Geometric conservation in curved spacetime and entropy

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This talk is based on

SA, Y. Hidaka, K. Kawana and K. Shimada, "Geometric conservation in curved spacetime and entropy", arXiv:2312.09712[hep-th]

Related references

SA, T. Onogi and S. Yokoyama, "Charge conservation, Entropy Current, and Gravitation", Int. J. Mod. Phys. A36 (2021)2150201.

SA, "Noether's 1st theorem with local symmetry", PTEP 2023(2023)1,013B03.

SA and K. Kawana, "Entropy and its conservation in expanding Universe", International Journal of Modern Physics A38 (2023) 2350072 [arXiv:2210.03323 [hep-th]].

SA, T. Onogi and T. Yamaoka, "Energies and a gravitational charge for massive particles in general relativity", [arXiv:2305.09849 [gr-qc]].

I. Introduction

Questions

Is there a covariantly conserved quantity in general relativity ?

If exists, what is its physical meaning ?

Is energy conserved in general relativity ?

spacetime matter

Einstein equation $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 2\kappa T_{\mu\nu}$$ $\kappa = 4\pi G_N$

Energy Momentum Tensor (EMT)

Bianchi identity for $R_{\mu\nu}$

$$\longrightarrow \nabla_\mu(\sqrt{-g} T^\mu{}_\nu) = \partial_\mu(\sqrt{-g} T^\mu{}_\nu) + \Gamma^\mu_{\mu\alpha}(\sqrt{-g} T^\alpha{}_\nu) - \Gamma^\alpha_{\mu\nu}(\sqrt{-g} T^\mu{}_\alpha) = 0$$

covariant conservation

but what we naively need for a conserved is $\partial_\mu(\sqrt{-g} T^\mu{}_\nu) = 0$

2nd and 3rd terms are obstructions.

The standard definitions

1. Einstein's (pseudo-)energy $\partial_\mu[\sqrt{-g}(T^\mu{}_\nu + t^\mu{}_\nu)] = 0$ $t^\mu{}_\nu$

$$E = \int_\Sigma d\Sigma_\mu (T^\mu{}_0 + t^\mu{}_0)$$

pseudo-tensor (non-covariant)

2. Quasi-local energy ADM, Komar, Bondi

$$E = \int_{r\to\infty} dS \,(\text{quasi-local energy})$$

absence of local energy density

There exists no covariant definition of conserved energy in general relativity, due to Noether's 2nd theorem.

I will not discuss this anymore in this talk, due to the limitation of time.

For more details, please take a look at

SA and T. Onogi, "Conserved non-Noether charge in general relativity: Physical definition vs. Noether's 2nd theorem", Int. J. Mod. Phys. A36 (2022) 2250129, In this talk, we propose a covariantly conserved quantity and discuss its physical meaning.

PPP2023では、重力系の保存量「重力荷」の存在を示し、それをエントロピーと考えると、膨張宇宙での熱力学的関係式を満たすことを示した。
今回は、「重力荷」を幾何的に表現し、完全流体の場合にはそれがエントロピーと一致することを示す。

Results

1. There exists a covariant and geometric conservation for a general class of energy momentum tensor in curved spacetime.

2. The geometric conserved charge becomes "entropy" for a perfect fluids.

content

I. ~~Introduction~~
II. Our set up
III. Conserved current and conserved charge
IV. Geometric conservation and entropy
V. Conclusion

II. Our set up

(which may not be found in textbooks)

1. Decomposition of energy momentum tensor

Energy Momentum Tensor (EMT)

In this talk, we consider the EMT in the following form:

$$T_{\mu\nu} = \varepsilon u_\mu u_\nu + P_{\mu\nu}, \quad P_{\mu\nu} u^\nu = u^\mu P_{\mu\nu} = 0$$

ε: energy density $\quad$ u^μ: a time-like unit vector $\quad$ $P_{\mu\nu}$: pressure tensor

This EMT, classified as the Hawking-Ellis type I, covers standard classical matters in 3+1 dimensions.

Hawking&Ellis 1973, Martin-Moruno&Visser 2018

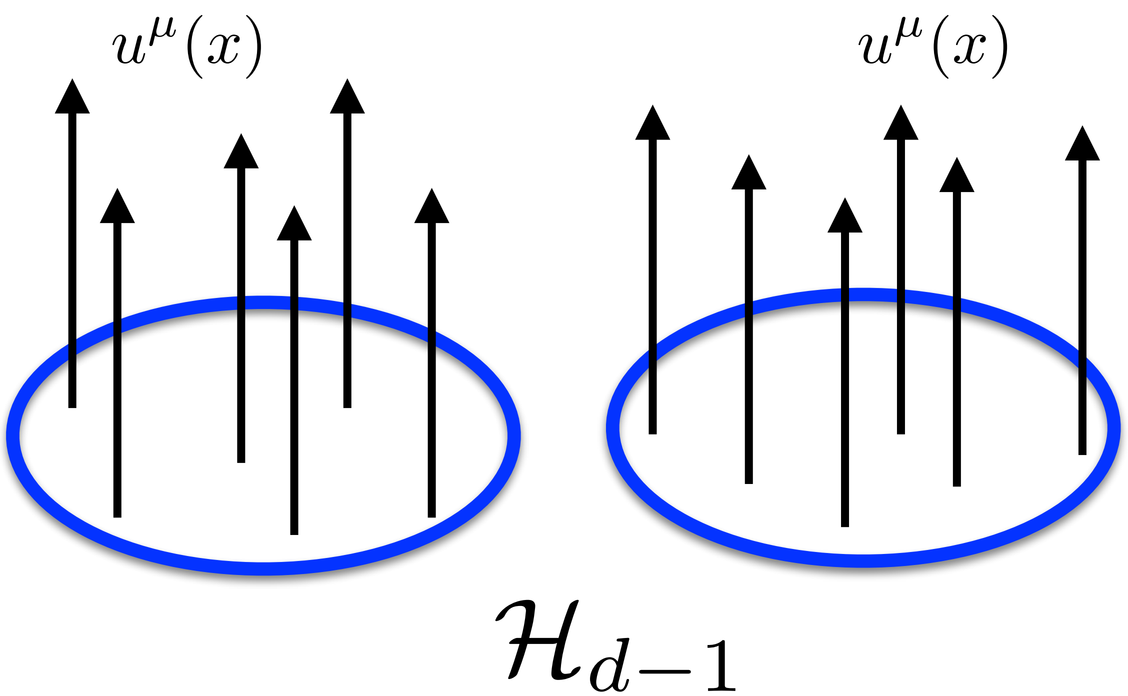
Conservation law $\nabla_\mu T^{\mu\nu} = 0$ is satisfied as mentioned before.

