押しつぶされた Kaluza-Klein BH 時空におけるプラズマ媒質中の光の時間遅延

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Introduction

5D squashed Kaluza-Klein black hole solutions asymptote to effective 4D spacetimes with twisted S^1 as extra dimension at infinity and represent fully 5D black holes near S^3 horizons.

- It could be regarded as candidates of realistic higher-dimensional models and would describe geometry around astrophysical compact objects.
- Quasinormal modes, thin accretion disk, x-ray reflection spectroscopy, gyroscope precession, strong gravitational lensing, light deflection, and black hole shadow have been discussed in this spacetime.

✓ In this work, we consider photon time delay in unmagnetized cold homogeneous plasma medium by charged static squashed Kaluza-Klein black holes in weak-field limit.

✓ We derive corrections of time delay to general relativity, which are related to size of extra dimension, charge of black hole, and ratio between plasma and photon frequencies.

Kaluza-Klein black holes with squashed S^3 horizons

(H. Ishihara and K. Matsuno)

$$
\begin{cases}
ds^2 = -F dt^2 + \dfrac{K^2}{F} d\rho^2 + \rho^2 K^2 (d\theta^2 + \sin^2\theta d\phi^2) + \dfrac{r_\infty^2}{4K^2} (d\psi + \cos\theta d\phi)^2 \\
A_\mu dx^\mu = \dfrac{\sqrt{3}Q}{2\rho} dt, \qquad F = 1 - \dfrac{2M}{\rho} + \dfrac{Q^2}{\rho^2}, \qquad K^2 = 1 + \dfrac{\rho_0}{\rho} \\
\rho_0 = \sqrt{M^2 - Q^2 + r_\infty^2/4} - M, \qquad M \geq Q > 0, \qquad r_\infty > 0
\end{cases}
$$

(M : Komar mass, Q : black hole charge, r_∞ : extra dimension size at infinity)

- $\rho \to \infty$: twisted constant S^1 fiber over 4D Minkowski spacetime
- Parameter ρ_0 : typical scale of transition from five dimensions to effective four dimensions
- $\rho_0 \to 0$ limit : 4D Reissner-Nordström black hole with twisted constant S^1 fiber

Photon motions in unmagnetized cold homogeneous plasma medium

$$\begin{cases} \mathcal{H} = \frac{1}{2}(g^{\mu\nu} p_\mu p_\nu + \omega_e^2) = 0 & \text{: Hamiltonian for photon} \\ \omega_e^2 = e^2 N_e/(\epsilon_0 m_e), \;\; N_e = \text{const} & \text{: electron plasma frequency} \end{cases}$$

- constants of motion: $\begin{cases} \omega_\infty := -p_t & \text{: frequency } (\omega_\infty > \omega_e) \\ L := p_\phi, \quad \text{and} \quad p_\psi & \text{: angular momenta} \end{cases}$

✓ Assumption: $p_\psi = 0$ (no momentum in extra dimensional direction)

- $\mathcal{H}_{\text{eff}} = \frac{1}{2}\left(-\frac{p_t^2}{F} + \frac{F}{K^2} p_\rho^2 + \frac{p_\theta^2}{\rho^2 K^2} + \frac{p_\phi^2}{\rho^2 K^2 \sin^2\theta} + \omega_e^2\right) = 0$: effective Hamiltonian

✓ We can concentrate on orbits with $\theta = \pi/2$ and $p_\theta = 0$.

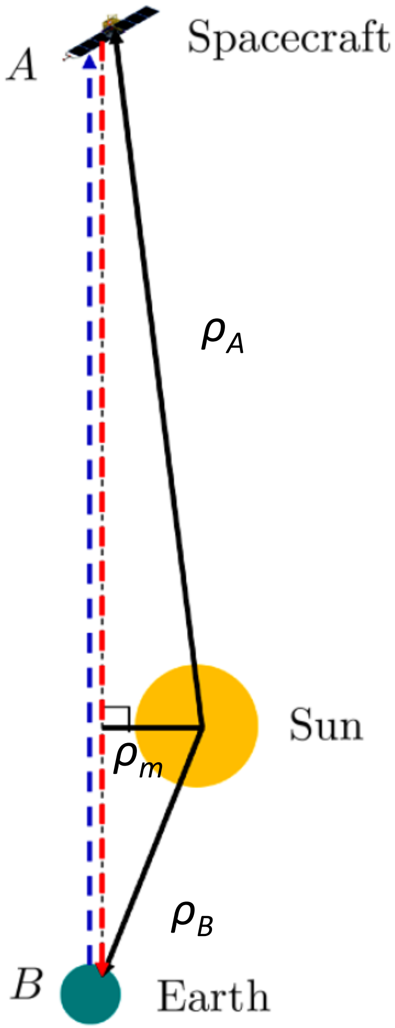
- $\frac{dt}{d\lambda} = \frac{\omega_\infty}{F}, \quad \frac{d\rho}{d\lambda} = \frac{F}{K^2} p_\rho, \quad \frac{d\phi}{d\lambda} = \frac{L}{\rho^2 K^2}$: Hamilton's equations

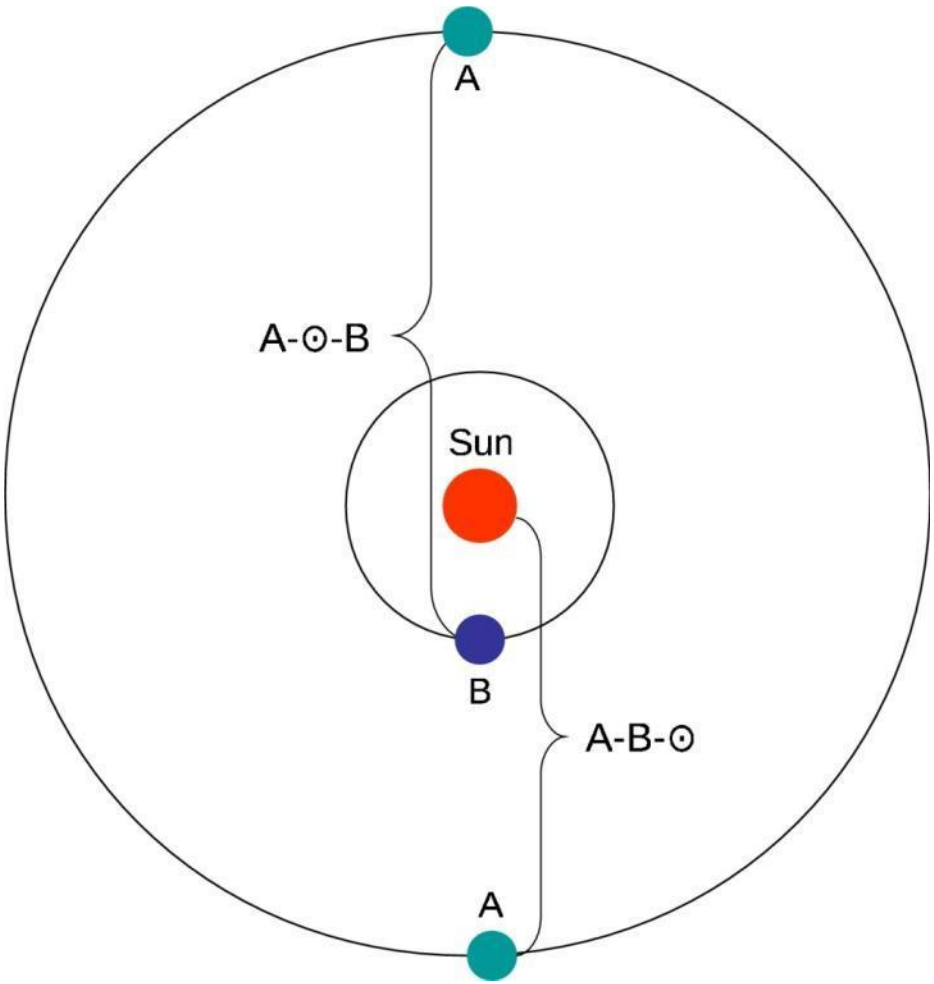
✓ Energy conservation equation:

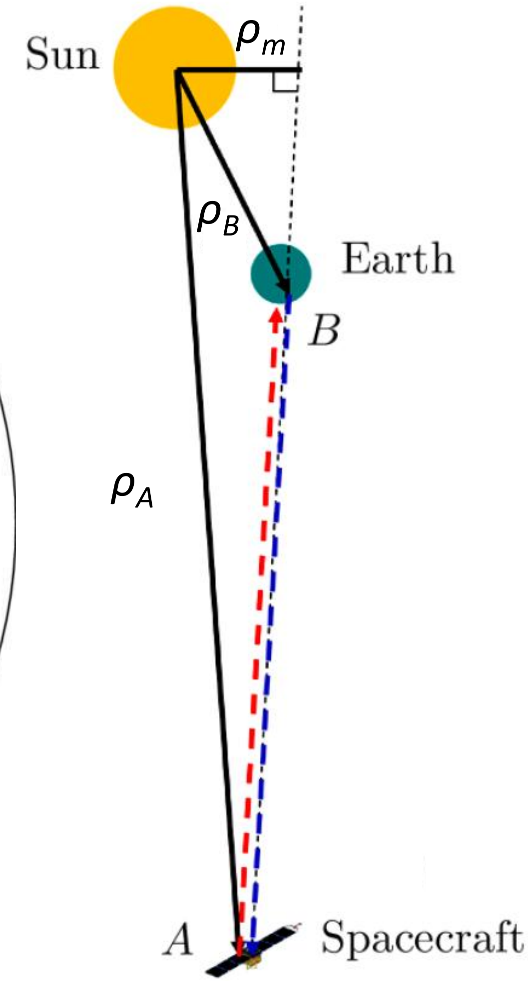
$$\left(1 + \frac{\rho_0}{\rho}\right)\left(\frac{d\rho}{d\lambda}\right)^2 + V_{\text{eff}} = \omega_\infty^2, \quad V_{\text{eff}} = \left(1 - \frac{2M}{\rho} + \frac{Q^2}{\rho^2}\right)\left(\omega_e^2 + \frac{L^2}{\rho(\rho + \rho_0)}\right)$$

Photon behaves as massive particle in 4D spherically symmetric spacetime.

Shapiro time delay







Superior conjunction

Inferior conjunction

- Accuracy and stability of laser-cooled atomic clocks have achieved at 10^{-15}~10^{-16} level.

Shapiro time delay in superior/inferior conjunctions

- Coordinate time interval of light ray emitting from observer to reflector and back to observer: $T(\rho_i) = \int_{\rho_m}^{\rho_i} \left|\frac{dt}{d\rho}\right| d\rho, \quad T_0(\rho_i) = \sqrt{\left(\rho_i^2 - \rho_m^2\right)\left(\sigma^2+1\right)}$

$$\begin{cases} \Delta t_{SC} = 2\left[T(\rho_A) + T(\rho_B) - (T_0(\rho_A) + T_0(\rho_B))\right] \\ \quad \simeq 4M\sqrt{\sigma^2+1}\left(1 - \frac{\sigma^2}{2} + \frac{\rho_0}{4M}\right)\log\left(\frac{4\rho_A\rho_B}{\rho_m^2}\right) \\ \quad + 2M\sqrt{\sigma^2+1}\left(1+\sigma^2+\frac{\rho_0}{2M}\right)\left(2 - \frac{\rho_m(\rho_A+\rho_B)}{\rho_A\rho_B}\right) \\ \Delta t_{IC} = 2\left[T(\rho_A) - T(\rho_B) - (T_0(\rho_A) - T_0(\rho_B))\right] \\ \quad \simeq 4M\sqrt{\sigma^2+1}\left(1 - \frac{\sigma^2}{2} + \frac{\rho_0}{4M}\right)\log\left(\frac{\rho_A}{\rho_B}\right) \\ \quad + 2M\sqrt{\sigma^2+1}\left(1+\sigma^2+\frac{\rho_0}{2M}\right)\frac{\rho_m(\rho_A-\rho_B)}{\rho_A\rho_B} \end{cases}$$

Observational effects in time delay in superior conjunction

- Difference between time delays in sq. KK spacetime and in 4D Schwarzschild spacetime:

$$\begin{cases} \dfrac{\Delta t_{\rm SC} - \Delta t_{\rm SCSch}}{\Delta t_{\rm SCSch}} \simeq \delta_{\rm SC1} - \delta_{\rm SC2} + \delta_{\rm SC3} \\ \Delta t_{\rm SCSch} = 4M - \dfrac{2M\rho_m(\rho_A+\rho_B)}{\rho_A\rho_B} + 4M\log\left(\dfrac{4\rho_A\rho_B}{\rho_m^2}\right) \end{cases}$$

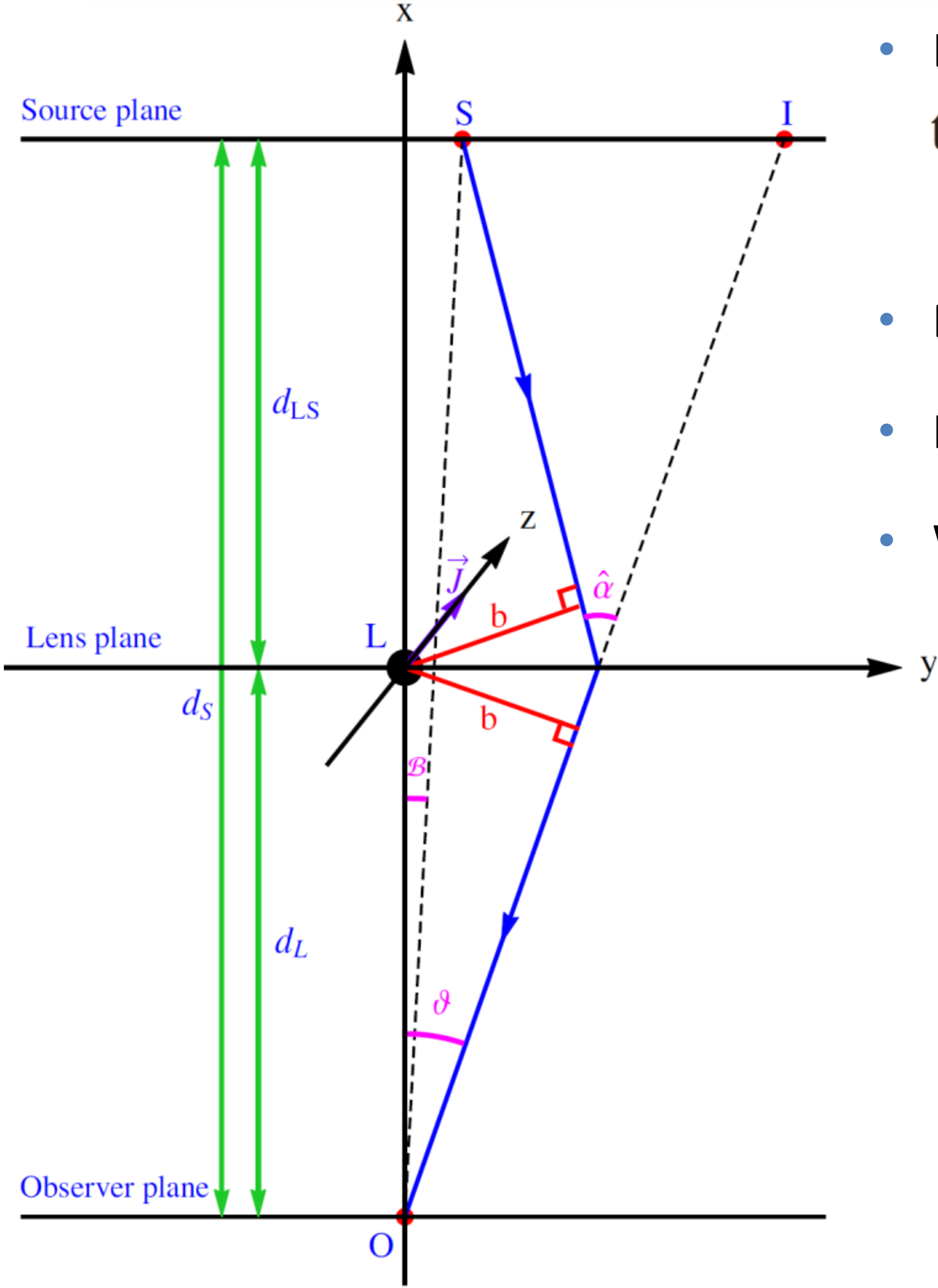
$$\begin{cases} \delta_{\rm SC1} = \dfrac{3\omega_e^2 M}{\omega_\infty^2 \Delta t_{\rm SCSch}}\left(2 - \dfrac{\rho_m(\rho_A+\rho_B)}{\rho_A\rho_B}\right) \simeq 10^{-6} \\ \delta_{\rm SC2} = \dfrac{Q^2}{2M\Delta t_{\rm SCSch}}\left(2 - \dfrac{\rho_m(\rho_A+\rho_B)}{\rho_A\rho_B} + \log\left(\dfrac{4\rho_A\rho_B}{\rho_m^2}\right)\right) \lesssim 10^{-7} \\ \delta_{\rm SC3} = \dfrac{r_\infty^2}{8M\Delta t_{\rm SCSch}}\left(2 - \dfrac{\rho_m(\rho_A+\rho_B)}{\rho_A\rho_B} + \log\left(\dfrac{4\rho_A\rho_B}{\rho_m^2}\right)\right) \simeq 10^{-18} \end{cases}$$

$$\Big(\rho_A = 40 \text{ au},\ \rho_B = 1 \text{ au},\ \rho_m = 1.5R_\odot,\ M = M_\odot,$$
$$\omega_\infty/(2\pi) \simeq 3\times 10^8 \text{ Hz},\ N_e \simeq 5\times 10^{10} \text{ m}^{-3},\ Q \lesssim 4\times 10^{17} \text{ C},\ r_\infty \simeq 0.1 \text{ mm}\Big)$$

- Accuracy and stability of optical clocks on ground have achieved at ~10^{-18} level.

✓ In near future, implement of planetary laser ranging and optical clocks would provide us more insights on such compact extra dimension.

