Phase diagram of QCD matter with magnetic field: domain-wall Skyrmion chain in chiral soliton lattice

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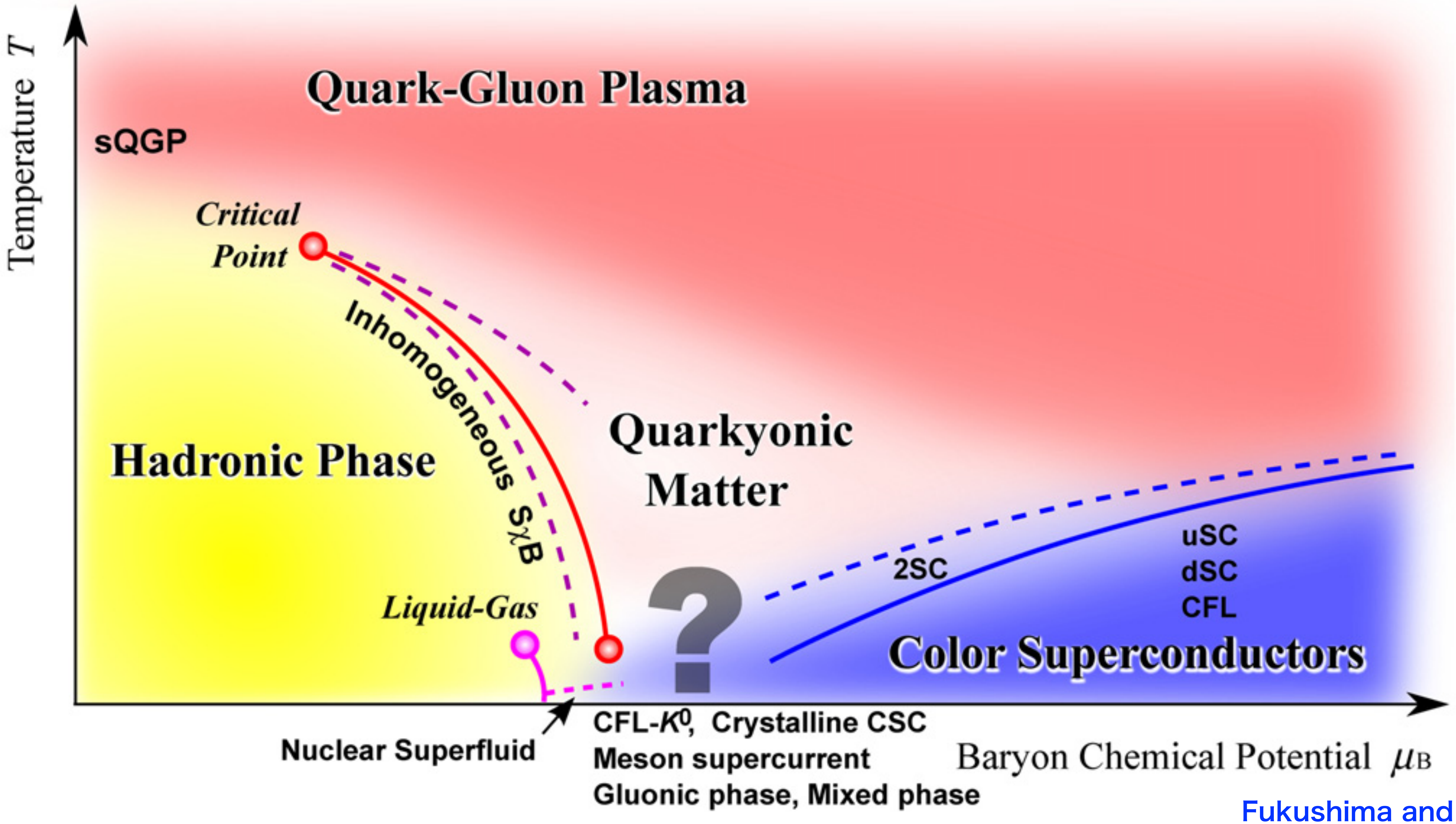
Compact stars in QCD phase diagram 2024/10/11

