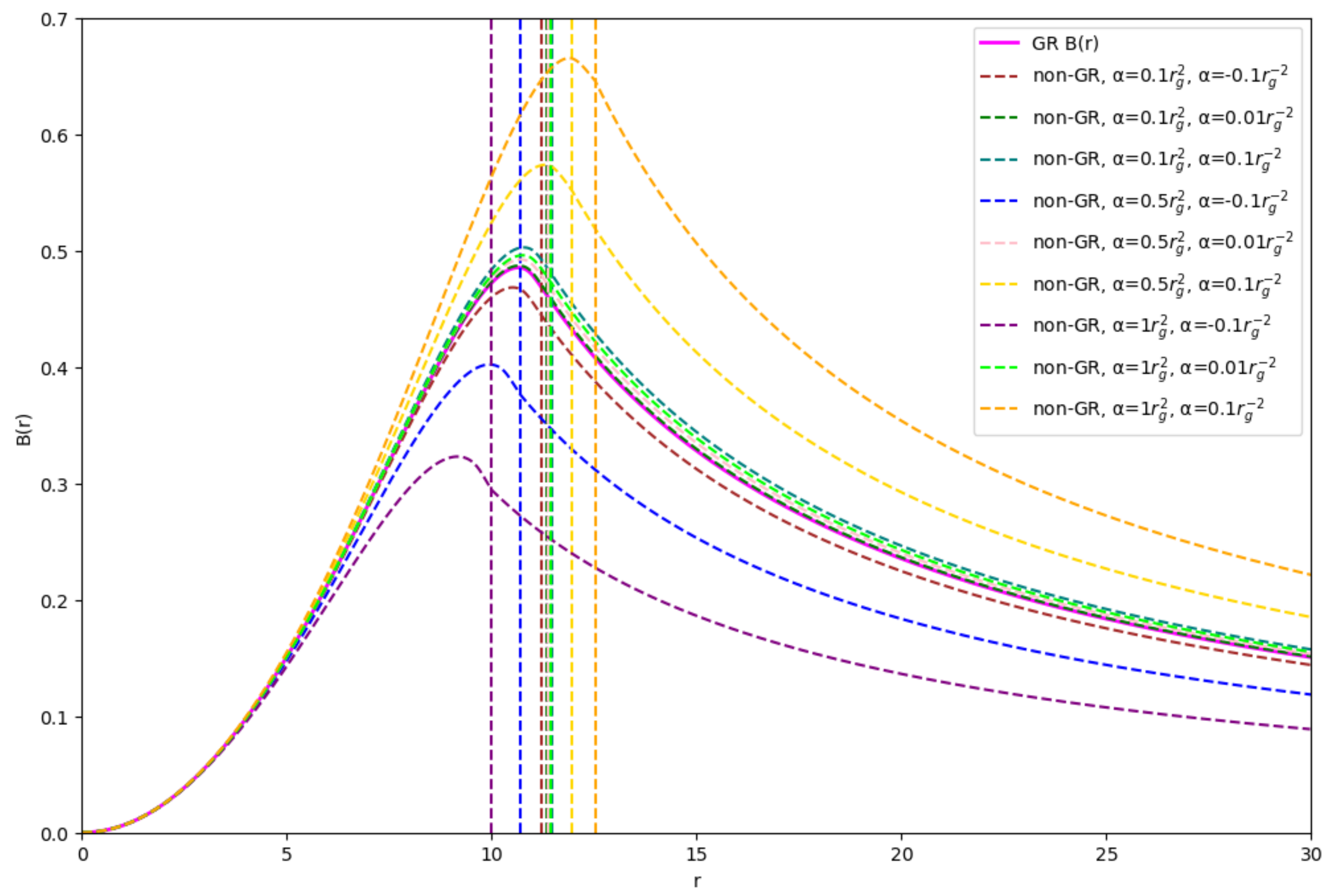
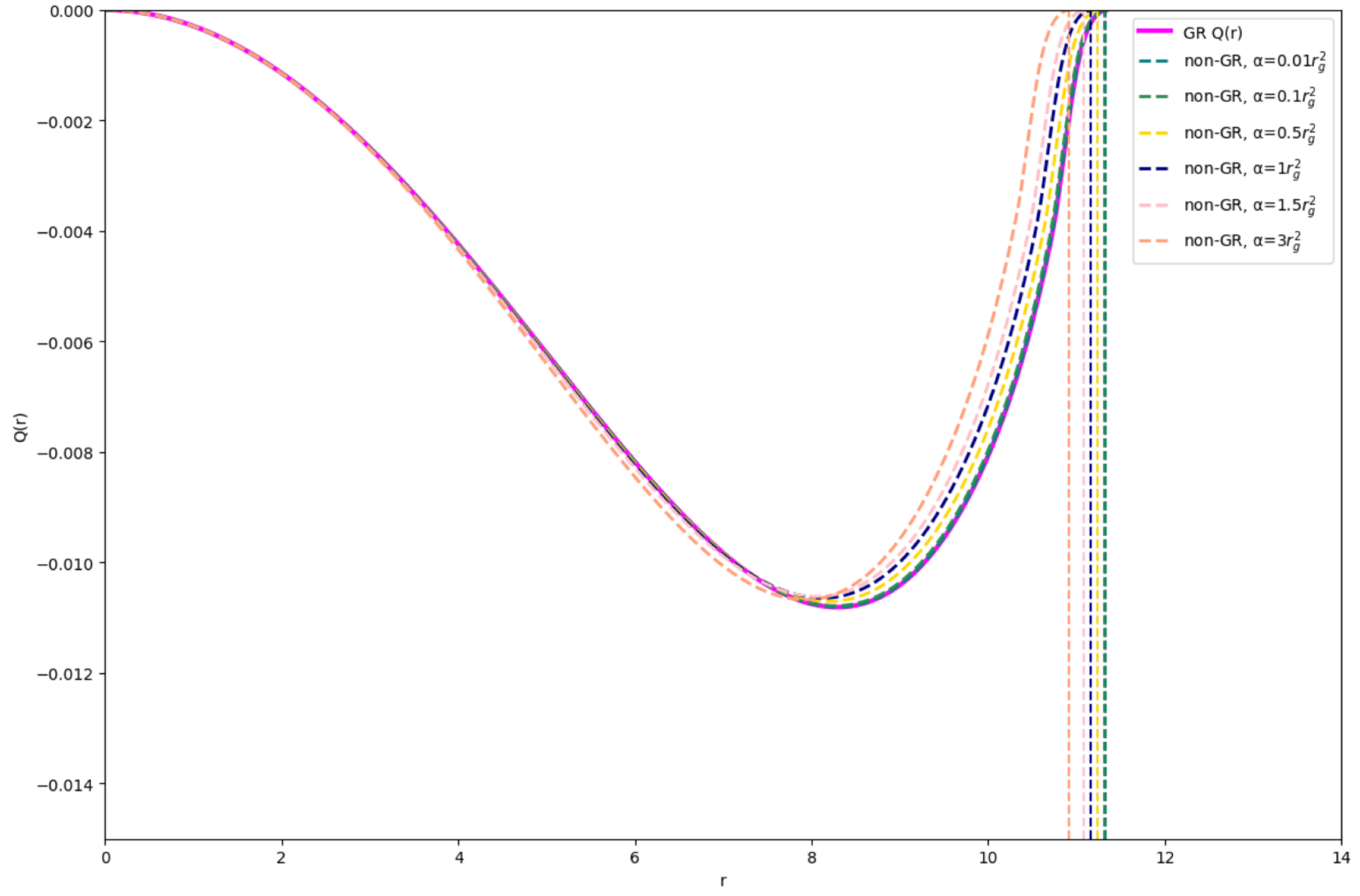
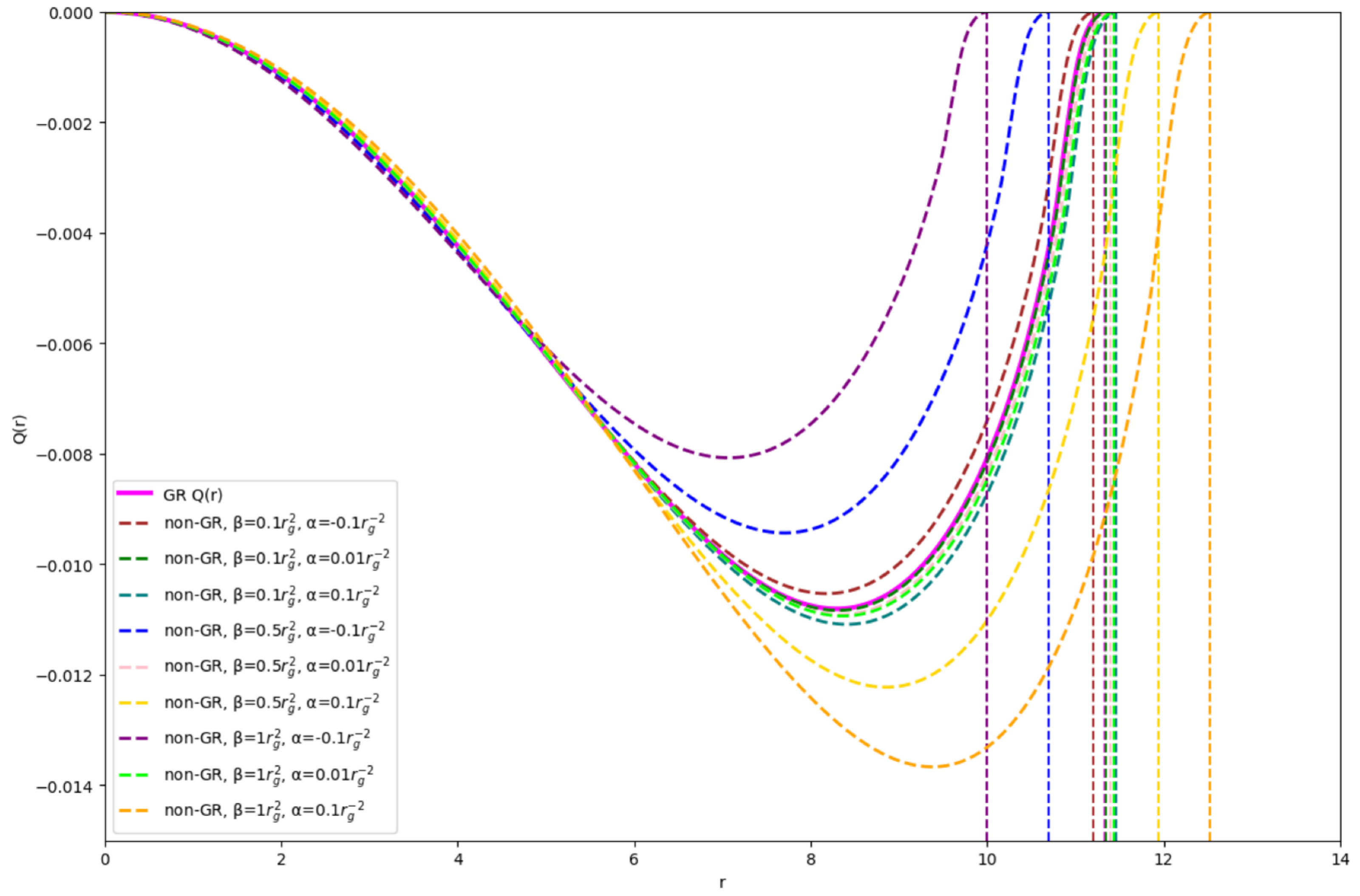


B(r) profile using $f(Q) = Q + \alpha e^{\beta Q}$

- Deviations in $Q(r)$ affect the $B(r)$ profile.
- In the Schwarzschild solution within the star's interior, $e^{B(r)} = 1 - \frac{2m(r)}{r}$. It means, deviations in $B(r)$ clearly affect the mass function $m(r)$.
- When $B(r)$ deviates due to $Q(r)$, the star can either accommodate more matter or less matter, impacting the total mass. Will be clearer after we see the mass-radius diagram.