Lens diagram of squashed Kaluza-Klein black hole



- Lens equation:

$$\tan\mathcal{B} = \tan\vartheta - D[\tan\vartheta + \tan(\hat{\alpha} - \vartheta)]$$

$$D = d_{LS}/d_S$$

- Einstein ring radius: $\vartheta_E \equiv \sqrt{4DM/d_L}$

- Impact parameter: $b = d_L \sin\vartheta$

- Weak-field gravitational time delay:

$$\begin{cases} \tau = T(R_S) + T(R_O) - \dfrac{d_S}{\cos\mathcal{B}} \\ R_S = \sqrt{d_{LS}^2 + d_S^2 \tan^2\mathcal{B}}\,, \quad R_O = d_L \end{cases}$$

$$\begin{cases} \beta \equiv \dfrac{\mathcal{B}}{\vartheta_E} \qquad \theta \equiv \dfrac{\vartheta}{\vartheta_E} \qquad \varepsilon \equiv \dfrac{\vartheta_E}{4D} \\ \theta = \theta_0 + \theta_1\varepsilon + \theta_2\varepsilon^2 + \mathcal{O}(\varepsilon^3) \end{cases}$$

Differential time delay in plasma medium

- Difference between scaled time delays of primary and secondary images:

$$\begin{cases} \Delta\hat{\tau} = \hat{\tau}_- - \hat{\tau}_+ = \Delta\hat{\tau}_0 + \Delta\hat{\tau}_1 \varepsilon + \mathcal{O}(\varepsilon^2) \\ \hat{\tau}_\pm = \tau_\pm/\tau_E, \quad \tau_E = d_L \vartheta_E^2/D \end{cases}$$

: natural lensing timescale

$$\begin{cases} \Delta\hat{\tau}_0 = \frac{\sqrt{\sigma^2+1}}{2}\left(\left(\theta_0^+\right)^2 - \left(\theta_0^-\right)^2 + \left(2 - \sigma^2 + \frac{\rho_0}{2M}\right)\log\left(\frac{\theta_0^+}{\theta_0^-}\right)\right) \\ \Delta\hat{\tau}_1 = \frac{\pi\sqrt{\sigma^2+1}}{16}\left[6\left(5 - \frac{Q^2}{M^2} + \frac{2\rho_0}{M}\right)\left(\frac{1}{\theta_0^-} - \frac{1}{\theta_0^+}\right) + \left(3\left(5+4\sigma^2\right) + \left(3+2\sigma^2\right)\left(\frac{2\rho_0}{M} - \frac{Q^2}{M^2}\right)\right)\right. \\ \qquad \left. \times\left(\frac{4 - 2\sigma^2 + 4\left(\theta_0^+\right)^2 + \rho_0/M}{\theta_0^+\left(4 + 2\sigma^2 + 4\left(\theta_0^+\right)^2 + \rho_0/M\right)} - \frac{4 - 2\sigma^2 + 4\left(\theta_0^-\right)^2 + \rho_0/M}{\theta_0^-\left(4 + 2\sigma^2 + 4\left(\theta_0^-\right)^2 + \rho_0/M\right)}\right)\right] \\ \theta_0^\pm = \frac{1}{2}\left(\sqrt{\beta^2 + 4 + 2\sigma^2 + \frac{\rho_0}{M}} \pm |\beta|\right) \end{cases}$$

: angular positions of positive and negative-parity images

➢ 4D Sch. case: $\Delta\hat{\tau}_{0\mathrm{Sch}} = \frac{|\beta|\sqrt{\beta^2+4}}{2} + \log\left(\frac{\sqrt{\beta^2+4}+|\beta|}{\sqrt{\beta^2+4}-|\beta|}\right), \quad \Delta\hat{\tau}_{1\mathrm{Sch}} = \frac{15\pi|\beta|}{16}$

Lensing by supermassive black hole Sgr A*

- Difference between differential time delays in sq. KK spacetime and in 4D Sch. one:

$$\tau_E \left(\Delta \hat{\tau}_0 - \Delta \hat{\tau}_{0\mathrm{Sch}} \right) \simeq \delta_{01} - \delta_{02} + \delta_{03}$$

$$\begin{cases} \delta_{01} = \dfrac{\tau_E \omega_e^2 \beta \sqrt{\beta^2+4}}{4\omega_\infty^2} \simeq 0.02 \text{ s} \\ \delta_{02} = \dfrac{\tau_E Q^2}{8M^2} \log \left(\dfrac{\sqrt{\beta^2+4}+\beta}{\sqrt{\beta^2+4}-\beta} \right) \lesssim 10^{-36} \text{ s} \\ \delta_{03} = \dfrac{\tau_E r_\infty^2}{32M^2} \log \left(\dfrac{\sqrt{\beta^2+4}+\beta}{\sqrt{\beta^2+4}-\beta} \right) \simeq 10^{-29} \text{ s} \end{cases}$$

$$\epsilon \tau_E \left(\Delta \hat{\tau}_1 - \Delta \hat{\tau}_{1\mathrm{Sch}} \right) \simeq \delta_{11} - \delta_{12} + \delta_{13}$$

$$\begin{cases} \delta_{11} = \dfrac{3\pi \tau_E \epsilon \beta \omega_e^2}{16\omega_\infty^2} \simeq 10^{-7} \text{ s} \\ \delta_{12} = \dfrac{33\pi \tau_E \epsilon \beta Q^2}{128M^2} \lesssim 10^{-39} \text{ s} \\ \delta_{13} = \dfrac{9\pi \tau_E \epsilon \beta r_\infty^2}{512M^2} \simeq 10^{-33} \text{ s} \end{cases}$$

$$\left(d_L = 8.2 \text{ kpc},\ d_{LS} = 0.01 \text{ kpc},\ \beta = 10,\ M \sim 4 \times 10^6 M_\odot, \right.$$
$$\left. \omega_\infty/(2\pi) \simeq 3 \times 10^8 \text{ Hz},\ N_e \simeq 5 \times 10^{10} \text{ m}^{-3},\ Q \lesssim 3 \times 10^8 \text{ C},\ r_\infty \simeq 0.1 \text{ mm} \right)$$

✓ With present accuracy of Very Long Baseline Interferometry (~10^{-12} s), only plasma effect on differential time delay would be measurable.

Observational effects in gravitational lensing

- Relative change in angular position of Einstein ring images: $\begin{cases} \theta_{\rm KK} = b/D_L \\ \theta_{\rm Sch} := \sqrt{4MD_{LS}/(D_L D_S)} \end{cases}$

$$\frac{\theta_{\rm KK} - \theta_{\rm Sch}}{\theta_{\rm Sch}} = \sqrt{\frac{1}{2}\left(1 + \frac{\rho_0}{2M} + \frac{1}{1 - \omega_e^2/\omega_\infty^2}\right)} - 1 \simeq \delta_1 - \delta_2 + \delta_3$$

$$\delta_1 := \frac{\omega_e^2}{4\omega_\infty^2}, \qquad \delta_2 := \frac{Q^2}{16M^2}, \qquad \delta_3 := \frac{r_\infty^2}{64M^2}$$

$\delta_1 \simeq 10^{-5}$ would be detected in near future radio spectra observations. $\left(\omega_\infty/(2\pi) \simeq 3 \times 10^8 \text{ Hz},\ N_e \simeq 5 \times 10^{10} \text{ m}^{-3}\right)$

$\delta_2 \lesssim 10^{-38}$ for Sgr A* would not appear to be relevant for present and near future observations. $\left(M \sim 4 \times 10^6 M_\odot,\ Q \lesssim 3 \times 10^8 \text{ C}\right)$

$$\delta_3 \simeq \begin{cases} 10^{-30} & : \text{Sgr A*} \left(\sim 4 \times 10^6 M_\odot,\ r_\infty \simeq 0.1\text{mm}\right) \\ 10^{-18} & : \text{stellar mass black hole} \left(\sim 10 M_\odot,\ r_\infty \simeq 0.1\text{mm}\right) \\ 10^{-5} & : \text{Earth mass primordial black hole} \left(\sim 3 \times 10^{-6} M_\odot,\ r_\infty \simeq 0.1\text{mm}\right) \end{cases}$$

✓ It would be challenging to detect extra dimension for primordial black holes in future observations.

Summary

We study motions of photons around astrophysical compact objects in unmagnetized cold homogeneous plasma medium in 5D charged static squashed Kaluza-Klein black hole spacetime.

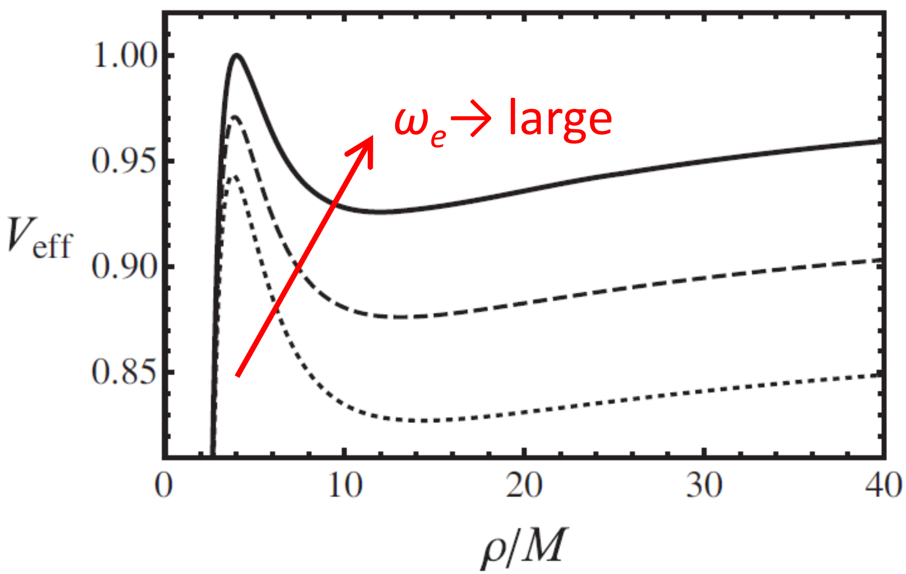
- We derive time delay of photon in weak-field limit with corrections by extra dimension, Maxwell field, and plasma.

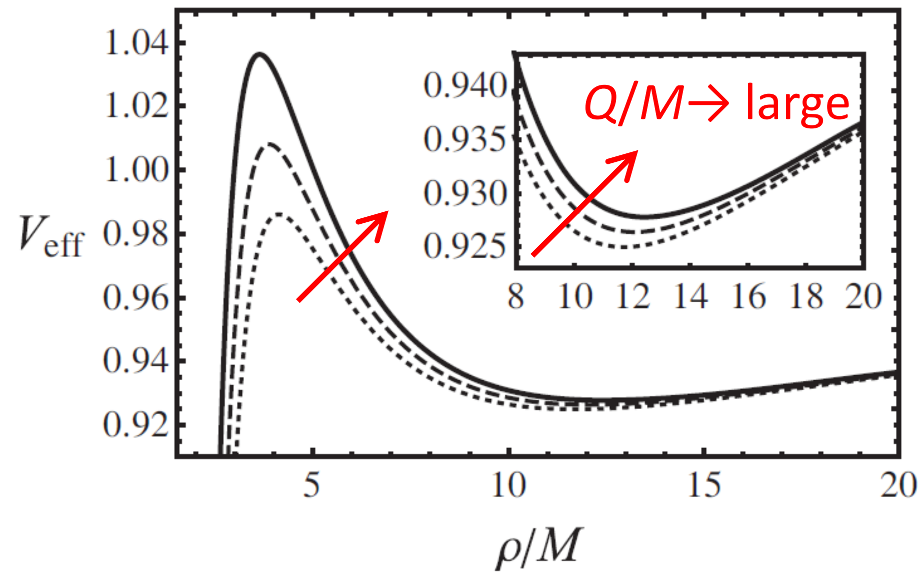
✓ Correction by extra dimension might be detected in future observations of Shapiro time delay by Sun and gravitational lensing by primordial black holes.

- If precise observations of these phenomena by compact objects agree with expected values of general relativity, it would require rigorous upper limits of size of extra dimension, or it would exclude squashed Kaluza-Klein metric for describing geometry around such objects.

Appendix

Photon motions in unmagnetized cold homogeneous plasma medium



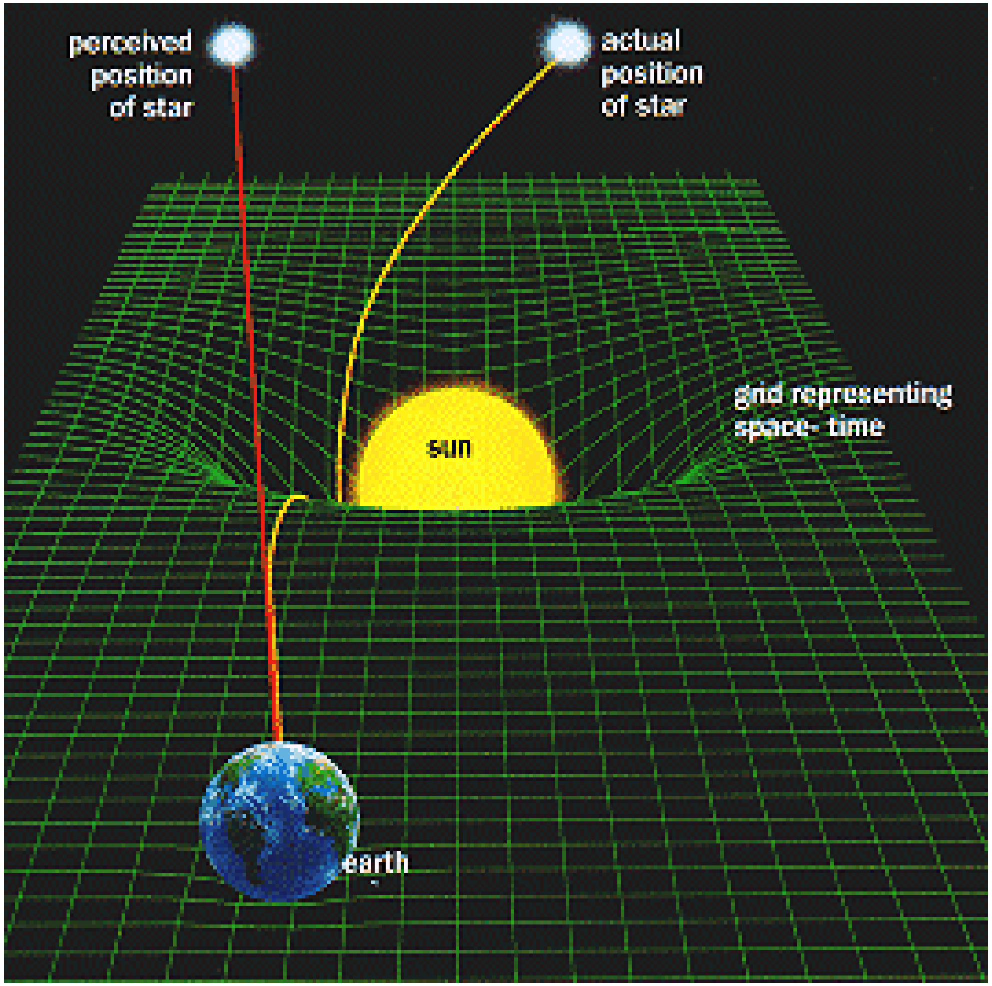


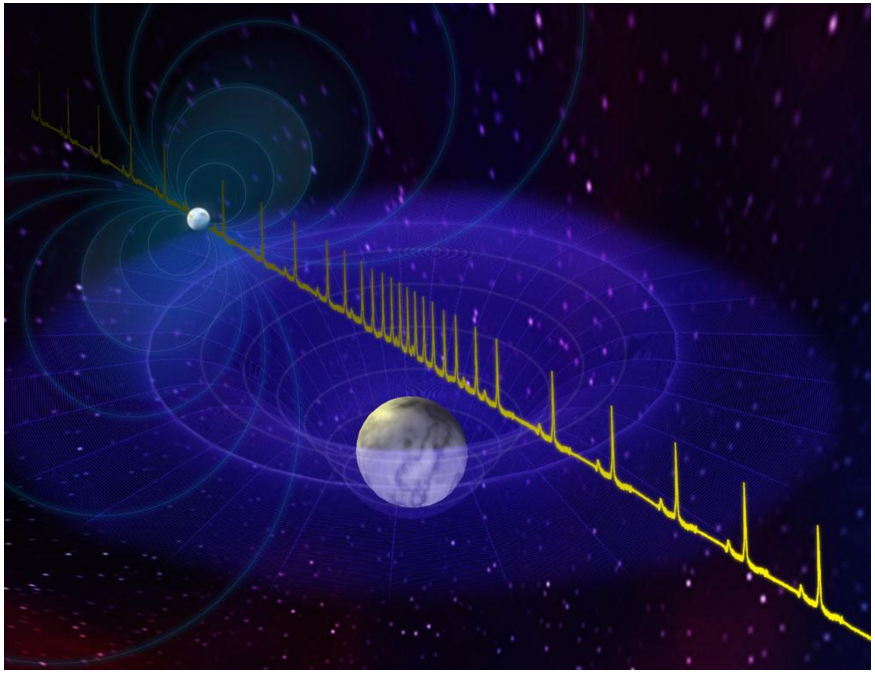
✓ Energy conservation equation:

$$\left(1+\frac{\rho_0}{\rho}\right)\left(\frac{d\rho}{d\lambda}\right)^2 + V_{\text{eff}} = \omega_\infty^2, \quad V_{\text{eff}} = \left(1-\frac{2M}{\rho}+\frac{Q^2}{\rho^2}\right)\left(\omega_e^2+\frac{L^2}{\rho(\rho+\rho_0)}\right)$$

Photon behaves as massive particle in 4D spherically symmetric spacetime.

Gravitational time delay





Photon time delay by sq. KK BH in plasma medium

- Photon traveling time propagating from distance of closest approach ρ_m to central object to arbitrary finite point $\rho_i \geq \rho_m$ of its trajectory:

$$
\begin{aligned}
T(\rho_i) &= \int_{\rho_m}^{\rho_i} \left| \frac{dt}{d\rho} \right| d\rho \\
&\simeq \sqrt{\left(\rho_i^2 - \rho_m^2\right)\left(\sigma^2+1\right)} + M\sqrt{\sigma^2+1}\left(1+\sigma^2+\frac{\rho_0}{2M}\right)\sqrt{\frac{\rho_i-\rho_m}{\rho_i+\rho_m}} \\
&\quad + 2M\sqrt{\sigma^2+1}\left(1-\frac{\sigma^2}{2}+\frac{\rho_0}{4M}\right)\log\left(\frac{\rho_i+\sqrt{\rho_i^2-\rho_m^2}}{\rho_m}\right) \\
&\quad + \frac{3M^2\sqrt{\sigma^2+1}}{2\rho_m}\left(5-\frac{Q^2}{M^2}+\frac{2\rho_0}{M}\right)\cos^{-1}\left(\frac{\rho_m}{\rho_i}\right) \\
&\quad - \frac{M^2\sqrt{\left(\sigma^2+1\right)\left(\rho_i^2-\rho_m^2\right)}}{2\rho_m(\rho_i+\rho_m)^2}\Big[\left(\sigma^2+1\right)\left(2\rho_i\left(2-\sigma^2\right)+\rho_m\left(5-\sigma^2\right)\right) \\
&\quad + \frac{\rho_0\left(3\rho_i+\rho_m\left(\sigma^2+4\right)\right)}{M} + \frac{\rho_0^2(2\rho_i+3\rho_m)}{4M^2}\Big], \qquad \sigma^2 := \frac{\omega_e^2}{\omega_\infty^2-\omega_e^2}
\end{aligned}
$$

Observational effects in time delay in inferior conjunction

- Difference between time delays in sq. KK spacetime and in 4D Schwarzschild spacetime:

$$\left\{\begin{array}{l} \dfrac{\Delta t_{\rm IC} - \Delta t_{\rm ICSch}}{\Delta t_{\rm ICSch}} \simeq \delta_{\rm IC1} - \delta_{\rm IC2} + \delta_{\rm IC3} \\ \Delta t_{\rm ICSch} = \dfrac{2M\rho_m(\rho_A - \rho_B)}{\rho_A\rho_B} + 4M\log\left(\dfrac{\rho_A}{\rho_B}\right) \end{array}\right.$$

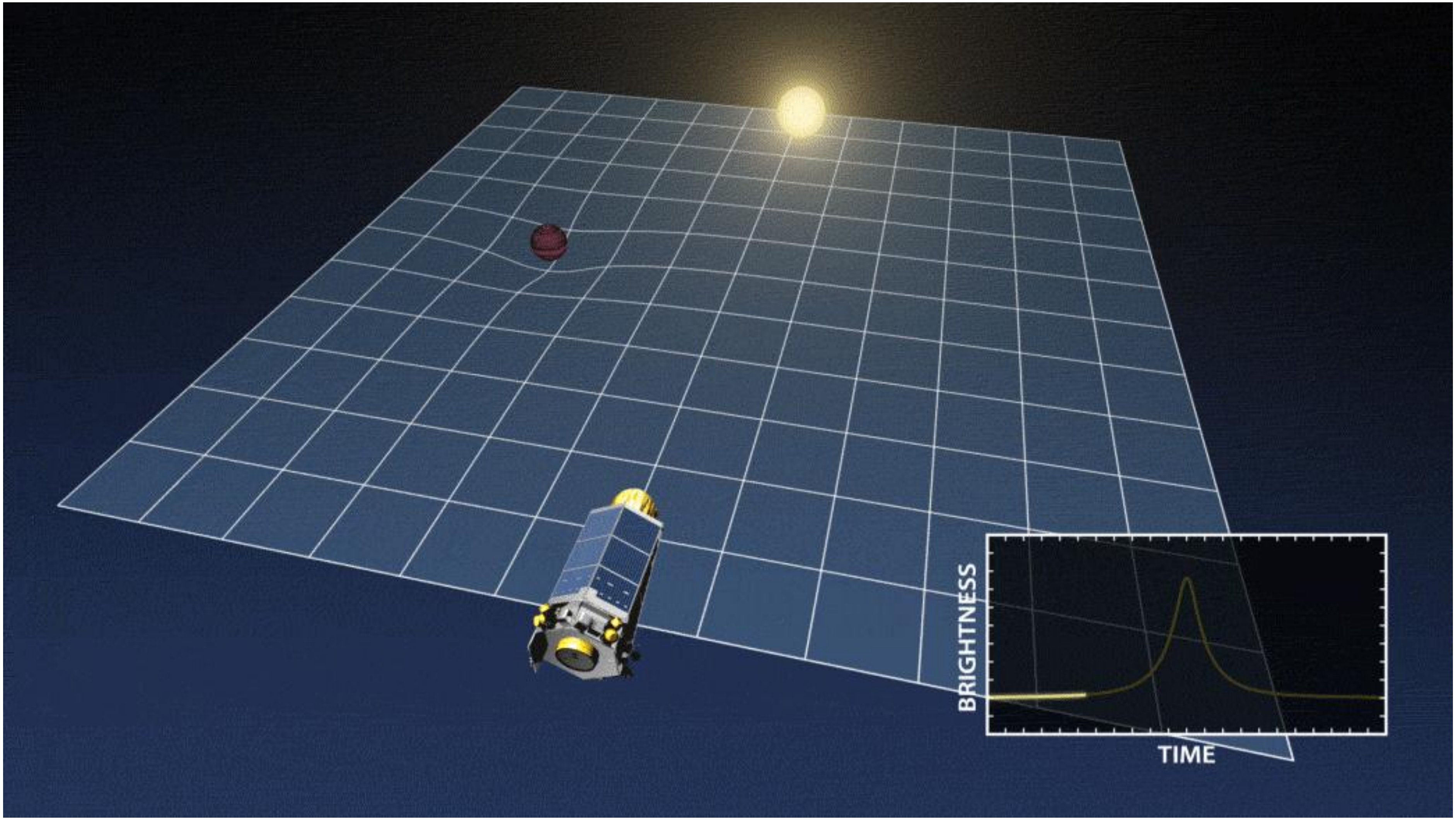
$$\left\{\begin{array}{l} \delta_{\rm IC1} = \dfrac{3\omega_e^2 M\rho_m(\rho_A - \rho_B)}{\omega_\infty^2\rho_A\rho_B\Delta t_{\rm ICSch}} \simeq 10^{-8} \\ \delta_{\rm IC2} = \dfrac{Q^2}{2M\Delta t_{\rm ICSch}}\left(\dfrac{\rho_m(\rho_A - \rho_B)}{\rho_A\rho_B} + \log\left(\dfrac{\rho_A}{\rho_B}\right)\right) \lesssim 10^{-7} \\ \delta_{\rm IC3} = \dfrac{r_\infty^2}{8M\Delta t_{\rm ICSch}}\left(\dfrac{\rho_m(\rho_A - \rho_B)}{\rho_A\rho_B} + \log\left(\dfrac{\rho_A}{\rho_B}\right)\right) \simeq 10^{-18} \end{array}\right.$$

$$\Big(\rho_A = 40\ \text{au},\ \rho_B = 1\ \text{au},\ \rho_m = 1.5R_\odot,\ M = M_\odot,$$
$$\omega_\infty/(2\pi) \simeq 3\times 10^8\ \text{Hz},\ N_e \simeq 5\times 10^{10}\ \text{m}^{-3},\ Q \lesssim 4\times 10^{17}\ \text{C},\ r_\infty \simeq 0.1\ \text{mm}\Big)$$

