素粒子物理学の進展2024 (PPP2024)
2024/08/23

トレースス保存則を用いたオービフォールド高次元理論における境界条件の完全な分類

Kota Takeuchi (Hiroshima U.)

Based on:

KT, T. Inagaki, PTEP2024, 033B03 (arXiv:2401.09809)

KT, T. Inagaki, PTEP2024, 063B04 (arXiv:2404.19411)

Today's Talk

What?

In gauge theories with S^1/Z_2 and T^2/Z_m orbifolded extra-spaces,

the classification of boundary conditions have been completed by TCLs.

Why? How?

01. **What:** Background & Set-Up

02. **Why:** The Arbitrariness Problem & Our Work

03. **How:** New Classification Method & Results

04. Summary & Future Work

Today's Talk

What?

In gauge theories with S^1/Z_2 and T^2/Z_m orbifolded extra-spaces,

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Why? How?

01

Background: Higher-dim gauge theory

Higher-dimensional gauge theory is a framework beyond the Standard Model (BSM)

ex.) Gauge-Higgs Unification (GHU) scenario is working well.

Hosotani (1983)
Hosotani (1989)

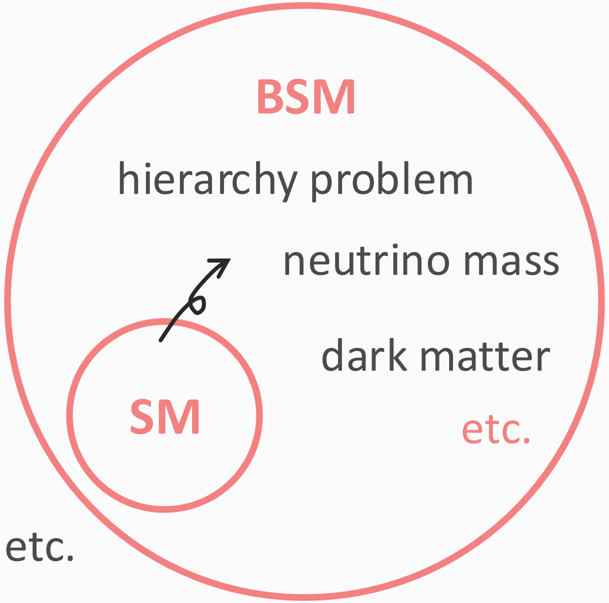
5D gauge field: $A_M = (A_\mu , A_5)$

Higgs (A_5)

No Higgs potential (No quadratic divergence)

Gauge SSB (Hosotani Mechanism)

Dark Matter, Strong CP, Baryon Asymmetry, Neutrino Mass etc.



Various Models

$(4+d)$-dimensional gauge theory:

$$S=\int d^4x d^d y\, \mathcal{L}_{4+d}=\int d^4x d^d y \left\{-\frac{1}{4}F_{MN}F^{MN}+\bar{\psi}\, i\Gamma^M(\partial_M - igA_M)\psi\right\}$$

1. Dimension d
2. Compactification
3. Boundary Conditions (BCs)
4. Gauge Group G

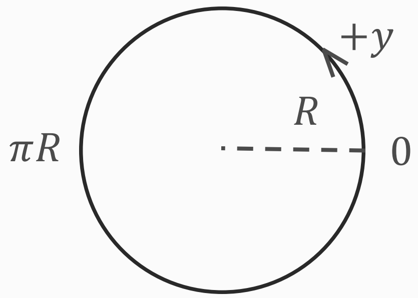
$U(1)$ on S^1	Hatanaka et al. (1998)
$SU(3)$ on S^1/Z_2	Kubo et al. (2002)
$U(3)\times U(3)$ on T^2/Z_2	Hosotani et al. (2005)
$U(3)$ on T^2/Z_3	Matsumoto et al. (2016)
$SU(6)$ on S^1/Z_2	Maru et al. (2022)

etc.

S^1/Z_2 Orbifold

S^1 sphere

cannot achieve chiral 4D theory

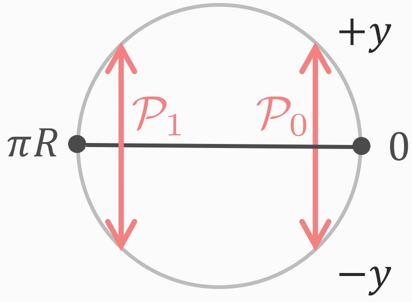


translation symmetry

$$\mathcal{T}: y \to y + 2\pi R$$

S^1/Z_2 orbifold

can achieve chiral 4D theory

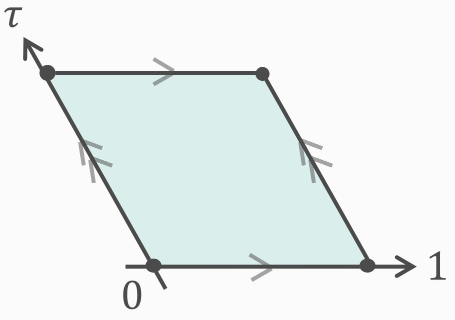


parity around the fixed points

$$\mathcal{P}_0: y \to -y \quad \mathcal{P}_1: y \to 2\pi R - y$$

T^2/Z_m Orbifold ($m = 2,3,4,6$)

T^2 torus

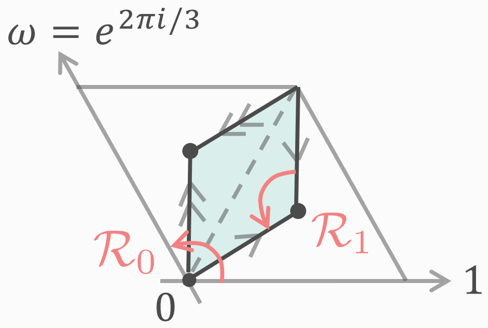


translation symmetry

$$\mathcal{T}_1 : z \to z + 1$$

$$\mathcal{T}_2 : z \to z + \tau$$

T^2/Z_3 orbifold



rotation around the fixed points

$$\mathcal{R}_0 : z \to \omega z$$

$$\mathcal{R}_1 : z \to \omega z + 1$$

Boundary Conditions (BCs)

Geometric Symmetry

$\mathcal{P}_0 : y \to -y$

$\mathcal{P}_1 : y \to 2\pi R - y$

→

Boundary conditions (BCs)

$A_\mu(x,-y) = P_0 A_\mu(x,y) P_0^\dagger$

$A_\mu(x,2\pi R - y) = P_1 A_\mu(x,y) P_1^\dagger$

The BCs are characterized by representation matrices.