Non-metricity profile using $f(Q) = Q + \alpha Q^2$



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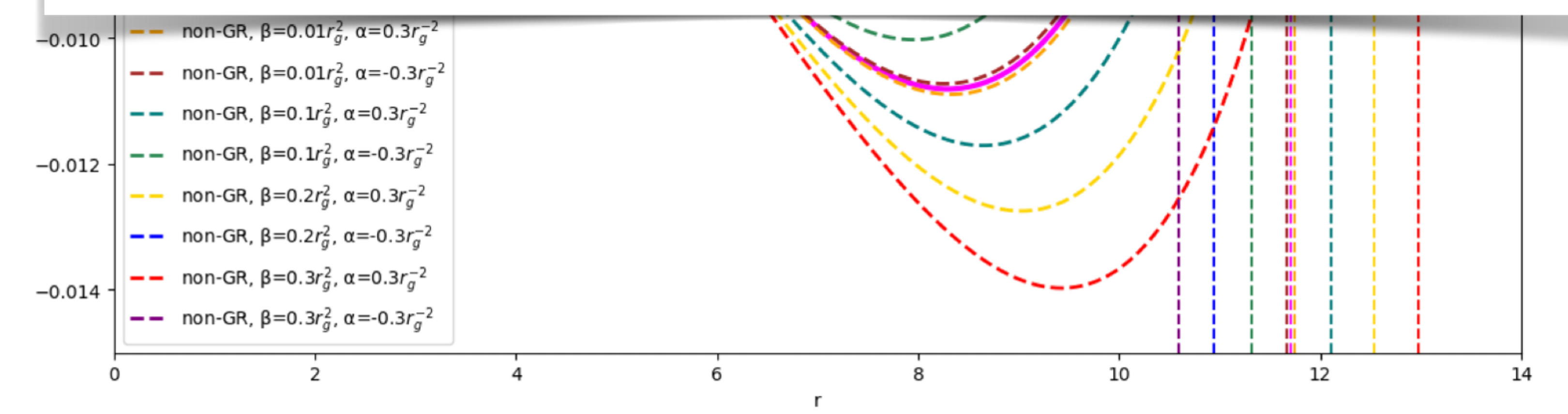
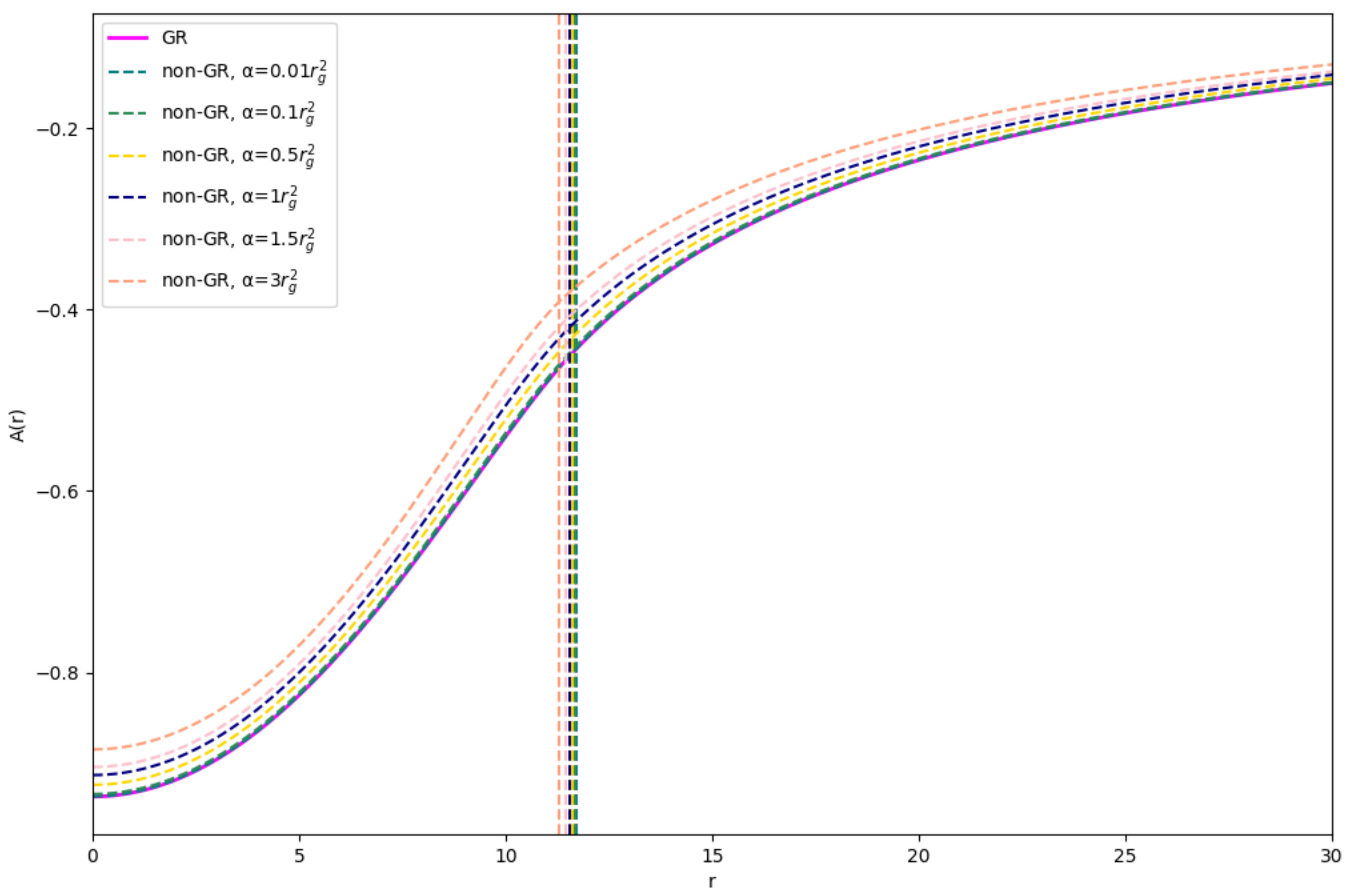
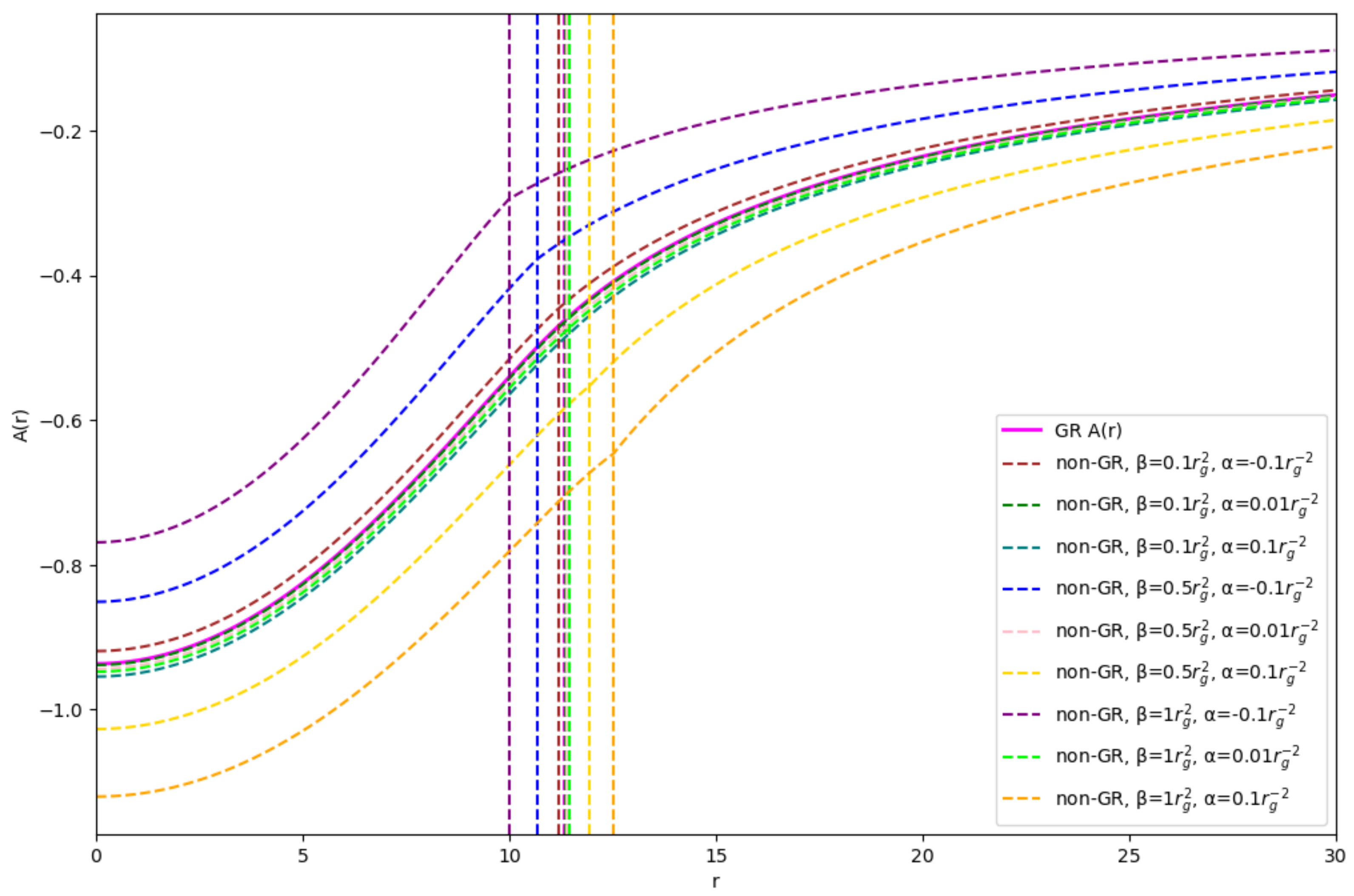