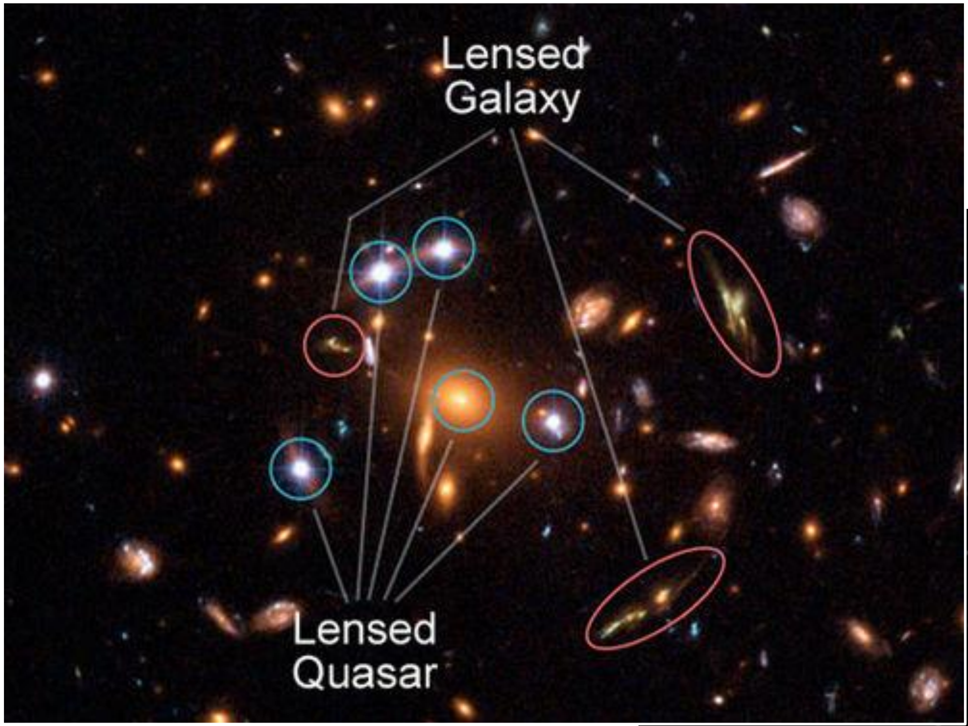
- Accuracy and stability of optical clocks on ground have achieved at ~10^{-18} level.

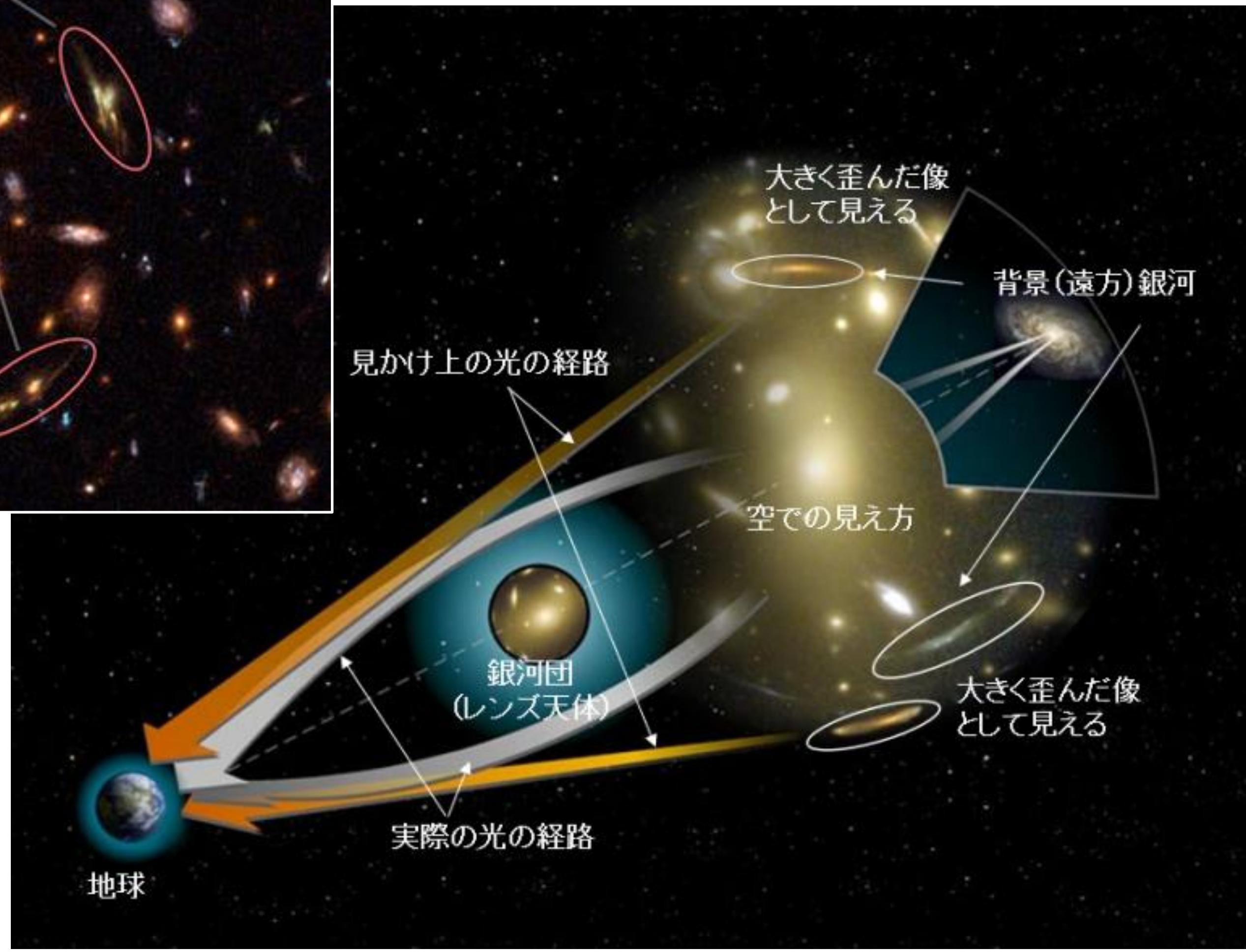
✓ In near future, implement of planetary laser ranging and optical clocks would provide us more insights on such compact extra dimension.

Gravitational lensing

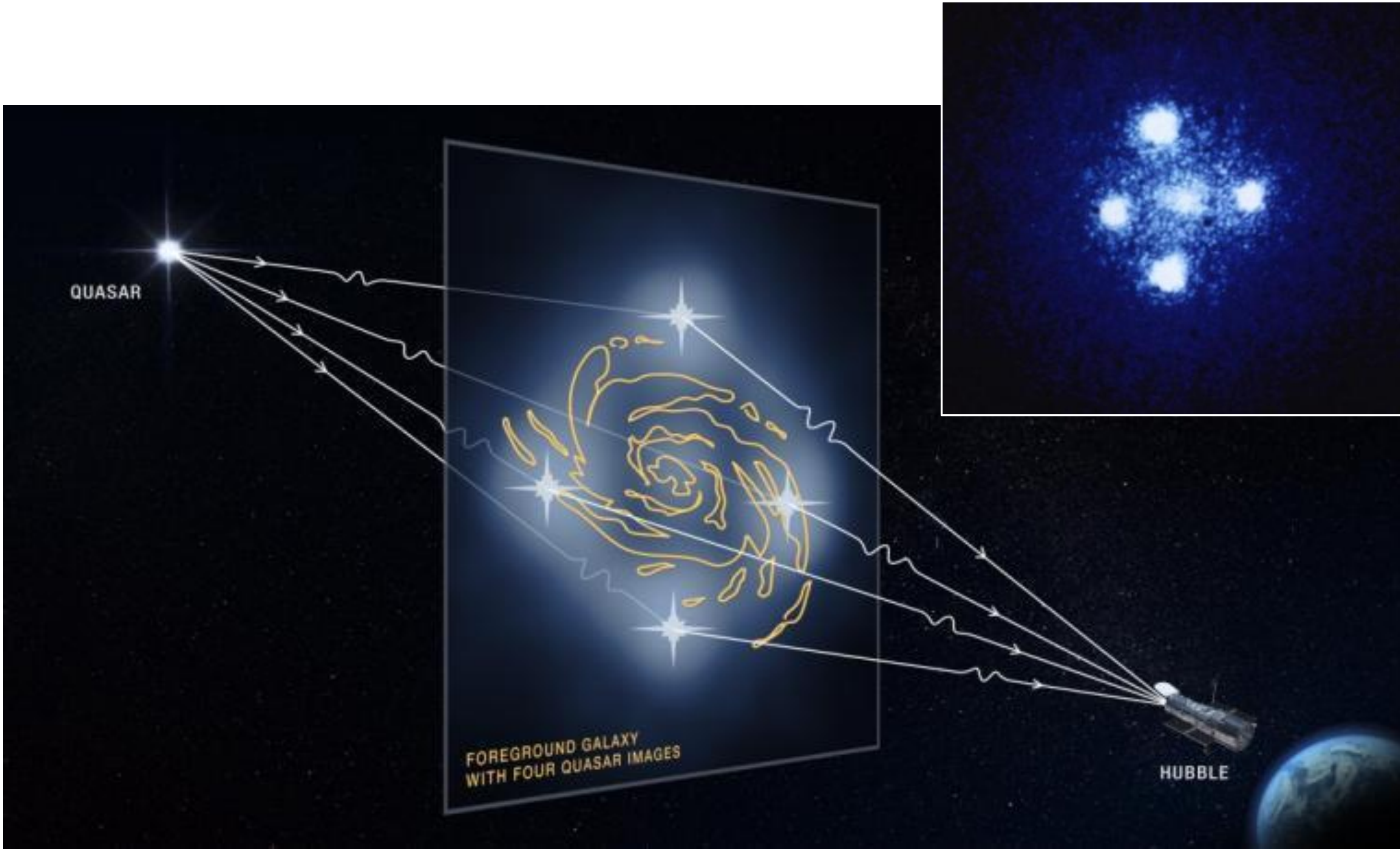


Gravitational lensing

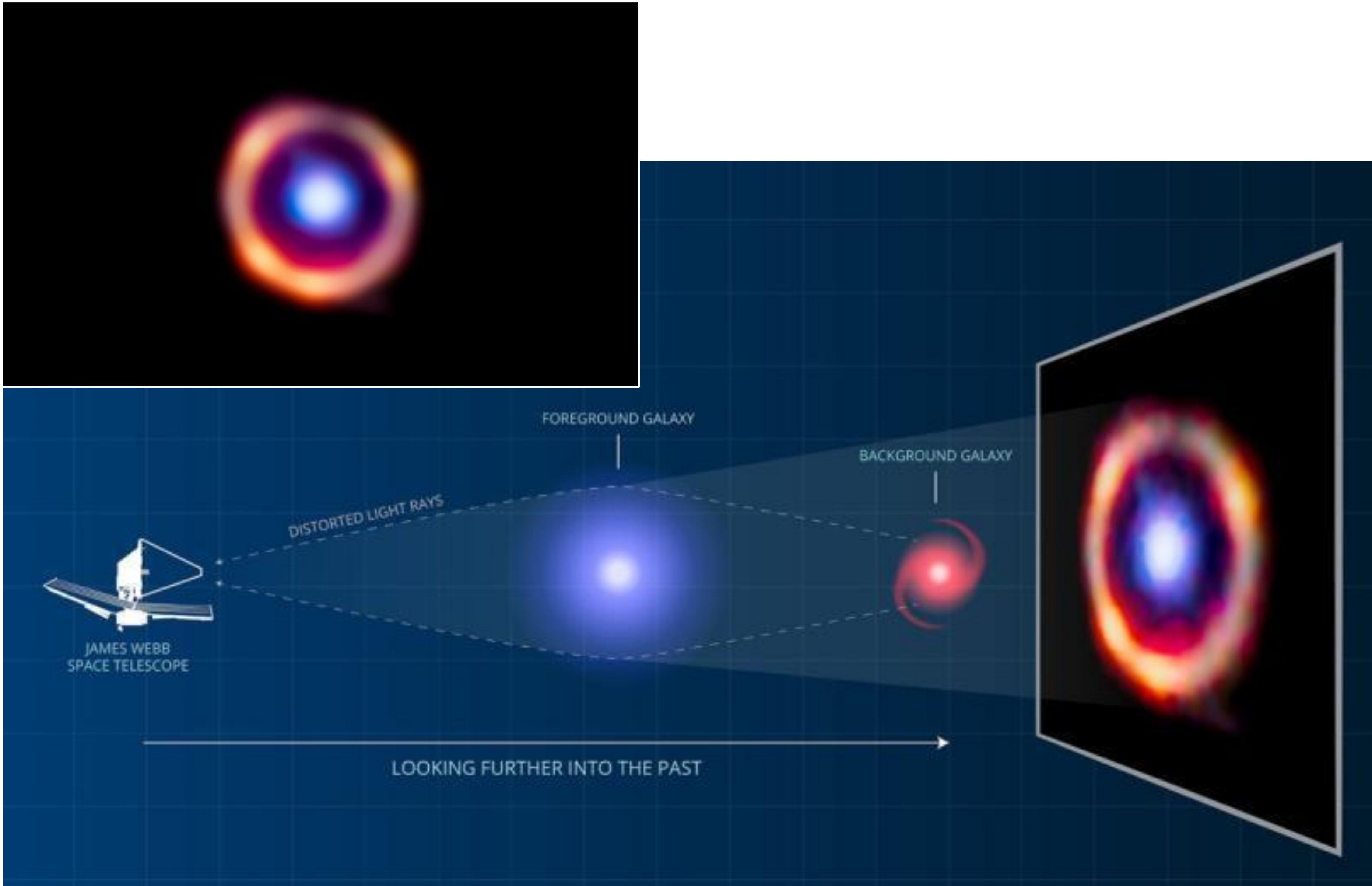




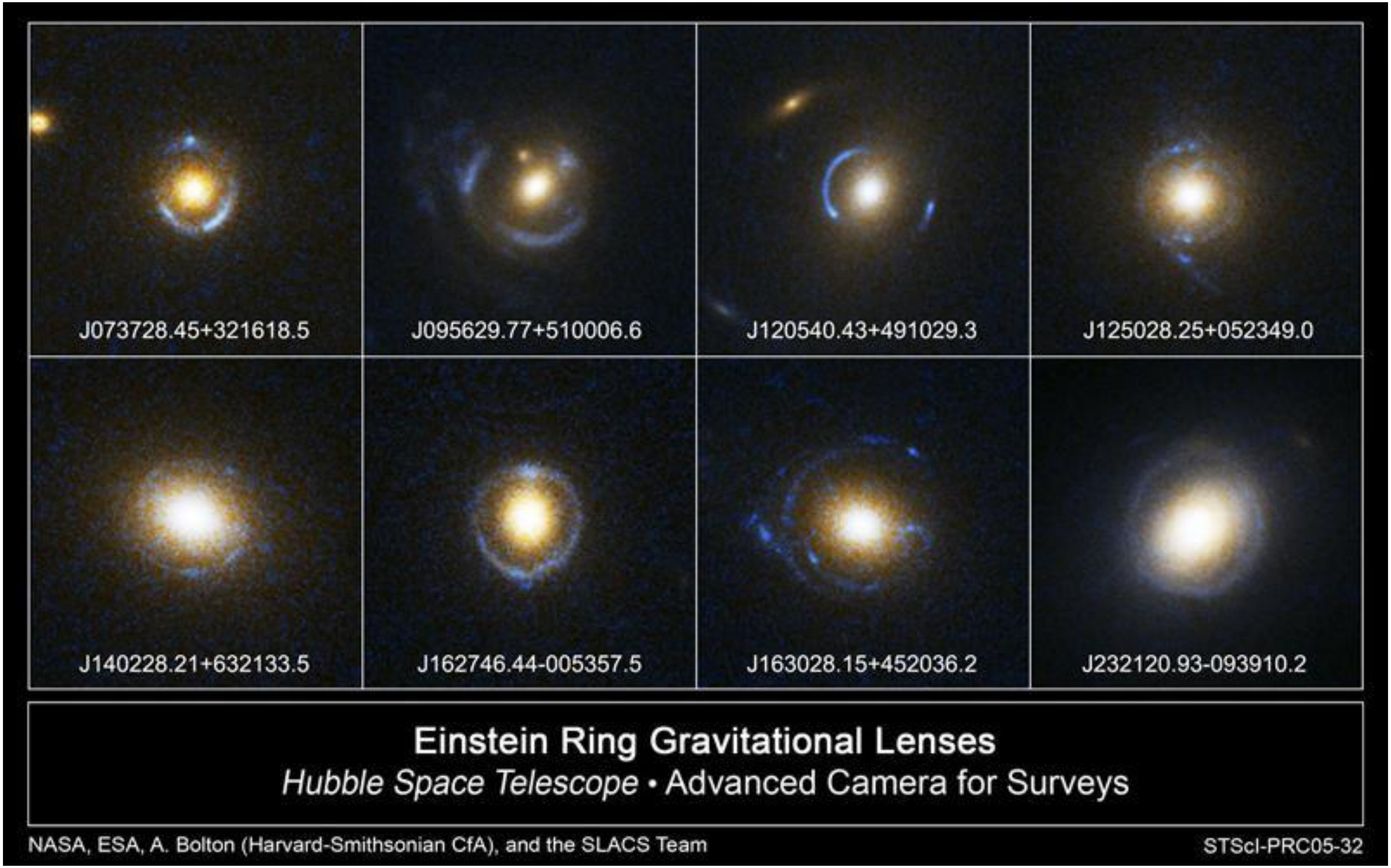
Einstein cross



Einstein ring



Einstein ring



Weak-field gravitational time delay in plasma medium

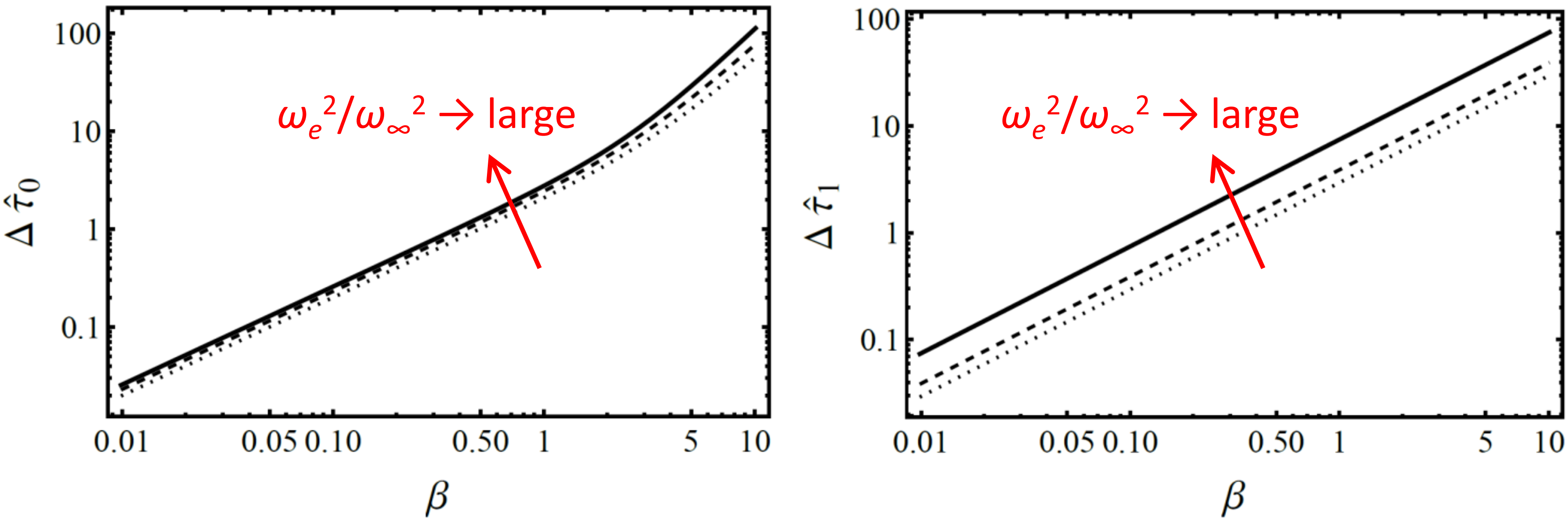
$$\tau = T(R_S) + T(R_O) - \frac{d_S}{\cos \mathcal{B}}$$: from source *S* to observer *O*

$$R_S = \sqrt{d_{LS}^2 + d_S^2 \tan^2 \mathcal{B}}, \quad R_O = d_L$$: radial coordinates