S^1/Z_2: (P_0, P_1)

$U(N)$ matrices

$P_0^2 = P_1^2 = 1$

Our Set-up

$(4+d)$-dimensional gauge theory:

$$S = \int d^4x d^d y \, \mathcal{L}_{4+d} = \int d^4x d^d y \left\{ -\frac{1}{4} F_{MN} F^{MN} + \bar{\psi} \, i\Gamma^M (\partial_M - igA_M)\psi \right\}$$

1. Dimension d → 1,2
2. Compactification → orbifold
3. Boundary Conditions (BCs)
4. Gauge Group G → $SU(N)$, $U(N)$

$d = 1$:

S^1/Z_2

(P_0, P_1)

$d = 2$: $(z = y^1 + \tau y^2)$

T^2/Z_m $(m = 2,3,4,6)$

(R_0, R_1) (R_2 for $m = 2$)

▶ Classification of BCs

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What?

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the classification of boundary conditions have been completed by TCLs.

Why? How?

02

Arbitrariness problem of BCs

ex.) $G = SU(3)$ on S^1/Z_2 Kubo et al. (2002)

model $\mathcal{A}$

$$P_0 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad P_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\mathcal{g}_{BC}^{(4D)} = SU(3)$$

$$m_\phi^{(n)} = \left(\frac{n}{R}, \frac{n}{R}, \frac{n}{R}\right)$$

model $\mathcal{B}$

$$P_0 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \quad P_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

$$\mathcal{g}_{BC}^{(4D)} = U(1) \times U(1)$$

$$m_\phi^{(n)} = \left(\frac{n}{R}, \frac{n+1/2}{R}, \frac{n+1/2}{R}\right)$$

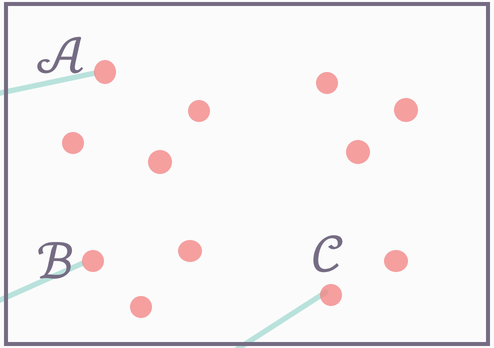
model $\mathcal{C}$

$$P_0 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \quad P_1 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$$

$$\mathcal{g}_{BC}^{(4D)} = U(1)$$

$$m_\phi^{(n)} = \left(\frac{n+1/2}{2R}, \frac{n+1/2}{2R}, \frac{n}{R}\right)$$

choices for BCs



Setting them by hand is not ideal.

Which BCs should be imposed?

↓

the Arbitrariness Problem of BCs

Gauge Transformations

Gauge Transformations for BCs:

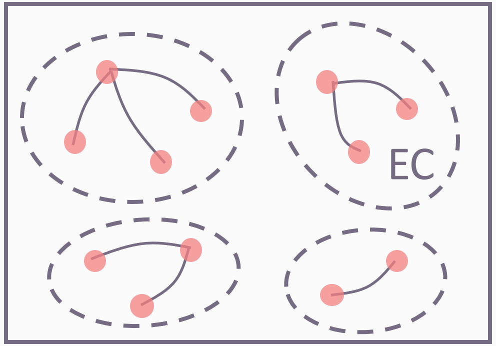
$$P_0' = \Omega(-y) P_0 \Omega^\dagger(y)$$

$$P_1' = \Omega(2\pi R - y) P_1 \Omega^\dagger(y)$$

$U(N)$ constant
$P_0'^2 = P_1'^2 = 1$

$U(N)$ constant
$P_0^2 = P_1^2 = 1$

choices for BCs



The connected BCs construct Equivalence Classes (ECs):

Haba et al.(2003)
Haba et al.(2004)

physical equivalent

$$(P_0', P_1') \sim (P_0, P_1)$$

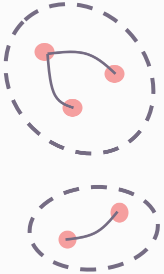
History of ECs research

- Hosotani (1989)
- Haba, Harada, Hosotani, Kawamura (2003)
- Haba, Hosotani, Kawamura (2004)
- Hosotani, Noda, Takenaga (2004)
- Kawamura, Kinami, Miura (2008)
- Kawamura, Miura (2009)
- Kawamura, Nishikawa (2020)
- Kawamura, Kodaira, Kojima, Yamashita (2023)

\# Which BCs are connected?

\# How is each class characterized?

\# How many ECs are there?



Motivation for classifying ECs:

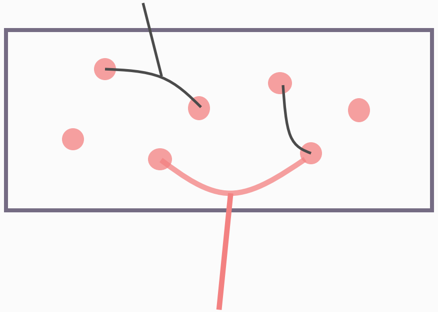
1. Progress in resolving the arbitrariness of BCs
2. Systematic understanding of models

Our Work

Previous Method

Classified by Finding $\Omega(y)$

$$\Omega(y) = \exp\left[ic_1 y T^a\right]$$



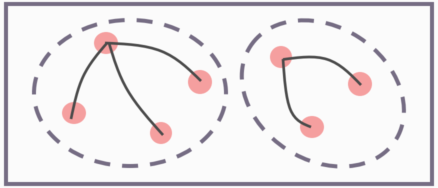
Other transformations?