We also assume that other conserved currents such as electric charge or baryon number exist:

$$\nabla_\mu N_i^\mu = 0 \qquad i = 1, 2 \cdots, f$$

2. Initial hyper-surface

First we pick up an initial space-like hyper-surface $\mathscr{H}_{d-1}$.　　”初期”時刻での部分宇宙



EMT is non-zero on $\mathscr{H}_{d-1}$:

$$\varepsilon(x) \neq 0 \text{ at } {}^\forall x \in \mathcal{H}_{d-1}$$

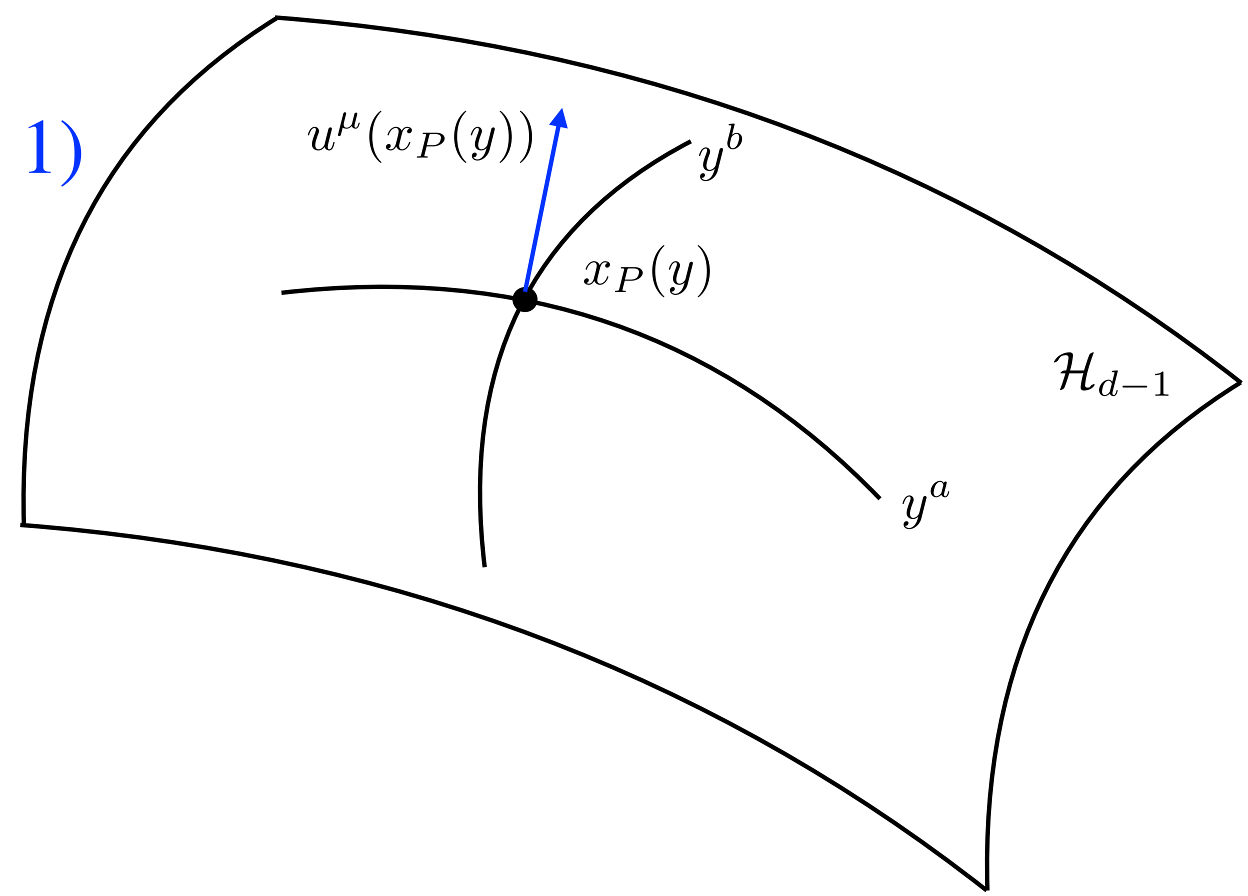
$\mathscr{H}_{d-1}$ may not be connected.

We introduce a coordinate y^a $(a = 1,2,\cdots,d-1)$ on $\mathscr{H}_{d-1}$.

$$\mathcal{H}_{d-1} = \{x_P^\mu(y) \mid y \in H_{d-1}\}$$

初期時刻での空間座標

H_{d-1} is a $d-1$ dimensional subspace of $\mathbb{R}^{d-1}$, which may not be necessarily connected.