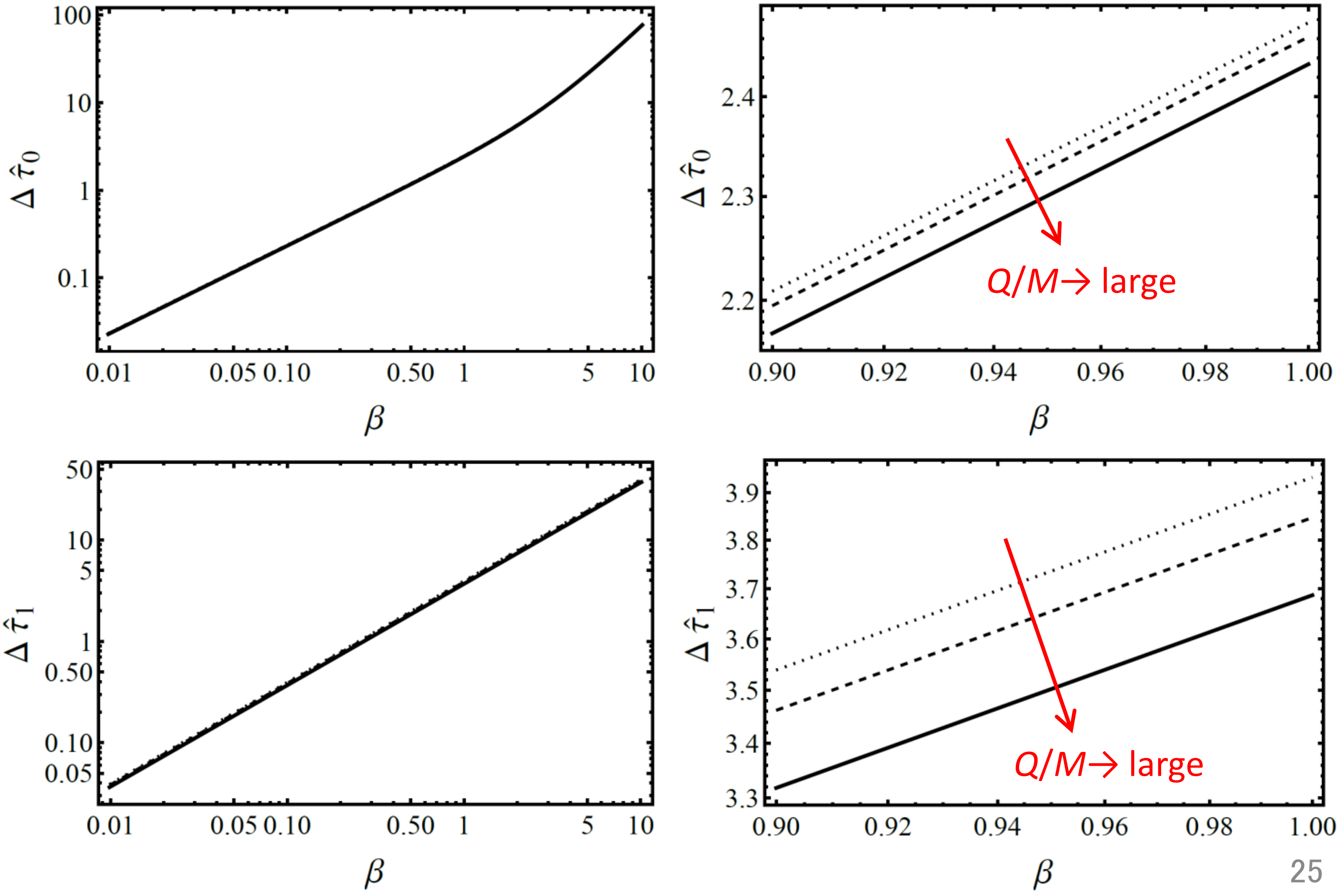
- Modified gravitational time delay of photon:

$$\tau \simeq d_S \left(\sqrt{\sigma^2+1} - 1\right) + \frac{2 d_L d_{LS} \sqrt{\sigma^2+1}}{d_S} \left[4\left(1+\sigma^2-\theta_0^2\right) + \frac{2\rho_0}{M} + 4\beta^2 \left(1 + \frac{d_{LS}\left(\sqrt{\sigma^2+1}-1\right)}{d_L\sqrt{\sigma^2+1}}\right) + \left(2\sigma^2 - 4 - \frac{\rho_0}{M}\right) \log\left(\frac{d_L \theta_0^2 \vartheta_E^2}{4 d_{LS}}\right)\right] \epsilon^2 + \frac{2 d_L d_{LS}\sqrt{\sigma^2+1}}{d_S \theta_0} \left(3\pi\left(5 - \frac{Q^2}{M^2} + \frac{2\rho_0}{M}\right) + 2\theta_1\left(2\sigma^2 - 4 - 4\theta_0^2 - \frac{\rho_0}{M}\right)\right)\epsilon^3$$

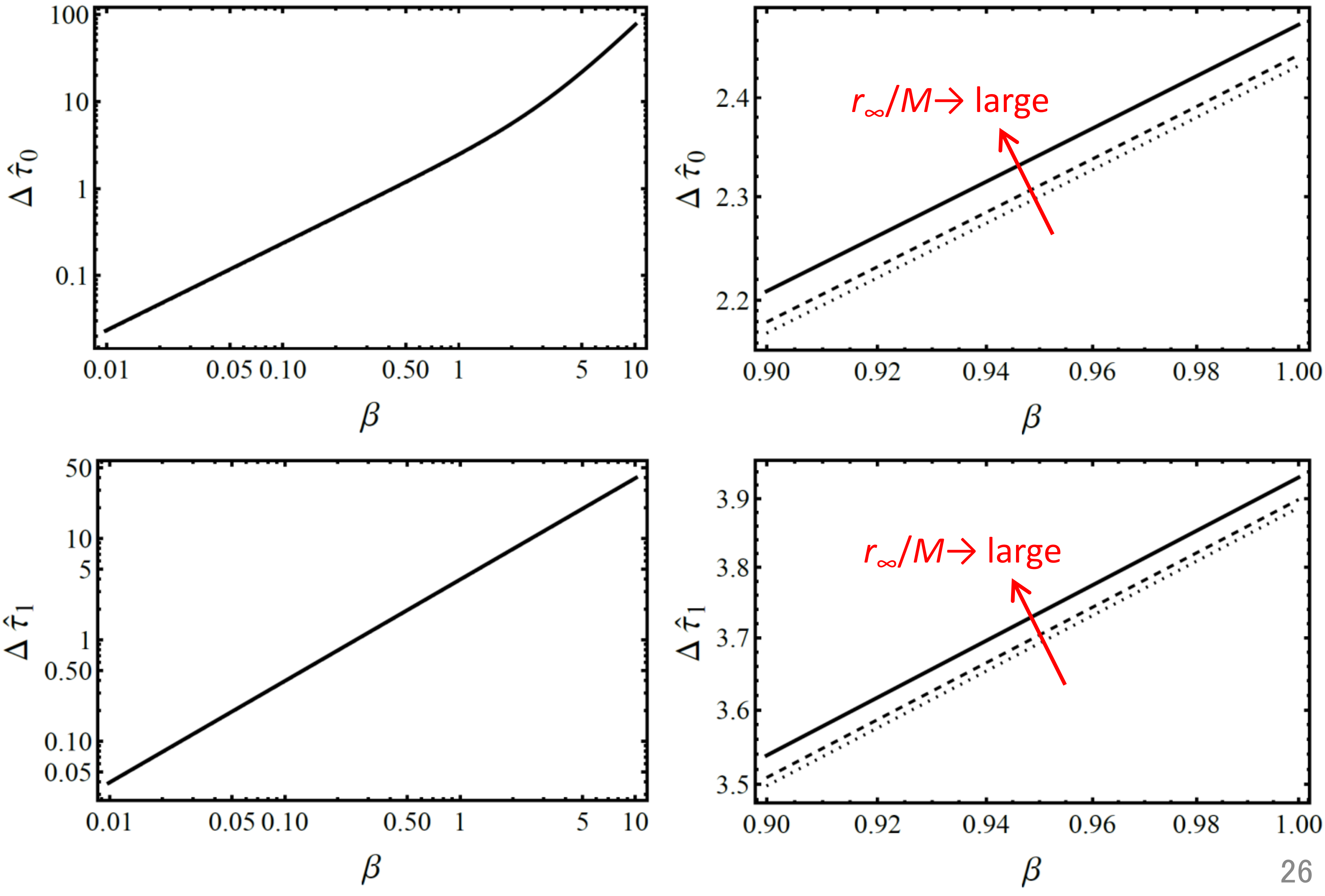
Differential time delay components ($Q/M = 0.03$, $r_\infty/M = 0.1$)



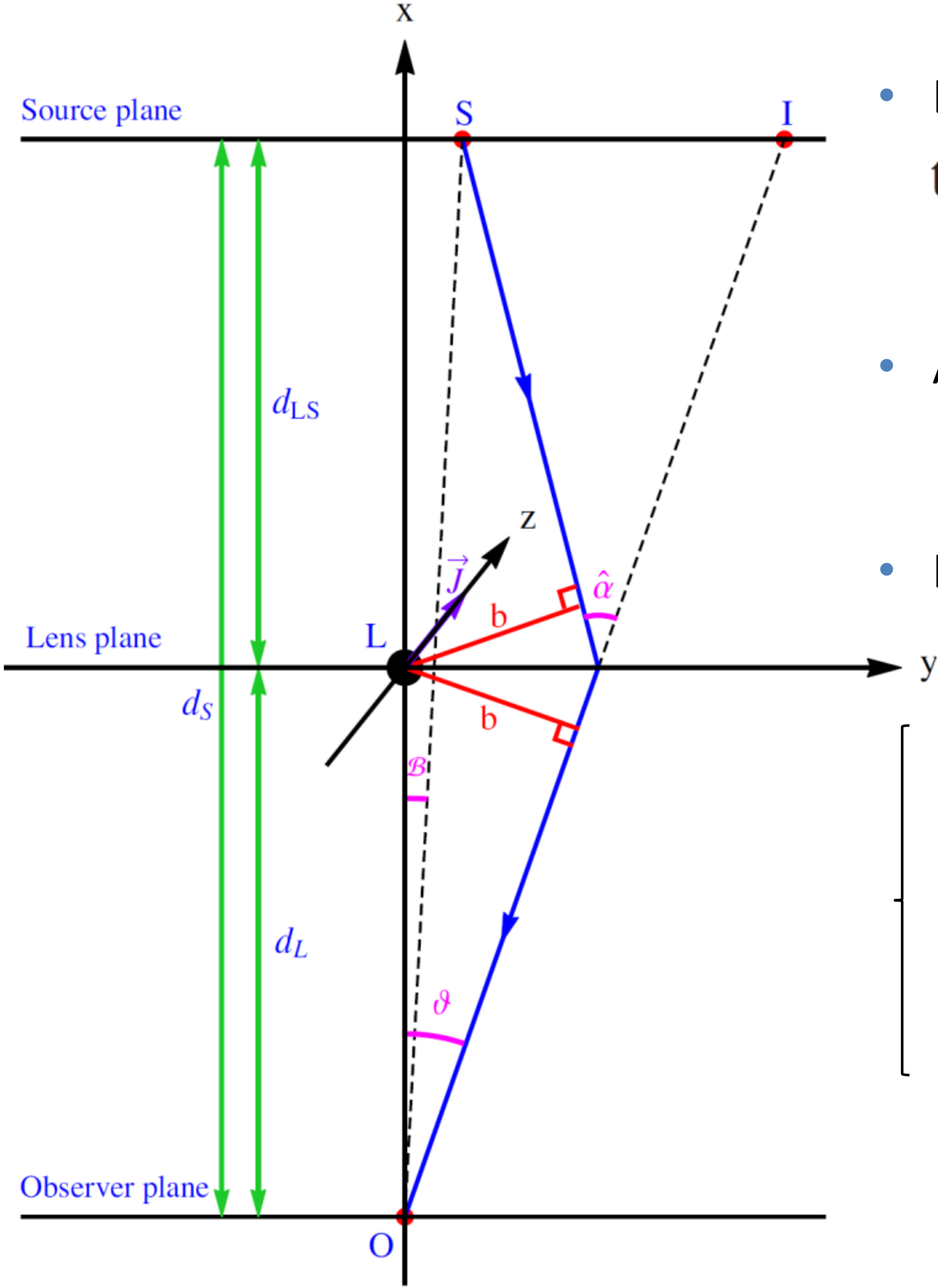
Differential time delay components ($\omega_e^2/\omega_\infty^2 = 0.5$, $r_\infty/M = 1$)



Differential time delay components ($\omega_e^2/\omega_\infty^2 = 0.5$, $Q/M = 0.03$)



Lens diagram of squashed Kaluza-Klein black hole



- Lens equation:

$$\tan \mathcal{B} = \tan \vartheta - D[\tan \vartheta + \tan(\hat{\alpha} - \vartheta)]$$

$$D = d_{LS}/d_S$$

- Angular Einstein ring radius:

$$\vartheta_E \equiv \sqrt{4DM/d_L}$$

- Impact parameter: $b = d_L \sin \vartheta$

$$\beta \equiv \frac{\mathcal{B}}{\vartheta_E} \qquad \theta \equiv \frac{\vartheta}{\vartheta_E} \qquad \varepsilon \equiv \frac{\vartheta_E}{4D}$$

$$\theta = \theta_0 + \theta_1 \varepsilon + \theta_2 \varepsilon^2 + \mathcal{O}(\varepsilon^3)$$

Light deflection by squashed KK BH in plasma medium

- Introducing coordinate $u := \rho^{-1}$, energy conservation equation of photon:

$$\begin{cases} \left(\frac{du}{d\phi}\right)^2 = \frac{1}{b^2} + \frac{2\sigma^2 M + \rho_0}{b^2} u - \left(1 + \frac{\sigma^2(Q^2 - 2M\rho_0)}{b^2}\right) u^2 + \left(2M - \frac{\sigma^2 Q^2 \rho_0}{b^2}\right) u^3 - Q^2 u^4 \\ b^2 := \frac{L^2}{\omega_\infty^2 - \omega_e^2}, \qquad \sigma^2 := \frac{\omega_e^2}{\omega_\infty^2 - \omega_e^2} \end{cases}$$

➢ Orbit of photon (b: impact parameter):

$$\begin{aligned} u(\phi) = {} & \frac{\cos\phi}{b} + \frac{(2\sigma^2 + 3)M - M\cos(2\phi) + \rho_0}{2b^2} \\ & + \frac{1}{16b^3}[((8\sigma^2(\sigma^2 + 6) + 37)M^2 + 24M\rho_0(\sigma^2 + 1) + 2\rho_0^2 - (8\sigma^2 + 9)Q^2)\cos\phi \\ & + (3M^2 + Q^2)\cos(3\phi) + 4(3M^2(4\sigma^2 + 5) + (2\sigma^2 + 3)(2M\rho_0 - Q^2))\phi\sin\phi] \\ & + O(M^3, \rho_0^3, M^2\rho_0, MQ^2, M\rho_0^2, Q^2\rho_0), \end{aligned}$$

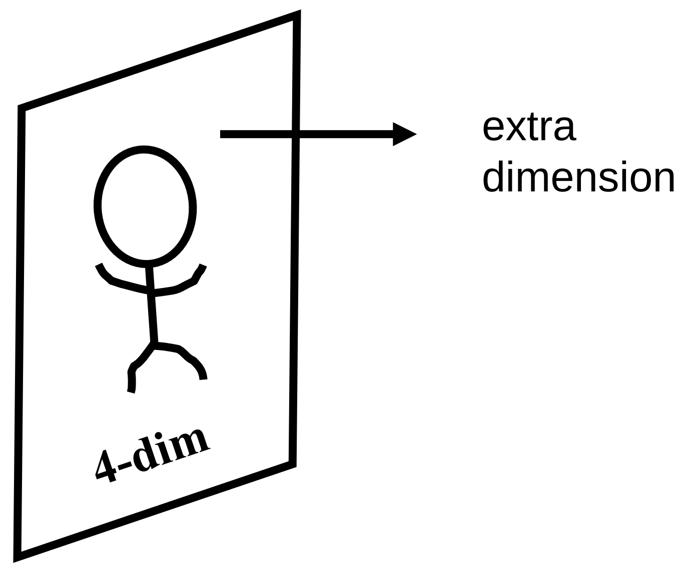
✓ Deflection angle of photon in weak-field limit:

$$\begin{aligned} \hat{\alpha} = {} & \frac{2M}{b}\left(1 + \frac{\rho_0}{2M} + \frac{1}{1 - \omega_e^2/\omega_\infty^2}\right) \\ & + \frac{3\pi M^2}{4b^2}\left(1 + \frac{4}{1 - \omega_e^2/\omega_\infty^2} + \frac{2M\rho_0 - Q^2}{3M^2}\left(1 + \frac{2}{1 - \omega_e^2/\omega_\infty^2}\right)\right) + O(b^{-3}) \end{aligned}$$

Introduction

String theory
and
Supergravity etc.