$$\Omega(y) \overset{?}{=} \exp\left[if^a(y)T^a\right]$$

Our New Method

By Trace Conservation Laws (TCLs)

Classified without specifying $\Omega(y)$



Takeuchi, Inagaki, PTEP2024, 033B03 (arXiv:2401.09809) → $S^1/Z_2, T^2/Z_3$

Takeuchi, Inagaki, PTEP2024, 063B04 (arXiv:2404.19411) → T^2/Z_m $(m = 2,3,4,6)$

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03

Diagonal set of BCs

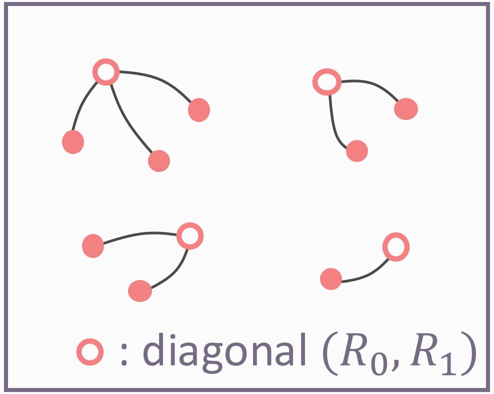
We classify the BCs on T^2/Z_3.

(R_0, R_1)

- # $U(N)$ matrices
- # $R_0^3 = R_1^3 = (R_1 R_0)^3 = 1$
- # eigenvalues $\omega, \omega^2, 1$ $\;(\omega = e^{2\pi i/3})$
- # simultaneously diagonalizable.

Kawamura et al.(2023)

T^2/Z_3



→ just examine diagonal (R_0, R_1) $\leftrightarrow$ diagonal (R'_0, R'_1)

$$R_0 = \begin{pmatrix} \omega & \\ & 1 \end{pmatrix} \; R_1 = \begin{pmatrix} \omega & \\ & 1 \end{pmatrix} \leftrightarrow R'_0 = \begin{pmatrix} \omega & \\ & \omega^2 \end{pmatrix} \; R'_1 = \begin{pmatrix} 1 & \\ & \omega^2 \end{pmatrix}$$

etc.

Candidates for connection

A lot of candidates are considered…

2×2

$$R_0 = (\omega, 1) \longleftrightarrow R_0 = (\omega, \omega^2)$$
$$R_1 = (\omega, 1) \longleftrightarrow R_1 = (1, \omega^2)$$

$$R_0 = (\omega, 1) \longleftrightarrow R_0 = (\omega, 1)$$
$$R_1 = (\omega, 1) \longleftrightarrow R_1 = (1, \omega)$$

etc.

3×3

$$R_0 = (\omega, \omega^2, 1) \longleftrightarrow R_0 = (\omega, \omega^2, 1)$$
$$R_1 = (\omega, \omega^2, 1) \longleftrightarrow R_1 = (\omega^2, \omega, 1)$$

$$R_0 = (\omega, \omega^2, 1) \longleftrightarrow R_0 = (\omega, \omega^2, 1)$$
$$R_1 = (\omega, \omega^2, 1) \longleftrightarrow R_1 = (\omega^2, 1, \omega)$$

etc.

4×4, 5×5, ……

Trace Conservation laws (TCLs)

BCs-connecting gauge transformation is the identity of z !

$U(N)$ constant
$R_0'^3 = 1$

$$R_0' = \Omega(\omega z) R_0 \, \Omega^\dagger(z)$$

$U(N)$ constant
$R_0^3 = 1$

general gauge transf.

$$\begin{aligned} \mathrm{tr} R_0' &= \mathrm{tr}\left[\Omega(\omega z)\, R_0\, \Omega^\dagger(z)\right] \\ &= \mathrm{tr}\left[\Omega^\dagger(z)\, \Omega(\omega z)\, R_0\right] \\ &\neq \mathrm{tr} R_0 \end{aligned}$$

$$\begin{aligned} \mathrm{tr} R_0' &= \mathrm{tr}\left[\Omega(\omega 0)\, R_0\, \Omega^\dagger(0)\right] \\ &= \mathrm{tr}\left[\Omega^\dagger(0)\, \Omega(0)\, R_0\right] \\ &= \mathrm{tr} R_0 \end{aligned}$$

→ $\forall z, \quad \mathrm{tr} R_0' = \mathrm{tr} R_0$

TCLs for Orbifolds

The number of TCLs

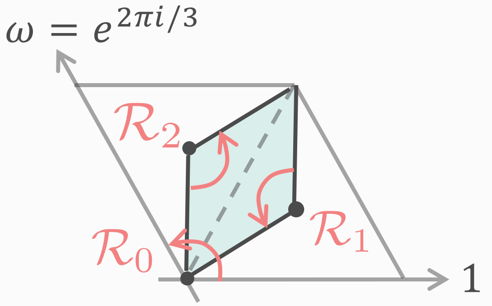
$$\mathrm{tr} R_0' = \mathrm{tr} R_0$$

$$\mathrm{tr} R_1' = \mathrm{tr} R_1$$

$$\mathrm{tr}(R_1' R_0') = \mathrm{tr}(R_1 R_0)$$

$\sim$

The number of fixed points



final point = initial point

$R_0' = \Omega(\omega z) R_0 \Omega^\dagger(z)$	$\omega z = z$	for $z = 0$
$R_1' = \Omega(\omega z + 1) R_1 \Omega^\dagger(z)$	$\omega z + 1 = z$	for $z = \frac{2+\omega}{3}$
$R_2' = \Omega(\omega z + 1 + \omega) R_2 \Omega^\dagger(z)$	$\omega z + 1 + \omega = z$	for $z = \frac{1+2\omega}{3}$

arXiv:2404.19411

Constraints from TCLs

Most patterns are prohibited by the TCLs!