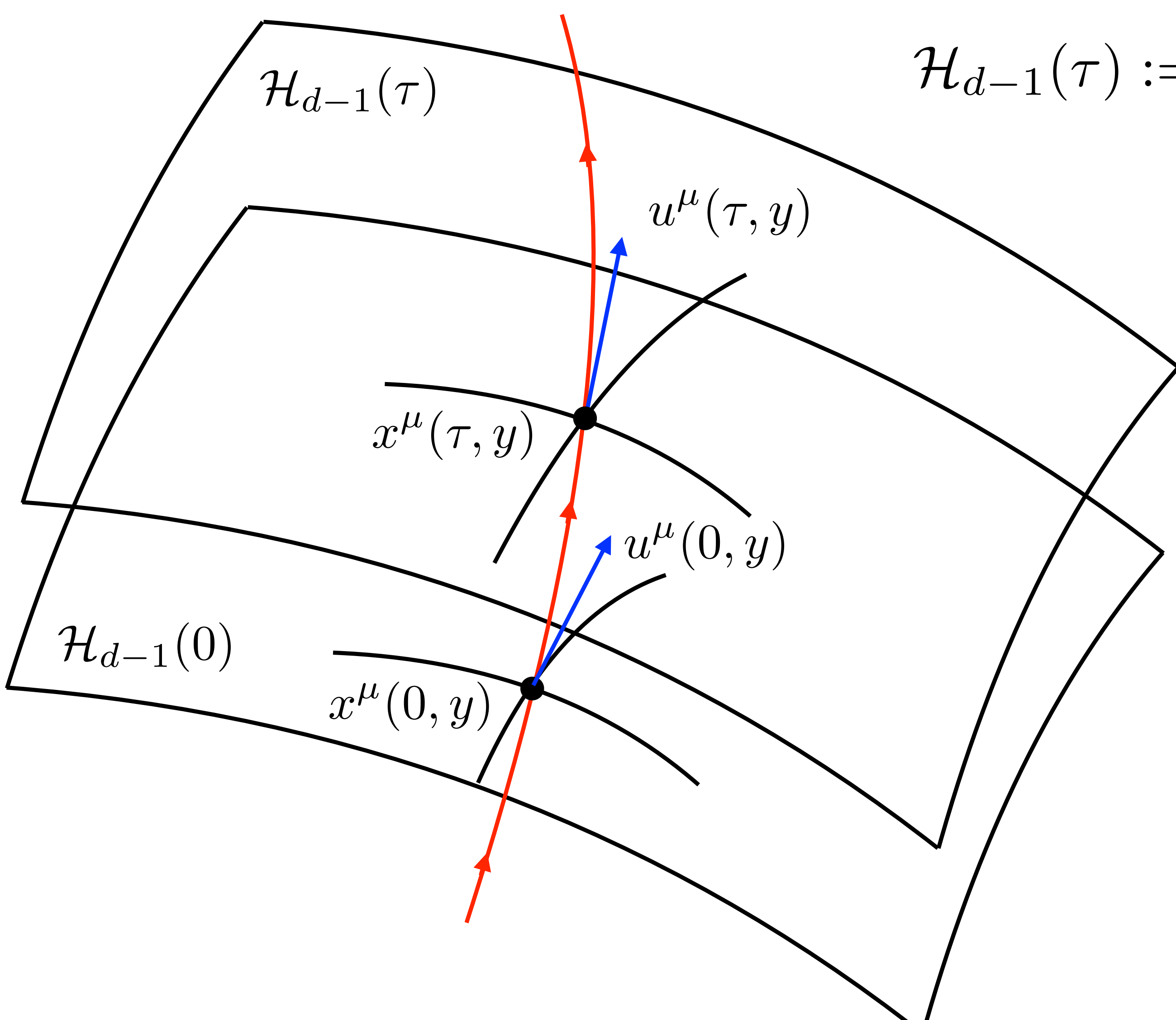
3. Time-like curves and a foliation of hyper-surfaces

We define a time-like curve $x^\mu(\tau, y)$ starting from an arbitrary point $x_P(y)$ on $\mathscr{H}_{d-1}$:

$$\frac{dx^\mu(\tau,y)}{d\tau} = u^\mu(x^\mu(\tau,y)), \quad x^\mu(0,y) = x^\mu_P(y) \quad \Rightarrow \quad x^\mu(\tau,y) = x^\mu_P(y) + \int_0^\tau d\eta\, u^\mu(x(\eta,y))$$

τ can be negative.

We can construct a foliation of hyper-surfaces $\mathscr{H}_{d-1}(\tau)$ using these time-like curves.



$$\mathcal{H}_{d-1}(\tau) := \{x^\mu(\tau,y) \mid {}^\exists\tau, {}^\forall y \in H_{d-1}\} \qquad \mathcal{H}_{d-1}(0) = \mathcal{H}_{d-1}$$

Assumption: $\nabla_\mu T^{\mu\nu} = 0$ implies $u^\mu(\tau, x)$ never end or emerge.

つまり$u^\mu(\tau, y)$上で$\epsilon(\tau, y)$はゼロにならない。

これはエネルギー条件($\varepsilon \geq 0$)より導かれる（であろう）。

Hereafter, we use simplified notations such as $u^\mu(\tau, y) := u^\mu(x(\tau, y))$.