我々の世界は**高次元**
余分な次元はコンパクト化されている



しかし、我々の世界が高次元であることはまだ未確認

高次元時空の検証

高次元の理論では

TeVスケールでの加速器実験(LHC)における、ミニブラックホール生成の可能性

⇒ **高次元理論が実験的に検証できるかも**

高次元のBlack Holeが注目されている

高次元時空の Black Objects

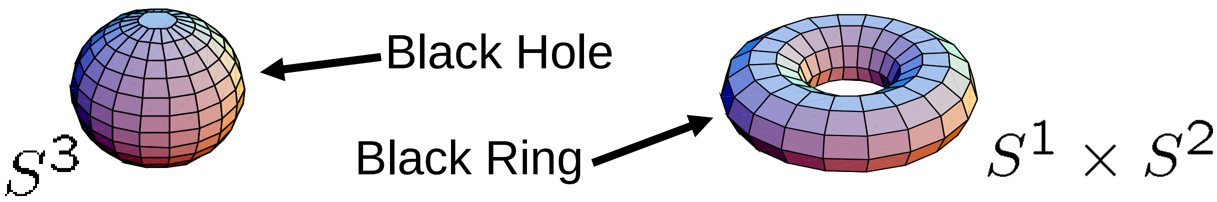
4次元で真空のEinstein理論

定常回転している漸近平坦な解

⟶ Kerr Black Hole 解のみ

5次元で真空のEinstein理論

同じ質量と角運動量を持つ、定常回転している漸近平坦な解が複数存在している

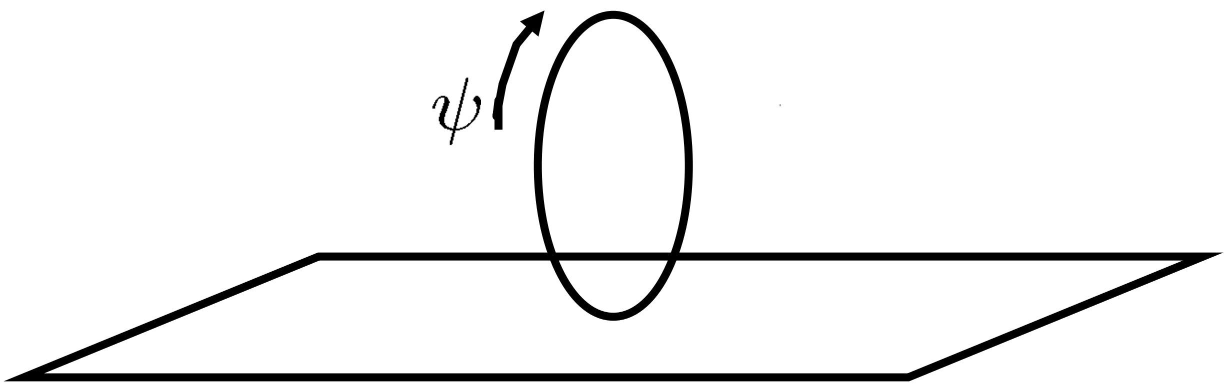


Kaluza-Klein black holes

Kaluza-Klein black holeとは？

コンパクトな余剰次元を持つblack hole

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\Omega_{S^2} + L^2(d\psi + \cos\theta d\phi)^2$$



4-dim Minkowski

Kaluza-Klein black holes の例

	M	Q	J_ϕ	J_ψ
Ishihara-Matsuno ('06)	○	○		
Dobiasch-Maison ('82)	○			○
Rasheed ('95)	○		○	○
Gaiotto-Strominger-Yin ('05)	○	○ $M = Q$		○
Elvang-Emparan-Mateos-Reall ('05)	○	○ $M = Q$		○
Nakagawa-Ishihara-Matsuno-Tomizawa ('08)	○	○		○

Charged Kaluza-Klein black holes

5次元Einstein-Maxwell理論

5-dim. Reissner-Nordstrom black hole

$$ds^2 = -f\,dt^2 + \frac{1}{f}dr^2 + r^2\Big[\underbrace{\frac{1}{4}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)}_{S^3}\Big]$$

Charged Kaluza-Klein black hole with squashed horizon ('06)

$$ds^2 = -f\,dt^2 + \frac{1}{f}\boxed{k^2}\,dr^2 + \frac{r^2}{4}\left[\boxed{k}\left(\boxed{\sigma_1^2 + \sigma_2^2}\right) + \sigma_3^2\right]$$

$$f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4}, \qquad k(r) = \frac{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}{(r_\infty^2 - r^2)^2}$$

$r = r_\infty$: spatial infinity

S^3 の metric-1

The metric of S^3

$$d\Omega_{S^3}^2 = d\chi^2 + \sin^2\chi\left(d\nu^2 + \sin^2\nu d\xi^2\right)$$
$$(0 < \chi < \pi,\ 0 < \nu < \pi,\ 0 < \xi < 2\pi)$$

↓

$$d\Omega_{S^3}^2 = \frac{1}{4}\left((d\theta^2 + \sin^2\theta d\phi^2) + (d\psi + \cos\theta d\phi)^2\right)$$
$$(0 < \theta < \pi,\ 0 < \phi < 2\pi,\ 0 < \psi < 4\pi)$$
$$= \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

S^3 の metric-1

The metric of S^3

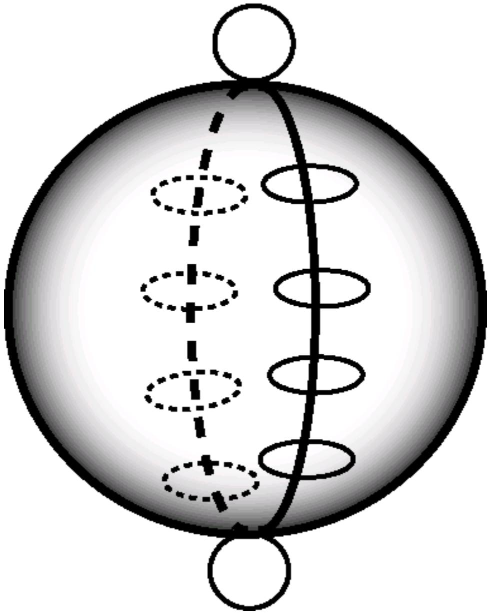
$$\sigma_1 = -\sin\psi d\theta + \cos\psi \sin\theta d\phi$$
$$\sigma_2 = \cos\psi d\theta + \sin\psi \sin\theta d\phi$$
$$\sigma_3 = d\psi + \cos\theta d\phi$$

$$\chi = \mathrm{ArcSin}\left(\sqrt{1-\cos^2\frac{\theta}{2}\cos^2\frac{\phi+\psi}{2}}\right)$$
$$\nu = \mathrm{ArcSin}\left(\frac{\sin\frac{\theta}{2}}{\sqrt{1-\cos^2\frac{\theta}{2}\cos^2\frac{\phi+\psi}{2}}}\right)$$
$$\xi = (\psi-\phi)/2$$

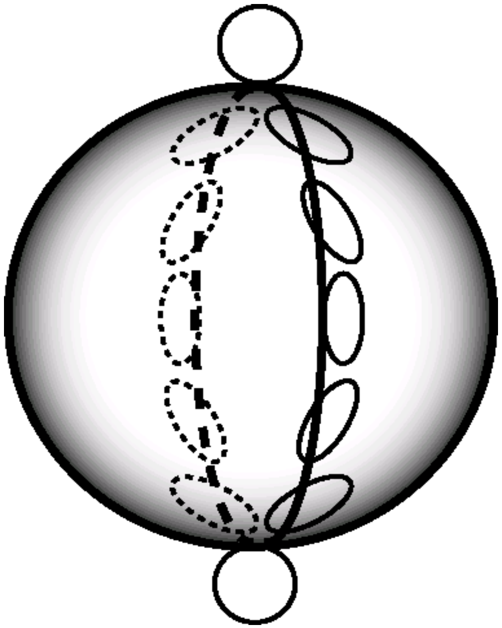
S^3 の metric-2

$$d\Omega^2_{S^3} = \frac{1}{4}\left((d\theta^2 + \sin^2\theta d\phi^2) + (d\psi + \cos\theta d\phi)^2 \right)$$

S2   S1



$S^2 \times S^1$



S^3

Charged Kaluza-Klein black holes

5次元Einstein-Maxwell理論

5-dim. Reissner-Nordstrom black hole

$$ds^2 = -f\,dt^2 + \frac{1}{f}dr^2 + r^2\Big[\underbrace{\frac{1}{4}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)}_{S^3}\Big]$$

Charged Kaluza-Klein black hole with squashed horizon ('06)

$$ds^2 = -f\,dt^2 + \frac{1}{f}\boxed{k^2}\,dr^2 + \frac{r^2}{4}\left[\boxed{k}\left(\boxed{\sigma_1^2 + \sigma_2^2}\right) + \sigma_3^2\right]$$

$$f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4}, \qquad k(r) = \frac{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}{(r_\infty^2 - r^2)^2}$$

$r = r_\infty$: spatial infinity

Squashing Transformation

T. Wang ('06)

$$ds^2 = -dt^2 + \frac{\Sigma}{\Delta}dr^2 + \frac{r^2+a^2}{4}[\sigma_1^2+\sigma_2^2+\sigma_3^2] + \frac{m}{r^2+a^2}\left(dt - \frac{a}{2}\sigma_3\right)^2$$

(5-dim Myers-Perry BH with two equal angular momenta)

↓ **"Squashing transformation"**

$$ds^2 = -dt^2 + \frac{\Sigma}{\Delta}k(r)^2 dr^2 + \frac{r^2+a^2}{4}[k(r)(\sigma_1^2+\sigma_2^2) + \sigma_3^2] + \frac{m}{r^2+a^2}\left(dt - \frac{a}{2}\sigma_3\right)^2$$

⟶ **Dobiasch-Maisonと一致する。**

Charged Rotating black holes-1

Chern-Simon項を含む5次元Einstein-Maxwell理論

$$S = \frac{1}{16\pi}\int d^5x\sqrt{-g}\Big[R - F^{\mu\nu}F_{\mu\nu} + \frac{2}{3\sqrt{3}}(\sqrt{-g})^{-1}\epsilon^{\mu\nu\rho\sigma\lambda}A_\mu F_{\nu\rho}F_{\sigma\lambda}\Big]$$

Chern-Simon項

⬇

Einstein eq.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\left(F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$$

Maxwell eq.

$$\nabla_\nu F^{\mu\nu} + \frac{1}{2\sqrt{3}}(\sqrt{-g})^{-1}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\nu\rho}F_{\sigma\lambda} = 0$$

Charged Rotating black holes-2

M. Cvetic and D. Youm ('96) , M. Cvetic, H. Lu and C. N. Pope ('04)

$$ds^2 = -\frac{w(r)}{h(r)}dt^2 + \frac{dr^2}{w(r)} + \frac{1}{4}r^2\left(\sigma_1^2 + \sigma_2^2 + h(r)(f(r)dt + \sigma_3)^2\right)$$

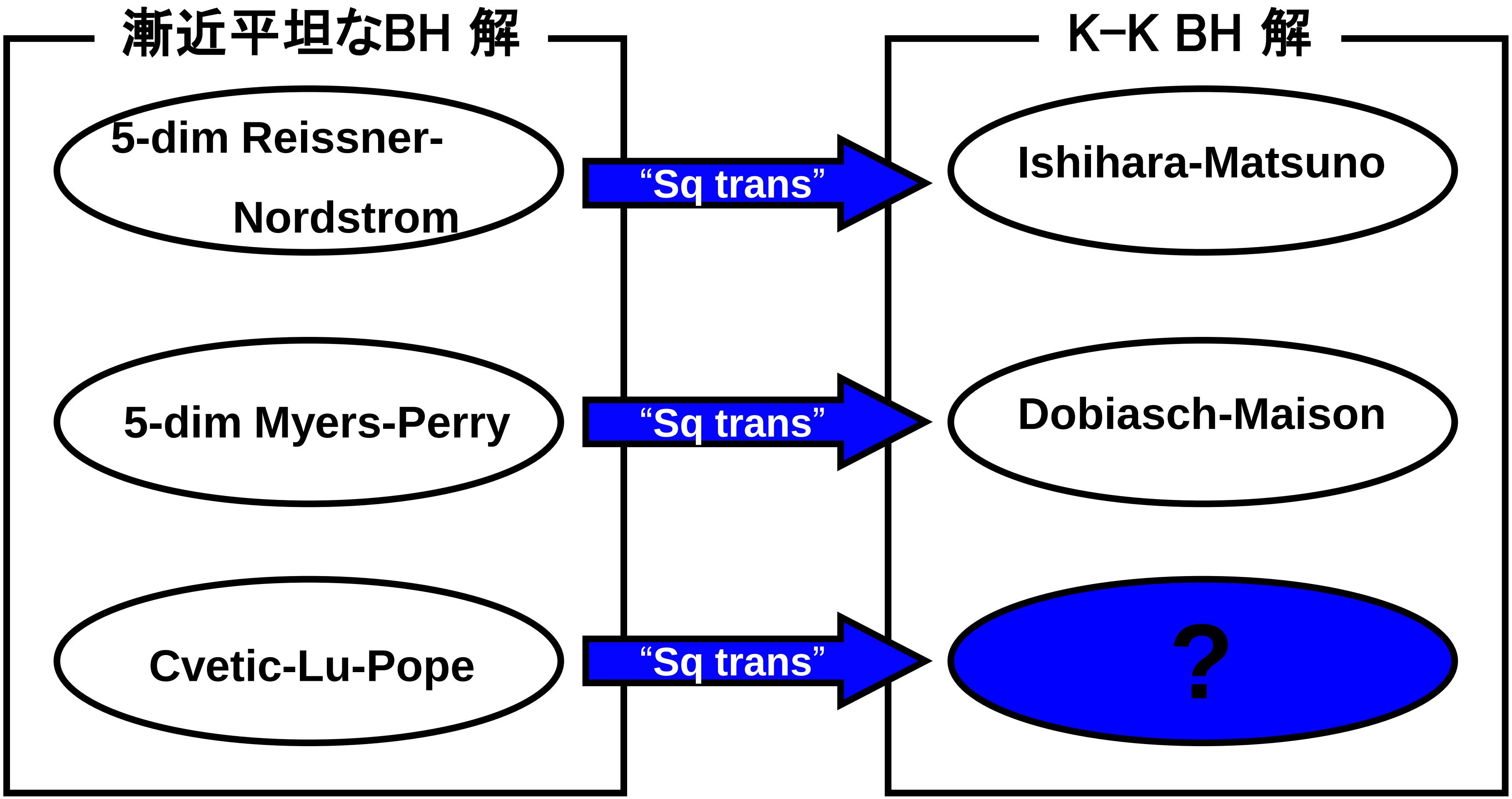
m : mass parameter

q : charge

a : angular momentum parameter

⇒ 先ほどの Myers-Perry と metric の形は全く同じ

解の生成



漸近平坦なBH 解
K−K BH 解
5-dim Reissner-Nordstrom
5-dim Myers-Perry
Cvetic-Lu-Pope
“Sq trans”
“Sq trans”
“Sq trans”
Ishihara-Matsuno
Dobiasch-Maison
?

Solutions

Chern-Simon項を含む5次元Einstein-Maxwell理論

$$ds^2 = -\frac{w(r)}{h(r)}dt^2 + \boxed{k(r)^2}\frac{dr^2}{w(r)} + \frac{1}{4}r^2\left(\boxed{k(r)}\boxed{(\sigma_1^2+\sigma_2^2)} + h(r)(f(r)dt+\sigma_3)^2\right)$$

m : mass parameter

q : charge

a : angular momentum parameter

r_∞ : spacial infinity

Solutions

$$w(r) = \frac{(r^2+q)^2 - 2(m+q)(r^2-a^2)}{r^4}$$
$$= \frac{(r^2-r_+^2)(r^2-r_-^2)}{r^4}$$
$$h(r) = 1 - \frac{a^2q^2}{r^6} + \frac{2a^2(m+q)}{r^4}$$
$$f(r) = -\frac{2a}{r^2h(r)}\left(\frac{2m+q}{r^2} - \frac{q^2}{r^4}\right)$$
$$k(r) = \frac{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}{(r_\infty^2 - r^2)^2}$$

$r_\pm$: **outer horizon**と**inner horizon**の位置

Parameter Region-1

解に含まれるparameter

⇒ m, q, a, r_∞

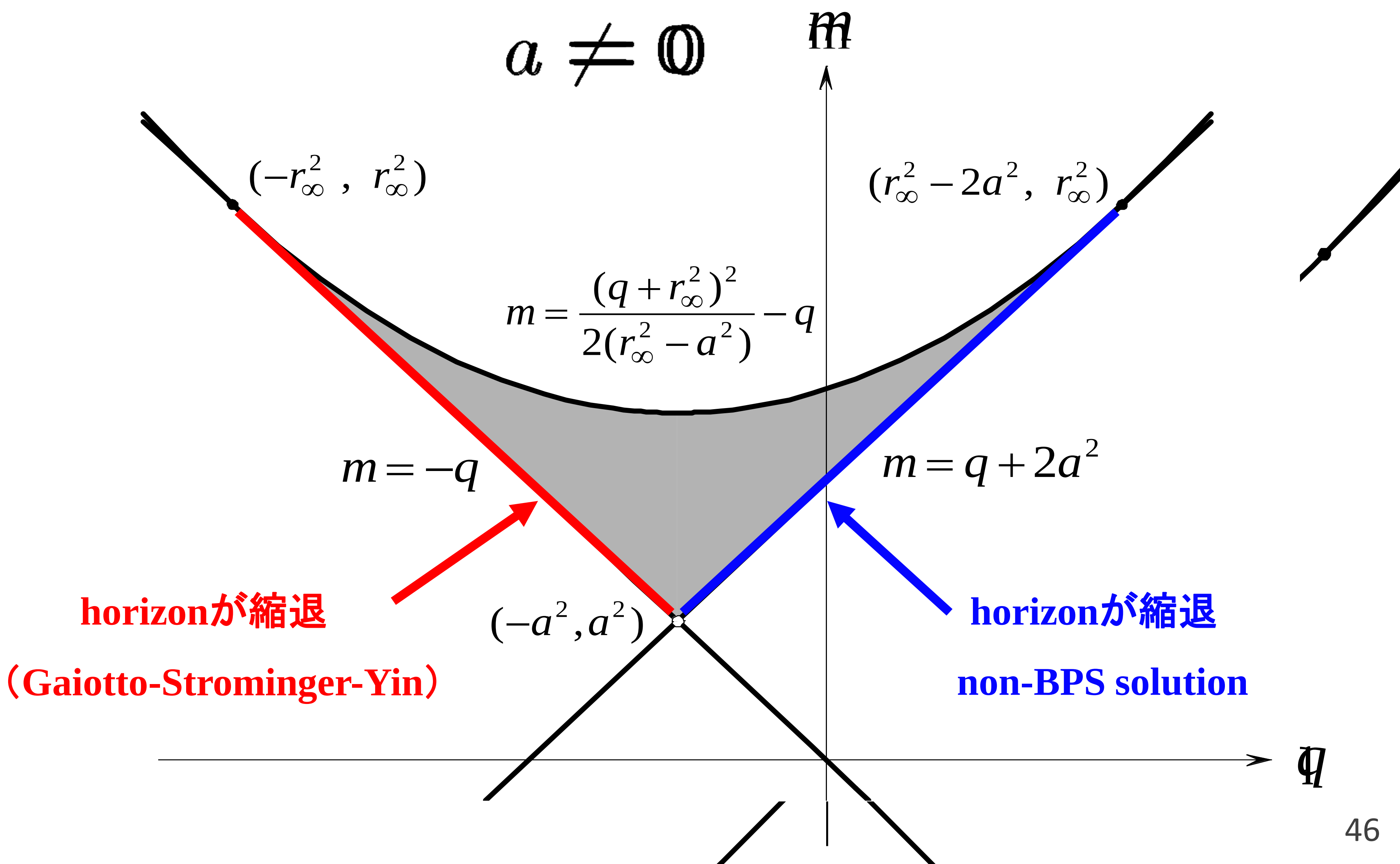
horizonが存在する

$$0 < r_- < r_+ < r_\infty$$

$$r_\pm^2 = m \pm \sqrt{(m+q)(m-q-2a^2)}$$

no CTCs (Closed Timelike Curves)

Parameter Region-2



漸近構造

$$\begin{cases} r \\ r \to r_\infty \end{cases} \Rightarrow \begin{cases} \rho = \rho_0 \dfrac{r^2}{r_\infty^2 - r^2} \\ \rho_0^2 = \dfrac{(r_\infty^2 + q)^2 - 2(m+q)(r_\infty^2 - a^2)}{4r_\infty^2} \\ \rho \to \infty \end{cases}$$

$$ds^2 \simeq -dt^2 + d\rho^2 + \rho^2(\sigma_1^2 + \sigma_2^2) + L^2\sigma_3^2$$