2×2

$R_0 = (\omega, 1)$	$\overset{\times}{\longleftrightarrow}$	$R_0 = (\omega, \omega^2)$
$R_1 = (\omega, 1)$	$\overset{\times}{\longleftrightarrow}$	$R_1 = (1, \omega^2)$
$R_{01} = (1, \omega^2)$	$\overset{\times}{\longleftrightarrow}$	$R_{01} = (\omega, \omega)$

$R_0 = (\omega, 1)$	$\overset{\circ}{\longleftrightarrow}$	$R_0 = (\omega, 1)$
$R_1 = (\omega, 1)$	$\overset{\circ}{\longleftrightarrow}$	$R_1 = (1, \omega)$
$R_{01} = (\omega^2, 1)$	$\overset{\times}{\longleftrightarrow}$	$R_{01} = (\omega, \omega)$

etc.

3×3

$R_0 = (\omega, \omega^2, 1)$	$\overset{\circ}{\longleftrightarrow}$	$R_0 = (\omega, \omega^2, 1)$
$R_1 = (\omega, \omega^2, 1)$	$\overset{\circ}{\longleftrightarrow}$	$R_1 = (\omega^2, \omega, 1)$
$R_{01} = (\omega^2, \omega, 1)$	$\overset{\times}{\longleftrightarrow}$	$R_{01} = (1, 1, 1)$

$R_0 = (\omega, \omega^2, 1)$	$\overset{\circ}{\longleftrightarrow}$	$R_0 = (\omega, \omega^2, 1)$
$R_1 = (\omega, \omega^2, 1)$	$\overset{\circ}{\longleftrightarrow}$	$R_1 = (\omega^2, 1, \omega)$
$R_{01} = (\omega^2, \omega, 1)$	$\overset{\circ}{\longleftrightarrow}$	$R_{01} = (1, \omega^2, \omega)$

etc.

→ $N \times N$ cases have been sufficiently classified by the TCLs!

Results: T^2/Z_3 Equiv Relations

$\mathrm{N} \times N$: Only allowed the repetitions of the permutation of 3 eigenvalues!

R_0: $(\omega, \omega^2, 1)$ $\longleftrightarrow$ $(\omega, \omega^2, 1)$ $\longleftrightarrow$ $(\omega, \omega^2, 1)$

R_1: $(\omega, \omega^2, 1)$ $\longleftrightarrow$ $(\omega^2, 1, \omega)$ $\longleftrightarrow$ $(1, \omega, \omega^2)$

$$\Omega(x^\mu, z) = \exp\left[-\frac{2\pi i}{3}\left(zY + \bar{z}Y^\dagger\right)\right], \qquad Y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Results: Equivalent Relations

$N \times N$: Only allowed the repetitions of the permutation of,

S^1/Z_2:

$$P_0 : (+1,-1) \quad P_1 : (+1,-1) \longleftrightarrow P'_0 : (+1,-1) \quad P'_1 : (-1,+1).$$

T^2/Z_3:

$$R_0 : (\omega,\omega^2,1) \quad R_1 : (\omega,\omega^2,1) \longleftrightarrow R_0 : (\omega,\omega^2,1) \quad R_1 : (\omega^2,1,\omega) \longleftrightarrow R_0 : (\omega,\omega^2,1) \quad R_1 : (1,\omega,\omega^2).$$

T^2/Z_2:

$$R_0 : (+,-) \quad R_1 : (+,-) \quad R_2 : (+,-) \longleftrightarrow R'_0 : (+,-) \quad R'_1 : (-,+) \quad R'_2 : (+,-)$$

$$R_0 : (+,-) \quad R_1 : (+,-) \quad R_2 : (+,-) \longleftrightarrow R'_0 : (+,-) \quad R'_1 : (+,-) \quad R'_2 : (-,+)$$

T^2/Z_4:

$$R_0 : (+1,-1,+i,-i) \quad R_1 : (+1,-1,+i,-i) \longleftrightarrow R'_0 : (+1,-1,+i,-i) \quad R'_1 : (-1,+1,-i,+i).$$

T^2/Z_6: No existence.

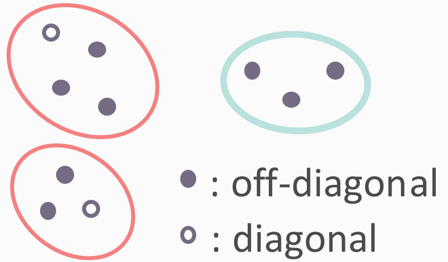
Results: The number of ECs

The total numbers of the diagonal and off-diagonal ECs with $G = SU(N)$ and $U(N)$

ex.) T^2/Z_4:

diagonal ECs: $\frac{1}{180}(N+1)(N+2)^2(N+3)(N^2+4N+15)$

off-diagonal ECs: $\frac{1}{1260}(N+3)(N+2)(N+1)N(N-1)(N^2+2N+13)$



	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$\cdots$
S^1/Z_2	$4+0$	$9+0$	$16+0$	$25+0$	$36+0$	$49+0$	$\cdots$
T^2/Z_2	$8+0$	$33+0$	$96+0$	$225+0$	$456+0$	$833+0$	$\cdots$
T^2/Z_3	$9+0$	$45+0$	$163+0$	$477+0$	$1197+0$	$2674+0$	$\cdots$
T^2/Z_4	$8+0$	$36+2$	$120+16$	$329+74$	$784+256$	$1680+732$	$\cdots$
T^2/Z_6	$6+0$	$21+3$	$56+20$	$126+81$	$252+252$	$462+663$	$\cdots$

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In gauge theories with S^1/Z_2 and T^2/Z_m orbifolded extra-spaces,

the classification of boundary conditions have been completed by TCLs.

Why? How?

04

01. **What:** Background & Set-Up
02. **Why:** The Arbitrariness Problem & Our Work
03. **How:** New Classification Method & Results
04. Summary & Future Work

Summary

In S^1/Z_2 and T^2/Z_m $(m = 2,3,4,6)$ orbifolded $SU(N)$ and $U(N)$ gauge theories,

Trace Conservation Laws have completed the classification of boundary conditions.