Non-metricity profile using $f(Q) = Q - \alpha \ln(1 - \beta Q)$

$A(r)$ profile and pressure distribution



A(r) profile using $f(Q) = Q + \alpha Q^2$



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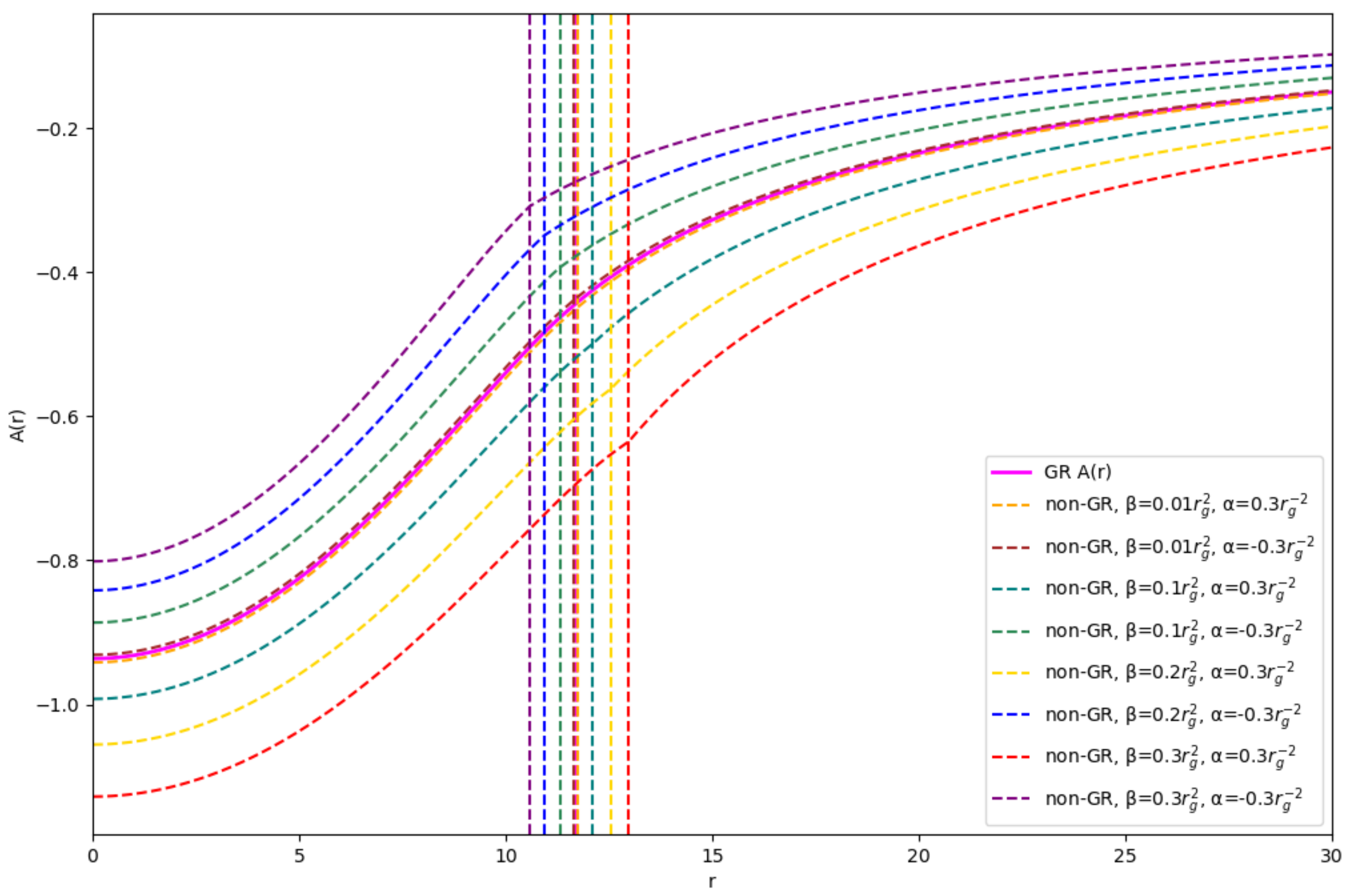
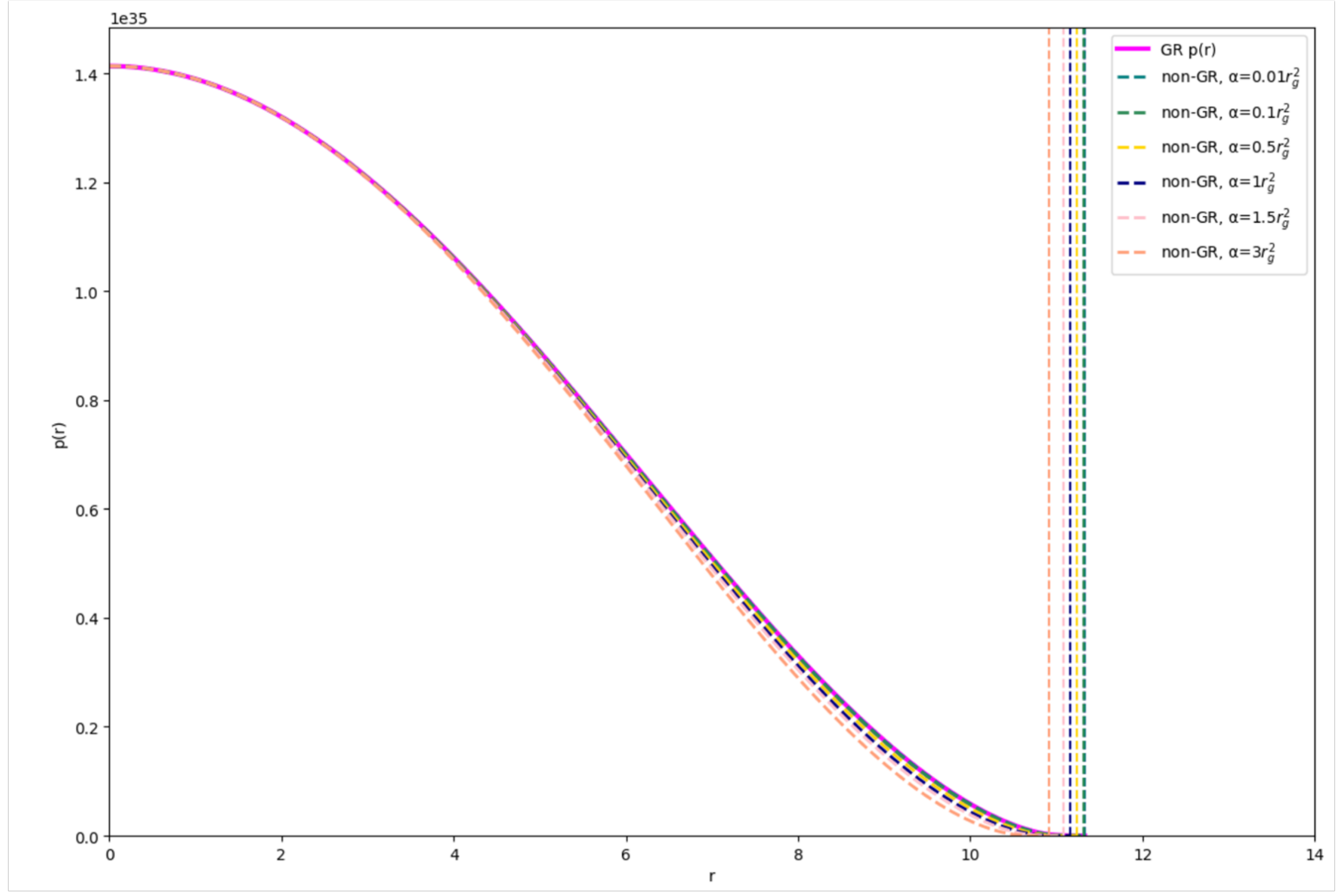
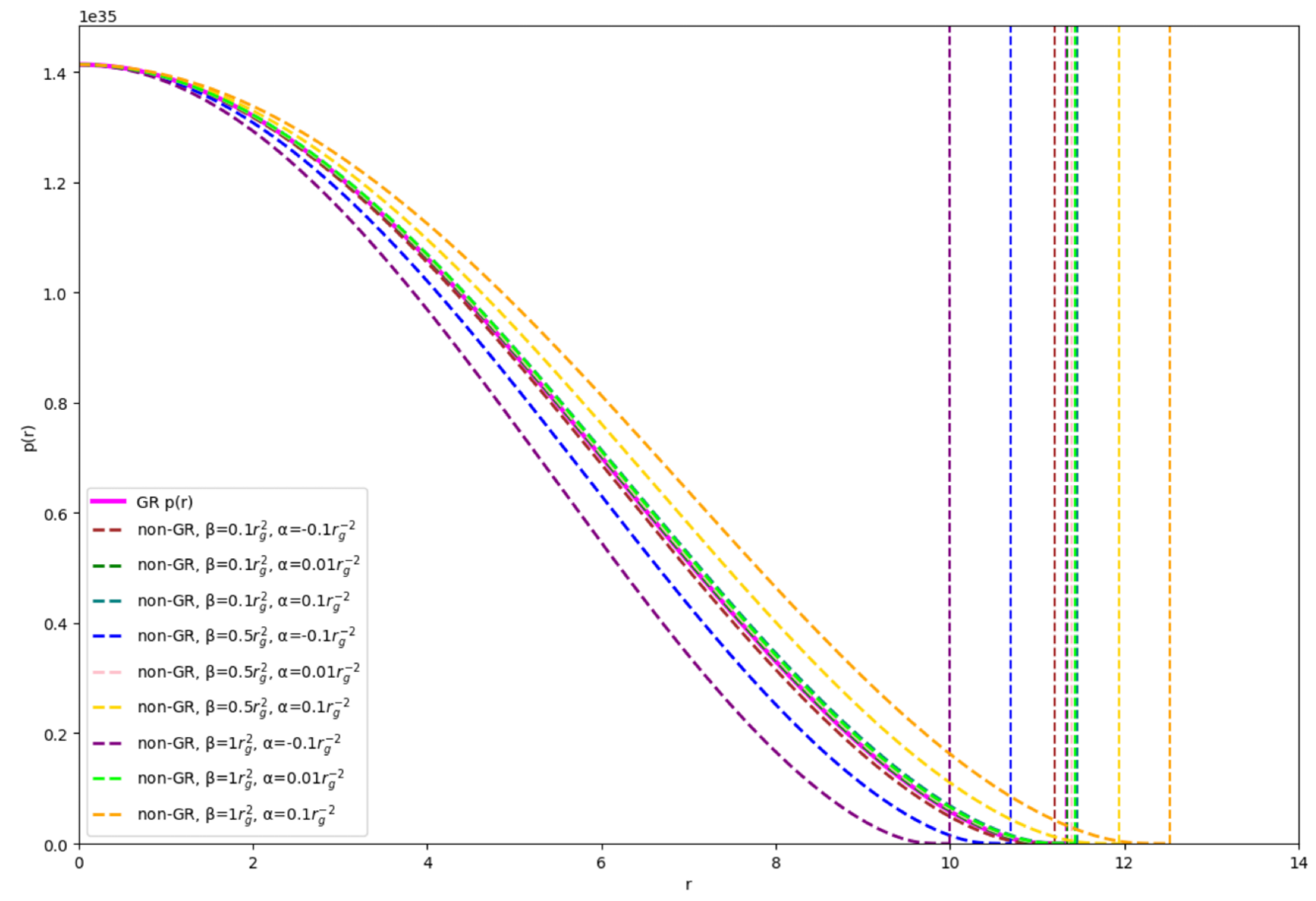


B(r) profile using $f(Q) = Q - \alpha \ln(1 - \beta Q)$



Pressure distribution using $f(Q) = Q + \alpha Q^2$



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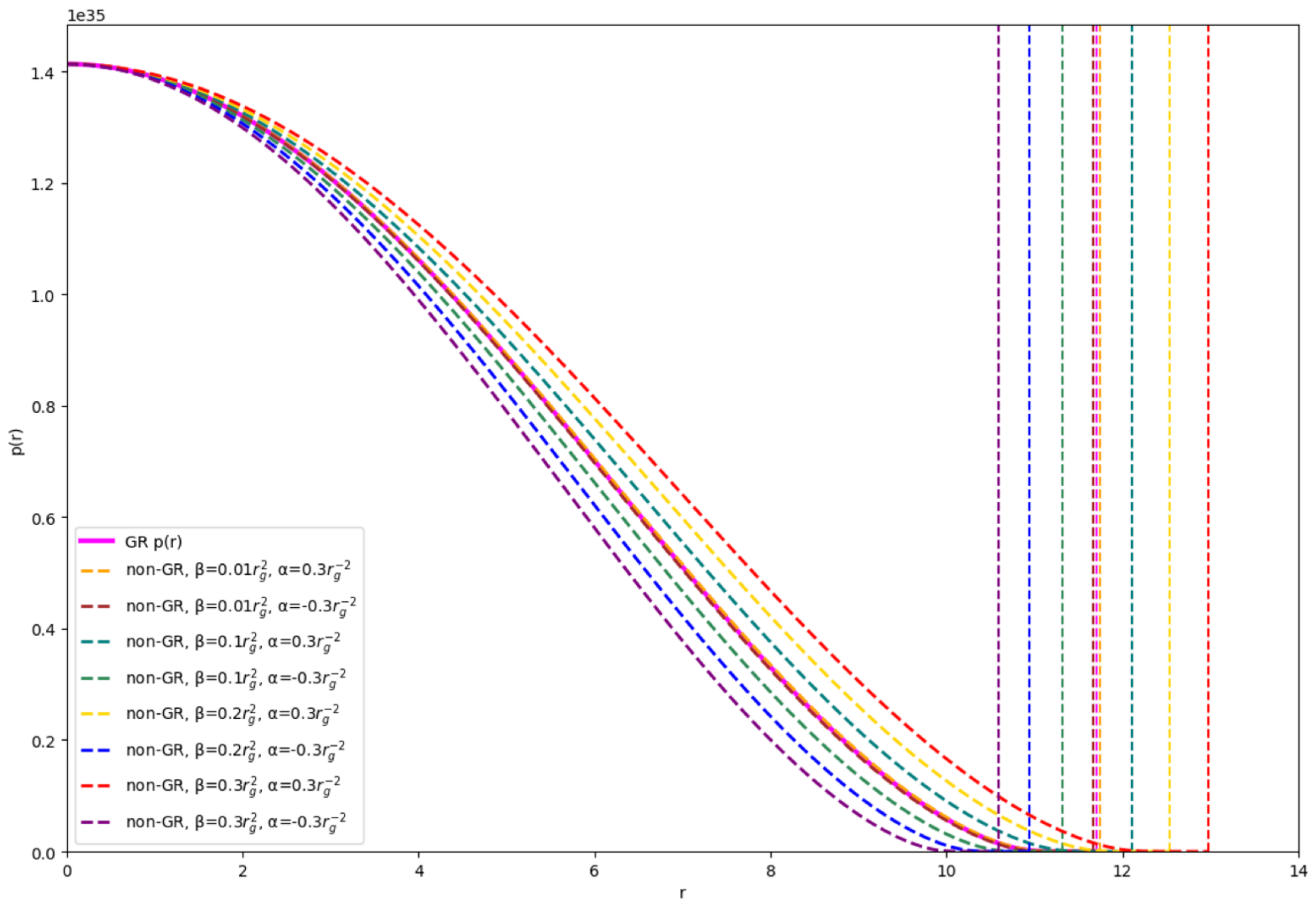
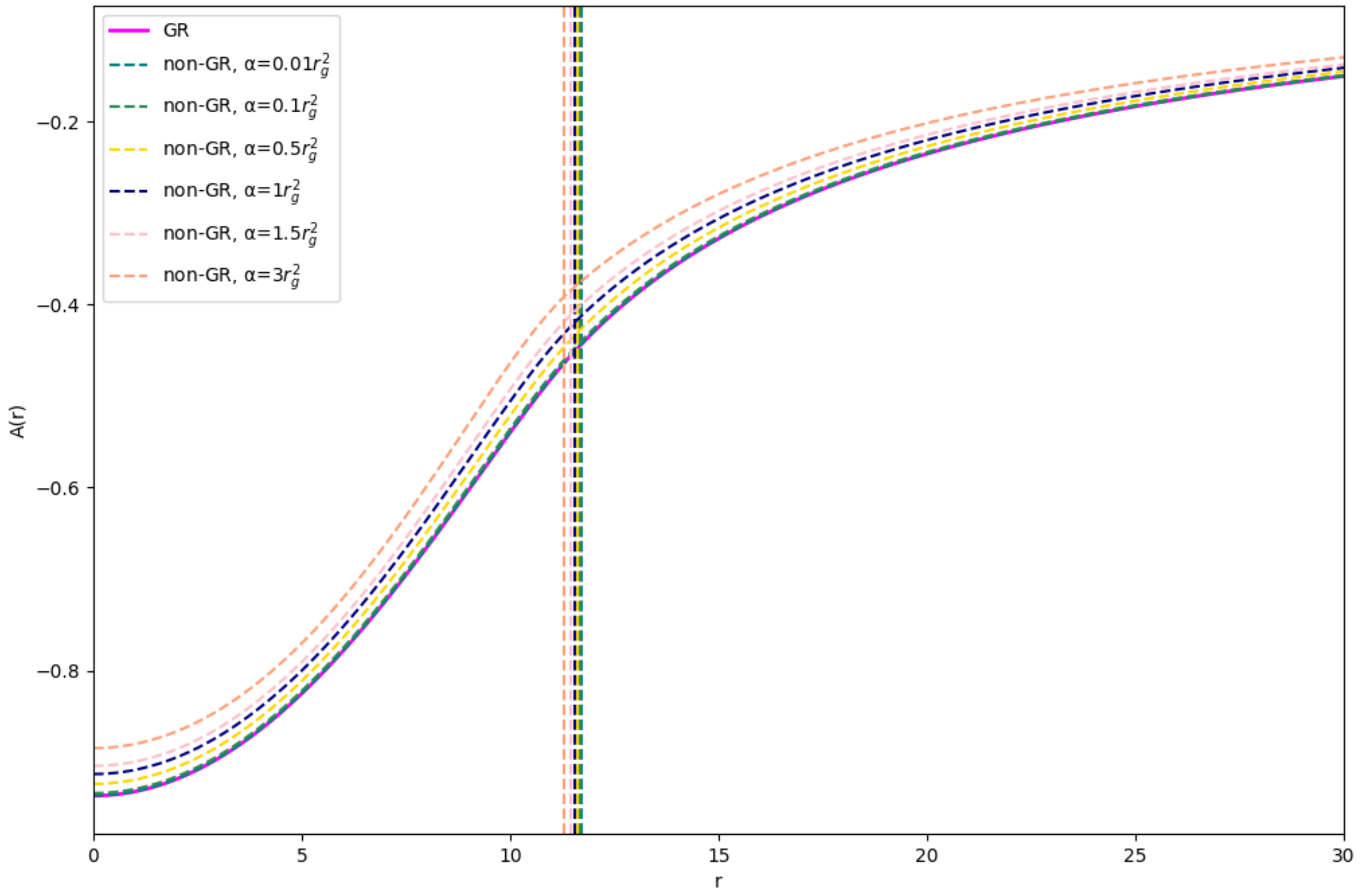
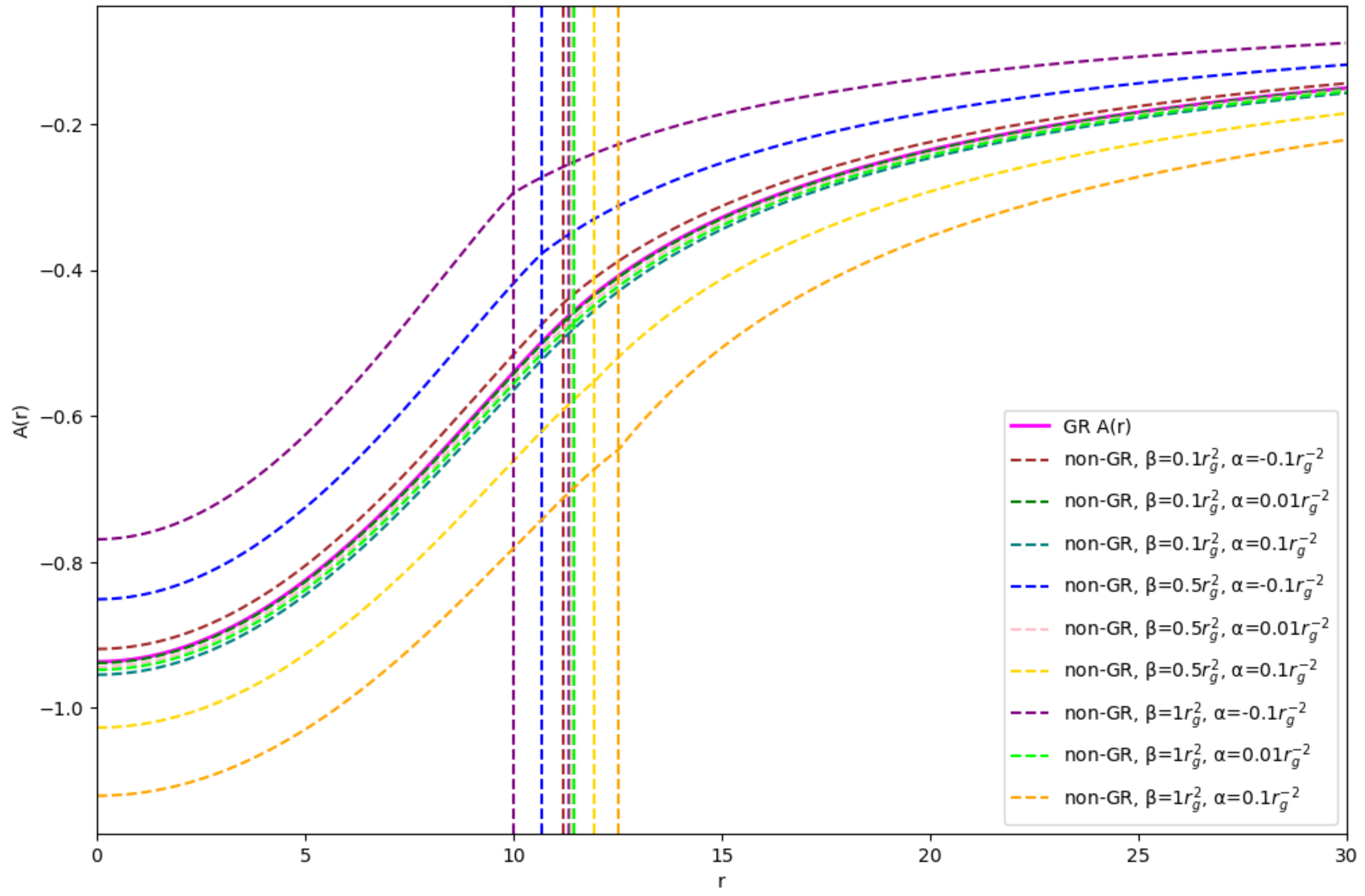