4-dim Minkowski

twisted S^1

$$L^2 = \frac{r_\infty^6 - a^2(q^2 - 2(m+q)r_\infty^2)}{4r_\infty^4}$$

保存量

解を特徴付ける保存量

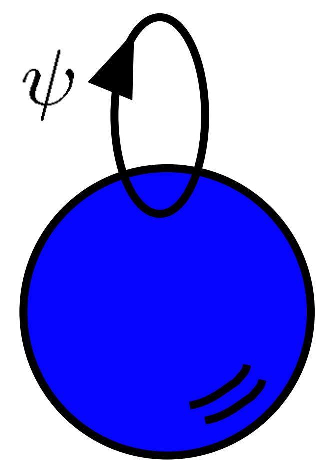
$$M_{\text{Komar}} = \pi \frac{2r_\infty^6(mr_\infty^2 - q^2) - 2a^4(m+q)q^2 - a^2(q^4 - 4mq^2r_\infty^2 + (4m^2 + 4mq + 3q^2)r_\infty^4)}{2r_\infty^2(r_\infty^6 - a^2(q^2 - 2(m+q)r_\infty^2))\rho_0} L$$

$$Q = -\frac{\sqrt{3}}{2}\pi q$$

$$J_\phi = 0$$

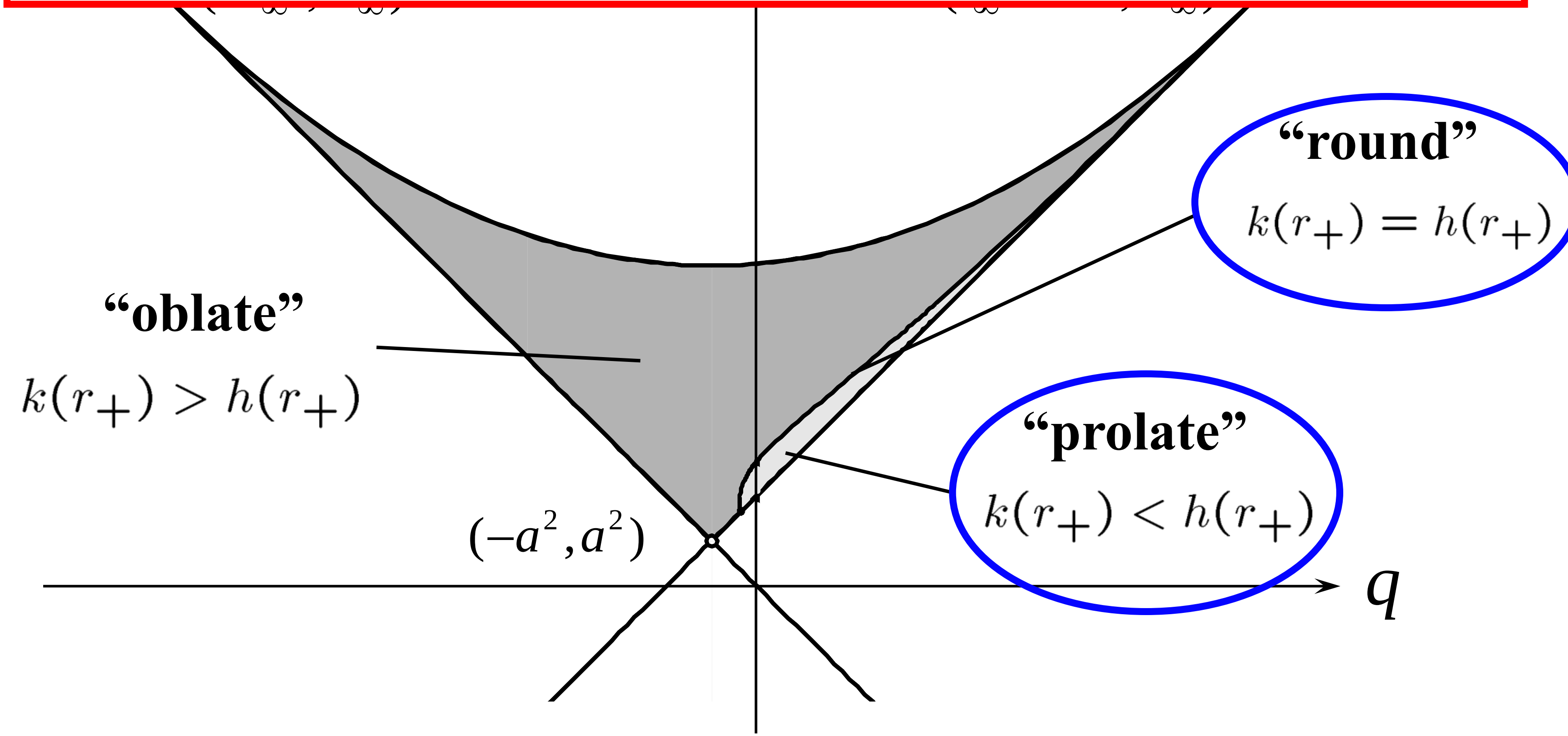
$$J_\psi = -\pi \frac{a(a^2q^3 + 3q^2r_\infty^4 - 2(2m+q)r_\infty^6)}{4r_\infty^4\sqrt{r_\infty^6 - a^2(q^2 - 2(m+q)r_\infty^2)}} L$$

余剰次元方向にのみ回転

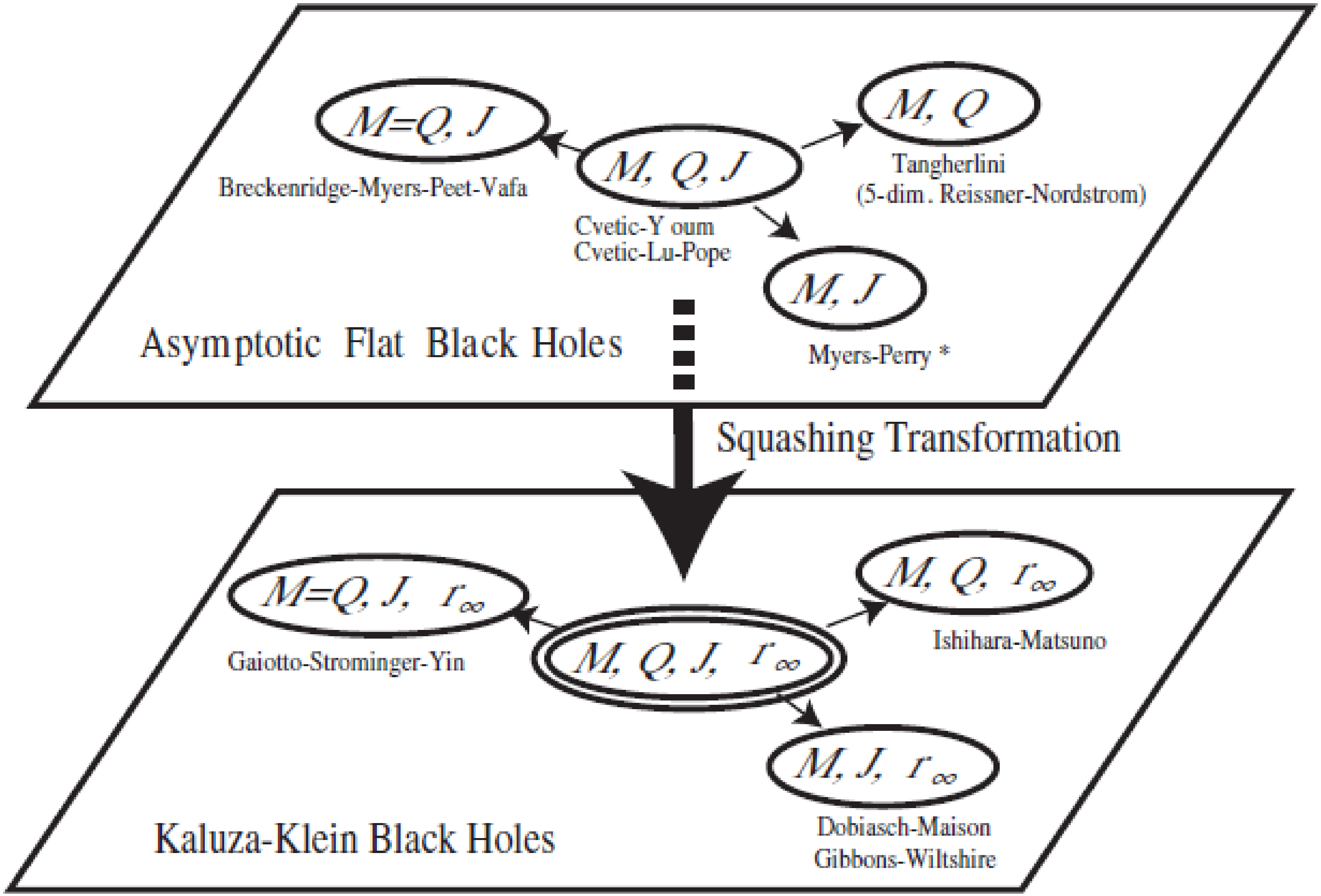


Geometry of Horizon

$$ds^2_{horizon} = \frac{r_+^2}{4}\left(k(r_+)d\Omega^2_{S^2} + h(r_+)(d\psi + \cos\theta d\phi)^2\right)$$



他の解との関係



M=Q, J
Breckenridge-Myers-Peet-Vafa
M, Q, J
Cvetic-Y oum
Cvetic-Lu-Pope
M, Q
Tangherlini
(5-dim. Reissner-Nordstrom)
M, J
Myers-Perry *
Asymptotic Flat Black Holes
Squashing Transformation
M=Q, J, r∞
Gaiotto-Strominger-Yin
M, Q, J, r∞
M, Q, r∞
Ishihara-Matsuno
M, J, r∞
Dobiasch-Maison
Gibbons-Wiltshire
Kaluza-Klein Black Holes

squashed Kaluza-Klein BHs の一般化 と 応用

- **squashed Kaluza-Klein ブラックホール解の一般化**
  - 回転パラメータを含むブラックホール解
  - 多体BPSブラックホール解
  - Dilaton場や非可換ゲージ場を含む重力理論におけるブラックホール解
  - …

- **厳密解が得られたことにより**
  - 安定性などの摂動的研究、熱力学、ホーキング放射、…
  - squashed Kaluza-Klein ブラックホールの周りの試験粒子の運動

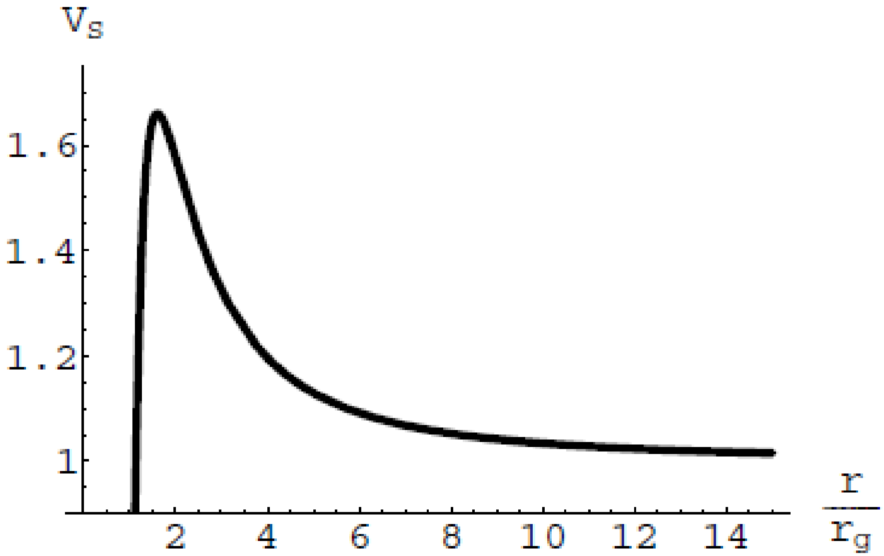
    (geodetic precession, light deflection, 重力レンズ, …)

  ⇒ **物理の観測結果との比較により具体的に検証可能な高次元時空モデル**

ブラックホール周辺の試験粒子の運動

5次元漸近平坦BH

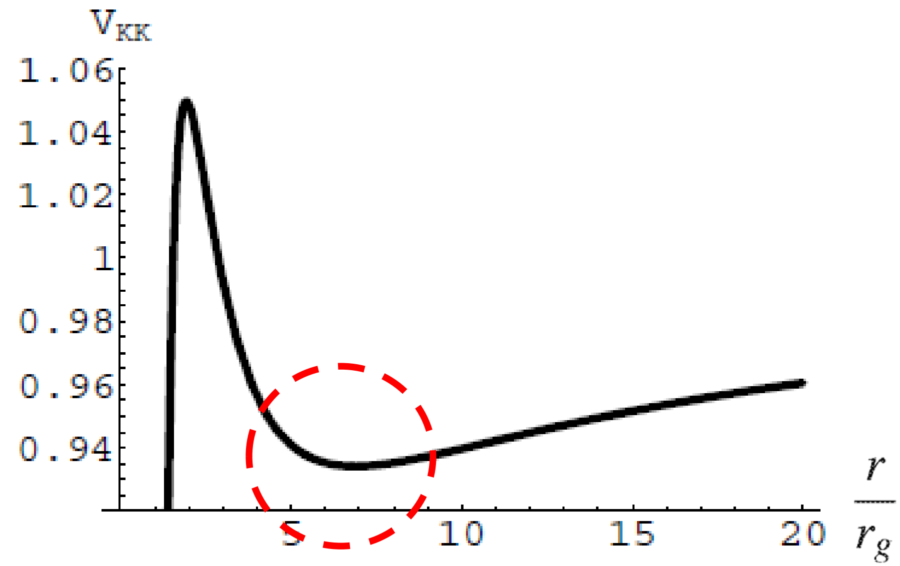
$$V_S(r) = \left(1 - \frac{r_g^2}{r^2}\right)\left(1 + \frac{L_S^2}{r^2}\right)$$



安定円軌道なし

5次元 squashed Kaluza-Klein BH

$$V_{KK}(r) = \left(1 - \frac{r_g}{r}\right)\left[1 + \frac{l^2}{r(r+r_0)}\right]$$

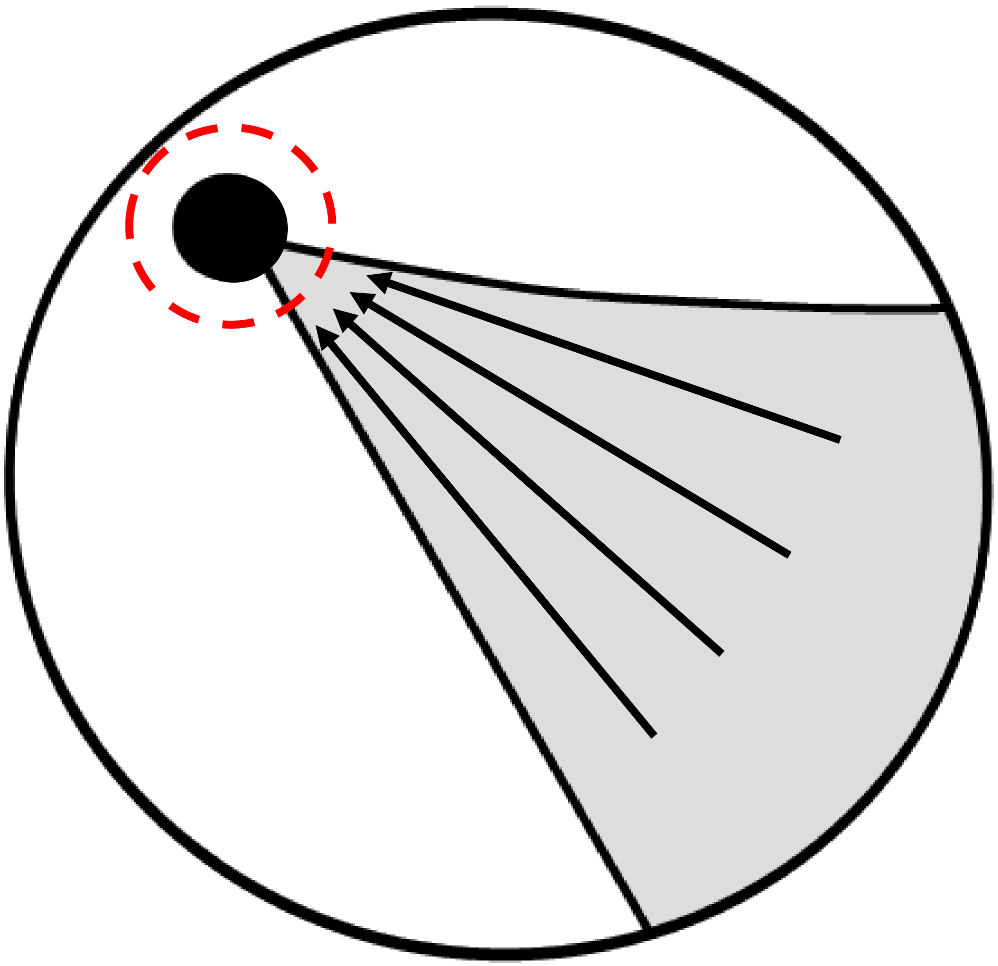


安定円軌道

5次元 squashed Kaluza-Klein BH: 現実の天体周辺を記述できる

⇒ 重力源周りの物理現象 (近日点移動等) に現れる高次元補正

$r_\pm \to r_\infty$limit-1



$$(q,m) \to \begin{cases} (-r_\infty^2, r_\infty^2) \\ \text{or} \\ (r_\infty^2 - 2a^2, r_\infty^2) \end{cases}$$

$$r_\pm \to r_\infty$$

ただし$\rho_\pm$は有限に保つ

$$\rho_\pm = \frac{r_\pm^2}{2r_\infty^2}\sqrt{\frac{r_\infty^2 - r_\mp^2}{r_\infty^2 - r_\pm^2}}$$

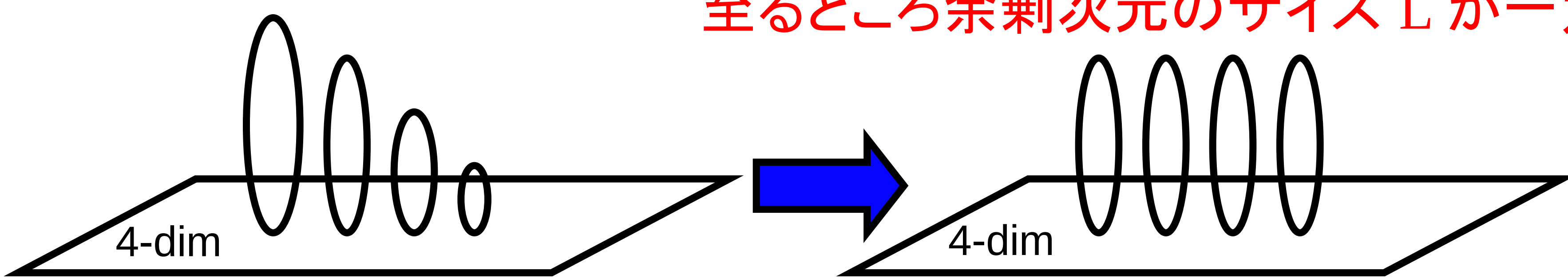
q

$r_{\pm} \to r_{\infty}$ limit-2

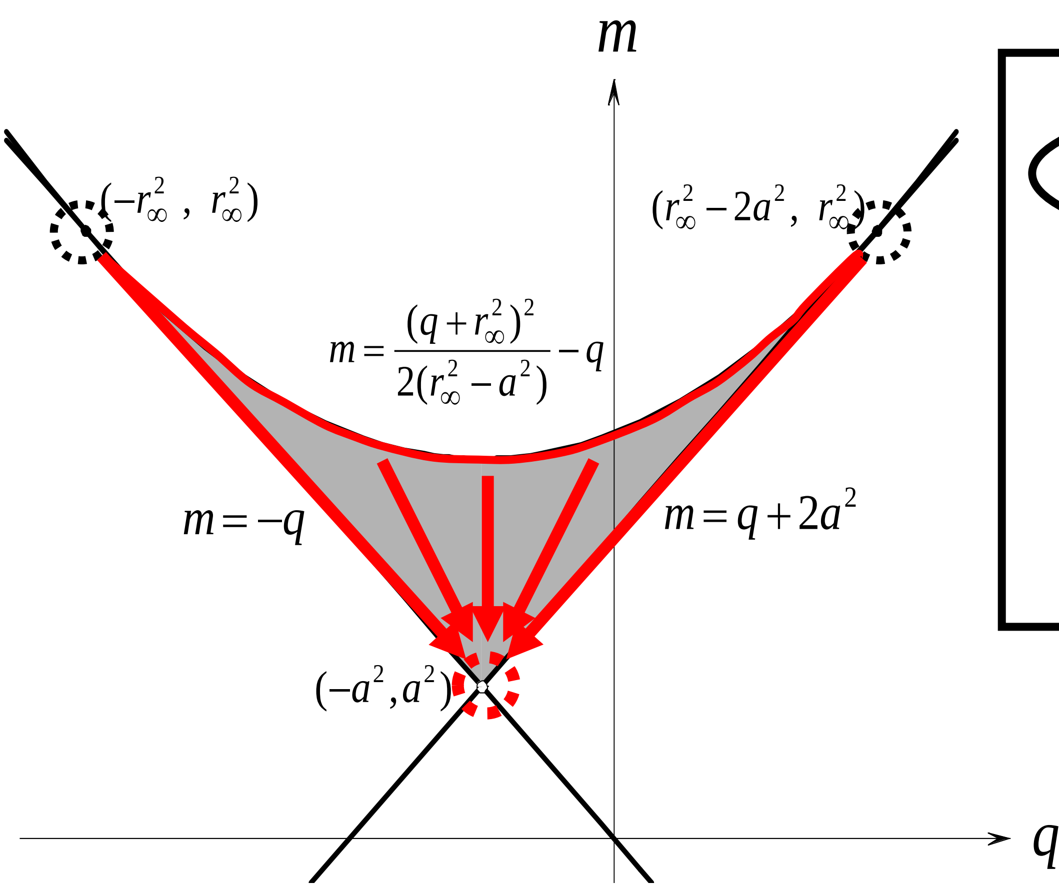
$(q, m) \to (-r_{\infty}^2, r_{\infty}^2)$ の場合

$$ds^2 = -\frac{4(\rho-\rho_+)(\rho-\rho_-)-a^2}{4\rho^2}dT^2 + \frac{d\rho^2}{\left(1-\frac{\rho_+}{\rho}\right)\left(1-\frac{\rho_-}{\rho}\right)} + \rho^2(\sigma_1^2+\sigma_2^2) + \frac{4\rho_+\rho_- - a^2}{4}\sigma_3^2 + a\frac{\sqrt{4\rho_+\rho_- - a^2}}{2\rho}dT\sigma_3,$$

至るところ余剰次元のサイズ L が一定!!