Previous work

	Classification
S^1/Z_2	○ (△)
T^2/Z_2	△
T^2/Z_3	△
T^2/Z_4	×
T^2/Z_6	×

Our work

	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$\cdots$
S^1/Z_2	$4+0$	$9+0$	$16+0$	$25+0$	$36+0$	$49+0$	$\cdots$
T^2/Z_2	$8+0$	$33+0$	$96+0$	$225+0$	$456+0$	$833+0$	$\cdots$
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Future Work

The arbitrariness problems of BCs is still under exploration…

TCLs have a wide range of applications:

- Other higher-dim orbifolds: T^n/Z_m etc.
- Other gauge groups: $SO(N)$ etc.
- Warped space-time: Randall-Sundrum

arXiv:2404.19411

Summary

In S^1/Z_2 and T^2/Z_m $(m = 2,3,4,6)$ orbifolded $SU(N)$ and $U(N)$ gauge theories,

Trace Conservation Laws have completed the classification of boundary conditions.

Thank you for listening!

	Classification
S^1/Z_2	$\bigcirc\,(\triangle)$
T^2/Z_2	$\triangle$
T^2/Z_3	$\triangle$
T^2/Z_4	$\times$
T^2/Z_6	$\times$

	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$\cdots$
S^1/Z_2	$4+0$	$9+0$	$16+0$	$25+0$	$36+0$	$49+0$	$\cdots$
T^2/Z_2	$8+0$	$33+0$	$96+0$	$225+0$	$456+0$	$833+0$	$\cdots$
T^2/Z_3	$9+0$	$45+0$	$163+0$	$477+0$	$1197+0$	$2674+0$	$\cdots$
T^2/Z_4	$8+0$	$36+2$	$120+16$	$329+74$	$784+256$	$1680+732$	$\cdots$
T^2/Z_6	$6+0$	$21+3$	$56+20$	$126+81$	$252+252$	$462+663$	$\cdots$

Follow-Up

25

27

05

SU(N) ECs = U(N) ECs

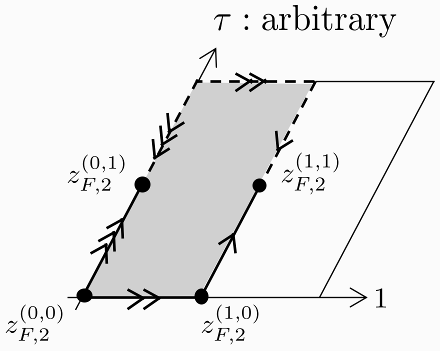
Since the sufficient classification of ECs has been completed, we can count the exact number of ECs in U(N) and SU(N) gauge theories on each 2D orbifold. First of all, we prove that the numbers of ECs in U(N) and SU(N) models are equal. This is because the rotation matrix R around the fixed point z_F is invariant under BCs-connecting U(1) gauge transformation:

$$R' = e^{ia_2} R\, e^{-ia_1} = e^{i(a_2 - a_1)} R = R, \tag{5.1}$$

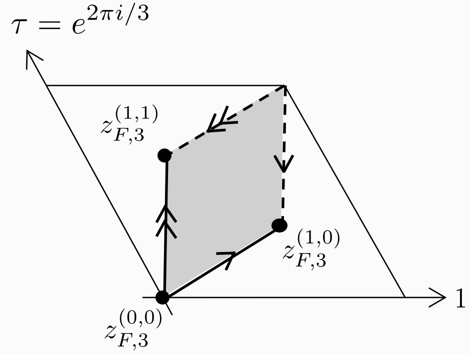
where $a_2 = f(e^{i\frac{2\pi}{m}}(z - z_F))$ and $a_1 = f(z - z_F)$ are U(1) parameters. R' is z-independent under BCs-connecting gauge transformations, so that the phase $(a_2 - a_1)$ must be constant. Since $a_2 - a_1 = 0$ at $z = z_F$, the phase globally vanishes and (5.1) is obtained. Actually, $\det \Omega(x^\mu, z) = 1$ is satisfied for all the essential transformation functions (3.12) on T^2/Z_2, (3.22) on T^2/Z_3, and (3.36) on T^2/Z_4, so that they can be applied to both U(N) and SU(N) models.

T^2/Z_m Orbifold ($m = 2,3,4,6$)

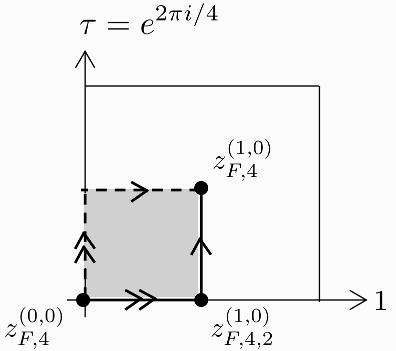
T^2/Z_2:



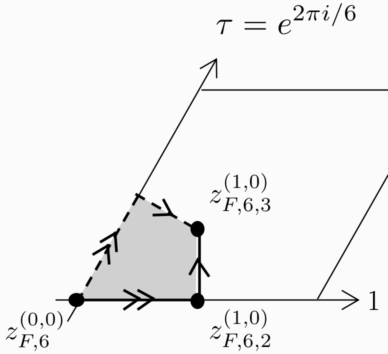
T^2/Z_3:



T^2/Z_4:



T^2/Z_6:



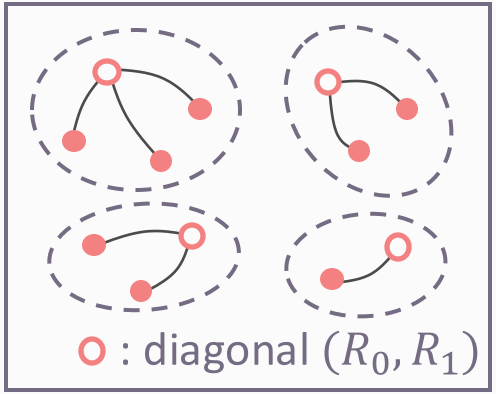
Diagonal ECs on T^2/Z_6

Consistent Conditions:

$$R_0^6 = 1, \quad R_1^3 = 1, \quad R_1 R_0 R_1 R_0 = 1$$

For diagonal matrices, $R_0^2 = R_1$

There is no non-trivial connection.