Pressure distribution using $f(Q) = Q - \alpha \ln(1 - \beta Q)$

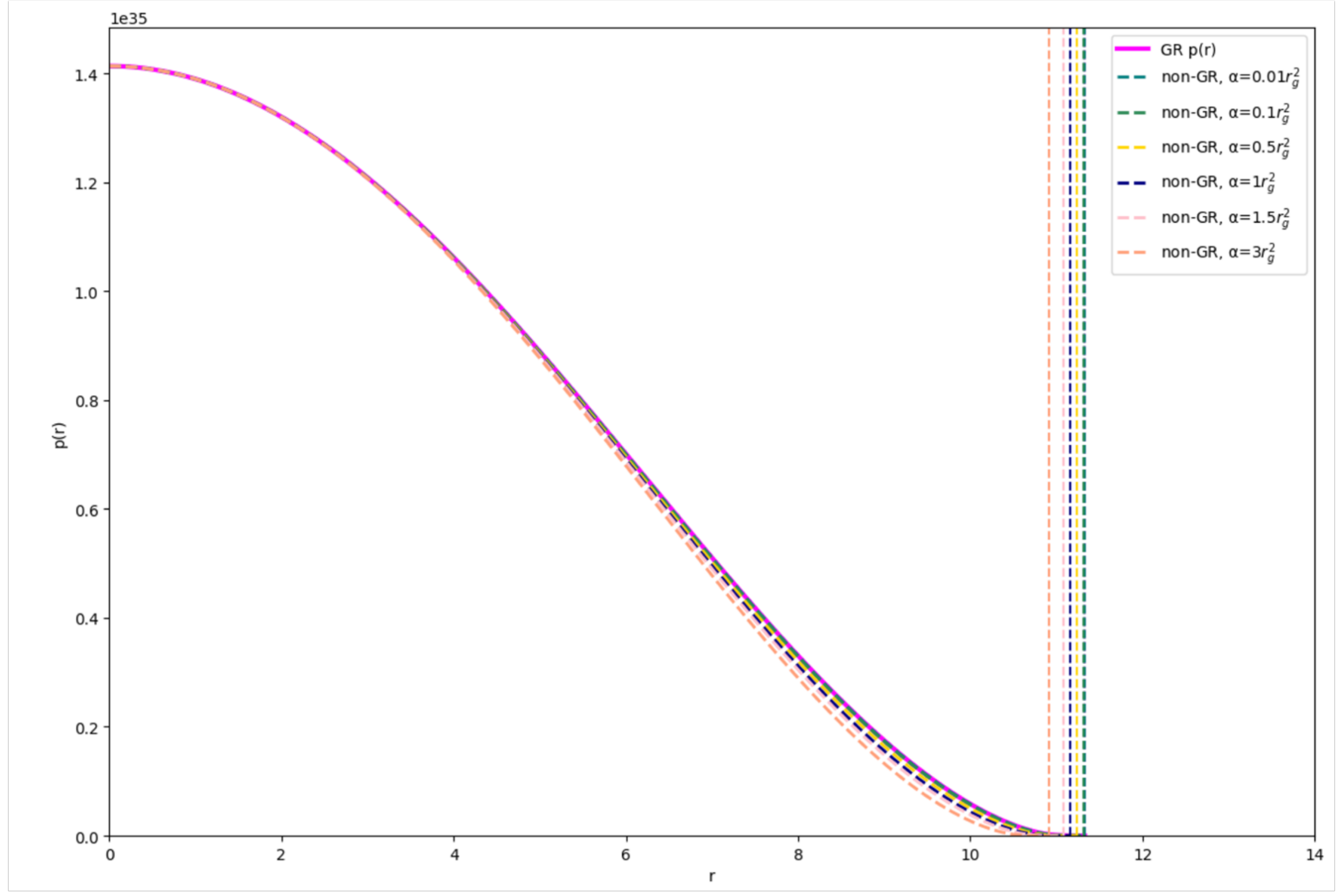
$A(r)$ profile and pressure distribution



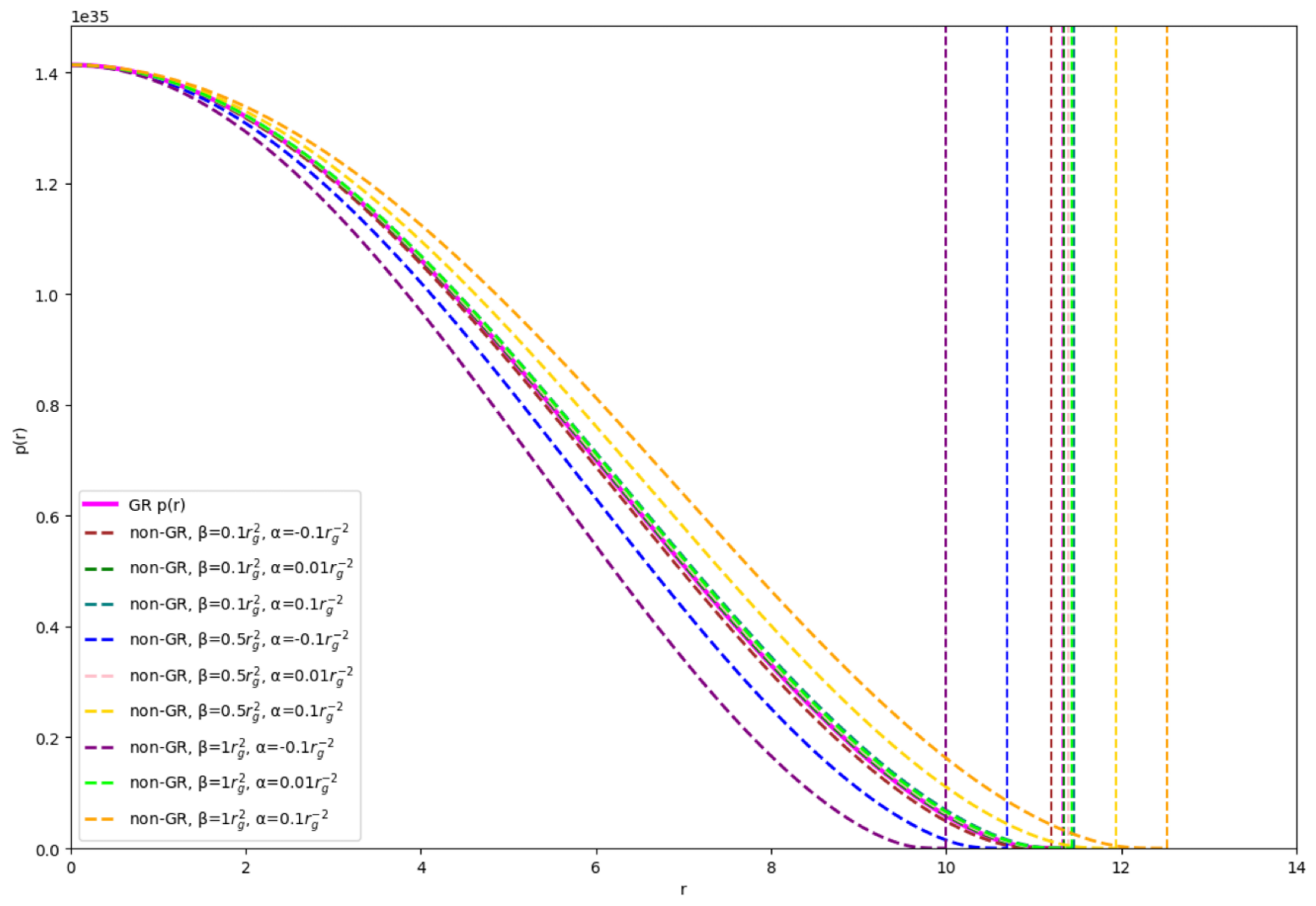
A(r) profile using $f(Q) = Q + \alpha Q^2$



