Charge Neutrality & Beta Equilibrium in Magnetic Dual Chiral Density Waves

Will Gyory[1]
Vivian de la Incera[2] Efrain Ferrer[2]

[1]Iowa State University

[2]Collaborators, University of Texas – Rio Grande Valley

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Talk Overview

This talk will have three parts:

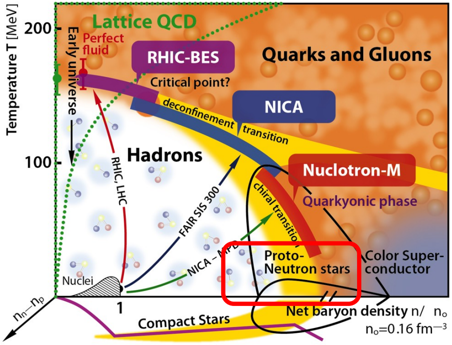
1. The old: Review of MDCDW
2. The new: Thermal fluctuations & stability
3. The cutting edge: Charge-neutral MDCDW

QCD Phase Diagram

We know...

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi,$$

but we don't know...

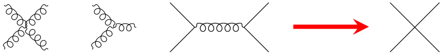


- Small μ: Lattice QCD
- Large μ or T: pQCD
- Intermediate μ, small T: Effective theories
  - Many models have found inhomogeneous phases to appear near this region (NJL, quarkyonic, large-N, color superconducting, ChPT)

e.g., Nickel PRD 80 (2009) 074025
Kojo et al NPA 843 (2010) 37
Deryagin et al IJ Mod Phys A 07 (1992) 659
Alford et al Rev Mod Phys 80 (2008) 1455

Chiral Density Wave Model

NJL Model



$$\mathcal{L}_{\text{NJL}} = \bar{\psi} i\gamma^\mu \partial_\mu \psi + G\left[(\bar{\psi}\psi)^2 + (\bar{\psi} i\gamma^5 \vec{\tau}\psi)^2\right]$$

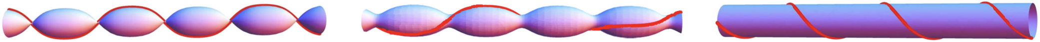
Chiral symmetry: $\psi \mapsto e^{-i\gamma^5 \vec{\tau}\cdot\vec{\theta}/2}\psi$

(Dual) Chiral Density Wave

- Define composite fields $\sigma = \bar{\psi}\psi$ and $\vec{\pi} = \bar{\psi} i\gamma^5 \vec{\tau}\psi$.
- Expand $\mathcal{L}$ about the DCDW ansatz:

$$\langle\sigma\rangle + i\langle\pi_z\rangle = \Delta e^{iqz} = -\frac{m}{2G} e^{i(2b)z}$$
$$\langle\pi_x\rangle = \langle\pi_y\rangle = 0$$

$$\mathcal{L}_{\text{MF}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m + \gamma^3\gamma^5\tau^3 b)\psi - \frac{m^2}{4G}$$



real kink crystal　　twisted kink crystal　　chiral density wave

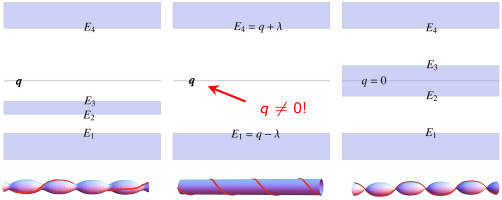
DCDW favored in 1+1 dimensions　　Başar et al PRD **79** (2009) 105012

Comparing Ansätze

Which ansatz is favored?

1+1 dimensions

- DCDW spectrum is asymmetric about zero



twisted kink crystal | chiral density wave | real kink crystal

- DCDW is favored in $1+1$ dimensions

3+1 dimensions, $B = 0$

- DCDW spectrum is symmetric

$$E_{\epsilon\zeta} = \epsilon\sqrt{\left(\zeta\sqrt{m^2 + k_{\parallel}^2} + b\right)^2 + k_{\perp}^2} \quad \epsilon, \zeta = \pm 1$$