$$R_0 = (\eta, \cdots, \eta, \eta^4, \cdots, \eta^4 \;|\; \eta^2, \cdots, \eta^2, \eta^5, \cdots, \eta^5 \;|\; +1, \cdots, +1, -1, \cdots, -1)$$
$$R_1 = (\eta^2, \cdots, \eta^2, \eta^2, \cdots, \eta^2 \;|\; \eta^4, \cdots, \eta^4, \eta^4, \cdots, \eta^4 \;|\; +1, \cdots, +1, +1, \cdots, +1)$$

$(\eta = e^{2\pi i/6})$

Detail History

Y. Hosotani, Annals of Physics, **190**, 233-253. (1989)

ECs on S^1

N. Haba, M. Harada, Y. Hosotani, and Y. Kawamura, Nuclear Physics B, **657**, 169-213. (2003)

ECs on S^1/Z_2

N. Haba, Y. Hosotani, and Y. Kawamura, Progress of Theoretical Physics, **111**, 265–289. (2004)

simul.diagonalizability on S^1/Z_2

Y. Hosotani, S. Noda, and K. Takenaga, Phys. Rev. D, **69**, 125014. (2004)

ECs on T^2/Z_2

Y. Kawamura, T. Kinami, and T. Miura, Progress of Theoretical Physics, **120**, 815-831. (2008)

ECs on T^2/Z_3

Y. Kawamura, T. Miura, Progress of Theoretical Physics, **122**, 847-864. (2009)

ECs on $(S^1/Z_2)^2, T^2/Z_N$

Y. Kawamura, Y. Nishikawa, International Journal of Modern Physics A, **35**, 2050206 (2020)

Y. Kawamura, E. Kodaira, K. Kojima, and T. Yamashita, Journal of High Energy Physics, 04 (2023) 113

simul.diagonalizability on $S^1/Z_2, T^2/Z_N$

Gauge Transformations

$$\psi(x, y_i - y) = P_i \gamma^5 \psi(x, y_i + y)$$

$$A_\mu(x, y_i - y) = P_i A_\mu(x, y_i + y) P_i$$

$$A_y(x, y_i - y) = -P_i A_y(x, y_i + y) P_i$$

$$(y_0 = 0,\ \ y_1 = \pi R)$$

unitary parity $N \times N$ matrices

$$P_i P_i^\dagger = P_i^2 = 1$$

↓

$$\psi'(x, y_i - y) = P_i' \gamma^5 \psi'(x, y_i + y)$$

$$A_\mu'(x, y_i - y) = P_i' A_\mu'(x, y_i + y) P_i'^\dagger - P_i' \,\partial_\mu\, P_i'^\dagger$$

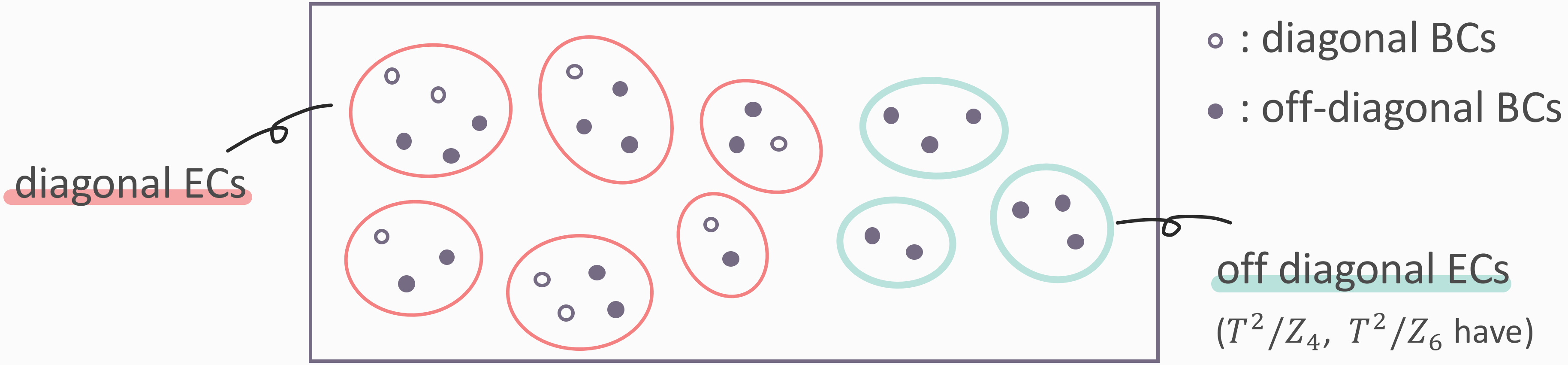
$$A_y'(x, y_i - y) = -P_i' A_y'(x, y_i + y) P_i'^\dagger - P_i' \left(-\partial_y\right) P_i'^\dagger$$

$$P_0' = \Omega(-y) P_0 \Omega^\dagger(y)$$
$$P_1' = \Omega(2\pi R - y) P_1 \Omega^\dagger(y)$$

Simultaneous diagonalizability

$S^1/Z_2,\ T^2/Z_2,\ T^2/Z_3$ → Any BCs can be simultaneously diagonalized

$T^2/Z_4,\ T^2/Z_6$ → One cannot be simultaneously diagonalized



TCLs for diagonal matrices

The number of TCLs is infinite, corresponding to the infinite number of fixed points.

→ For diagonal matrices, there are only a few independent TCLs.