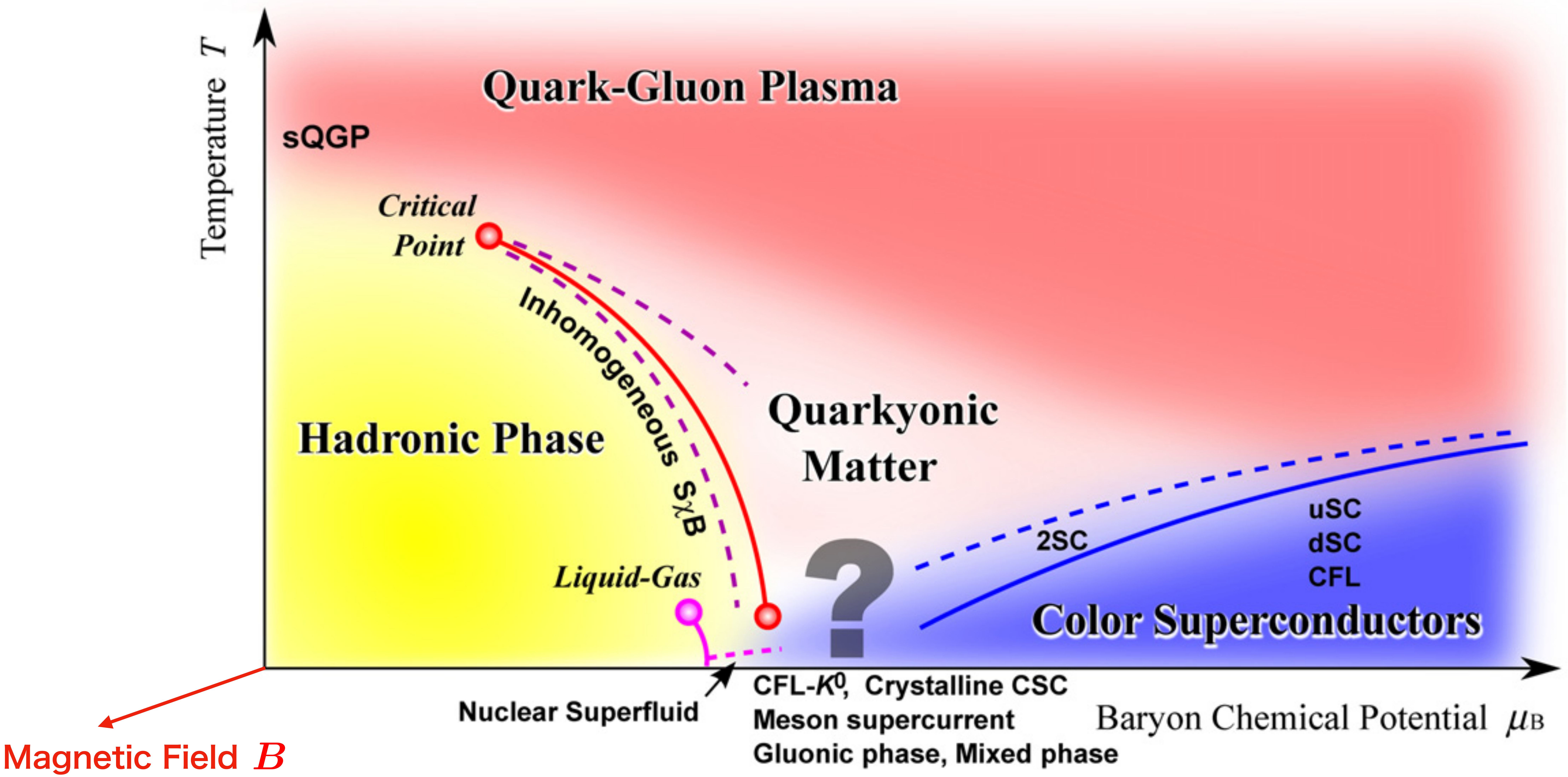
JHEP 12 (2023) 032

QCD phase diagram



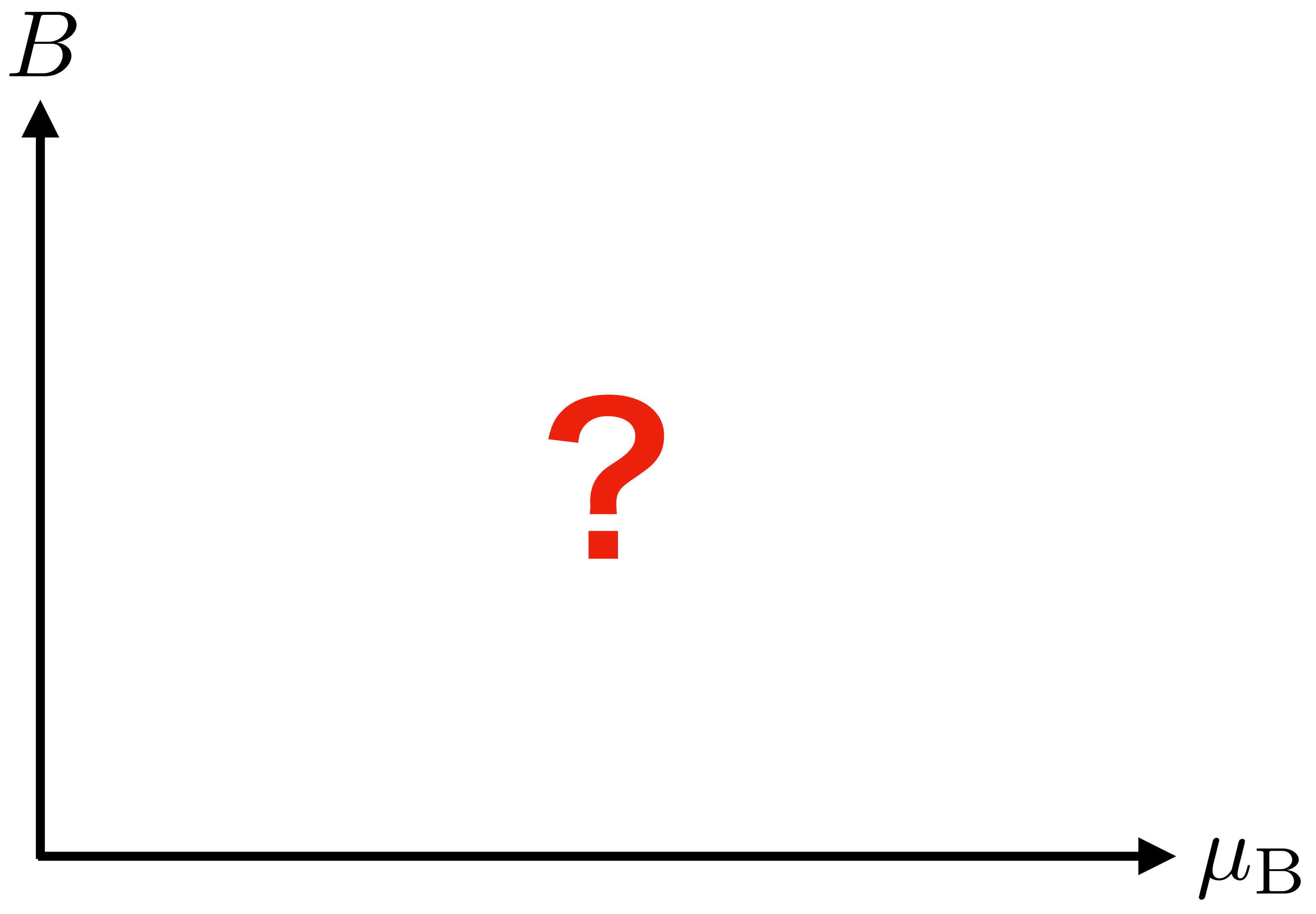
Fukushima and Hatsuda (2008)

QCD phase diagram with B



Questions

- How is the phase diagram modified by B and μ_B?
  - Consider the region where B and μ_B are small, and chiral perturbation theory is valid.
  - Temperature and isospin chemical potential are not considered.