Black String Limit-1



$\rho_\pm$ を有限に保ちながら $r_\pm \to r_\infty$

＋

$L \to 0$

ただし $L\psi$ は有限に保つ

Black String Limit-2

metric

$$ds^2 = -\left(1-\frac{\rho_+}{\rho}\right)dt^2 + \frac{d\rho^2}{\left(1-\frac{\rho_+}{\rho}\right)\left(1-\frac{\rho_-}{\rho}\right)} + \rho^2 d\Omega^2_{S^2} + \left(1-\frac{\rho_-}{\rho}\right)d\psi^2$$

$$\mathbf{A} = \frac{\sqrt{3}}{2}\sqrt{\rho_+\rho_-}\cos\theta d\phi$$

⇒ magnetically charged Black String !!

(Horowitz-Strominger ('91))

Summary

- Chern-Simon項を含む 5次元 Einstein-Maxwell 理論における Kaluza-Klein Black Hole 解をこれまでに多数構成した。
- 今回の時空では余剰次元方向にのみ角運動量を持つ。つまり Black Hole は余剰次元方向にのみ回転しており、4次元方向には回転していない。
- 構成した解とすでに知られている解との関係を調べた。
- $\rho_\pm$ を有限に保ちながら$r_\pm \to r_\infty$ という極限をとると、余剰次元のサイズがいたるところで一定になる。
- その極限からさらに magnetically charged Black String への極限が存在する。

Periapsis shift by squashed Kaluza-Klein black hole

- Energy conservation equation of neutral massive test particle:

$$\begin{cases} \left(\frac{du}{d\phi}\right)^2 = \frac{1}{b^2} + \frac{2\sigma^2 M + \rho_0}{b^2} u - \left(1 + \frac{\sigma^2 (Q^2 - 2M\rho_0)}{b^2}\right) u^2 + \left(2M - \frac{\sigma^2 Q^2 \rho_0}{b^2}\right) u^3 - Q^2 u^4 \\ b^2 = \frac{L^2}{E^2 - 1}, \quad \sigma^2 = \frac{1}{E^2 - 1} \end{cases}$$

➢ Bound orbit of massive particle (*a*: semimajor axis, *e*: eccentricity):

$$u(\phi) = \frac{1 + e\cos(\gamma\phi)}{a(1-e^2)} + \frac{3M(2+e^2) + 4\rho_0 - 2Q^2/M - e^2 M \cos(2\phi)}{2a^2(1-e^2)^2}$$
$$+ O\left(M^2, \rho_0^2, Q^2, M\rho_0, \frac{\rho_0 Q^2}{M}, \frac{Q^4}{M^2}\right), \quad \gamma := 1 - \frac{2(3M + \rho_0) - Q^2/M}{2a(1-e^2)}$$

✓ Periapsis shift angle of massive particle in weak-field limit:

$$\delta\phi_{\mathrm{KK}} = \frac{6\pi M}{a(1-e^2)}\left(1 + \frac{\rho_0}{3M} - \frac{Q^2}{6M^2}\right) + O\left(a^{-2}\right)$$

Observational effects in periapsis shift

- Difference between shift angles in sq. KK spacetime and in 4D Schwarzschild spacetime:

$$\frac{\delta\phi_{\mathrm{KK}} - \delta\phi_{\mathrm{Sch}}}{\delta\phi_{\mathrm{Sch}}} \simeq -\tilde{\delta}_1 + \tilde{\delta}_2, \quad \tilde{\delta}_1 := \frac{Q^2}{3M^2}, \quad \tilde{\delta}_2 := \frac{r_\infty^2}{24M^2}$$

$\tilde{\delta}_1 \lesssim 10^{-6},\ \tilde{\delta}_2 \simeq 10^{-16}$: Mercury around Sun
$\left(Q \lesssim 4 \times 10^{17}\ \mathrm{C},\ r_\infty \simeq 0.1\ \mathrm{mm}\right)$

$\tilde{\delta}_1 \lesssim 10^{-5},\ \tilde{\delta}_2 \simeq 10^{-5}$: LAGEOS II satellite around Earth
$\left(Q \lesssim 3 \times 10^{12}\ \mathrm{C},\ r_\infty \simeq 0.1\ \mathrm{mm}\right)$

$\tilde{\delta}_1 \lesssim 10^{-37},\ \tilde{\delta}_2 \simeq 10^{-29}$: S2 around Sgr A*
$\left(Q \lesssim 3 \times 10^{8}\ \mathrm{C},\ r_\infty \simeq 0.1\ \mathrm{mm}\right)$

✓ Correction by extra dimension might be detected in future observations around Earth.

Derivation of squashed Kaluza-Klein BHs-1

We assume a five-dimensional static spacetime with SO(3)× U(1) symmetry.

$$\begin{cases} ds^2 = -h^2(x)dt^2 + N^2(x)dx^2 + a^2(x)\sigma_1^2 + b^2(x)(\sigma_2^2 + \sigma_3^2) \\ d\sigma^i = \frac{1}{2}C^i_{jk}\sigma^j \wedge \sigma^k, \quad C^1_{23} = C^2_{31} = C^3_{12} = 1 \end{cases}$$

✓ From Maxwell equation, $F = \frac{u}{ab^2}(hdt) \wedge (Ndx)$

➢ If we take $N=hab^2$, EOM from Einstein equation:

$$\begin{cases} 2\left(2\frac{a'}{a} + \frac{b'}{b}\right)\frac{b'}{b} + 2\left(\frac{a'}{a} + 2\frac{b'}{b}\right)\frac{h'}{h} - (2b^2 - \frac{1}{2}a^2)a^2h^2 = -\frac{3}{2}u^2h^2 \\ \left(\frac{a'}{a}\right)' - \frac{1}{2}a^4h^2 = -\frac{1}{2}u^2h^2, \\ \left(\frac{b'}{b}\right)' - (b^2 - \frac{1}{2}a^2)a^2h^2 = -\frac{1}{2}u^2h^2, \\ \left(\frac{h'}{h}\right)' = u^2h^2. \end{cases}$$

Derivation of squashed Kaluza-Klein BHs-2

✓ If metric has a horizon; $h \to 0$, $a \to$ finite, $b \to$ finite at $x \to \infty$, we obtain

$$h = \frac{\alpha}{u \sinh \alpha x}, \qquad a^2 = \frac{|u| \sinh \alpha x}{\sinh \alpha (x - x_a)}, \qquad b^2 = \frac{|u| \sinh \alpha x \sinh \alpha (x - x_a)}{\sinh^2 \alpha (x - x_b)}$$

➢ Introducing a coordinate and parameters:

$$r^2/4 = a^2(x), \qquad r_\pm^2 := 4u e^{\pm \alpha x_a}, \qquad r_\infty^2 := \frac{4|u| \sinh \alpha x_b}{\sinh \alpha (x_b - x_a)}$$

we have the metric of squashed Kaluza-Klein black holes:

$$\begin{cases} ds^2 = -f dt^2 + \dfrac{k^2}{f} dr^2 + \dfrac{r^2}{4} \left[k(\sigma_1^2 + \sigma_2^2) + \sigma_3^2 \right] \\ f(r) := \dfrac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4}, \quad k(r) := \dfrac{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}{(r_\infty^2 - r^2)^2} \end{cases}$$

5D charged squashed Kaluza-Klein black holes

(H. Ishihara, K.M)

$$\begin{cases} ds^2 = -F dt^2 + \dfrac{K^2}{F} d\rho^2 + \rho^2 K^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) + \dfrac{r_\infty^2}{4K^2} \left(d\psi + \cos\theta d\phi\right)^2 \\ A_\mu dx^\mu = \pm \dfrac{\sqrt{3\rho_+\rho_-}}{2\rho} dt, \quad F = \dfrac{(\rho-\rho_+)(\rho-\rho_-)}{\rho^2}, \quad K^2 = \dfrac{\rho+\rho_0}{\rho} \end{cases}$$

$$\begin{cases} r_\infty^2 = 4(\rho_+ + \rho_0)(\rho_- + \rho_0) & \text{: compact extra dimension size at infinity} \\ \rho_+ \geq \rho_- \geq 0, \ \rho_- + \rho_0 > 0 & \text{: parameter regions } (\rho_\pm\text{: horizons}) \end{cases}$$

➢ Mass, Charge, Hawking temperature, Entropy :

$$\begin{cases} M = \dfrac{\pi r_\infty}{G_5}(\rho_+ + \rho_-) = \dfrac{\rho_+ + \rho_-}{2G_4}, \quad |Q| = \dfrac{2\pi r_\infty}{G_5}\sqrt{\rho_+\rho_-} = \dfrac{\sqrt{\rho_+\rho_-}}{G_4} \\ T_{\mathrm{KK}} = \dfrac{\kappa}{2\pi} = \dfrac{\rho_+ - \rho_-}{4\pi\rho_+\sqrt{\rho_+(\rho_+ + \rho_0)}}, \quad S_{\mathrm{KK}} = \dfrac{\mathcal{A}}{4G_5} = \dfrac{\pi\rho_+\sqrt{\rho_+(\rho_+ + \rho_0)}}{G_4} \end{cases}$$

✓ Smarr-type formula :

$$M - 2QA_+/\sqrt{3} = 2T_{\mathrm{KK}}S_{\mathrm{KK}}, \quad |A_+| = \sqrt{3\rho_-/\rho_+}/2$$

Horizon and asymptotic structures

- Squashed S^3 horizons in the form of Hopf bundle :

$$ds_3^2 = \rho_\pm^2 K^2(\rho_\pm)\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{r_\infty^2}{4K^2(\rho_\pm)}(d\psi + \cos\theta d\phi)^2$$

$$\begin{cases} \rho_+ = M + \sqrt{M^2 - Q^2} & : \text{ outer horizon} \\ \rho_- = M - \sqrt{M^2 - Q^2} & : \text{ inner horizon} \end{cases}$$

- Surface gravity on outer horizon of black hole :

$$\kappa = \frac{\rho_+ - \rho_-}{2\rho_+\sqrt{\rho_+(\rho_+ + \rho_0)}}$$

- At infinity ($\rho \to \infty$) :

$$ds^2 \simeq -dt^2 + d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{r_\infty^2}{4}(d\psi + \cos\theta d\phi)^2$$

A twisted S^1 fiber bundle over 4D Minkowski spacetime

(r_∞ : size of compactified extra dimension at infinity)

Physical meanings of parameter ρ_0

✓ For observer $\rho_0 \ll \rho_\pm \lesssim \rho$,
4D Reissner-Nordstrom black hole with a twisted constant S^1 fiber :

$$ds^2 \simeq -\frac{(\rho-\rho_+)(\rho-\rho_-)}{\rho^2}dt^2 + \frac{\rho^2}{(\rho-\rho_+)(\rho-\rho_-)}d\rho^2 + \rho^2 d\Omega_{S^2}^2 + \frac{r_\infty^2}{4}(d\psi + \cos\theta d\phi)^2$$

✓ For observer $\rho_\pm \lesssim \rho \ll \rho_0$,
5D Reissner-Nordstrom black hole with round S^3 horizons :

$$ds^2 \simeq -\frac{(r^2-r_+^2)(r^2-r_-^2)}{r^4}dt^2 + \frac{r^4}{(r^2-r_+^2)(r^2-r_-^2)}dr^2 + r^2 d\Omega_{S^3}^2 .$$

➤ Parameter ρ_0 gives typical scale of transition from five dimensions to effective four dimensions.

今後の展望

Squashing Transformationに関して

- 宇宙項入りの解に対してこのような変換が存在するのか？
- 角運動量が等しくない漸近平坦な解に対しての変換は存在するのか？

より一般的な解を構成する

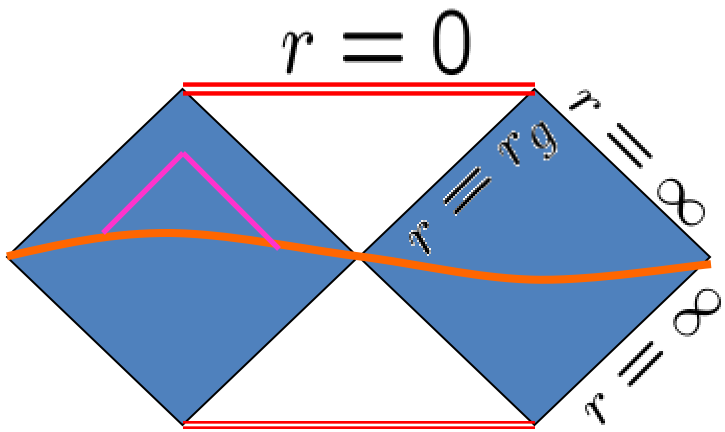
- 今回構成した解に宇宙項を入れたもの、等

電荷を持つ5次元ブラックホールにおける地平線の変形と特異点

松野 研、石原 秀樹
(大阪市立大学)

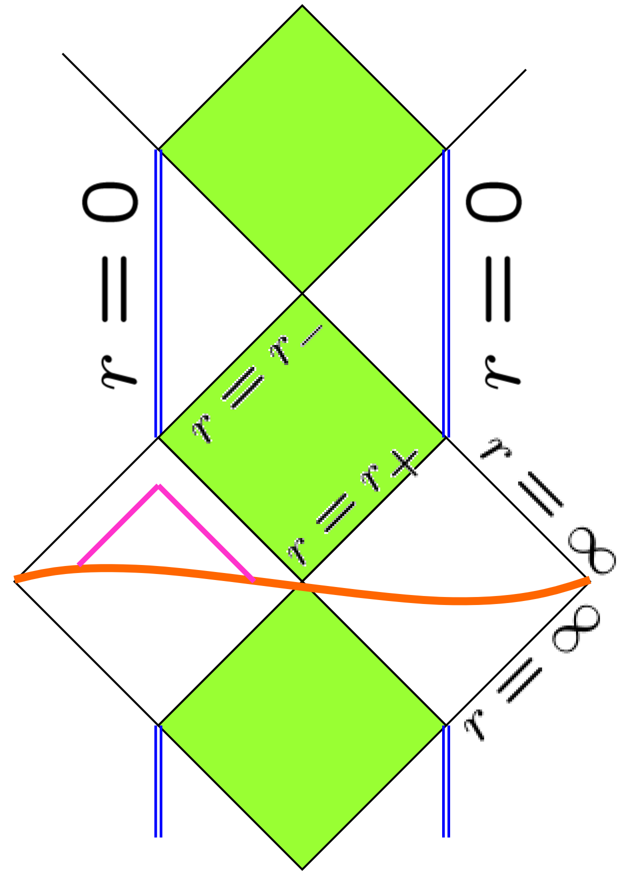
導入

- ブラックホールの構造



シュバルツシルトBH
(Globally Hyperbolic)

- Structure of RN-BH
  is unstable for some
  field perturbations.



RN-BH

{ Not Globally Hyperbolic
  2つの時空を繋ぐ"トンネル"

BH内特異点の安定性

RN-BH解や Sch. BH解: 球対称な地平線

- 現実的な系を考えると

銀河内のBH
非等方宇宙内のBH
BH & 星の連星系

} 球対称の破れ

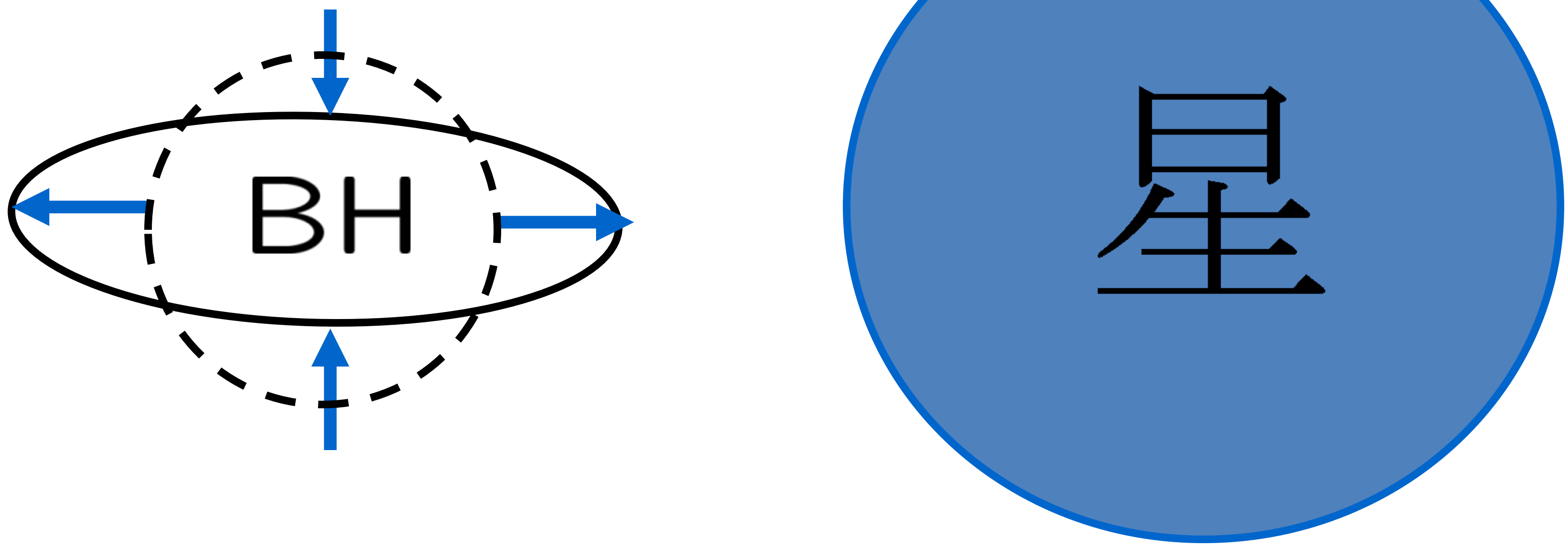
↓

地平線が球からゆがむ

このとき**BH内特異点はどう変化するか**

地平線の外の時空

- 例)BHと星の連星系



- 地平線近傍は一様な“潮汐力”
- 外部時空は星とBHの間の関係に大きく依存

→地平線より内側のみ扱う

BHと一様宇宙

- RN-BH解や Sch. BH解
  地平線の
  外側:静的かつ非一様
  内側:動的かつ一様

↔ 時間的座標と空間的座標の入れ替わり

↓

空間的に一様で非等方な宇宙

+RN-BH解と対応する電場

を考える

RN-BHの内部

$$r > r_+ \begin{cases} ds^2 = -h^2(r)dt^2 + h^{-2}(r)dr^2 + r^2(d\Omega_2)^2, \\ h^2(r) = 1 - 2mr^{-1} + Q^2 r^{-2} \\ r_\pm = m \pm \sqrt{m^2 - Q^2} : \text{見かけの地平線} \end{cases}$$

↓ 領域 $r_- < r < r_+$ で座標変換

$$r < r_+ \quad ds^2 = -d\hat{\tau}^2 + \hat{u}^2(\hat{\tau})d\hat{r}^2 + \hat{h}^2(\hat{\tau})(d\Omega_2)^2,$$

空間的に一様非等方な4次元宇宙

$$\left.\begin{array}{l} r \to r_\pm \\ h \to 0 \\ h^{-1} \to \infty \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} \hat{\tau} \to 0 \\ \hat{u} \propto \hat{\tau} \to 0 \\ \hat{h} \to \mathrm{const} \end{array}\right.$$

対称性(Killing vec.)の数

- 静的球対称4次元時空($R^1 \times R^1$)$\times S^2$

  時間(中で空間)1個+球 S^2 (SO(3)) 3個=4個

  ↓

  球対称性やぶる:SO(2)=1個 ⟶ 計2個

  空間次元数よりも少なく、非一様!