4. "3+1" decomposition (standard)

We consider a new coordinate (3+1 decomposition) as $y^A = (y^0, y^a)$ with $\tau = f(y^0)$:

$$\tilde{g}_{AB} dy^A dy^B = -N^2 (dy^0)^2 + h_{ab}(dy^a + N^a dy^0)(dy^b + N^b dy^0)$$

$$\tilde{g}_{AB} = \begin{pmatrix} -N^2 + N_a N^a, & N_b \\ N_a, & h_{ab} \end{pmatrix} \qquad \tilde{g}^{AB} = \frac{1}{N^2} \begin{pmatrix} -1, & N^b \\ N^a, & N^2 B^{ab} \end{pmatrix}$$

$$N := \sqrt{(f')^2 + N_a N^a} \qquad N_a := g_{\mu\nu} u^\mu e^\nu_a \qquad B^{ab} := g^{\mu\nu} \frac{\partial y^a}{\partial x^\mu} \frac{\partial y^b}{\partial x^\nu} = h^{ab} - \frac{N^a N^b}{N^2}$$

$$h_{ab} := g_{\mu\nu} e^\mu_a e^\nu_b \qquad e^\nu_a := \frac{\partial x^\mu}{\partial y^a}$$

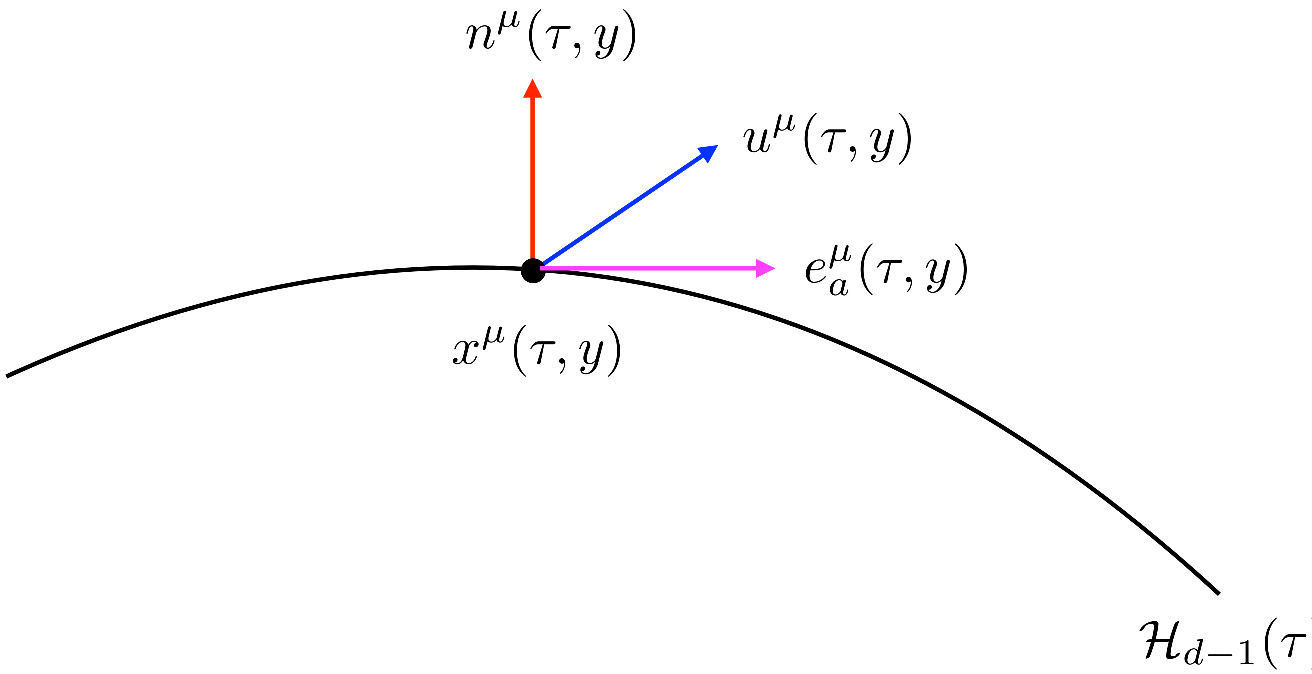
N: laps, N^a: shift

In the new coordinate, the unit vector normal to $\mathscr{H}_{d-1}(\tau)$ is given by

$$\tilde{n}_A = \frac{\partial x^\mu}{\partial y^A} n_\mu = -N\,\delta^0_A$$

On the other hand, u^μ is expressed as

$$\tilde{u}^A = \frac{\partial y^A}{\partial x^\mu} u^\mu = \frac{\partial y^A}{\partial x^\mu} \frac{\partial x^\mu}{\partial \tau} = \frac{\partial y^A}{\partial \tau} = \frac{\delta^A_0}{f'}$$



Therefore $\quad u \cdot n = \tilde{u} \cdot \tilde{n} = -\frac{N}{f'}$

5. Evaluation of $K := \nabla_\mu u^\mu$ (non-standard)

For a latter use, we calculate $K = \nabla_\mu u^\mu$ as

$$K = \partial_\mu u^\mu + \Gamma^\nu_{\mu\nu} u^\mu = \partial_\mu u^\mu + u^\mu \partial_\mu \ln \sqrt{-g} \qquad g := \det g_{\mu\nu}$$

In the y^A coordinate, we have $\tilde{u}^A = \dfrac{\delta^A_0}{f'} \longrightarrow \partial_A \tilde{u}^A = -\tilde{u}^A \partial_A \ln f' = -\partial_\tau \ln f'$

$$\tilde{g} := \det \tilde{g}_{AB} = -N^2 h \longrightarrow \tilde{u}^A \partial_A \ln \sqrt{-g} = \partial_\tau \ln(N\sqrt{h})$$

$$\tilde{g}_{AB} = \begin{pmatrix} -N^2 + N_a N^a, & N_b \\ N_a, & h_{ab} \end{pmatrix} \qquad h := \det h_{ab}$$