S^2/Z_2: $\mathrm{tr}P'_0 = \mathrm{tr}P_0, \quad \mathrm{tr}P'_1 = \mathrm{tr}P_1$

T^2/Z_2: $\mathrm{tr}R'_0 = \mathrm{tr}R_0, \quad \mathrm{tr}R'_1 = \mathrm{tr}R_1, \quad \mathrm{tr}R'_2 = \mathrm{tr}R_2, \quad \mathrm{tr}(R'_0R'_1R'_2) = \mathrm{tr}(R_0R_1R_2)$

T^2/Z_3: $\mathrm{tr}R'_0 = \mathrm{tr}R_0, \quad \mathrm{tr}R'_1 = \mathrm{tr}R_1, \quad \mathrm{tr}(R'_1R'_0) = \mathrm{tr}(R_1R_0)$

T^2/Z_4: $\mathrm{tr}R'_0 = \mathrm{tr}R_0, \quad \mathrm{tr}R'_1 = \mathrm{tr}R_1, \quad \mathrm{tr}R'^2_0 = \mathrm{tr}R^2_0, \quad \mathrm{tr}(R'_0R'_1) = \mathrm{tr}(R_0R_1)$

T^2/Z_6: $\mathrm{tr}R'_0 = \mathrm{tr}R_0, \quad \mathrm{tr}R'^2_0 = \mathrm{tr}R^2_0, \quad \mathrm{tr}R'^3_0 = \mathrm{tr}R^3_0$

Results: off-diagonal ECs

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The TCLs show the existence of off-diagonal ECs, which consist only of off-diagonal matrices.

$$T^2/Z_4: \quad r_0 = i^a \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad r_1 = i^a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1}$$

$$T^2/Z_6: \quad r_0 = \eta^b \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad r_1 = \eta^{2b} \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2}i \\ \frac{\sqrt{3}}{2}i & -\frac{1}{2} \end{pmatrix}, \tag{2}$$

$$r_0 = \eta^c \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad r_1 = \eta^{2c} \begin{pmatrix} -\frac{1}{3}\omega^2 & +\frac{2}{3}\omega & +\frac{2}{3} \\ +\frac{2}{3}\omega^2 & -\frac{1}{3}\omega & +\frac{2}{3} \\ +\frac{2}{3}\omega^2 & +\frac{2}{3}\omega & -\frac{1}{3} \end{pmatrix}, \tag{3}$$

The TCLs also indicate whether the off-diagonal ECs are independent of each other.

Results: The number of ECs

The exact numbers of ECs in $SU(N)$ and $U(N)$ gauge theories:

S^2/Z_2: $(N+1)^2$

T^2/Z_2: $\frac{1}{3}(N+1)^2(N^2+2N+3)$

T^2/Z_3: $\frac{1}{80}(N+1)(N+2)(N^4+6N^3+25N^2+48N+40)$ ←

T^2/Z_4: $\frac{1}{1260}(N+1)(N+2)(N+3)(N^2+4N+7)(N^2+4N+30)$

T^2/Z_6: ${}_{N+5}C_5+\gamma_N^{(6)}$ (solved but too detailed)

# of Total ECs

$\alpha_N^{(4)}-\beta_N^{(4)}+\gamma_N^{(4)}$

# of diag BCs

$\alpha_N^{(4)} = {}_{N+7}C_7$

# of equivalence relations

$$\beta_N^{(4)} = \sum_{k=4}^{N} {}_{k-1}C_3 \cdot {}_{N-k+3}C_3$$

# of off-diag ECs

$$\gamma_N^{(4)} = \sum_{l=1}^{[N/2]} 2\cdot\alpha_{N-2l}^{(4)}$$

The number of ECs (T^2/Z_6)

$$\gamma_N^{(6)} = \sum_{m=1}^{[N/3]} 2\cdot\alpha_{N-3m} + 3\cdot\alpha_{N-2} + \sum_{m=1}^{[(N-2)/3]} 2\cdot 3\cdot\alpha_{N-2-3m} + \sum_{l=2}^{[N/2]} (3+{}_3C_2\cdot{}_{l-1}C_1)\alpha_{N-2l} + \sum_{m=1}^{[N/3]}\sum_{l=2}^{[(N-3m)/2]} 2\cdot(3+{}_3C_2\cdot{}_{l-1}C_1)\alpha_{N-2l-3m}$$

$$= \begin{cases} \frac{1}{483840}N(3N^7+72N^6+2282N^5+19908N^4 \\ \quad +36372N^3-91392N^2+61968N+781632) & \text{for } N=0 \\ \frac{1}{483840}(N+5)(N-1)(3N^6+60N^5+2057N^4 \\ \quad +11980N^3-1263N^2-26440N+159523) & \text{for } N=1 \\ \frac{1}{483840}(N+4)(3N^7+60N^6+2042N^5+11740N^4 \\ \quad -10588N^3-49040N^2+258128N-250880) & \text{for } N=2 \\ \frac{1}{483840}(N+3)(3N^7+63N^6+2093N^5+13629N^4 \\ \quad -4515N^3-77847N^2+293619N-110565) & \text{for } N=3 \\ \frac{1}{483840}(N+2)(3N^7+66N^6+2150N^5+15608N^4 \\ \quad +5156N^3-101704N^2+265376N+250880) & \text{for } N=4 \\ \frac{1}{483840}(N+1)(3N^7+69N^6+2213N^5+17695N^4 \\ \quad +18677N^3-110069N^2+170147N+600145) & \text{for } N=5 \\ & \pmod 6. \end{cases}$$

The consistency conditions

From the above discussion, the consistency conditions on T^2/Z_m ($m = 2, 3, 4, 6$) are listed as

$$[\hat{\mathcal{T}}_i, \hat{\mathcal{T}}_j] = 0, \quad \left(\hat{\mathcal{T}}_1^{n_1}\hat{\mathcal{T}}_2^{n_2}\hat{\mathcal{R}}_0\right)^2 = \hat{\mathcal{I}}, \quad \text{for } m = 2, \tag{12}$$

$$\begin{aligned} &[\hat{\mathcal{T}}_i, \hat{\mathcal{T}}_j] = 0, \quad \left(\hat{\mathcal{T}}_1^{n_1}\hat{\mathcal{T}}_2^{n_2}\hat{\mathcal{R}}_0\right)^3 = \hat{\mathcal{I}}, \\ &\hat{\mathcal{T}}_1\hat{\mathcal{T}}_2\hat{\mathcal{T}}_3 = \hat{\mathcal{I}}, \qquad \text{for } m = 3, \end{aligned} \tag{13}$$