- Single-mod. real kink is favored

Abuki et al PRD 85 (2012) 074002
Nickel PRD 80 (2009) 074025

3+1 dimensions, $B > 0$

- Effective dimensional reduction of LLL $\Rightarrow$ asymmetric spectrum

$$E_{\epsilon}^{0} = \epsilon\sqrt{m^2 + k^2} + b \quad \epsilon = \pm 1$$

- Favored ansatz unknown for $B > 0$

MDCDW Spectrum & Free Energy

- MDCDW = **Magnetic** DCDW
- Add B field in z-direction.
  - $\partial_\mu \to D_\mu = \partial_\mu + ie_f A_\mu$
  - $A = (0, 0, Bx, 0)$
- Diagonalizing the Hamiltonian gives:

$$E^0_\epsilon = \epsilon\sqrt{m^2 + k^2} + b$$

$$E^\ell_{\epsilon\xi} = \epsilon\sqrt{\left(\xi\sqrt{m^2 + k^2} + b\right)^2 + 2|e_f B|\ell}$$

- Asymmetry of LLL is related to nontrivial topology.

$$\Omega = \frac{m^2}{4G} + \sum_f \left(\Omega^f_{\text{vac}} + \Omega^f_{\text{anom}} + \Omega^f_\mu + \Omega^f_T\right)$$

$$\Omega^f_{\text{vac}} = \frac{1}{4\sqrt{\pi}}\frac{N_c|e_f B|}{(2\pi)^2}\int dk \sum_{n\xi\epsilon}\int_{1/\Lambda^2}^\infty \frac{ds}{s^{3/2}} e^{-sE^2}$$

$$\Omega^f_{\text{anom}} = -\frac{N_c|e_f B|}{(2\pi)^2} 2b\mu$$

$$\Omega^f_\mu = -\frac{1}{2}\frac{N_c|e_f B|}{(2\pi)^2}\int dk \sum_{n\xi\epsilon}\left(|E-\mu| - |E|\right)_{\text{reg}}$$

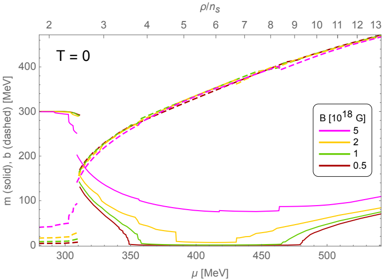
$$\Omega^f_T = -\frac{1}{2}\frac{N_c|e_f B|}{(2\pi)^2}\int dk \sum_{n\xi\epsilon}\frac{2}{\beta}\ln\left(1 + e^{-\beta|E-\mu|}\right)$$

$$\langle\hat{N}\rangle_{\text{top.}} = \underbrace{\eta_H(0)}_{\text{Atiyah-Patodi-Singer invariant}} = \underbrace{\lim_{s\to 0^+}\sum_k \text{sgn}(E_k)|E_k|^{-s}}_{\text{spectral asymmetry}} = \underbrace{-\frac{|e_f|}{(2\pi)^2}\int d^3x\, \mathbf{B}\cdot\nabla\theta(x)}_{\text{topological term}} \qquad \theta(x) = qz$$

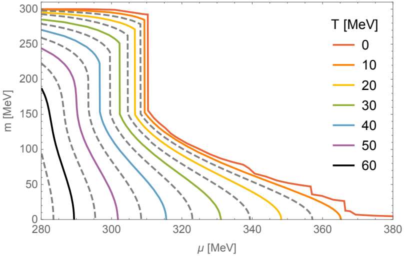
Frolov et al PRD 82 (2010) 076002 Tatsumi et al PLB 743 (2015) 66–70

Solutions at Finite B and T

m and b vs. μ at various B:

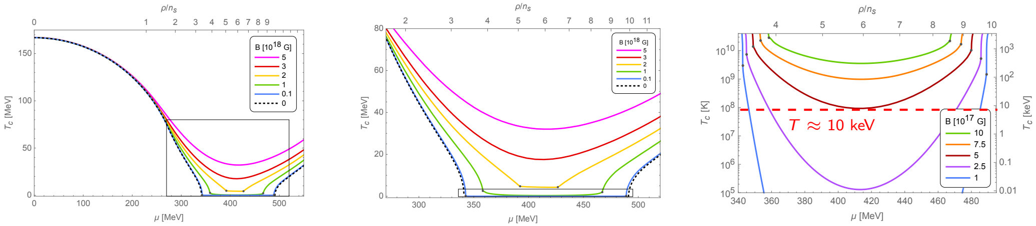


m vs. μ at various T, $B = 1.5 \times 10^{18}$ G:



- B favors condensate.
- We showed that "lifting" of m curves due to B is mainly a topological effect (more later).
- Even for smaller B, condensate is large in region of interest for cold NSs.
- When $B > 0$, small "remnant mass" at intermediate μ.
- T disfavors condensate.
- Inhom. at all $\mu > 0$!