- Skyrmion plays an important role to determine the phase structure.
  - Since pions do not carry baryon number, nothing seems to happen even if μ_B is considered.

Chiral perturbation theory

- **Order parameter is the chiral condensate:** $\langle \bar{q} q \rangle = |\langle \bar{q} q \rangle| \Sigma$

- **Nambu-Goldstone boson:** $\Sigma = \exp(\mathrm{i}\sigma_a \phi_a)\,, \quad \phi_a \equiv \pi_a / f_\pi$

- **Effective Lagrangian:** $$\mathcal{L}_{\mathrm{ChPT}} = \frac{f_\pi^2}{4} \mathrm{tr}\left(D_\mu \Sigma D^\mu \Sigma^\dagger\right) - \frac{f_\pi^2 m_\pi^2}{4}(2 - \Sigma - \Sigma^\dagger)$$

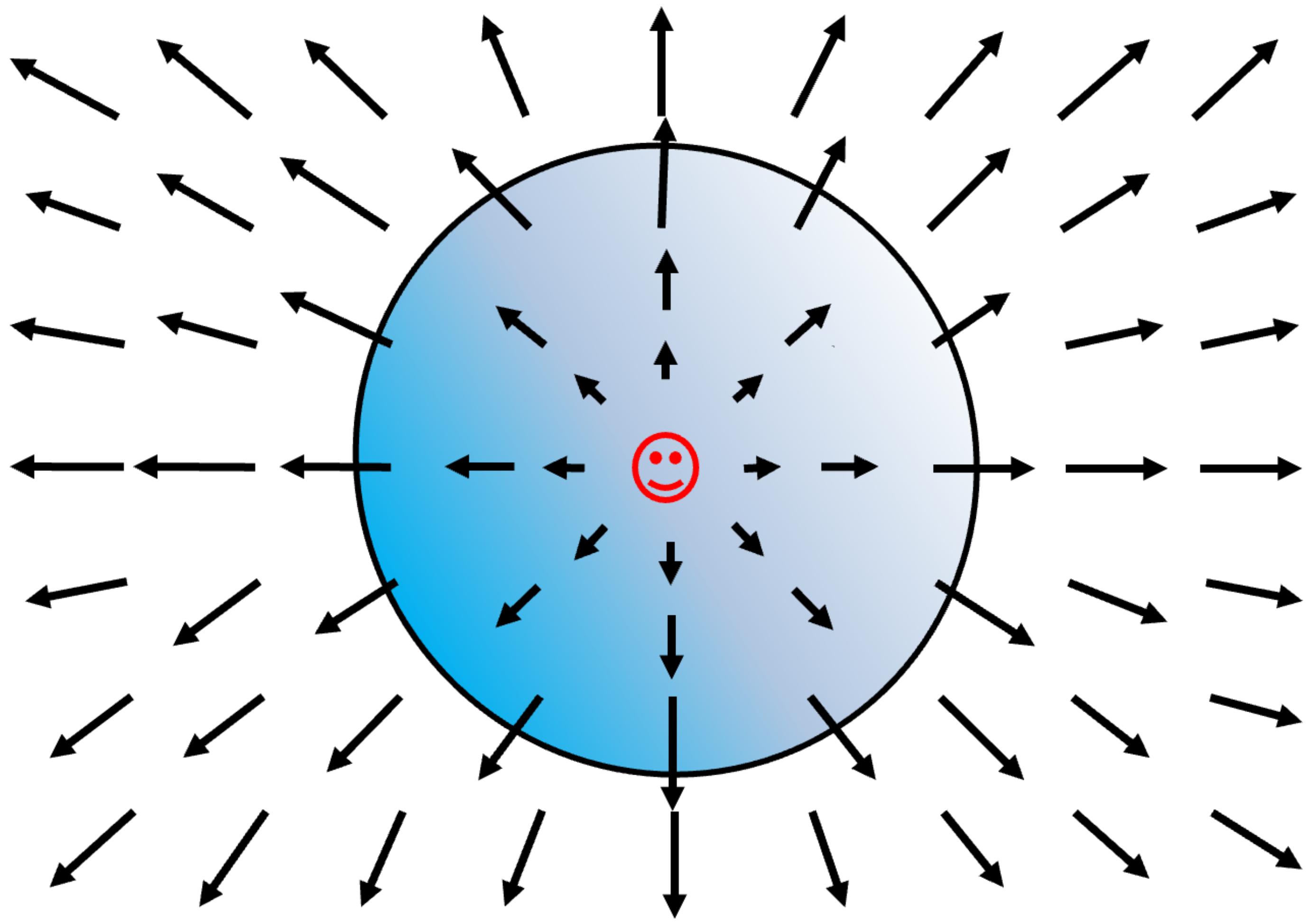
$$D_\mu \Sigma = \partial_\mu \Sigma + \mathrm{i} A_\mu [Q, \Sigma]\,, \quad Q = \mathrm{diag}(2/3, -1/3)$$

Skyrmion

- **Can the baryons be made by pions (rather than quarks)?**

- **Baryon as soliton = Skyrmion**

Skyrme (1961)



Hedgehog

- **Topological charge :**

$$j^{\mu}_{\mathrm{B}} = \frac{\epsilon^{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{tr}(\Sigma\partial_\nu\Sigma^\dagger\Sigma\partial_\alpha\Sigma^\dagger\Sigma\partial_\beta\Sigma^\dagger)$$