$$\begin{aligned} &[\hat{\mathcal{T}}_i, \hat{\mathcal{T}}_j] = 0, \quad \left(\hat{\mathcal{T}}_1^{n_1}\hat{\mathcal{T}}_2^{n_2}\hat{\mathcal{R}}_0\right)^4 = \hat{\mathcal{I}}, \\ &\left(\hat{\mathcal{T}}_1^{n_1}\hat{\mathcal{T}}_2^{n_2}\hat{\mathcal{R}}_0^2\right)^2 = \hat{\mathcal{I}}, \quad \hat{\mathcal{T}}_1\hat{\mathcal{T}}_2\hat{\mathcal{T}}_3\hat{\mathcal{T}}_4 = \hat{\mathcal{I}}, \\ &\hat{\mathcal{T}}_1\hat{\mathcal{T}}_3 = \hat{\mathcal{T}}_2\hat{\mathcal{T}}_4 = \hat{\mathcal{I}}, \qquad \text{for } m = 4, \end{aligned} \tag{14}$$

$$\begin{aligned} &[\hat{\mathcal{T}}_i, \hat{\mathcal{T}}_j] = 0, \quad \left(\hat{\mathcal{T}}_1^{n_1}\hat{\mathcal{T}}_2^{n_2}\hat{\mathcal{R}}_0\right)^6 = \hat{\mathcal{I}}, \\ &\left(\hat{\mathcal{T}}_1^{n_1}\hat{\mathcal{T}}_2^{n_2}\hat{\mathcal{R}}_0^3\right)^2 = \hat{\mathcal{I}}, \quad \left(\hat{\mathcal{T}}_1^{n_1}\hat{\mathcal{T}}_2^{n_2}\hat{\mathcal{R}}_0^2\right)^3 = \hat{\mathcal{I}}, \\ &\hat{\mathcal{T}}_1\hat{\mathcal{T}}_2\hat{\mathcal{T}}_3\hat{\mathcal{T}}_4\hat{\mathcal{T}}_5\hat{\mathcal{T}}_6 = \hat{\mathcal{I}}, \quad \hat{\mathcal{T}}_1\hat{\mathcal{T}}_4 = \hat{\mathcal{T}}_2\hat{\mathcal{T}}_5 = \hat{\mathcal{T}}_3\hat{\mathcal{T}}_6 = \hat{\mathcal{I}}, \\ &\hat{\mathcal{T}}_1\hat{\mathcal{T}}_3\hat{\mathcal{T}}_5 = \hat{\mathcal{T}}_2\hat{\mathcal{T}}_4\hat{\mathcal{T}}_6 = \hat{\mathcal{I}}, \qquad \text{for } m = 6, \end{aligned} \tag{15}$$

where n_1 and n_2 are integers and $i, j = 1, 2, \cdots, m$. We emphasize again that $\hat{\mathcal{T}}_i$ is expressed as $\hat{\mathcal{T}}_i = \hat{\mathcal{R}}_0^{i-1}\hat{\mathcal{T}}_1\hat{\mathcal{R}}_0^{1-i}$ for $m = 3, 4, 6$. Most of the conditions are not independent. The basic consistency conditions are summarized in Table 1 (see Appendix B). The BCs on T^2/Z_m always satisfy the basic consistency conditions.[2]

The **basic** consistency conditions

Table 1. The basic consistency conditions.

T^2/Z_m	the basic consistency conditions			
T^2/Z_2	$[\hat{\mathcal{T}}_1, \hat{\mathcal{T}}_2] = 0,$	$(\hat{\mathcal{R}}_0)^2 = \hat{\mathcal{I}},$	$(\hat{\mathcal{T}}_1\hat{\mathcal{R}}_0)^2 = \hat{\mathcal{I}},$	$(\hat{\mathcal{T}}_2\hat{\mathcal{R}}_0)^2 = \hat{\mathcal{I}}$
T^2/Z_3	$[\hat{\mathcal{T}}_1, \hat{\mathcal{T}}_2] = 0,$	$(\hat{\mathcal{R}}_0)^3 = \hat{\mathcal{I}},$	$\hat{\mathcal{T}}_1\hat{\mathcal{T}}_2\hat{\mathcal{T}}_3 = \hat{\mathcal{I}}$	
T^2/Z_4	$[\hat{\mathcal{T}}_1, \hat{\mathcal{T}}_2] = 0,$	$(\hat{\mathcal{R}}_0)^4 = \hat{\mathcal{I}},$	$\hat{\mathcal{T}}_1\hat{\mathcal{T}}_3 = \hat{\mathcal{I}}$	
T^2/Z_6	$(\hat{\mathcal{R}}_0)^6 = 1,$	$\hat{\mathcal{T}}_1\hat{\mathcal{T}}_3\hat{\mathcal{T}}_5 = \hat{\mathcal{I}},$	$\hat{\mathcal{T}}_1\hat{\mathcal{T}}_4 = \hat{\mathcal{I}}$	

Compactification

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Higher-dim theory

$$S_{\text{scalar}} = \int d^4x dy \, \frac{1}{2}\partial_M \phi \partial^M \phi$$

↓

Compactification
Boundary Conditions

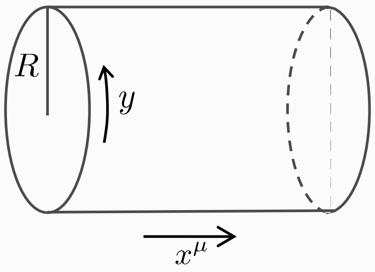
↓

4-dim theory

S^1 compactification

$$\phi(x, y + 2\pi R) = \phi(x, y)$$

$$\phi(x, y) = \sum_{n=-\infty}^{\infty} \phi^{(n)}(x) e^{i(ny/R)}$$



$$= \int d^4x \left\{ \frac{1}{2}\partial_\mu \phi^{(0)} \partial^\mu \phi^{(0)} + \sum_{n\neq 0} \frac{1}{2}\partial_\mu \phi^{(n)} \partial^\mu \phi^{(n)} - \frac{1}{2}\left(\frac{n}{R}\right)^2 \phi^{(n)2} \right\}$$

~measurable

~ Planck scale