Critical Temperature



- Strong B field increases T_c in region relevant to compact stars.
- Cold NS: 2–3.5 n_s, $T \sim$ keV scale / Hot NS: $n > 3.5 n_s$, $T \sim 10$ MeV scale.
- At $B > 2 \times 10^{18}$ G, T_c is large at all densities.
- At $B > 5 \times 10^{17}$ G, even remnant T_c is O(cold NS) at all densities.
- Possible applications for old NS and young short-lived remnant NS.

Part II: Thermal Stability

LP Instability

- Inhom. condensates usually suffer from Landau-Peierls (LP) instability.
  - Fluctuations erase long-range order at any $T > 0$.
  - SSB for translations $\Rightarrow$ NG phonons $u(x)$, $\perp$ soft modes $\Rightarrow$ IR div. in $\langle u^2 \rangle$.
  - Erases long-range order because $\langle M \rangle = e^{-\langle q^2 u^2 \rangle /2} M_0$.
- BUT this theorem does not necessarily apply when a B field is present.
  - Because explicit breaking of SO(3) creates new terms in the phonon spectrum.

> *"This leads to an interesting observation that inhomogeneous condensates in QCD under magnetic fields could be stable against fluctuations."*
>
> – Hidaka et. al., PRD **92**, 034003

> *"... and this is the case for MDCDW." (paraphrased)*
>
> – Ferrer & Incera, PRD **102**, 014010

GL Expansion

- Computing the exact free energy is computationally expensive. But we can approximate it using a generalized Ginzburg-Landau (GL) expansion.
- In (generalized) GL theory, we systematically expand free energy in powers of the condensate field (and its derivatives).

$$\Omega_{\rm GL}^{(6)} = \alpha_{2,0}|M|^2 - i\frac{\beta_{3,1}}{4}\left[M^*(\hat{B}\cdot\nabla M) - (\hat{B}\cdot\nabla M^*)M\right] + \alpha_{4,0}|M|^4 + \frac{\alpha_{4,2}}{4}|\nabla M|^2 - i\frac{\beta_{5,1}}{4}|M|^2\left[M^*(\hat{B}\cdot\nabla M) - (\hat{B}\cdot\nabla M^*)M\right]$$
$$+ i\frac{\beta_{5,3}}{16}\left[(\nabla^2 M^*)\hat{B}\cdot\nabla M - \hat{B}\cdot\nabla M^*(\nabla^2 M)\right] + \alpha_{6,0}|M|^6 + \frac{\alpha_{6,2}}{4}|M|^2|\nabla M|^2 + \frac{\alpha_{6,4}}{16}|\nabla^2 M|^2$$

- Expansion only includes structures allowed by symmetry. $B > 0$ explicitly breaks SO(3) symmetry, allowing for new structures labeled by β.
- After applying the DCDW ansatz $M = me^{i2bz}$, it reduces to an ordinary expansion in m and b.

$$\Omega = \alpha_{2,0}m^2 + \beta_{3,1}bm^2 + \alpha_{4,0}m^4 + \alpha_{4,2}b^2m^2 + \beta_{5,1}bm^4$$
$$+ \beta_{5,3}b^3m^2 + \alpha_{6,0}m^6 + \alpha_{6,2}b^2m^4 + \alpha_{6,4}b^4m^2$$

- β coefficients label terms with n_b odd ($\Leftrightarrow$ LLL, topology).