  - R^3 surrounds the configuration space of the pions S^3 : $\pi_3(S^3)$

ChPT w/ topological terms

- **Baryon current couples to U(1)$_\mathrm{B}$ gauge field:** $\mathcal{L}_\mathrm{B} = -A^\mu_\mathrm{B} j_{\mathrm{B}\mu}\,, \quad A^\mu_\mathrm{B} = (\mu_\mathrm{B}, \mathbf{0})$

- **“trial and error” U(1)$_\mathrm{em}$ gauging while preserving baryon number conservation.**

$$j^\mu_\mathrm{B} = \underbrace{-\frac{\epsilon^{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{tr}\{L_\nu L_\alpha L_\beta}_{\text{Skyrimon charge}} - \underbrace{3ie\partial_\nu\left[A_\alpha Q(L_\beta + R_\beta)\right]}_{\mathrm{U(1)_{em}}\text{ gauged part}}\}$$

$$L_\mu \equiv \Sigma\partial_\mu\Sigma^\dagger\,,\ R_\mu \equiv \partial_\mu\Sigma^\dagger\Sigma$$

$$Q = \mathrm{diag}(2/3, -1/3)$$

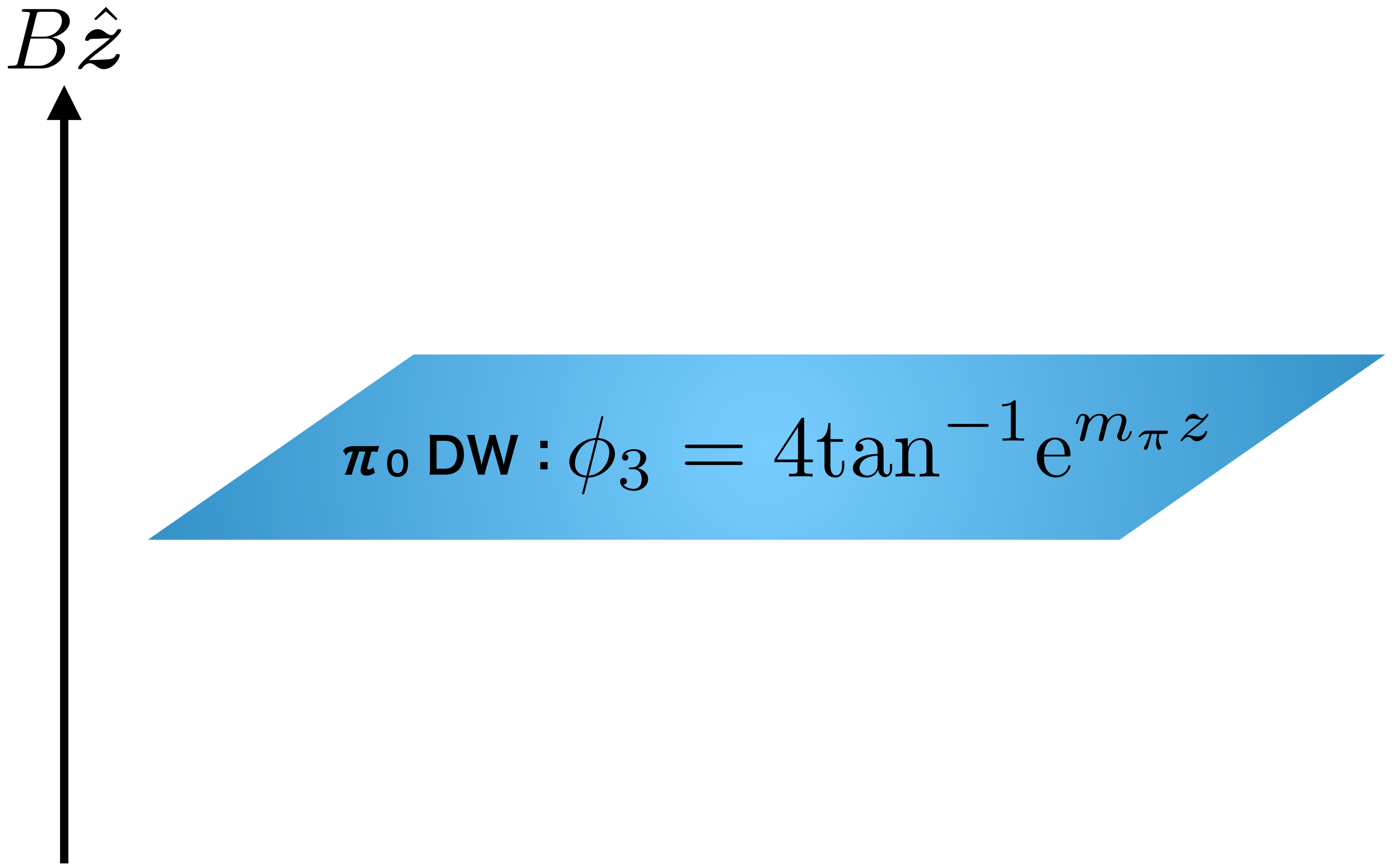
Son and Stephanov (2008); Goldstone and Wilczek (1981); Witten (1983)

Chiral Soliton Lattice

- **SG theory w/ total derivative**

$$\mathcal{H} = \frac{f_\pi^2}{2}(\partial_z \phi_3)^2 + f_\pi^2 m_\pi^2 (1 - \cos\phi_3) - \frac{e\mu_{\rm B}}{4\pi^2} B \partial_z \phi_3$$

- **Consider only π_0 :** $\Sigma = {\rm e}^{{\rm i}\phi_3 \tau_3}$
- **z-dependence of ϕ_3 is nontrivial.**