Therefore, we finally obtain

$$K = -\partial_\tau \ln f' + \partial_\tau \ln N\sqrt{h} = \partial_\tau \ln \left[\frac{N}{f'} \sqrt{h} \right] = \partial_\tau \ln[(-n \cdot u)\sqrt{h}]$$

$$u \cdot n = \tilde{u} \cdot \tilde{n} = -\frac{N}{f'}$$

$$g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu}\left(\frac{\partial x^\mu}{\partial \tau}f'(y^0)dy^0 + \frac{\partial x^\mu}{\partial y^a}dy^a\right)\left(\frac{\partial x^\nu}{\partial \tau}f'(y^0)dy^0 + \frac{\partial x^\nu}{\partial y^a}dy^b\right) \qquad \frac{\partial x^\mu}{\partial \tau} = u^\mu$$

$\tilde{n}_A \propto \delta^0_A$ ⟶ 規格化 $\tilde{n}_A = -N\delta^0_A$ $\quad \tilde{n}^A = \frac{1}{N}(\delta^A_0 - N^A)$ future-directed

定義より $\quad \Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left[\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right]$

$$\Gamma^\nu_{\mu\nu} = \frac{1}{2}g^{\nu\alpha}\partial_\mu g_{\nu\alpha} = \frac{1}{2g}\partial_\mu g = \partial_\mu \ln\sqrt{-g}$$

III. Conserved current and conserved charge

(Our proposal)

1. Construction of conserved current

Conserved current

refinement of the proposal in Aoki, Onogi & Yokoyama 2021

We construct the conserved current from the EMT using $u^\mu(x)$ as

$$J^\mu(x) := T^\mu{}_\nu(x)\underbrace{\zeta(x)u^\nu(x)}_{\zeta^\nu(x)} = -\varepsilon(x)\zeta(x)u^\mu(x)$$

This definition is coordinate independent.

We determine $\zeta(x)$ in order to satisfy the conservation law $\nabla_\mu J^\mu = 0$, which reads

$$\nabla_\mu J^\mu(x) = -u^\mu\partial_\mu(\zeta\varepsilon) - \zeta\varepsilon K = -\frac{d}{d\tau}(\zeta\varepsilon) - \zeta\varepsilon K = 0 \qquad \frac{dx^\mu}{d\tau}\frac{\partial}{\partial x_\mu} = \frac{d}{d\tau}$$

$$\zeta(\tau, y)\varepsilon(\tau, y) = \zeta(0, y)\varepsilon(0, y)\exp\left[-\int_0^\tau d\eta\, K(\eta, y)\right] = \zeta(0, y)\varepsilon(0, y)\frac{(n\cdot u)\sqrt{h}(0, y)}{(n\cdot u)\sqrt{h}(\tau, y)}$$

where we use $\quad K := \nabla_\mu u^\mu = \frac{\partial}{\partial\tau}\log\left\{-(u\cdot n)\sqrt{h}\right\}$

Therefore the conserved current is determined as

$$J^\mu(\tau, y) = -\zeta(0, y)\varepsilon(0, y)n(0, y)\cdot u(0, y)\sqrt{h(0, y)}\frac{u^\mu(\tau, y)}{n(\tau, y)\cdot u(\tau, y)\sqrt{h(\tau, y)}}$$

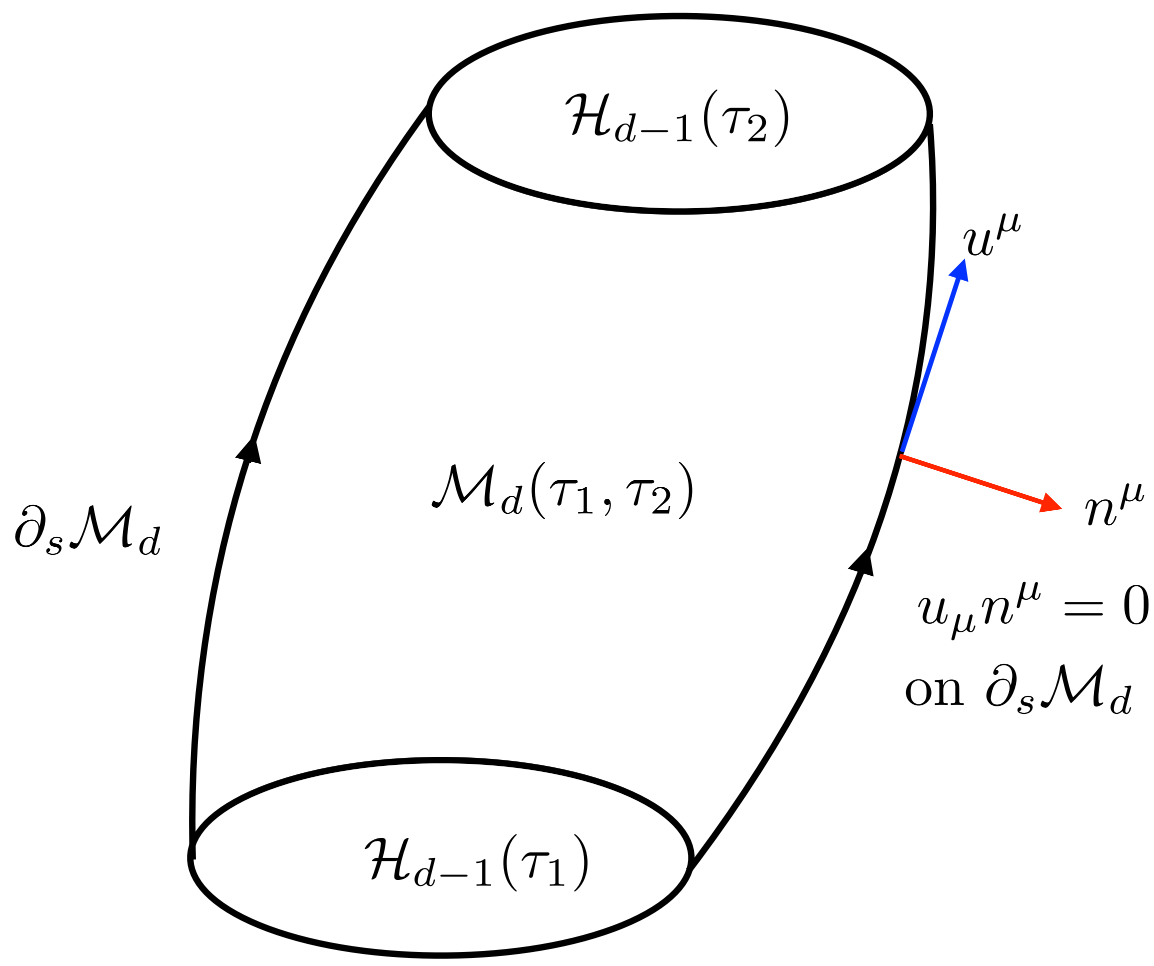
2. Conservation law and conserved charge (standard)