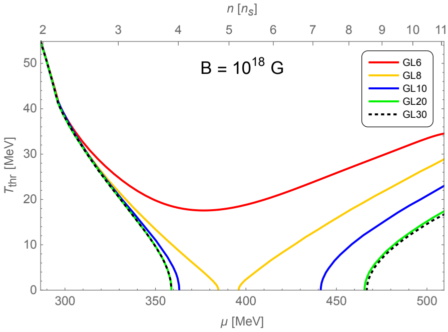
LP Instability

- Perturb $M(z)$ with phonon fluctuation $M(z+u(x))$ in GL exp. to $O(u^2)$.
- $\mathcal{F}(M(z+u)) = \mathcal{F}_0 + v_z^2(\partial_z\theta)^2 + v_\perp^2(\partial_\perp\theta)^2 +$ higher terms, $\theta = mqu$.
  - v_z and $v_\perp$ are combinations of GL coefficients, e.g., $v_\perp^2 = \alpha_{4,2} + b\beta_{5,3} + m^2\alpha_{6,2} + 2b^2\alpha_{6,4}$
- Solve for velocities ~~and coefs. of higher terms~~.
- $\langle q^2u^2\rangle = \frac{T}{2(2\pi)^2}\sum_n \int_0^\infty dk_\perp k_\perp \int_{-\infty}^{+\infty} dk_z \frac{1}{m^2(\omega_n^2+v_z^2k_z^2+v_\perp^2k_\perp^2+\zeta^2k^4)}$.
- Higher terms are negligible in the infrared and can be ignored—this is key.
- In the infrared we find $\langle q^2u^2\rangle \simeq \frac{T}{16m|v_zv_\perp|}$, which diverges when $v_\perp = 0$ (LP instability). But $v_\perp \neq 0$ thanks to β coefficients.
- Stationary EQ: $0 = \alpha_{4,2} + m^2\alpha_{6,2} + 2b^2\alpha_{6,4}$ $\longrightarrow$ topology ($\beta_{i,j} \neq 0$) eliminates LP instability!
- And we can then compute the threshold temperature T_{thr}, which we define as the temperature where $\langle M\rangle \mapsto e^{-1}M_0$.

Ferrer & de la Incera PRD 102 (2020) 014010

Threshold Temperature

- But there's a problem: 6th order isn't good enough.



- We needed to go to ⩾20th order to get reliable results.
- How did we do that? (20th order expansion has 100 coefficients!)
- We generalized the preceding calculation to arbitrary order in the GL expansion.

$$v_z^2 = \sum_{n=2}^{N} \sum_{n_q} c_{n,n_q} m^{n-n_q-2} q^{n_q-2} \frac{n_q(n_q-1)}{2}$$

$$v_\perp^2 = \sum_{n=2}^{N} \sum_{n_q} c_{n,n_q} m^{n-n_q-2} q^{n_q-2} \left\lfloor \frac{n_q}{2} \right\rfloor .$$

Fluctuations at Nth Order

$$\Omega^{(N)} = \sum_{n=2,4,\dots}^{N} \sum_{n_q=0,2,\dots}^{n-2} a_{n,n_q} |M|^{n-2-n_q} |\nabla^{n_q/2} M|^2 + \sum_{n=3,5,\dots}^{N-1} \sum_{n_q=1,3,\dots}^{n-2} b_{n,n_q} |M|^{n-2-n_q} i^{n_q-1} \operatorname{Im}[(\nabla^{n_q-1} M^*)(\hat{z} \cdot \nabla M)]$$

$$|\nabla^k M|^2 = m^2 q^{2k} |\hat{z} + \nabla u|^{2k}, \qquad k = 0, 1, 2, \dots$$

$$\operatorname{Im}[(\nabla^k M^*)(\hat{z} \cdot \nabla M)] = m^2 q^{k+1} (-i)^k |\hat{z} + \nabla u|^k (1 + \partial_z u), \qquad k = 0, 2, 4, \dots .$$

$$|\hat{z} + \nabla u|^k = 1 + k\partial_z u + \frac{k}{2} |\nabla u|^2 + k\left(\frac{k}{2} - 1\right)(\partial_z u)^2.$$

$$\mathcal{F}^{(N)} = \sum_{n=2}^{N} \sum_{n_q} c_{n,n_q} m^{n-n_q} q^{n_q} \left[1 + n_q \partial_z u + \frac{n_q(n_q-1)}{2} (\partial_z u)^2 + \left\lfloor \frac{n_q}{2} \right\rfloor (\partial_\perp u)^2 \right]$$