π_0 DW : $\phi_3 = 4\tan^{-1} {\rm e}^{m_\pi z}$

- **EOM :** $\partial_z^2 \phi_3 = m_\pi^2 \sin\phi_3$

- **Energy :** $E = \int_{-\infty}^{\infty} {\rm d}z\, \mathcal{H} = 8 m_\pi^2 f_\pi - \frac{e\mu_{\rm B} B}{2\pi}$

- **Critical B :** $B_{\rm CSL} = \frac{16\pi m_\pi f_\pi^2}{e\mu_{\rm B}}$

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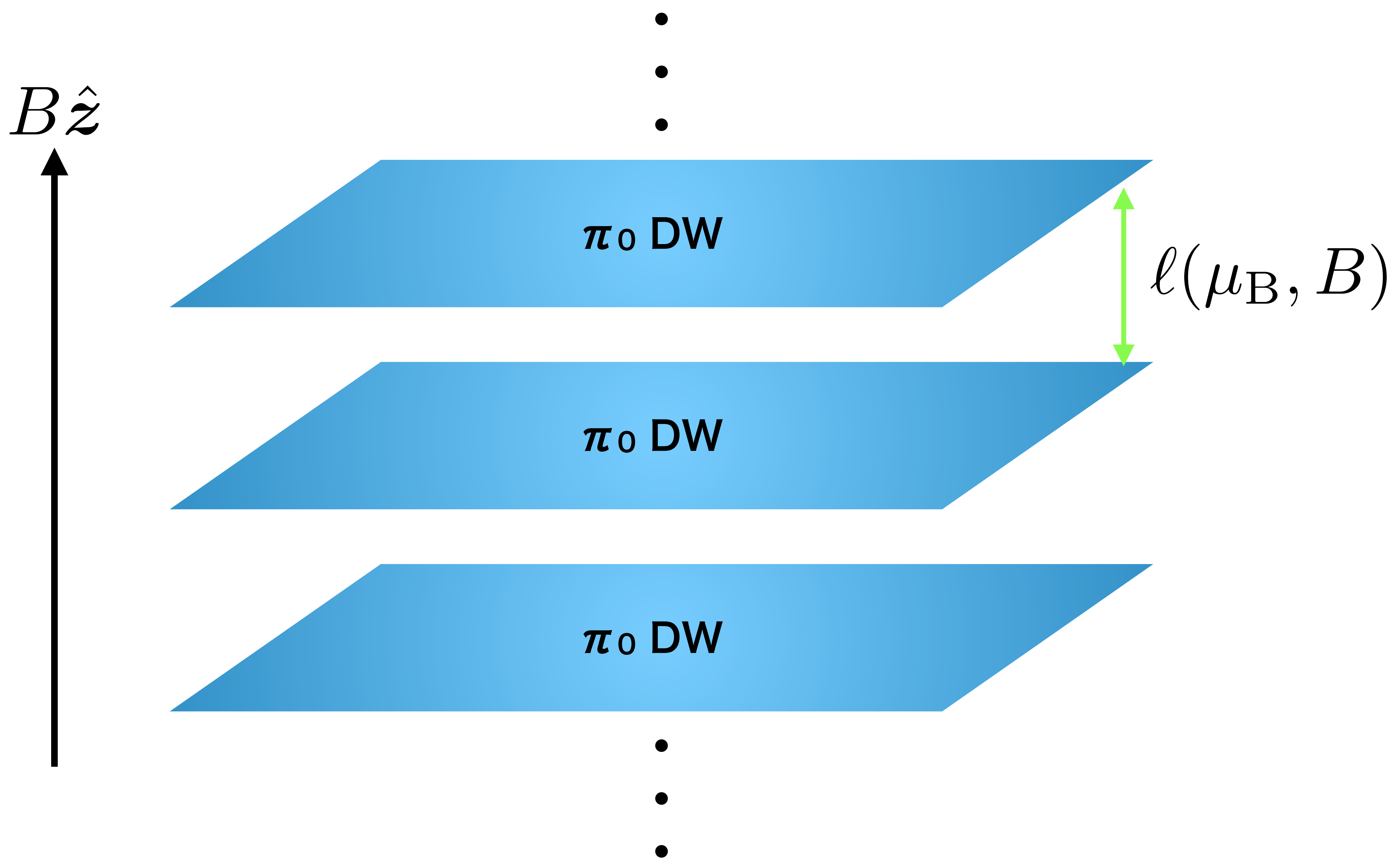


- Pack many DWs in ground state!
- Impossible to pack due to the repulsive force.

Dautry and Nyman (1979); Hatsuda (1986); Son and Stephanov (2008); Nishiyama, Kawasawa and Tatsumi (2015); Brauner and Yamamoto (2017)
See also talks by Evans, Incera, Gyory.

Non-Abelian soliton

- **The single soliton:**

$$\Sigma_0 = \mathrm{e}^{\mathrm{i}\sigma_3\theta}\,,\;\; \theta = 4\tan^{-1}\mathrm{e}^{m_\pi z}$$