We consider a foliation of $\mathscr{H}_{d-1}(\tau)$ as $\quad \mathcal{M}_d(\tau_1,\tau_2) := \{\mathcal{H}_{d-1}(\tau) \mid \tau_1 \le \tau \le \tau_2\}$

Integral of conservation law on $\mathscr{M}_d(\tau_1, \tau_2)$

$$0 = \int_{\mathcal{M}_d(\tau_1,\tau_2)} d^d x\, \sqrt{-g}\nabla_\mu J^\mu \;= Q(\mathcal{H}_{d-1}(\tau_2)) - Q(\mathcal{H}_{d-1}(\tau_1)) + \int_{\partial_s \mathcal{M}_d} d\Sigma_\mu J^\mu$$

where we define $\quad Q(\mathcal{H}_{d-1}(\tau)) := \int_{\mathcal{H}_{d-1}(\tau)} d\Sigma_\mu\, J^\mu$



By construction, the current is zero on $\partial_s\mathscr{M}_d$ as

$$d\Sigma_\mu\, J^\mu \propto d\Sigma_\mu u^\mu \propto n_\mu u^\mu = 0$$

$$Q(\mathcal{H}_{d-1}({}^\forall\tau_2)) = Q(\mathcal{H}_{d-1}({}^\forall\tau_1)) := Q$$

conserved charge

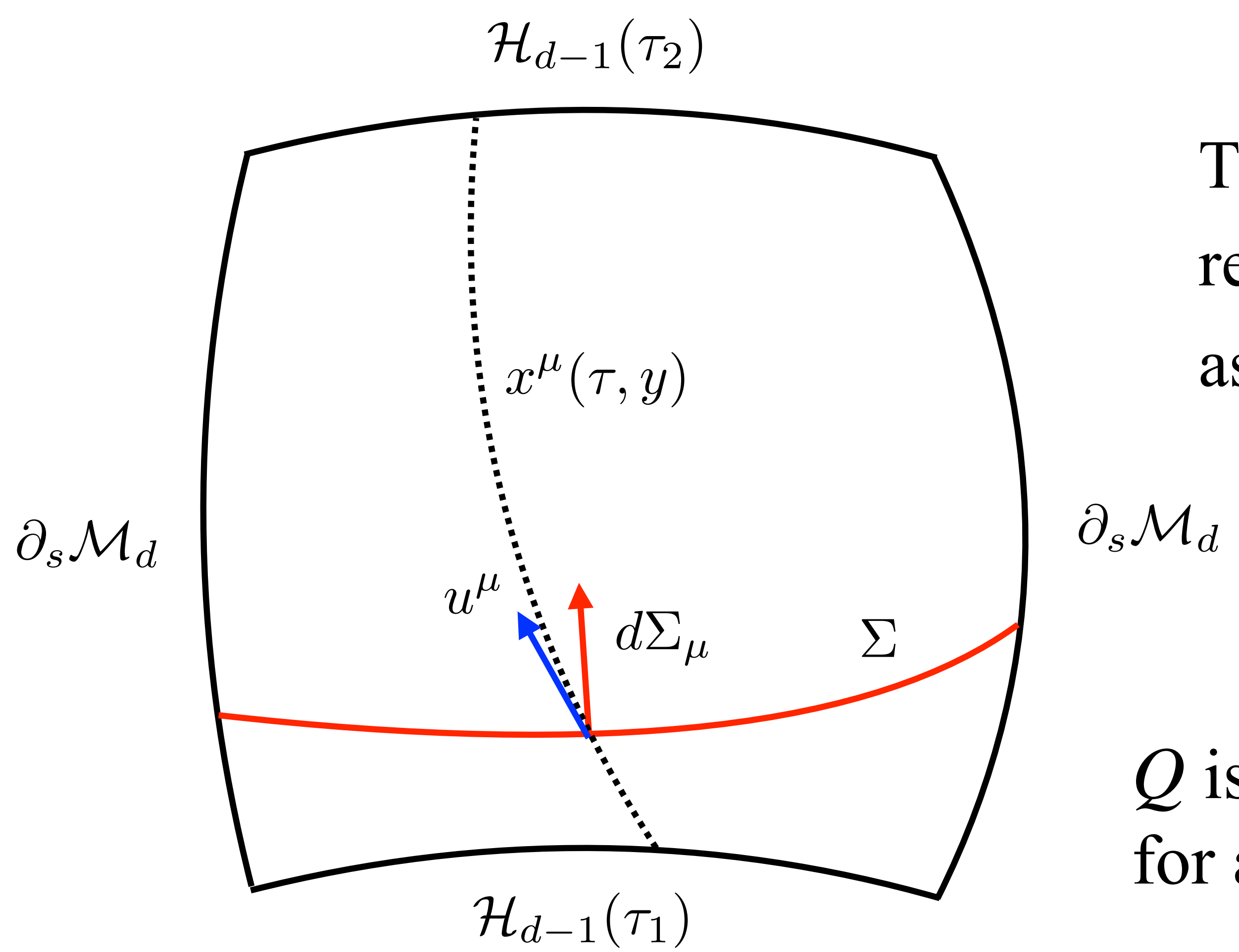
3. Explicit form of the conserved charge