A(r) profile using $f(Q) = Q + \alpha e^{\beta Q}$

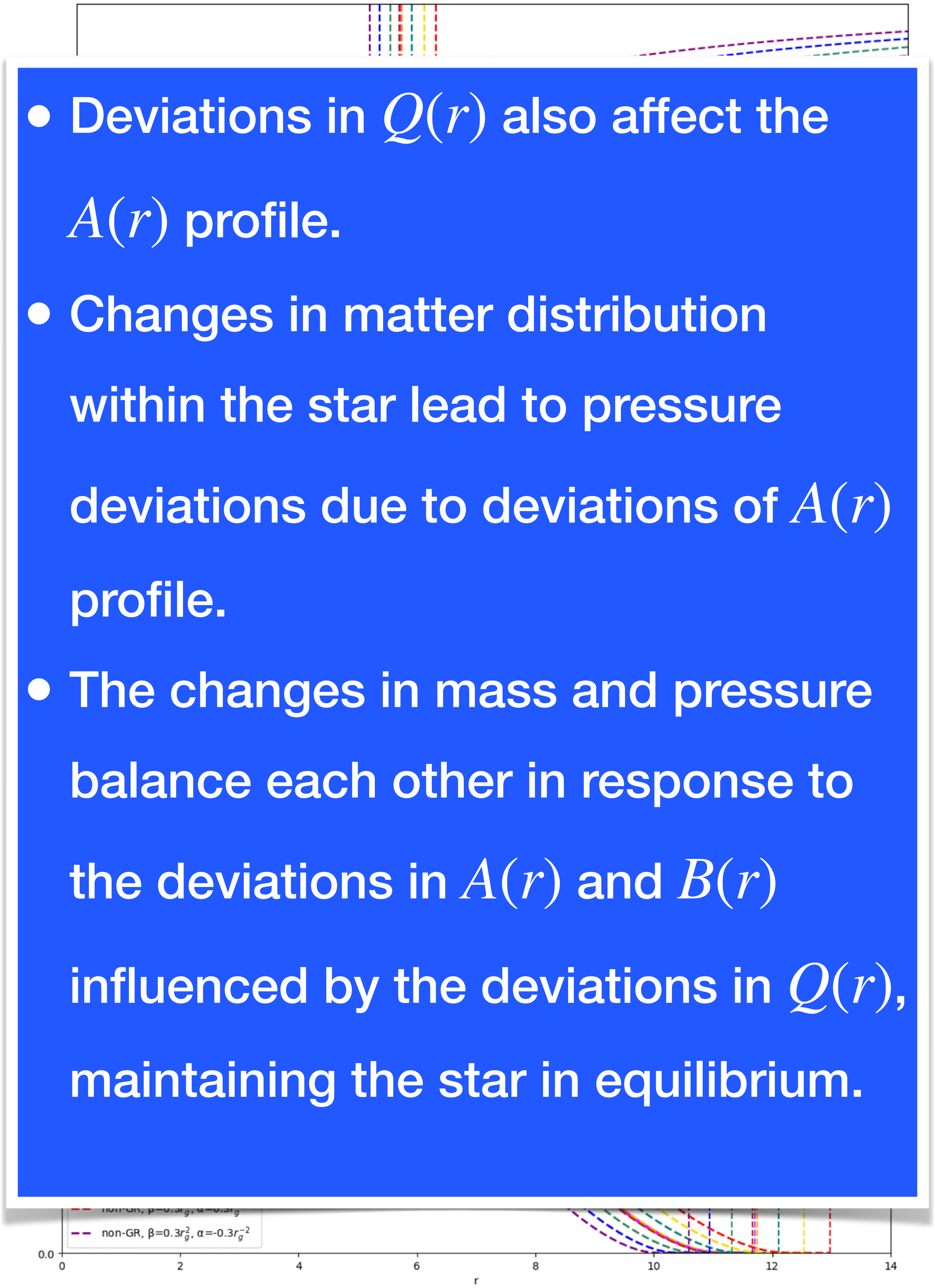


Pressure distribution using $f(Q) = Q + \alpha Q^2$



Pressure distribution using $f(Q) = Q + \alpha e^{\beta Q}$

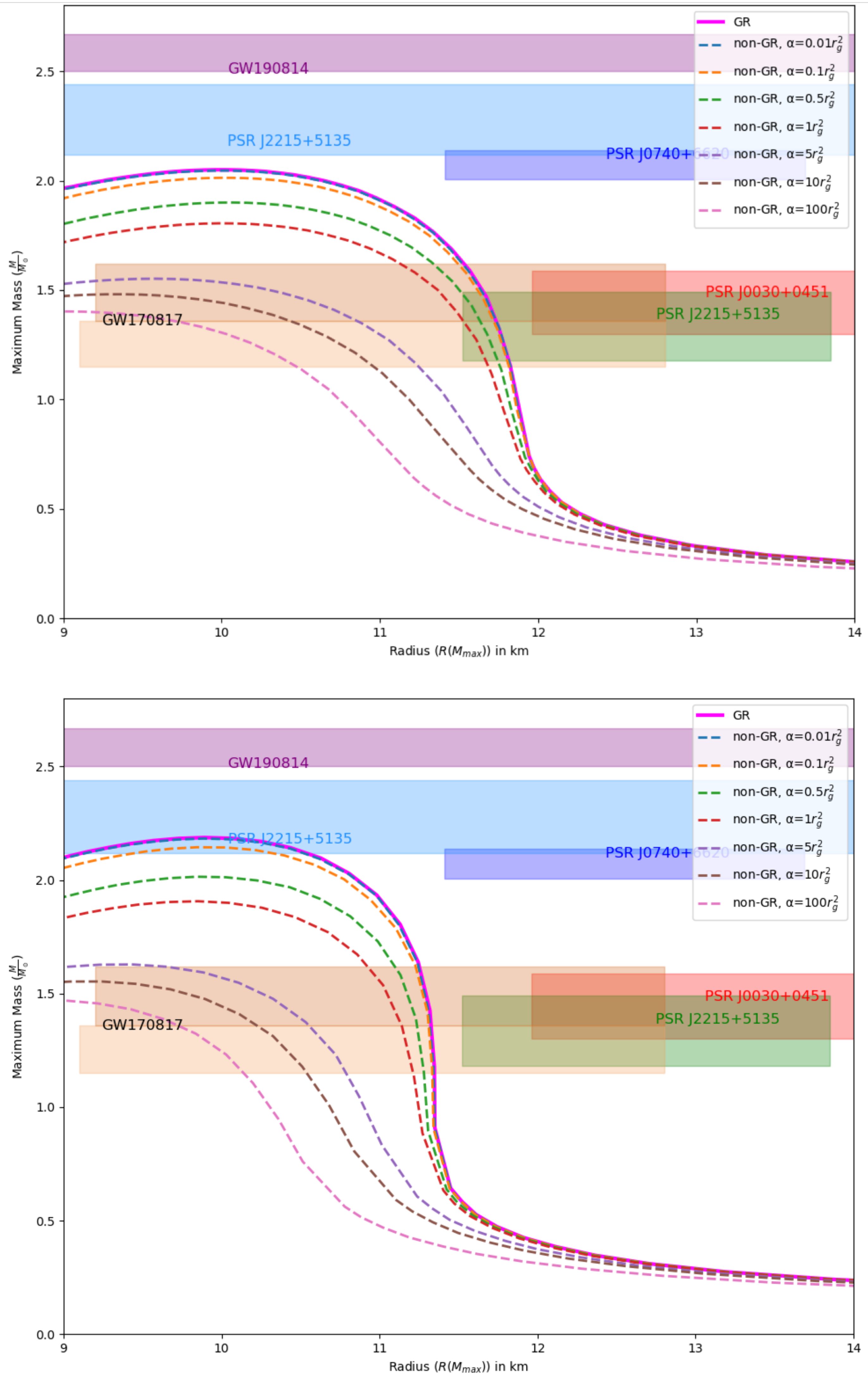
- Deviations in $Q(r)$ also affect the $A(r)$ profile.
- Changes in matter distribution within the star lead to pressure deviations due to deviations of $A(r)$ profile.
- The changes in mass and pressure balance each other in response to the deviations in $A(r)$ and $B(r)$ influenced by the deviations in $Q(r)$, maintaining the star in equilibrium.



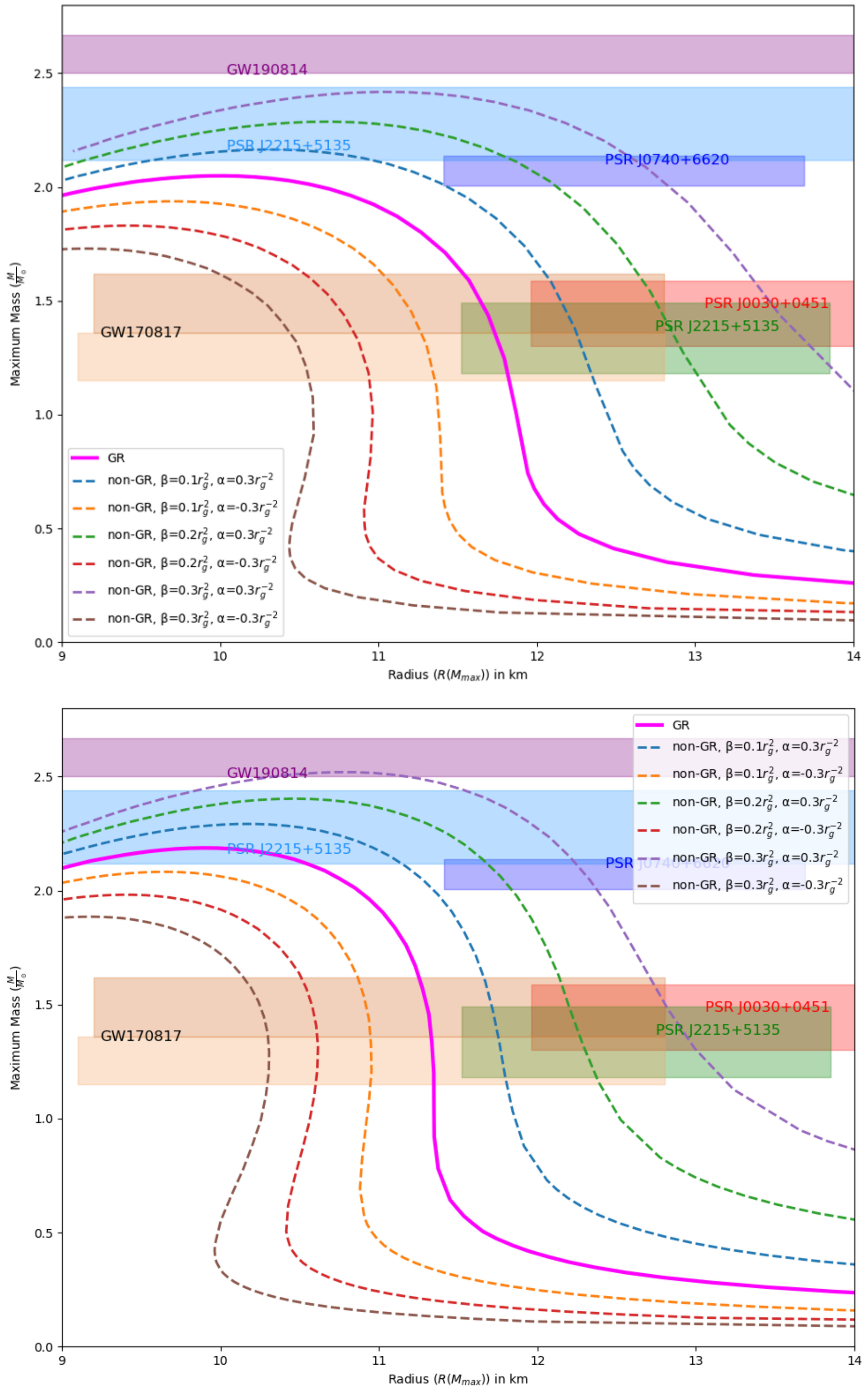
Pressure distribution using $f(Q) = Q - \alpha \ln(1 - \beta Q)$