Finding GL Coefficients

- How do we find the coefficients?
- In principle: Compute $\alpha_{n_b+n_m,n_b} = \frac{1}{n_m!n_b!}\partial_b^{n_b}\partial_m^{n_m}\Omega\big|_{m=b=0}$.
- Infeasible to compute by hand at high GL orders.
- Instead, we found an all-orders formula. Tricks:
  - Write $\Omega = C\int dkF(k)$.
  - Bring ∂'s under $\int$ and use repeated IBP.
  - Euler-Maclaurin formula

$$\alpha_{n_m+n_b,n_b} \sim \frac{\delta_{2,n}}{4G} + \frac{1}{(n_m/2)!n_b!}\sum_f \sum_{p=0,2,4,\ldots}^{\infty} \bar{\gamma}^{f,(p)}_{n_m+n_b,n_b}$$

$$\beta_{n_m+n_b,n_b} = \frac{1}{(n_m/2)!n_b!}\sum_f \bar{\gamma}^{f,(1)}_{n_m+n_b,n_b}$$

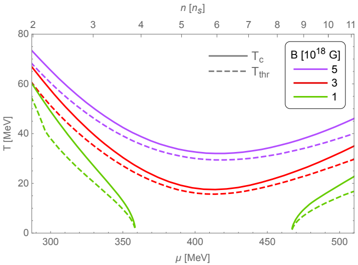
$$\bar{\gamma}^{f,(p)}_{n_m+n_b,n_b} = \begin{cases} \frac{N_c}{(2\pi)^2}\left(-\frac{1}{2}+\mu^2+\frac{1}{3}\pi^2T^2\right) & n_m=2,\ n_b=p=0 \\ \frac{N_c|e_fB|^p}{(2\pi)^2}\frac{B_p^-}{p!}\frac{2^{1-(n_m/2)}}{(n_m-2)!!}\frac{(n-2)!!}{(n+2p-4)!!}I_{n+2p-4} & \text{otherwise} \end{cases}$$

$$I_p^{(T=0)} = \begin{cases} -\gamma-2\ln(2\mu) & p=0 \\ \mathrm{Re}\left[(i\sqrt{2})^{p+2}(p-2)!!+\frac{2(p-1)!}{\mu^p}\right] & p=1,2,3,\ldots \end{cases}$$

$$I_p^{(T>0)} = \begin{cases} -\gamma+2\ln\left(\frac{\beta}{4\pi}\right)-2\,\mathrm{Re}\,\psi\left(\frac{1}{2}+i\frac{\beta\mu}{2\pi}\right) & p=0 \\ \mathrm{Re}\left[(i\sqrt{2})^{p+2}(p-2)!!-2\left(-\frac{i\beta}{2\pi}\right)^p\psi^{(p)}\left(\frac{1}{2}+i\frac{\beta\mu}{2\pi}\right)\right] & p=1,2,3,\ldots \end{cases}$$

- β coefs. are exactly proportional to B, whereas α coefs. are constant + $O(|eB|^2)$.
- Can show that powers of $m^{n_m}b^{n_b}$ in GL expansion are all divided by a power of μ or Λ.

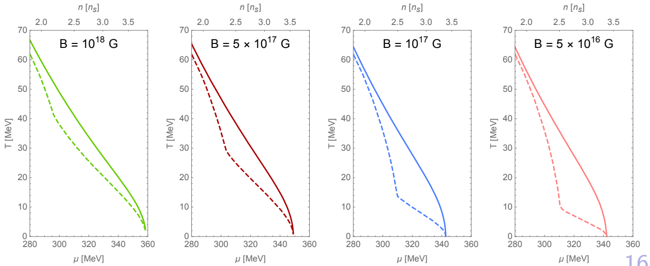
T_c vs T_{thr}



- For $B \sim 10^{18}$ G, T_{thr} is a large fraction of T_c.
- For $B \sim 10^{17}$ G, T_{thr} is still large compared to cold NSs ($T \sim$ keV).
- LP instability returns as $B \to 0$.

Ferrer, Gyory, & de la Incera,

arXiv:2307.05621.