- 空間的に一様な5次元宇宙($R^1 \times R^1 \times S^3$)

  空間1個+球 S^3 (SO(4)) 6個=7個

  ↓

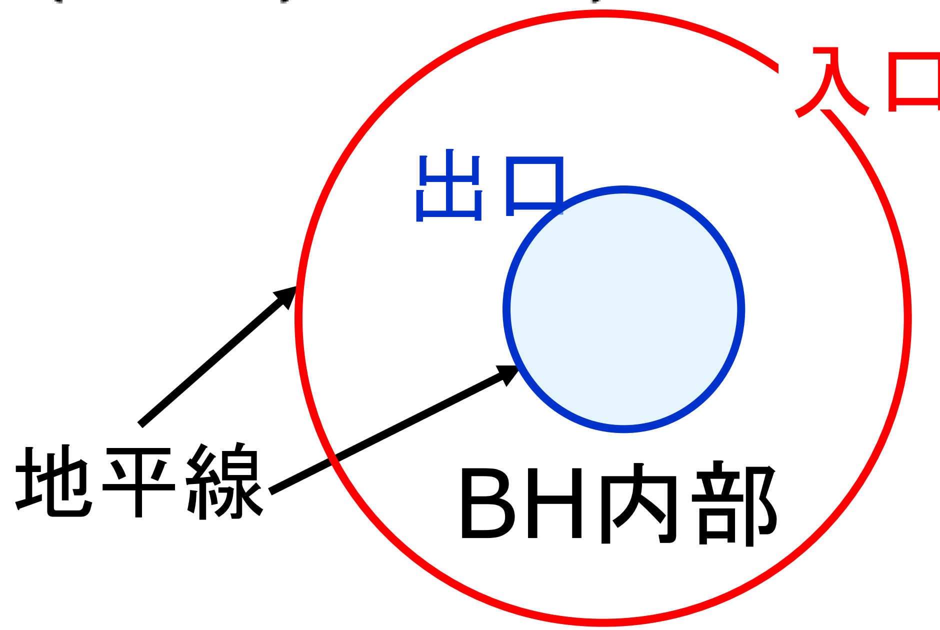
  球対称性やぶる:SO(3)=3個 ⟶ 計4個

  空間次元数と等しく、一様のまま!

計量の具体形と初期条件

$$\begin{cases} ds^2 = -dt^2 + f^2(t)dX^2 + \underbrace{a^2(t)(\sigma^1)^2 + b^2(t)(\sigma^2)^2 + c^2(t)(\sigma^3)^2}_{S^3} \\ \quad R^1 \times R^1 \quad \times \\ d\sigma^i = -\frac{1}{2}C^i_{jk}\sigma^j \wedge \sigma^k,\ C^1_{23} = C^2_{31} = C^3_{12} = 1, \\ \quad \text{(ビアンキIX型 } (= S^3) \text{ と同様)} \end{cases}$$

→トンネルの入口が
存在する為の
“地平線条件”
を初期条件



Einstein-Maxwell系

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \text{ : Einstein eq.} \\ \\ F^{\mu\nu}{}_{;\nu} = 0, \ \ F_{[\mu\nu;\rho]} = 0 \text{ : Maxwell eq.} \end{cases}$$

$$(\mu, \nu = 0, 1, 2, 3, 4)$$

Maxwell方程式

$$
\begin{cases}
\left[\partial_0 F_{\mu\nu} + \left(\frac{\partial_0 a_\mu}{a_\mu} + \frac{\partial_0 a_\nu}{a_\nu}\right) F_{\mu\nu}\right] \epsilon^{0\mu\nu\alpha\beta} \\
\quad -\frac{1}{2}\left(C^\sigma_{\rho\mu}\frac{a_\sigma}{a_\mu}F_{\sigma\nu} + C^\sigma_{\rho\nu}\frac{a_\sigma}{a_\nu}F_{\mu\sigma}\right)\epsilon^{\rho\mu\nu\alpha\beta} = 0 \\
\partial_0 F^{0\nu} + \sum_{\mu\neq\nu\neq\rho}\left(\frac{\partial_0 a_\rho}{a_\rho}F^{0\nu} + \frac{1}{6}C^\nu_{\mu\rho}\frac{a_\nu}{a_\rho}F^{\mu\rho} + \frac{1}{3}C^\mu_{\mu\rho}\frac{a_\mu}{a_\rho}F^{\rho\nu}\right) = 0
\end{cases}
$$

$$(a_\mu) = (1, a, b, c, f)$$

$(F_{0i}, F_{jk}),\ (F_{0i}, F_{j4}),$
$(F_{04}, F_{ij}),\ (F_{ij}, F_{k4}) \neq 0$ → RN-BH 同様 $F_{04} \neq 0, F_{12} = 0$
のみ許される。

$$\therefore \boxed{F_{04} = -F^{04} = -E/(abc),\ \ E \equiv \text{const} > 0}$$

Einstein方程式

$$\begin{cases} 2(a'/a)' = \left[(b^2-c^2)^2 - a^4 + U^2\right] f^2 \\ 2(b'/b)' = \left[(c^2-a^2)^2 - b^4 + U^2\right] f^2 \\ 2(c'/c)' = \left[(a^2-b^2)^2 - c^4 + U^2\right] f^2 \\ (f'/f)' = -U^2 f^2 \end{cases}$$

$$\begin{bmatrix} ' = abcf\frac{d}{dt} \\ U^2 \equiv 4E^2/3 \end{bmatrix}$$

$$\frac{a'b'}{ab} + \frac{b'c'}{bc} + \frac{c'a'}{ca} + \frac{f'}{f}\left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right)$$

$$-\frac{f^2}{4}\left[a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 + c^2a^2) + 3U^2\right] = 0$$

[地平線条件]

$t \to t_i$ で $a, b, c \to$ 一定 かつ $f \propto t \to 0$

軸対称な一般解(b=c)

$$ds^2_{\mathsf{Ax}} = -\frac{a^2 b^4 f^2}{(C_4)^2} d\tilde{\tau}^2 + a^2(\sigma^1)^2 + b^2\left[(\sigma^2)^2 + (\sigma^3)^2\right] + f^2 dX^2$$

$$a^2(\tilde{\tau}) = C_1 U \cosh(\tilde{\tau}) \mathrm{sech}(C_1\tilde{\tau} + C_2)$$

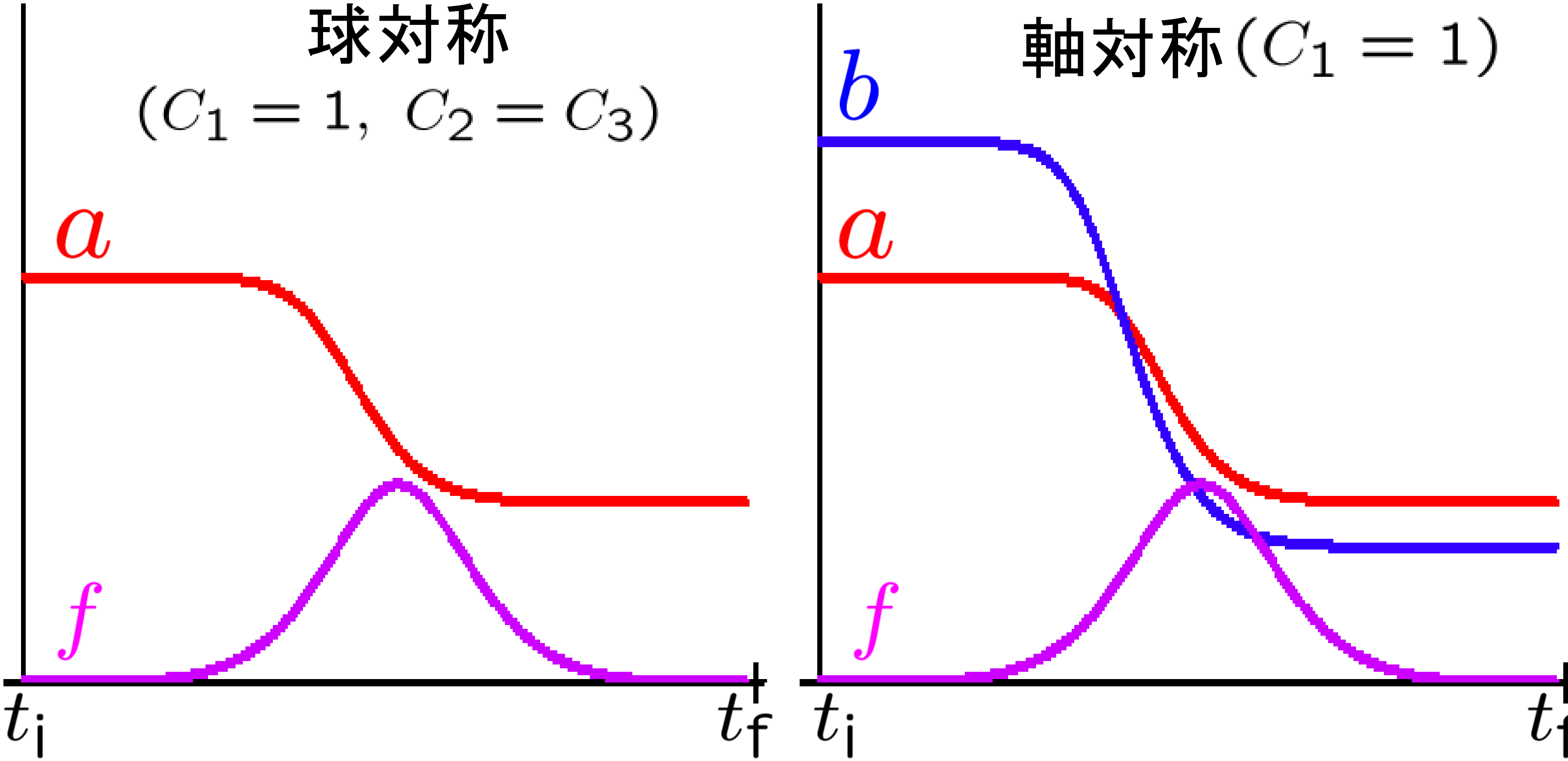
$$b^2(\tilde{\tau}) = \frac{U}{4}\left(C_1 + \frac{3}{C_1}\right)\cosh(\tilde{\tau})\cosh(C_1\tilde{\tau} + C_2)\mathrm{sech}^2\left(\frac{\sqrt{(C_1)^2+3}}{2}\tilde{\tau} + C_3\right)$$

$$f^2(\tilde{\tau}) = \left(\frac{C_4}{U}\mathrm{sech}(\tilde{\tau})\right)^2 \quad (-\infty \leq \tilde{\tau} \leq \infty)$$

$$(C_i = \mathrm{const}\ (i = 1,2,3,4)\ dt = ab^2 f d\tilde{\tau}/C_4)$$

特に、
$C_1 = 1,\ C_2 = C_3$ で $a = b = c$ となって
球対称解(RN-BH)に

軸対称解の振る舞い



$t = t_\mathrm{f}$ でも再び地平線条件満足

→ 地平線を越えた外の時空に解析接続

解析接続(軸対称の場合)

- $a^2 = r^2/4$ として地平線を越えた接続

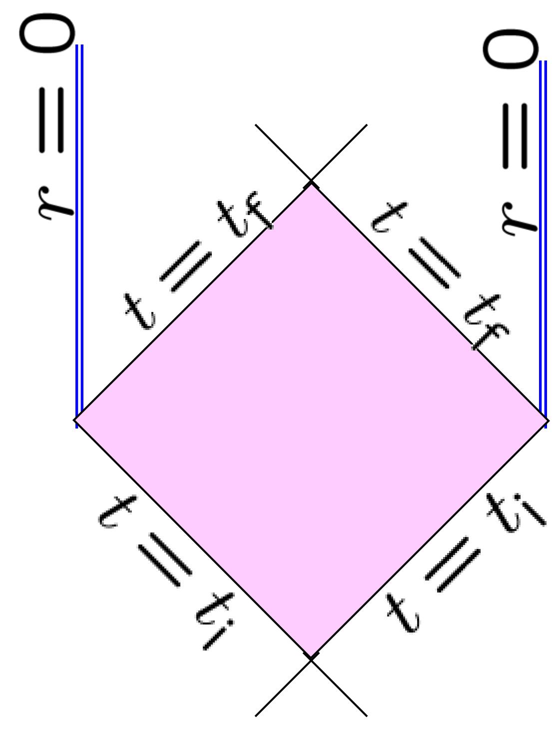
$$ds^2 = -\tilde{f}^2(r)d\tilde{t}^2 + \left[\frac{4U \sinh C_2}{\tilde{g}(r)r^2}\right]^4 \frac{dr^2}{\tilde{f}^2(r)}$$

$$+\frac{r^2}{4}(\sigma^1)^2 + \left[\frac{2U \sinh C_2}{\tilde{g}(r)r}\right]^2 \left[(\sigma^2)^2 + (\sigma^3)^2\right]$$

$$\tilde{f}^2(r) \equiv 1 - 8U \cosh(C_2)r^{-2} + 16U^2 r^{-4}$$

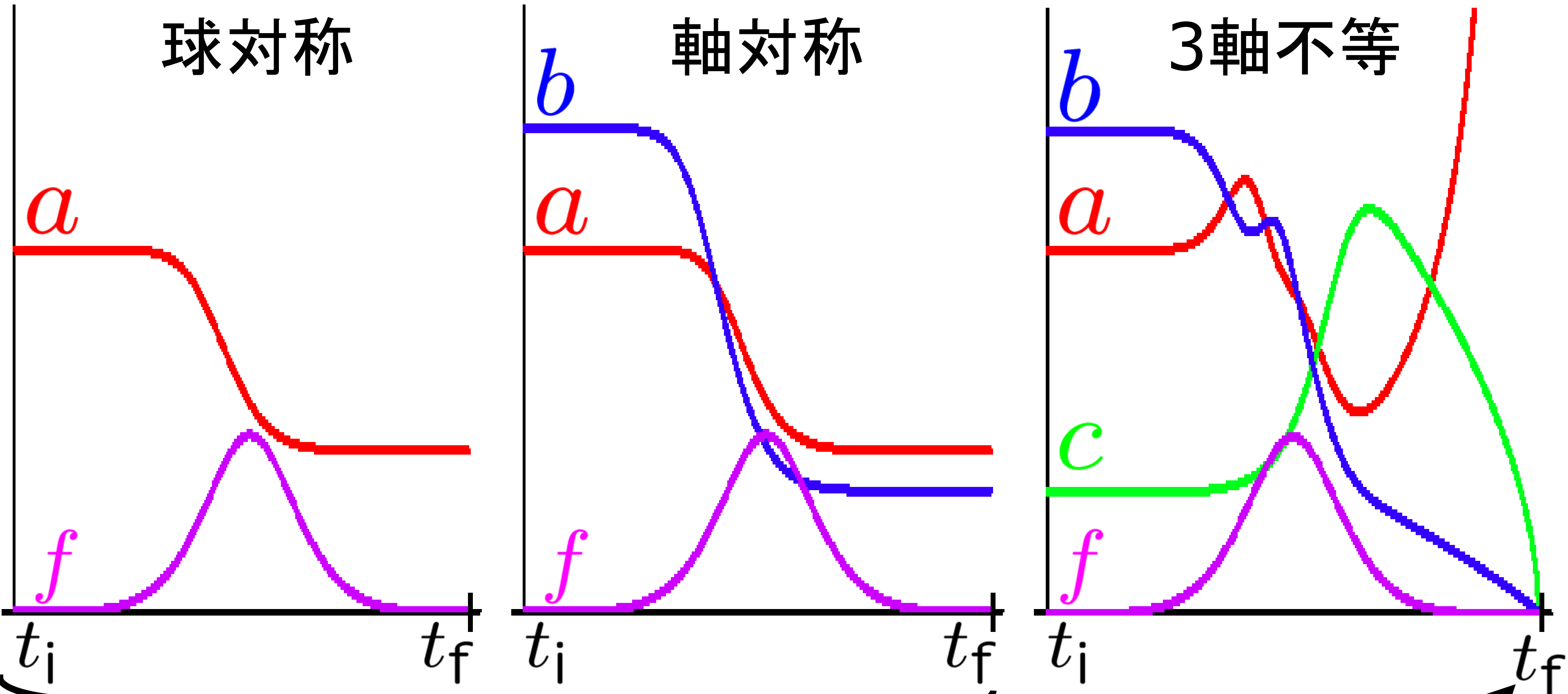
$$\tilde{g}(r) \equiv \sinh(C_2 - C_3) + 4U \sinh(C_3)r^{-2}$$

$$\tilde{t} \equiv C_4 X/U \sinh C_2$$



→ r=0 に 時間的特異点

3軸不等の場合



$t = t_f$ でも地平線条件満足

→ 2つの時空をつなぐ

トンネル的構造

トンネルが行き止まりに

ビアンキIX型宇宙との比較

- Solve eq. of $f(\tau)$ and change variables as

$$A \equiv a\sqrt{f},\ B \equiv b\sqrt{f},\ C \equiv c\sqrt{f},\ dt = ABC/\sqrt{f}d\tau,$$

then Einstein eq. are

$$f(\tau) = \pm C_1 \mathrm{sech}\,(C_1\tau + C_2)/U,\ {}' \equiv d/d\tau$$

$$2(A'/A)' = (B^2 - C^2)^2 - A^4$$
$$2(B'/B)' = (C^2 - A^2)^2 - B^4$$
$$2(C'/C)' = (A^2 - B^2)^2 - C^4$$

Same as vac.Bianchi IX

$$\frac{A'B'}{AB} + \frac{B'C'}{BC} + \frac{C'A'}{CA} - \frac{1}{4}\left[A^4 + B^4 + C^4 - 2(A^2B^2 + B^2C^2 + C^2A^2)\right] - \frac{3(C_1)^2}{4} = 0$$

Velocity　　Potential

But this term leads end of oscillation!

3軸不等の場合(議論)

- $t = t_\mathrm{i}$で地平線条件満足

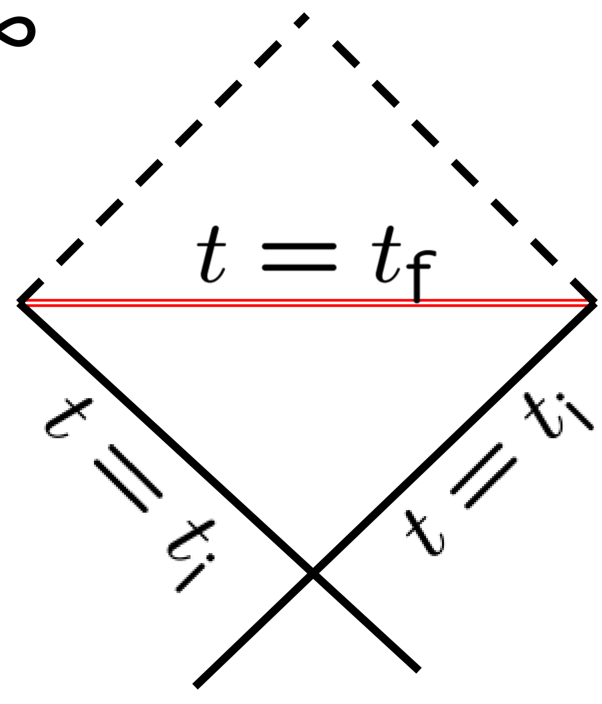
その後、$t = t_\mathrm{f}$ で

$(\mathrm{Velocity})/(\mathrm{Potential}) \longrightarrow \infty$

(Pot.無視可の
漸近的速度優勢解)

∴空間的特異点を含む

5次元Kasner解！



まとめ

- “地平線条件”を初期条件とする空間的に一様な5次元宇宙をRN-BH解の内部とみなし、球対称地平線の変形による構造変化を考えた
- 地平線が
  - ●軸対称:時空をつなぐトンネル的構造に変化なし
  - ●3軸不等:トンネルに、Sch. BH解と同様の行き止まり(空間的特異点)が出来る

現実的な宇宙において
トンネル的構造を持つBHは一般的でない