Mass-Radius Relation

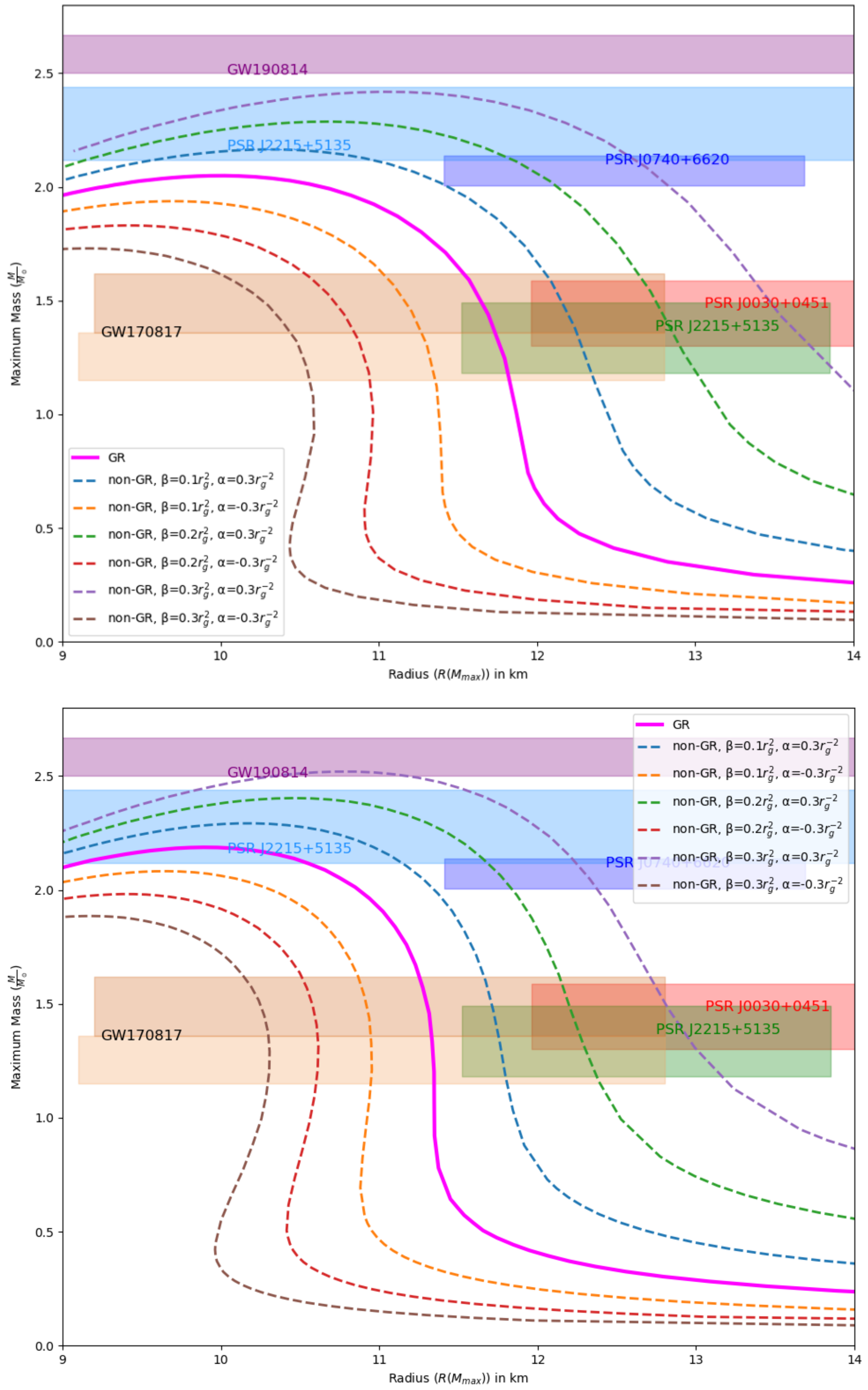
Using observational constraint as GW190814, GW170817, PSR J2215+5135, PSR J0740+6620, and PSR J0030+0451.



Mass-Radius Relation using SLy and APR4 EoS for $f(Q) = Q + \alpha Q^2$

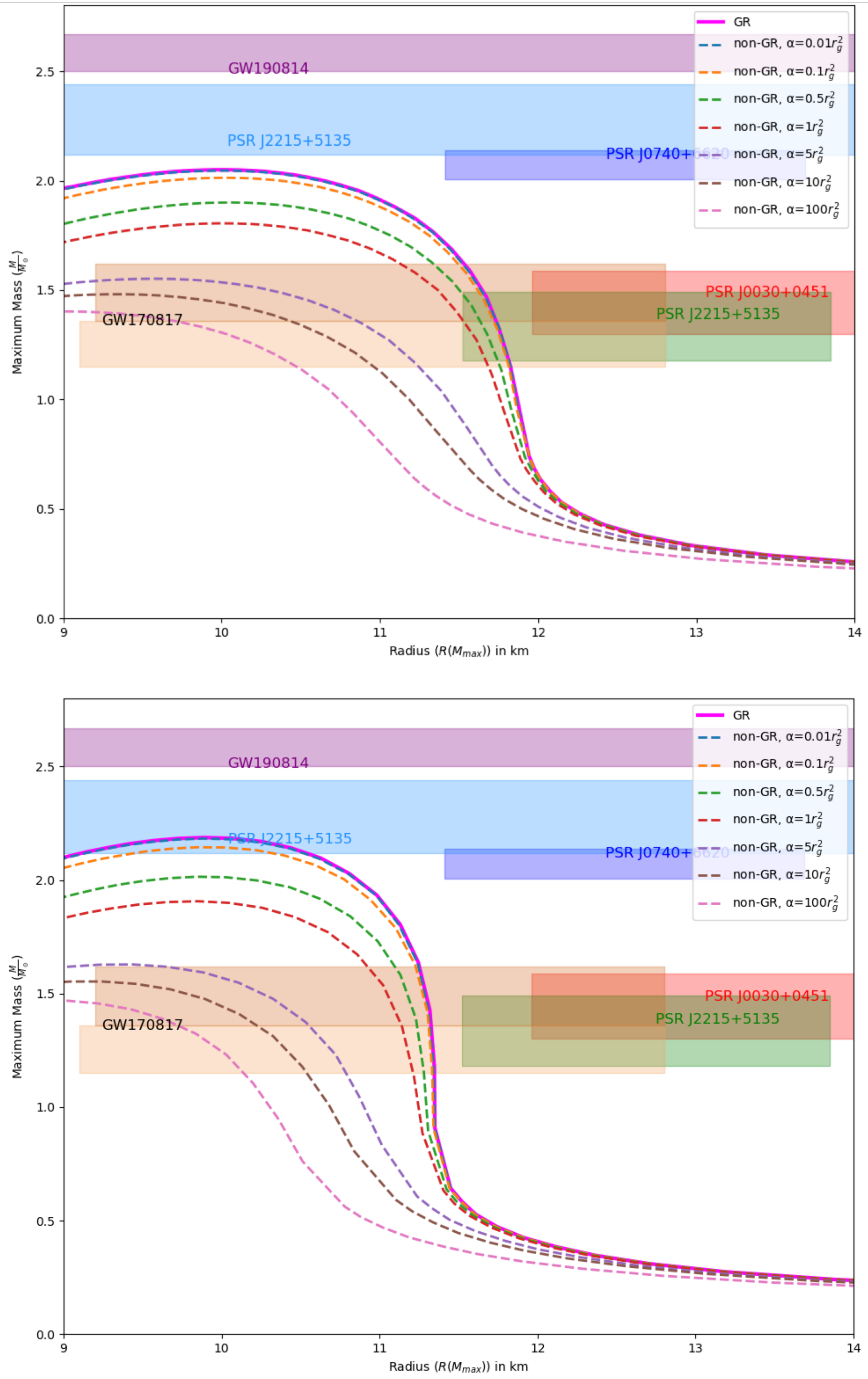


Mass-Radius Relation using SLy and APR4 EoS for $f(Q) = Q + \alpha e^{\beta Q}$

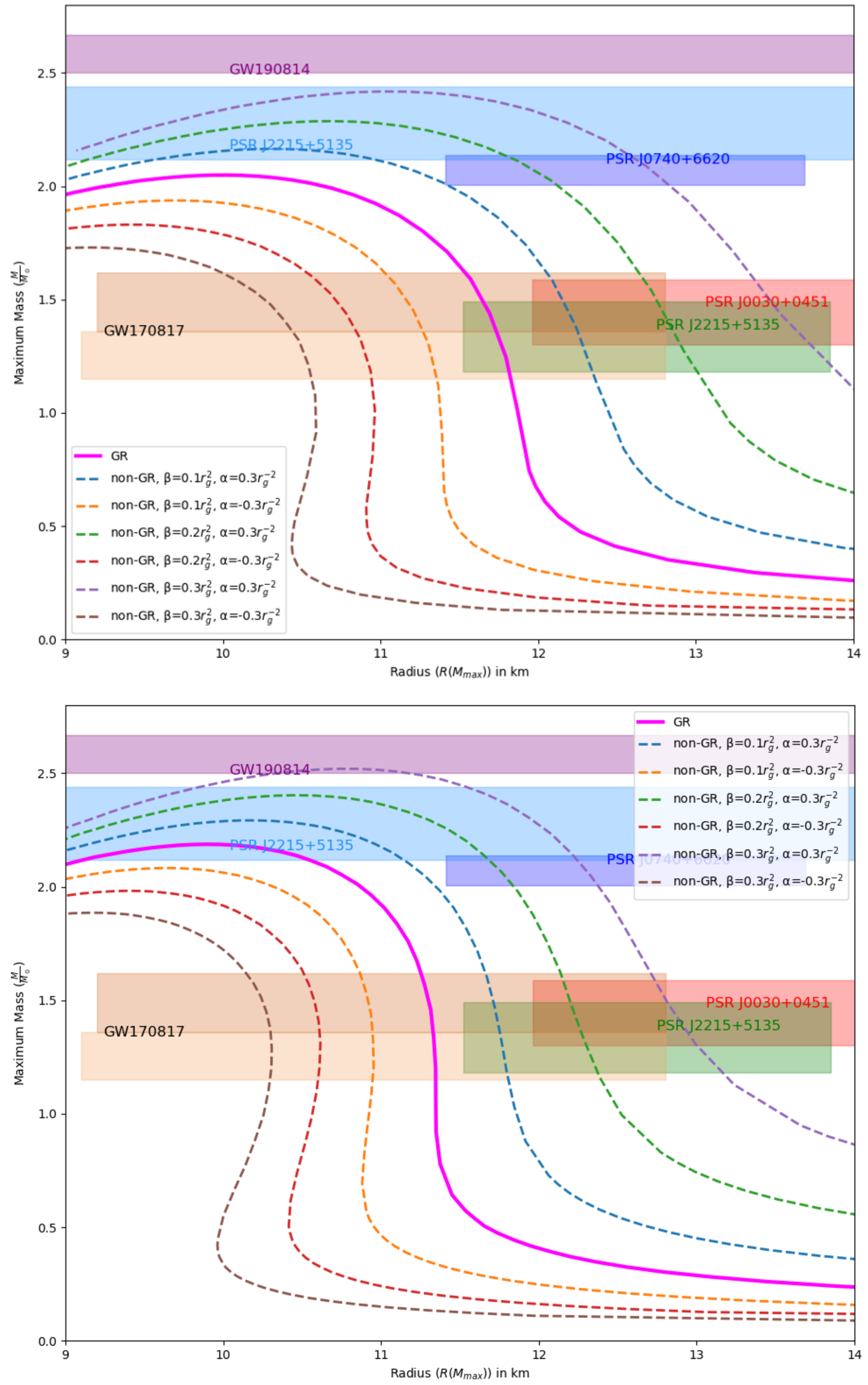


Mass-Radius Relation using SLy and APR4 EoS for $f(Q) = Q - \alpha \ln(1 - \beta Q)$

Mass-Radius Relation



Mass-Radius Relation using SLy and APR4 EoS for $f(Q) = Q + \alpha Q^2$



Mass-Radius Relation using SLy and APR4 EoS for $f(Q) = Q + \alpha e^{\beta Q}$

- Quadratic model can't accomodate larger NS (consistent with R.-H. Lin, *et. al., Phys. Rev. D, 2022* that use polytropic EoS)
- For exponential and logarithmic model, they can accomodate larger and smaller NS.
- In both model, using $\alpha = 2r_g^{-2}$, we can get massive stars $> 2.75M_\odot$ at small β values such as $\beta = 0.1r_g^2$.
- For $\alpha > 2r_g^{-2}$, the star will become unstable and collapse.

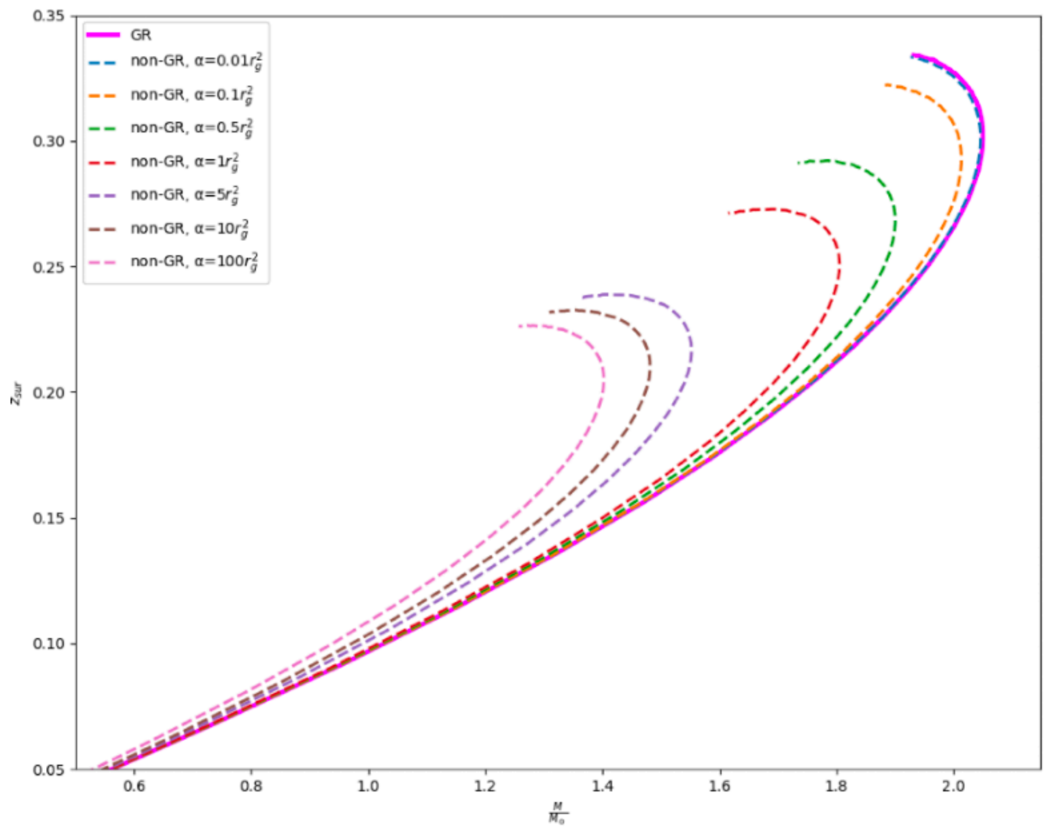
Mass-Radius Relation using SLy and APR4 EoS for $f(Q) = Q - \alpha \ln(1 - \beta Q)$