Part III:
Charge Neutrality

Theory

$$d \leftrightarrow ue\bar{\nu}.$$

$$\mu_d = \mu_u + \mu_e.$$

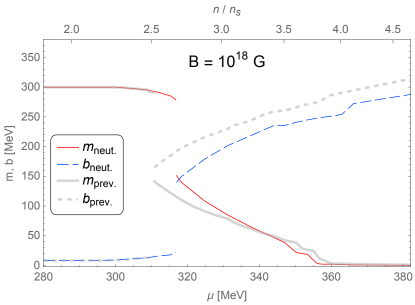
$$n_i = \frac{\partial \Omega}{\partial \mu_i}, \qquad i = u, d, e,$$

$$\frac{2}{3} n_u - \frac{1}{3} n_d - n_e = 0.$$

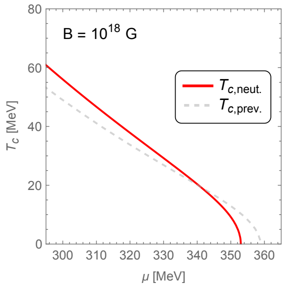
$$\frac{\partial \Omega}{\partial \mu_e} = 0$$

$$\mu_u = \mu - \frac{2}{3}\mu_e$$
$$\mu_d = \mu + \frac{1}{3}\mu_e$$

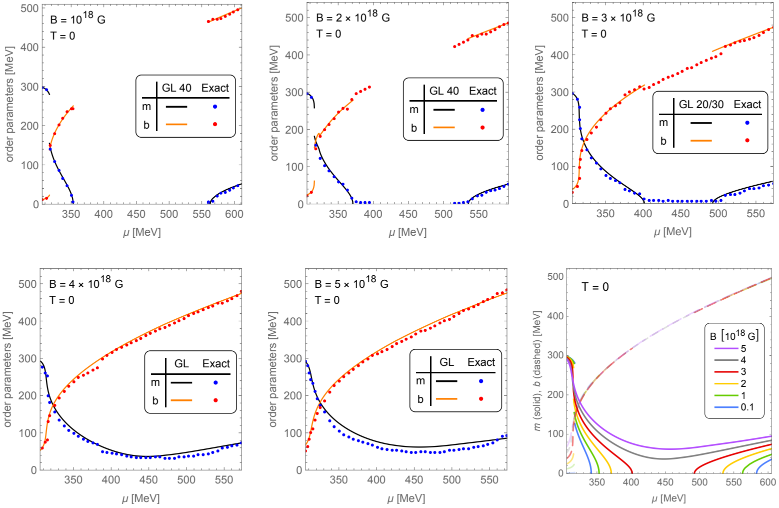
Neutrality Results I



- No major qualitative differences from the previous case.
- Quantitative differences are relatively small.
- Why the "crossover" behavior in m?

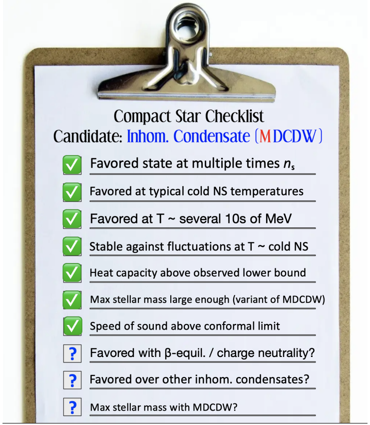


Neutrality Results II



Conclusion

- MDCDW favored over wide range of T & μ.
- Nth-order GL formulas enable many calculations, including T_c and T_{thr}.
- Magnetic field affects condensate via complex combination of mechanisms.
  - Symmetry breaking, dimensional reduction, topology, discretized HLL.
- Topology plays central role in making MDCDW robust against large T and μ, including thermal stability.
- Increasingly plausible candidate for NS matter.
- Next steps: Compute EoS & SoS from neutral MDCDW results.



The End

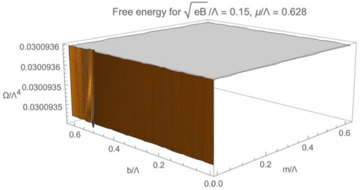
Thanks!

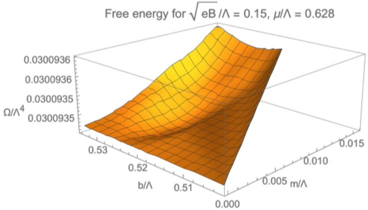
Remnant Mass

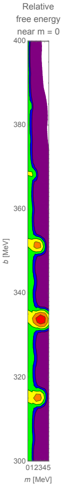
- Small $m > 0$ found at small T and large B.
- Comes from singularities in $\partial_{m^2}\Omega\big|_{m=0,b=\mu_n}$.
- Crude calculation:

$$\Omega_\mu^{f,n>0} \sim -\int dk \sum_{\xi=\pm1} 2(\mu - E)\theta(\mu - E)\big|_{\epsilon=+1} \qquad E_{\xi,\epsilon}^n = \epsilon\sqrt{(\xi\sqrt{m^2+k^2}+b)^2 + 2|e_f B|n}$$