$$J^\mu(\tau,y) = -\zeta(0,y)\varepsilon(0,y)n(0,y)\cdot u(0,y)\sqrt{h(0,y)}\frac{u^\mu(\tau,y)}{n(\tau,y)\cdot u(\tau,y)\sqrt{h(\tau,y)}}$$

$$Q(\mathcal{H}_{d-1}(\tau)) := \int_{\mathcal{H}_{d-1}(\tau)} d\Sigma_\mu\, J^\mu \qquad d\Sigma_\mu = -d^{d-1}y\sqrt{h}n_\mu$$

$$Q(\mathcal{H}_{d-1}(\tau)) = -\int_{H_{d-1}} d^{d-1}y\sqrt{h(\tau,y)}\,n_\mu(\tau,y)J^\mu(\tau,y) = \int_{H_{d-1}} d^{d-1}y\,\zeta(0,y)\varepsilon(0,y)\,n_\mu(0,y)u^\mu(0,y)\sqrt{h(0,y)}$$

The charge is indeed τ independent, and thus conserved.



The charge Q takes the same value even if we replace $\mathscr{H}_{d-1}(\tau)$ with an arbitrary hyper-surface Σ as in the left figure.

$$Q(\mathcal{H}_{d-1}(\tau)) = Q(\Sigma) = Q$$

Q is the conserved charge of Noether's 1st Theorem for a global translation generated by $\zeta^\mu = \zeta u^\mu$.

SA PTEP 2023(2023)1,013B03.

IV. Geometric conservation and entropy

(trivial conservation but non-trivial interpretation)

1. (No memory) initial condition for ζ

$$J^\mu(\tau,y) = -\zeta(0,y)\varepsilon(0,y)n(0,y)\cdot u(0,y)\sqrt{h(0,y)}\frac{u^\mu(\tau,y)}{n(\tau,y)\cdot u(\tau,y)\sqrt{h(\tau,y)}}$$

The conserved current depends on an initial value $\zeta(0,y)$.

We take "no memory" initial condition : $\zeta(0,y)\varepsilon(0,y)n(0,y)\cdot u(0,y)\sqrt{h(0,y)} = 1$

(We take the y-coordinate Cartesian or similar.)

geometric conserved current (no memory at $\tau = 0$)

$$J^\mu(\tau,y) = -\frac{u^\mu(\tau,y)}{n(\tau,y)\cdot u(\tau,y)\sqrt{h(\tau,y)}}$$

geometric conserved charge ("gravitational charge")

$$Q = \int_{H_{d-1}} d^{d-1}y$$

These are coordinate independent, but depend on your choice of coordinate τ and y^a .

Q is invariant under the volume preserving diffeo. of y^a .

(An existence of) the conserved current looks "trivial".

As we will show, however, $J^\mu =$ "entropy current" in the case of perfect fluids.

$$-\sqrt{h(\tau,y)}\,n_\mu(\tau,y)J^\mu(\tau,y)=\zeta(0,y)\varepsilon(0,y)n(0,y)\cdot u(0,y)\sqrt{h(0,y)}\frac{n(\tau,y)\cdot u(\tau,y)\sqrt{h(\tau,y)}}{n(\tau,y)\cdot u(\tau,y)\sqrt{h(\tau,y)}}$$

余因子行列 $M^{-1}_{AB}=\frac{\tilde{M}_{AB}}{\det M},\quad \tilde{M}_{AB}=\det\left[\text{cofactor of } M_{AB}\right]$ $\quad \tilde{g}^{\tau\tau}=\tilde{g}^{-1}_{\tau\tau}=\frac{\det B^{ab}}{\det \tilde{g}^{AB}}$

$$\nabla_\mu J^\mu\propto\varepsilon^{\mu\cdots\alpha_i\cdots\alpha_{d-1}}\nabla_\mu\partial_{\alpha_i}y^{a_i}=\varepsilon^{\mu\cdots\alpha_i\cdots\alpha_{d-1}}\left\{\partial_\mu\partial_\alpha y^{a_i}-\Gamma^\beta_{\mu\alpha_i}\partial_\beta y^{a_i}\right\}=0$$

反対称　　　　対称

$$J^\mu\partial_\mu y^a\propto\varepsilon^{\mu\cdots\alpha_i\cdots\alpha_{d-1}}\partial_\mu y^a\partial_{\alpha_i}y^{a_i=a}=0$$