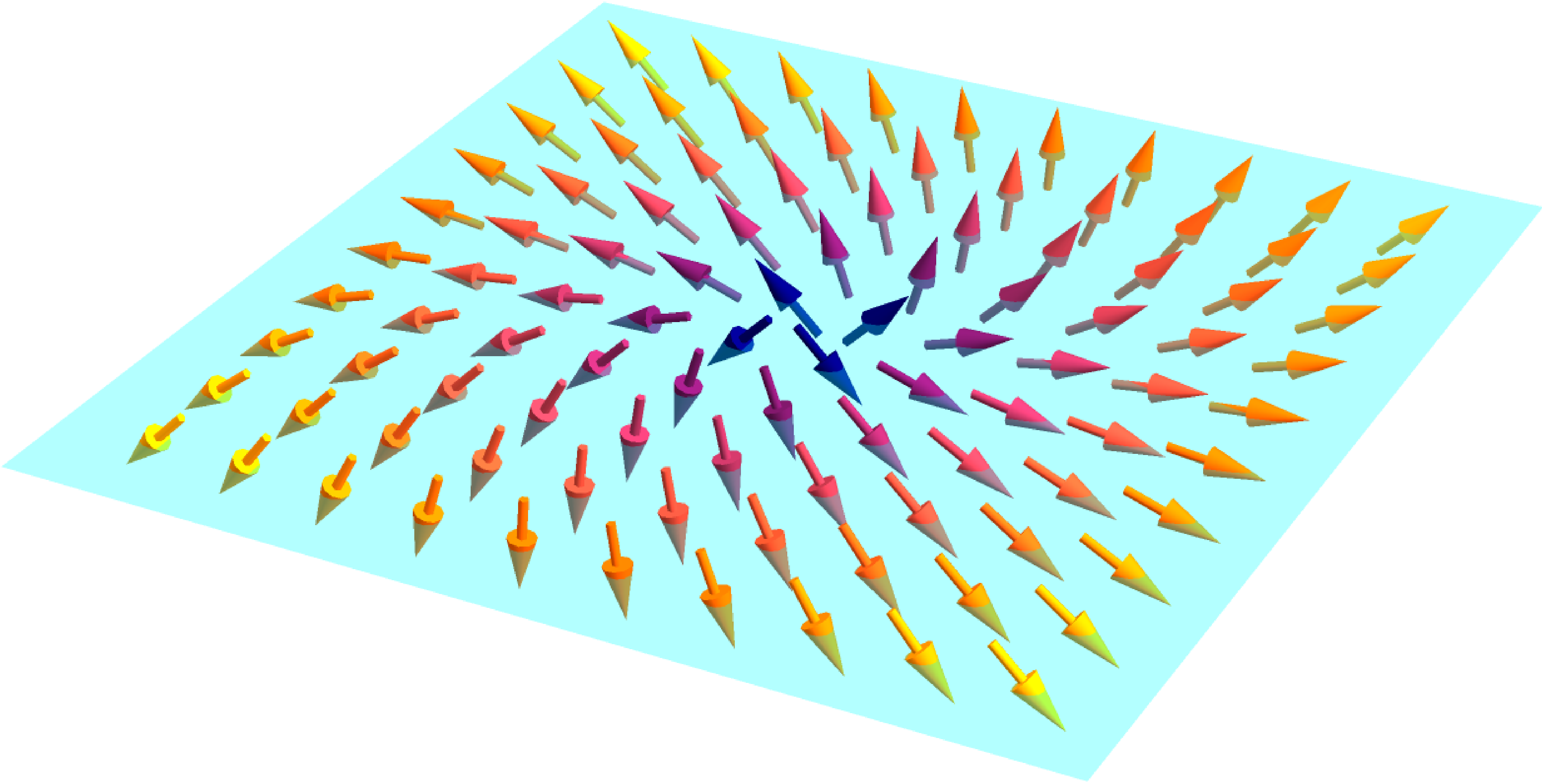
- **More general solution :**

$$\Sigma = g\Sigma_0 g^\dagger = \exp(\mathrm{i}\theta g\tau_3 g^\dagger)$$

- **SSB of SU(2)$_V$ → U(1)**

- **Σ_0 is invariant under** $g = \mathrm{e}^{\mathrm{i}\tau_3\theta}$

S^2 moduli on the domain wall



- **The collective coordinate :** $\phi \in \mathbb{C}^2\,, \qquad \phi^\dagger\phi = 1$

Nitta (2015); Eto and Nitta (2015)

$$g\sigma_3 g^\dagger = 2\phi\phi^\dagger - 1$$

EFT for S^2 moduli

- **Effective Lagrangian :** $\mathcal{L}_{\rm EFT} = \mathcal{L}_{\rm const} + \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm topo}$ Eto, KN and Nitta, JHEP 12 (2023) 032
- **Kinetic term :** $\mathcal{L}_{\rm kin} = \underline{\mathcal{C}(\ell)}[(\phi^\dagger D_\alpha \phi)^2 + D^\alpha \phi^\dagger D_\alpha \phi]$
- **Topological terms :** $\mathcal{L}_{\rm topo} = -2\mu_{\rm B} q + \frac{e\mu_{\rm B}}{2\pi}\epsilon^{03jk}\partial_j[A_k(1-n_3)]$