Compactness (C) and Surface gravitational redshift (z_s)

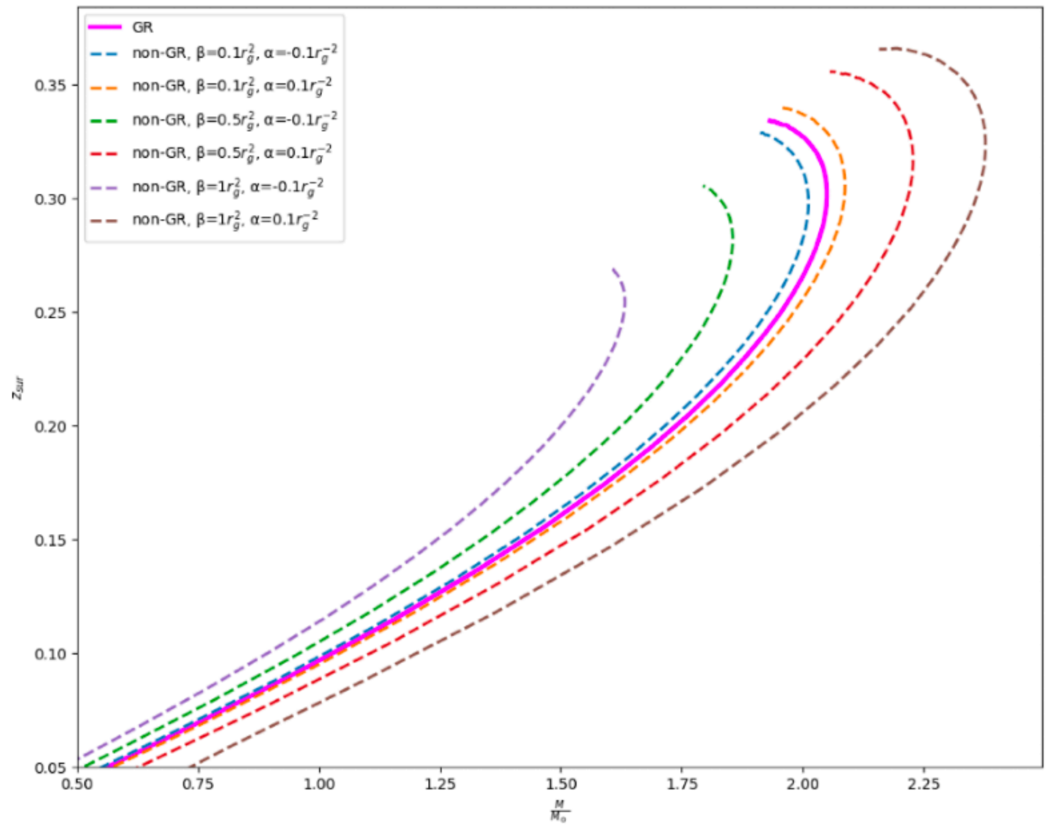
$f(Q)$ Model	α	β	$\mathcal{M}_{Max}$	$\mathcal{R}$	$\mathcal{C}$	z_s
$Q+\alpha Q^2$	GR		2.05	9.995	0.205	0.302
	0.01		2.046	9.989	0.205	0.302
	0.1		2.013	10.044	0.200	0.291
	0.5		1.900	10.032	0.189	0.268
	1		1.805	10.029	0.180	0.250
	5		1.552	9.545	0.163	0.218
	10		1.481	9.346	0.158	0.209
	100		1.402	8.987	0.156	0.206
$Q+\alpha e^{\beta Q}$	-0.1	0.1	2.012	9.8895	0.203	0.297
		0.5	1.855	9.460	0.196	0.282
		1	1.632	8.947	0.182	0.254
	GR		2.05	9.995	0.205	0.302
	0.1	0.1	2.087	10.098	0.207	0.306
		0.5	2.227	10.531	0.211	0.315
		1	2.378	11.053	0.215	0.325
	0.5	0.1	2.238	10.515	0.213	0.320
		0.2	2.426	11.048	0.220	0.336
		0.3	2.616	11.663	0.224	0.346
	1	0.1	2.429	11.078	0.219	0.334
		0.2	2.826	12.294	0.230	0.361
$Q+\alpha\ln(1-\beta Q)$	-0.3	0.1	1.937	9.702	0.200	0.291
		0.2	1.831	9.406	0.195	0.280
		0.3	1.73	9.164	0.189	0.268
	GR		2.05	0.995	0.205	0.302
	0.3	0.1	2.165	10.310	0.210	0.313
		0.2	2.288	10.690	0.214	0.322
		0.3	2.418	11.027	0.219	0.334
	0.5	0.1	2.244	10.529	0.213	0.320
		0.2	2.459	11.172	0.220	0.336
		0.3	2.706	12.009	0.225	0.348
	1	0.1	2.447	11.206	0.218	0.332
		0.2	2.968	12.964	0.229	0.358

Table I. Table of Neutron Stars Parameter using SLy EoS

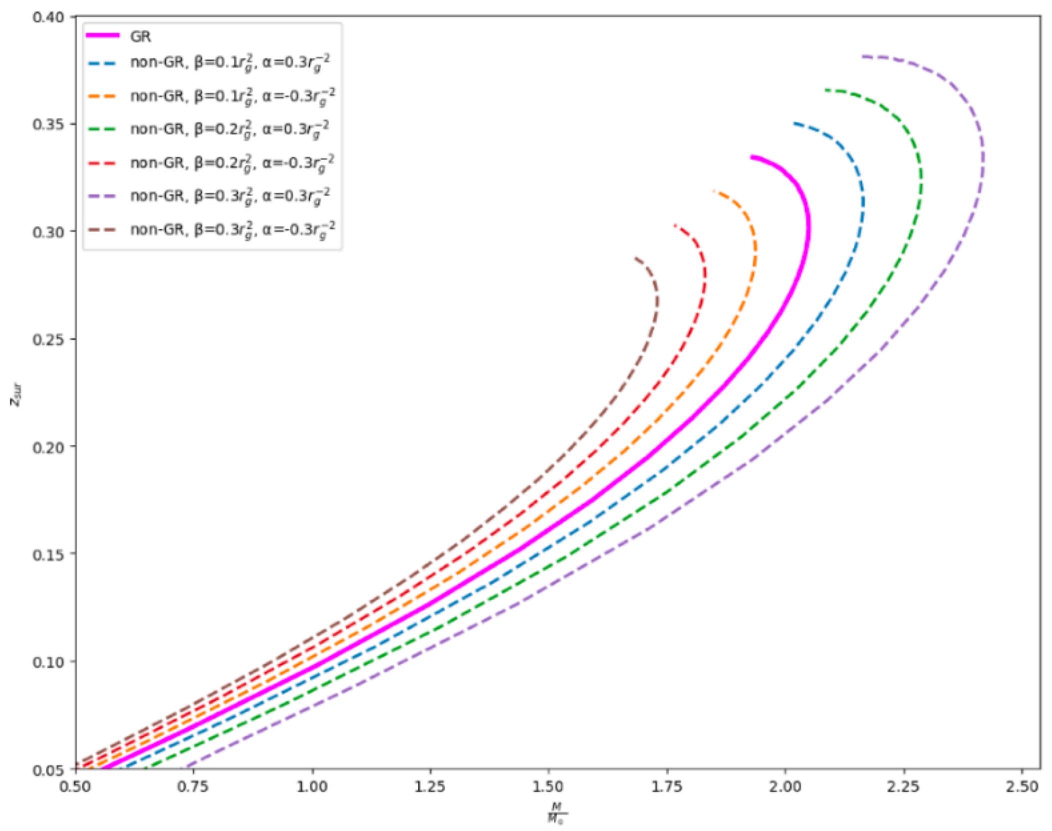
Satisfy constraint from Buchdal Limit for $C \leq \frac{4}{9}$ and $z_s \leq 2$



(a) $f(Q) = Q + \alpha Q^2$



(b) $f(Q) = Q + \alpha e^{\beta Q}$



(c) $f(Q) = Q + \alpha \log(1 - \beta Q)$

Figure 12. The surface gravitational redshift (z_s)