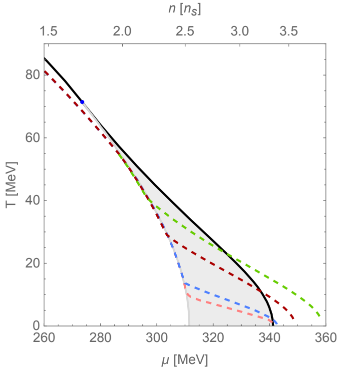
$$\partial_{m^2}\Omega_\mu^{f,n,\xi=-1}\big|_{m=0} \sim \int_{b-\mu_n}^{b+\mu_n} dk \frac{1}{\sqrt{(k-b)^2 + 2|e_f B|n}}(1 - b/k) \qquad \mu_n = \sqrt{\mu^2 - 2|e_f B|n}$$







Kinks in T_{thr} Curves



- Gray = inhom. phase at $B = 0$.
- No phonon fluctuations for homogeneous phase when $B = 0$.
- Tthr at $B = 0$ should lie exactly on gray-white boundary.
- Similar behavior near boundary even when $B > 0$.

Finding GL Coefficients

$$\Omega^f_{\mathrm{vac},\mu,T} = -\frac{1}{2}\frac{N_c|e_f B|}{(2\pi)^2}\int dk \sum_{\ell\xi\epsilon}\Big[F_\alpha(\epsilon E) + \epsilon F_\beta(\epsilon E)\Big]$$

$$F_\alpha(x) = -\frac{\Lambda}{\sqrt{\pi}}e^{-x^2/\Lambda^2} + x\,\mathrm{erfc}(x/\Lambda) + \sum_{\zeta=\pm 1}\frac{1}{\beta}\ln\left(1 + e^{-\beta(x+\zeta\mu)}\right)$$

$$F_\beta(x) = \sum_{\zeta=\pm 1}\frac{1}{\beta}(-\zeta)\ln\left(1 + e^{-\beta(x+\zeta\mu)}\right).$$

Lemma. For integers $n, p \geqslant 0$, with n odd, and real $\ell > 0$,

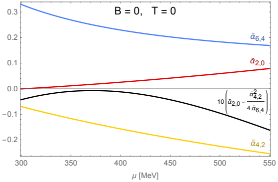
$$\int_{-\infty}^{+\infty} dk\,\frac{1}{k}\partial_k^n(k^{-1}\partial_k)^p f\left(\sqrt{k^2 + a\ell}\right) = \frac{(n-1)!!}{(n+2p-1)!!}\int_{-\infty}^{+\infty} dk\,\frac{1}{k}\partial_k^{n+2p} f\left(\sqrt{k^2 + a\ell}\right).$$

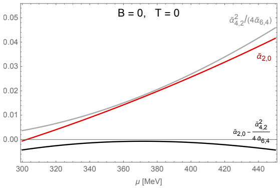
$$I_p = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} dk\,\frac{1}{k}\partial_k^{1+p}F_\alpha(k), & p = 0, 2, 4, \ldots \\ \displaystyle\int_{-\infty}^{+\infty} dk\,\frac{1}{k}\partial_k^{1+p}F_\beta(k), & p = 1, 3, 5, \ldots. \end{cases}$$

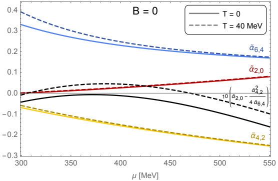
Analysis from GL Coefficients

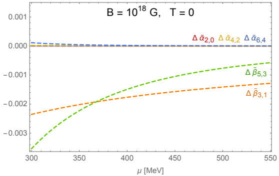
$$\left.\frac{\partial \Omega_{GL,6}}{\partial (m^2)}\right|_{m=0} = \alpha_{2,0} + \beta_{3,1} b + \alpha_{4,2} b^2 + \beta_{5,3} b^3 + \alpha_{6,4} b^4$$

$$\left.\frac{\partial \Omega^{B=0}_{GL,6}}{\partial (m^2)}\right|_{m=0,\, b=b_0} = \alpha_{2,0} - \frac{\alpha_{4,2}^2}{4\alpha_{6,4}}$$