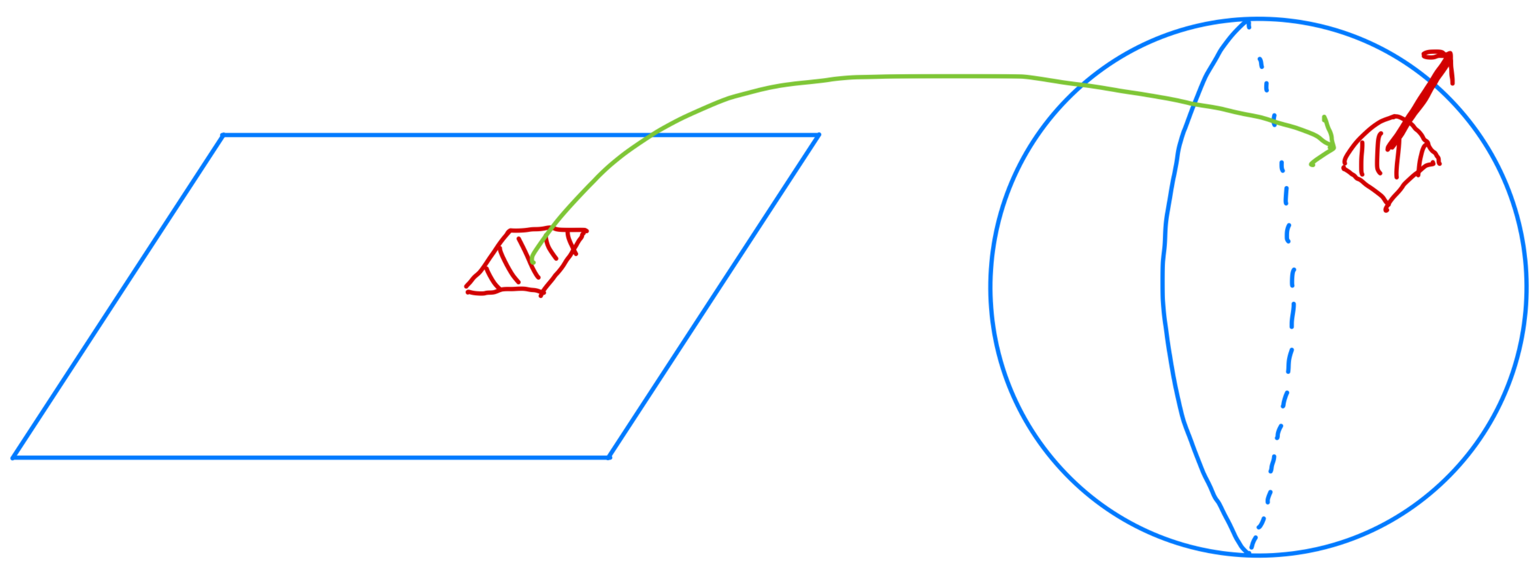
O(3) nonlinear sigma model

$n_a \equiv \phi^\dagger \sigma_a \phi \quad |\boldsymbol{n}| = 1$

- The red term stabilizes the configuration with finite k!

$\pi_2(S^2)$ topological charge (counting how many times xy plane covers S2 moduli)

$$k = \int \mathrm{d}^2x\, q$$
$$= \frac{1}{4\pi}\int \boldsymbol{n}\cdot\left(\frac{\partial\boldsymbol{n}}{\partial x}\mathrm{d}x \times \frac{\partial\boldsymbol{n}}{\partial y}\mathrm{d}y\right)$$
$$\in \mathbb{Z}$$



Bogomol’nyi bound

- **Baby Skyrmion naturally appears when minimizing the Hamiltonian.**

$$\mathcal{H}_{\rm DW} = \frac{\mathcal{C}(\kappa)}{4}\partial_i \boldsymbol{n}\cdot\partial_i \boldsymbol{n} + 2\mu_{\rm B} q - \frac{e\mu_{\rm B}}{2\pi}\epsilon^{03jk}\partial_j\left[A_k(1-n_3)\right]$$

- **Completing the square of the kinetic term is useful!**

$$(\partial_i \boldsymbol{n})^2 = \frac{1}{2}\underbrace{(\partial_i \boldsymbol{n} \pm \epsilon_{ij}\boldsymbol{n}\times\partial_j \boldsymbol{n})^2}_{=0} \pm 8\pi q \geq \pm 8\pi q$$

→ **BPS equation** → **Baby Skyrmion!**

- **Critical $\boldsymbol{\mu}_{\rm B}$:**

$$E_{\rm DW} \geq \underline{2\pi\mathcal{C}(\kappa)|k| + 2\mu_{\rm B}k} - \underline{\frac{e\mu_{\rm B}}{2\pi}\int d^2x\,\epsilon^{03jk}\partial_j\left[A_k(1-n_3)\right]}$$

$$\mu_{\rm B} > \pi\mathcal{C}(\kappa)$$

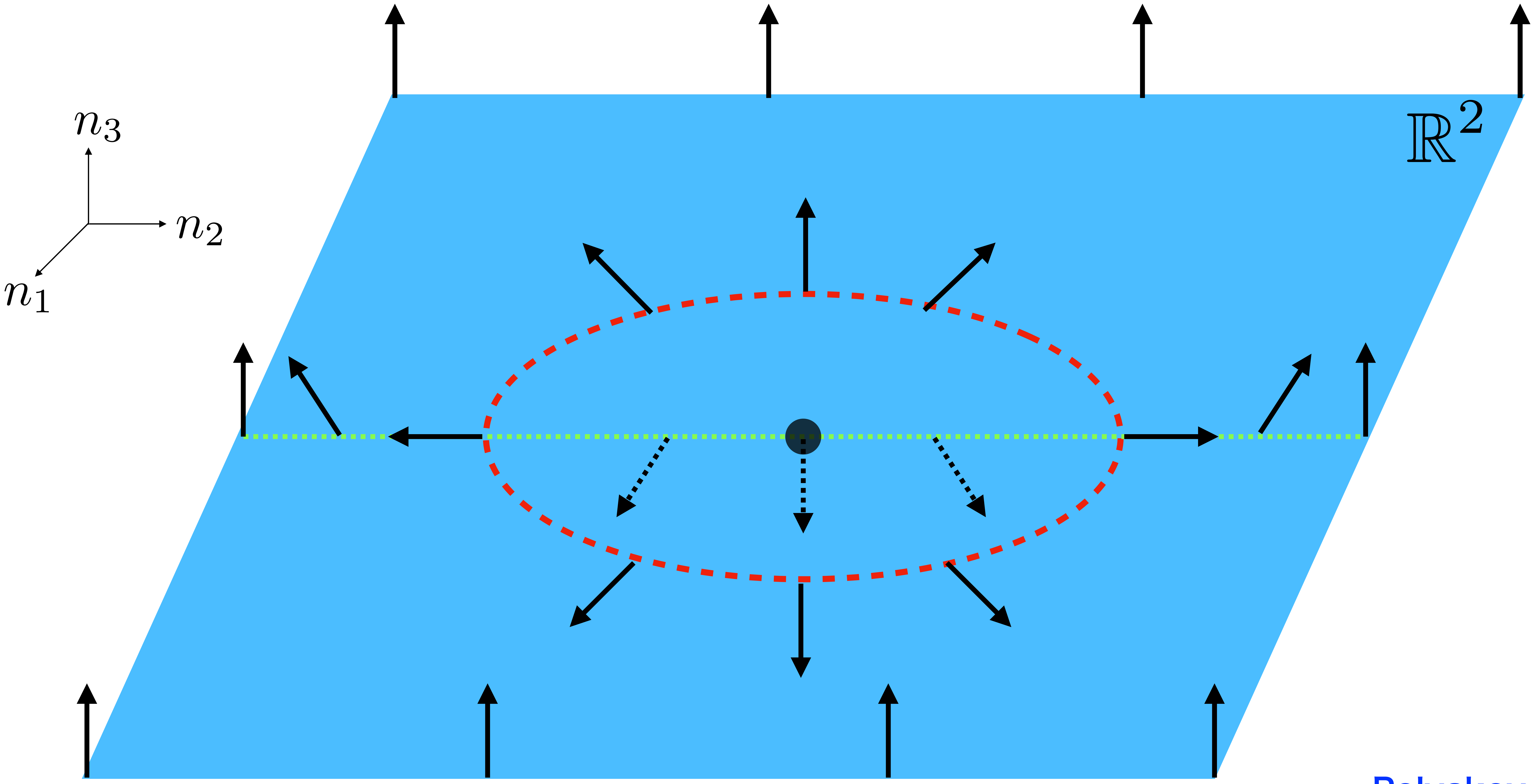
The total energy is negative $\mu > \mu_c$, and baby Skyrmion appears in the ground state!