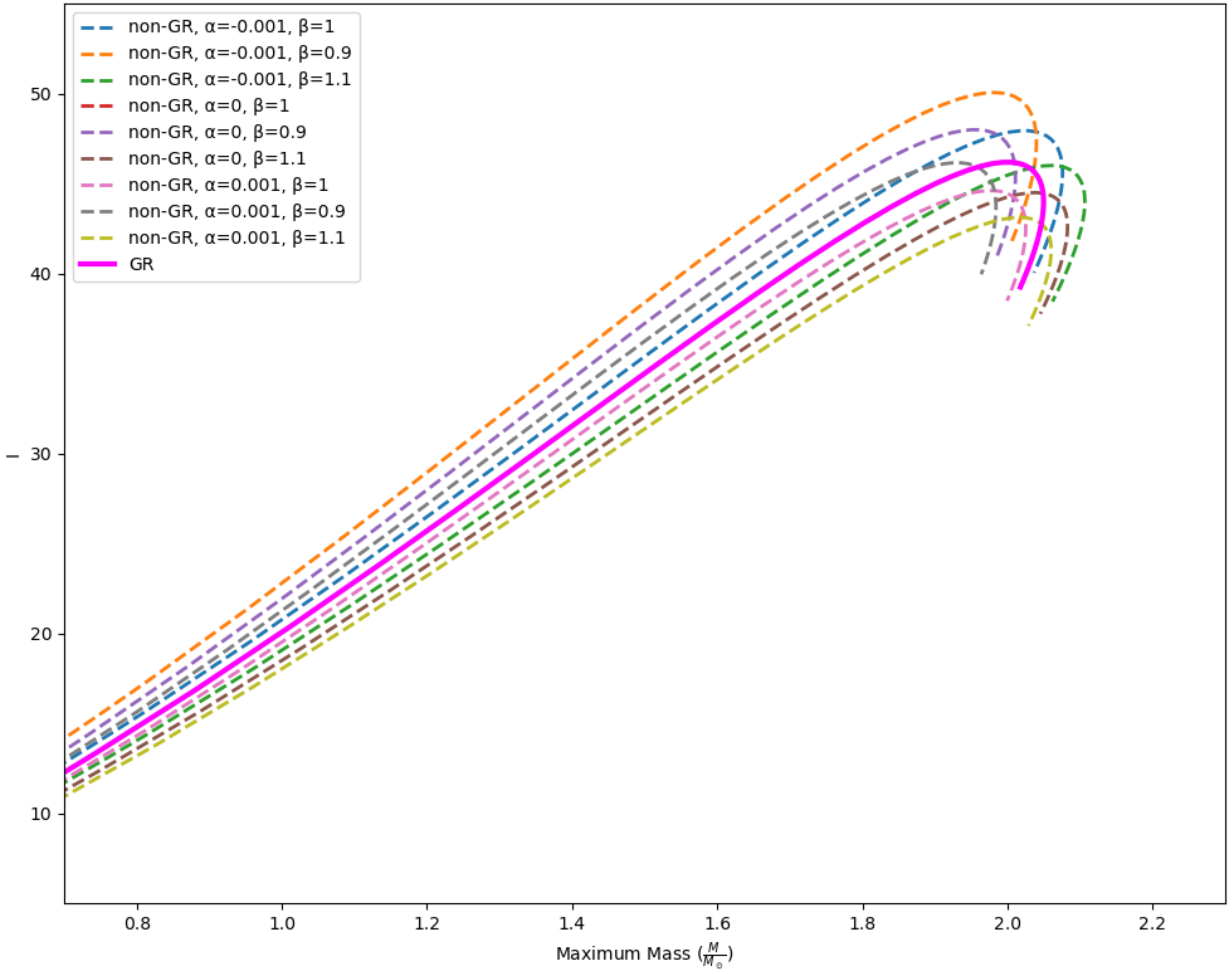
Work in progress

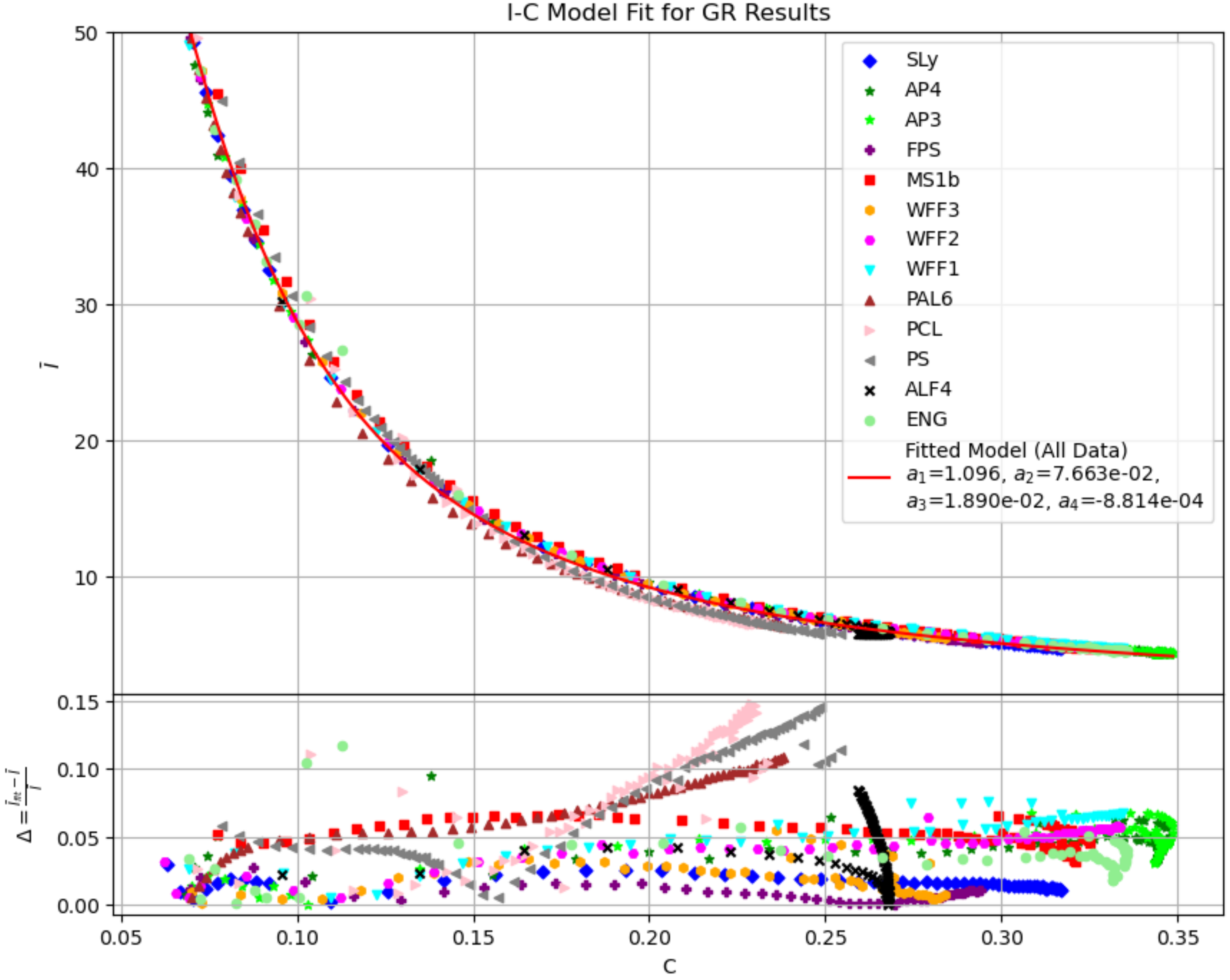
Slowly Rotating NS

- We also tried to extend the calculation using slowly rotating case with metric as
- $ds^2 = -e^{A(r)}dt^2 + e^{B(r)}dr^2 + r^2(d\theta^2 + sin^2\theta(d\phi - (\omega(r,\theta) + \mathcal{O}(\Omega^3))dt)^2)$
- From this, we tried to see the signature of $f(Q)$ under dimensionless moment inertia and compactness ($\bar{I} - C$) universal relation that can be expressed as (C. Breu and L. Rezolla, 2016)
- $$\bar{I} = \frac{I}{MR^2} = a_1C^{-1} + a_2C^{-2} + a_3C^{-3} + a_4C^{-4}$$
- where $a_{1,2,3,4}$ are fitting constant from scattering plot for each $\bar{I} - C$ plot under different EoS.
- Here, we have tried to use 12 EoSs for getting $\bar{I} - C$ universal relation.

$\bar{I} - C$ Relation

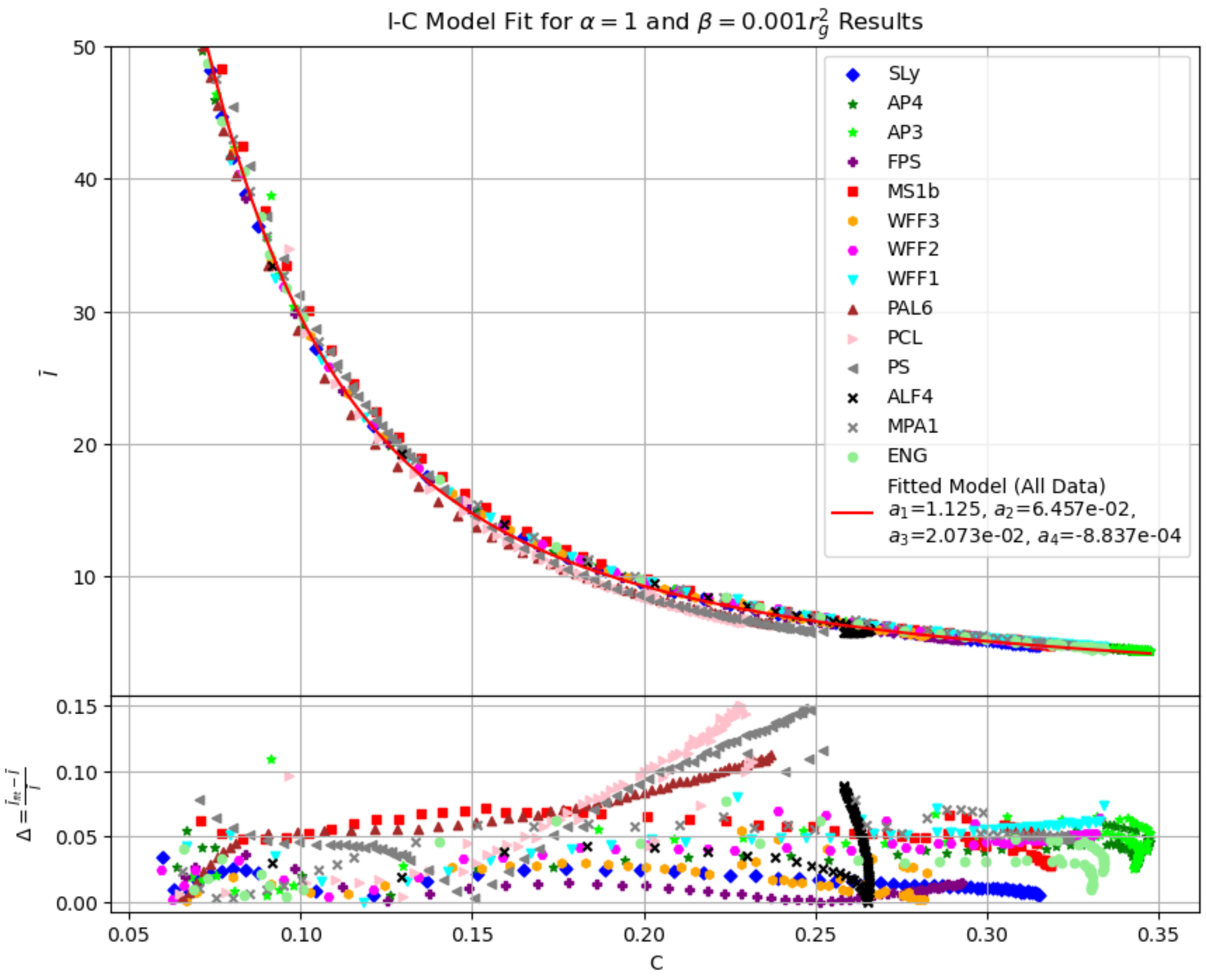
Here we are still using simple linear $f(Q) = \alpha Q + \beta$ model.

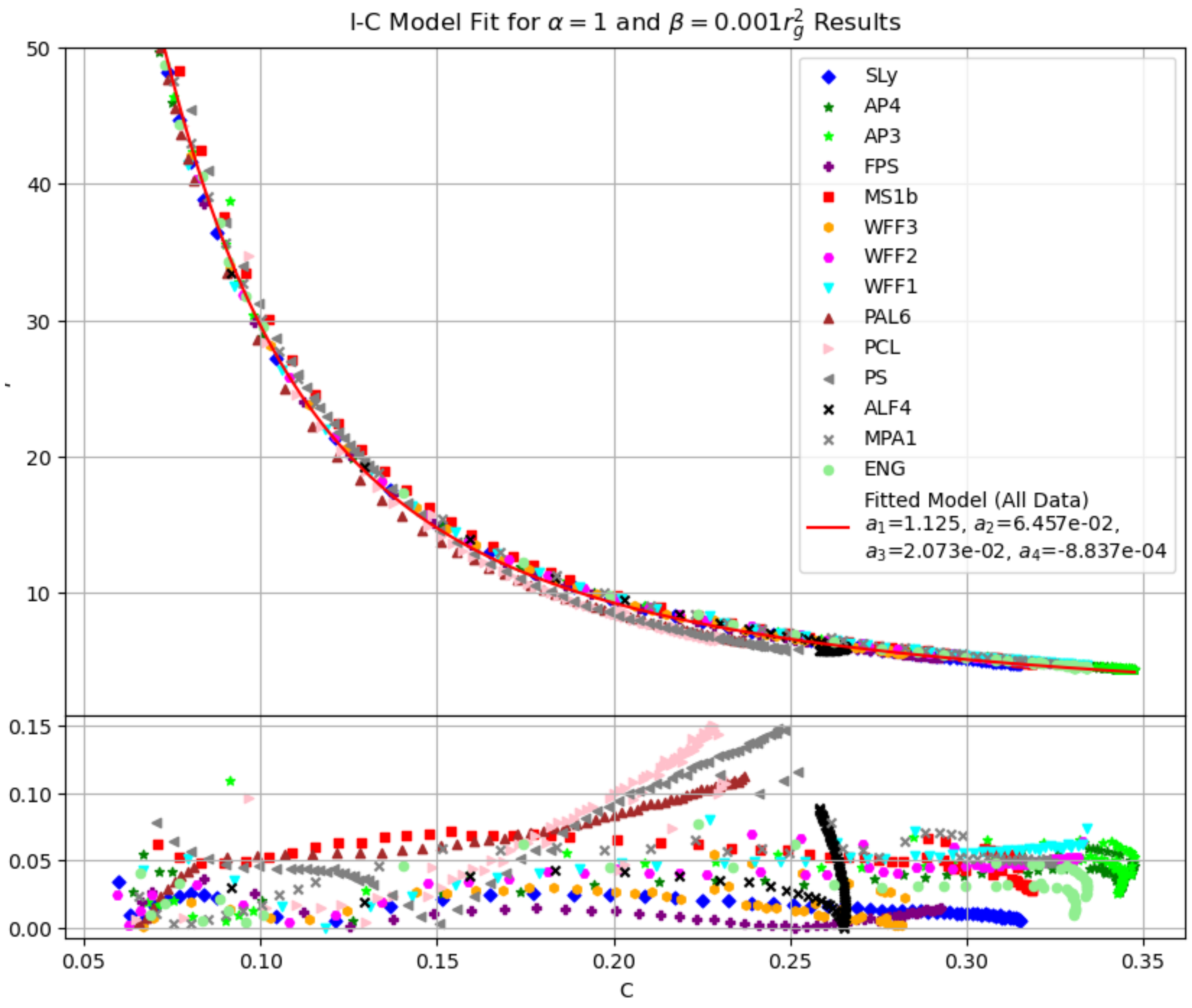




$\bar{I} - C$ Relation

Here we are still using
simple linear $f(Q) = \alpha Q + \beta$ model.





$\bar{I} - C$ Relation

$\bar{I} - C$ Relation

We can see the signature of $f(Q)$ gravity affect. Expecting larger deviations if we use non-linear $f(Q)$.

Summary and future work

- **Summary**
  - The Tolman-Oppenheimer-Volkoff (TOV) equation was derived within the context of Covariant f(Q) Gravity and the calculation show how Q affects the internal geometry of the star, which in turn affects the density, pressure, and overall stability of the neutron star. This enables the star to accommodate more matter and withstand a heavier mass.
  - For $f(Q) = Q + \alpha Q^2$ model, neutron stars lose their mass for positive α, meanwhile for negative α, we cant generate stable stars. For the exponential and the logarithmic model, neutron stars gain mass as α increase, and lose their mass as α decrease. Both models can satisfy observational constraints up to the possible most massive neutron star, GW190814, by tuning both parameters.
  - From $\bar{I} - C$ relation, using linear $f(Q) = \alpha Q + \beta$ model, we can show signature of $f(Q)$, expecting clearer signature using non-linear model.
- **Future work**
  - Continue for non-linear case for slowly rotating and calculating also rapidly rotating NS (Staykov, et.al., *PRD* 93 (2016))
  - Analyze the tidal deformation properties and NS cooling mechanism for thermal properties within the framework of $f(Q)$ (Doneva, et.al., *PRD* 92 (2015), Dohi, et.al, *PTEP* 9 (2021))
  - Extend to Scalar mode Quadrupole and GWs under NS Binary scenario (Narang, et.al. *JCAP* 03 (2023), and Inagaki, et.al., *PRD* 108 (2023))

Thank you very much!