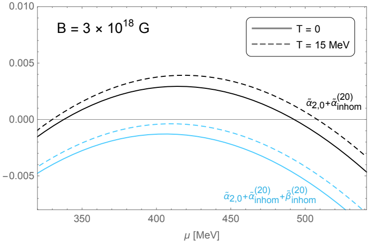




$$\left.\frac{\partial \Omega_{GL,20}}{\partial (m^2)}\right|_{m=0} = \alpha_{2,0} + \alpha^{(20)}_{\text{inhom}} + \beta^{(20)}_{\text{inhom}}$$

$$\alpha^{(20)}_{\text{inhom}} = \alpha_{4,2} b^2 + \cdots + \alpha_{20,18} b^{18}$$

$$\beta^{(20)}_{\text{inhom}} = \beta_{3,1} b + \cdots + \beta_{19,17} b^{17}$$

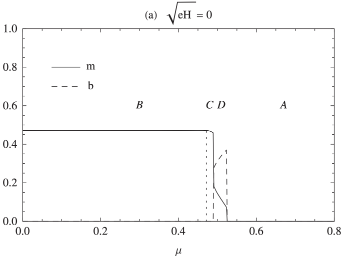


- Contribution from β coefficients (topological) pushes curves black curves down to blue curves, favoring m.
- α coefs. responsible for behavior at $B = 0$, β coefs. responsible for changes due to $B > 0$.

Solutions at $B = T = 0$

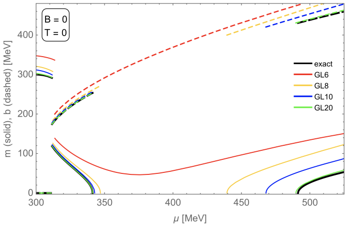
Previous results

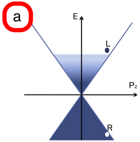
Frolov et al PRD 82 (2010) 076002



Our results

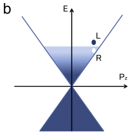
Gyory & de la Incera PRD 106 (2022) 016011





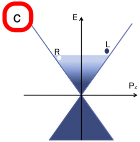
quark-antiquark

Kojo et al
NPA 843 (2010) 37



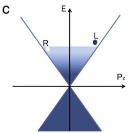
quark-hole

$(P_{\text{tot}} = 0)$



quark-hole

$(P_{\text{tot}} \neq 0)$



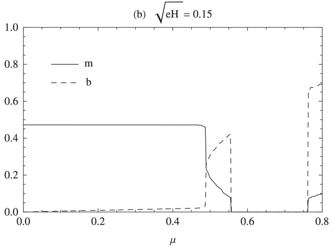
quark-quark

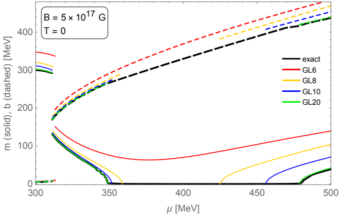
- GL approx. very good at order 20.
- μ separates naturally into four regions.
- First-order transition at $\mu = 311$ MeV.
- Second-order transitions at $\mu = 341$ MeV and 491 MeV.
- Chiral symmetry restored over region III, at intermediate μ.
- Condensate at large μ, even when $B = 0$.

Exact vs. GL Solutions at Finite B and T

Previous results

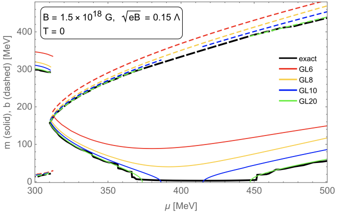
Frolov et al PRD 82 (2010) 076002

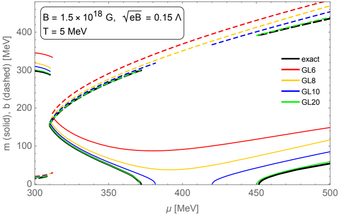




Our results

Gyory & de la Incera PRD 106 (2022) 016011





- GL approx. remains accurate at finite B and T.
- $b > 0$ for all $\mu > 0$!
- When $B > 0$, small "remnant mass" at intermediate μ.
- B favors condensate.
- T disfavors condensate.
- GL gives a smoothed average of exact solutions, without remnant mass.