Some constraints on the lump

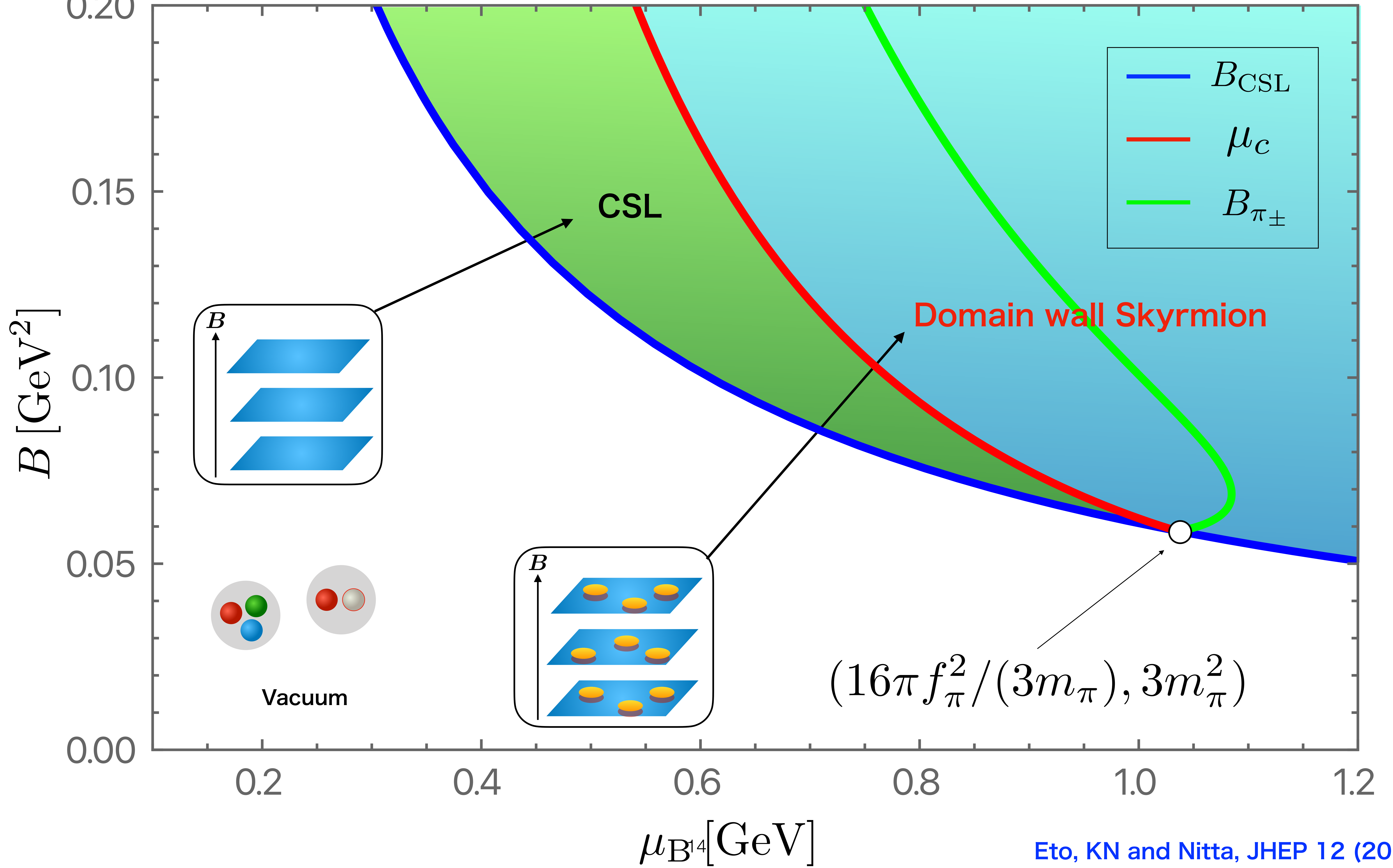
Eto, KN and Nitta, JHEP 12 (2023) 032

Baby Skyrmion

- **Configuration on DW surrounding S² :** ↑ = n_a $\boldsymbol{n^2} = 1$



Polyakov and Belavin (1975)



Eto, KN and Nitta, JHEP 12 (2023) 032

Summary

- Pions couple to baryon as Skyrmion.

- At B>B_c, the stack of π_0 DWs is energetically stable.

- At $\mu > \mu_c$, baby Skyrmion appears on π_0 DWs.

Thank you for your attention!