反対称　　対称

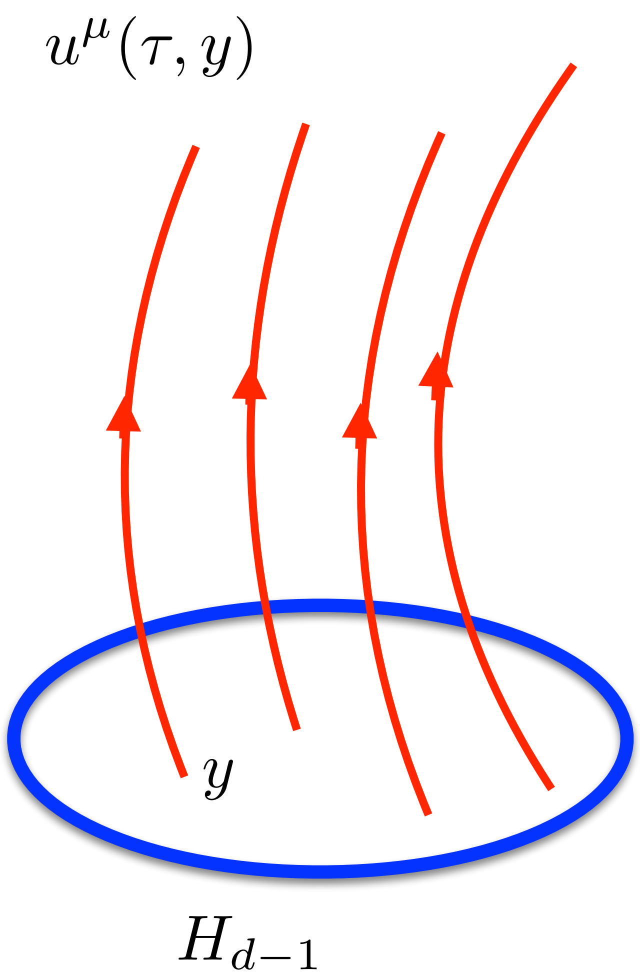
$$\frac{\partial\sqrt{\det B^{ab}}}{\partial g_{\mu\nu}}=\frac{b}{2}B_{ab}\frac{\partial\sqrt{B}^{ab}}{\partial g_{\mu\nu}}=-\frac{b}{2}B_{ab}g^{\mu\alpha}g^{\nu\beta}\partial_\alpha y^a\partial_\beta y^b$$

$$T^\mu{}_\nu=(F-F_z z)\delta^\mu_\nu+(F_z z-F_b b)(\delta^\mu_\nu+u^\mu u_\nu)=(F-F_b b)\delta^\mu_\nu+(F_z z-F_b b)u^\mu u_\nu$$

2. Geometric conserved current

An essence of the geometric conservation law is an existence of time-like vectors $u^\mu(\tau, y)$, generated by time-like curves $x^\mu(\tau, y)$, which never end or emerge as

$$u^\mu(\tau, y) \neq 0 \text{ at } {}^\forall\tau \text{ for } y \in H_{d-1} \qquad u^\mu(\tau, y) = 0 \text{ at } {}^\forall\tau \text{ for } y \notin H_{d-1}$$



$$\tilde{g}^{AB} = \frac{1}{N^2}\begin{pmatrix} -1, & N^b \\ N^a, & N^2 B^{ab} \end{pmatrix} \qquad B^{ab} = g^{\mu\nu}\frac{\partial y^a}{\partial x^\mu}\frac{\partial y^b}{\partial x^\nu}$$

A formula for cofactor $\dfrac{\det B^{ab}}{\det \tilde{g}^{AB}} = \tilde{g}_{00} = -(f')^2$ implies

$$b := \sqrt{\det B^{ab}} = \frac{f'}{\sqrt{-\tilde{g}}} = -\frac{1}{(u \cdot n)\sqrt{h}}$$

$$\tilde{g} := \det \tilde{g}_{AB} = -N^2 h$$

Geometric current

$$J^\mu(\tau, y) = -\frac{u^\mu(\tau, y)}{n(\tau, y) \cdot u(\tau, y)\sqrt{h(\tau, y)}}$$

$$J^\mu = b u^\mu$$

Here we can forget the original "set-up".

Other representation

$$J^\mu = bu^\mu = -\frac{f'}{\sqrt{-\tilde{g}}}u^\mu = -\frac{f'}{\sqrt{-\tilde{g}}}\delta^\mu_\nu u^\nu$$

$$\delta^\mu_\nu = -\frac{\epsilon^{\mu\rho_1\cdots\rho_{d-1}}\epsilon_{\nu\rho_1\cdots\rho_{d-1}}}{(d-1)!} = -\frac{\epsilon^{\mu\rho_1\cdots\rho_{d-1}}\partial_{\rho_1}y^{A_1}\cdots\partial_{\rho_{d-1}}y^{A_{d-1}}\tilde{\epsilon}_{AA_1\cdots A_{d-1}}}{(d-1)!}\frac{\partial y^A}{\partial x^\nu}$$

$$\Rightarrow \quad J^\mu = -\frac{1}{(d-1)!}\frac{f'}{\sqrt{-\tilde{g}}}e^{\mu\alpha_1\cdots\alpha_{d-1}}\tilde{e}_{Aa_1\cdots a_{d-1}}\partial_{\alpha_1}y^{a_1}\cdots\partial_{\alpha_{d-1}}y^{a_{d-1}}u^\nu\partial_\nu y^A$$

$$= -\frac{1}{(d-1)!}\frac{1}{\sqrt{-\tilde{g}}}e^{\mu\alpha_1\cdots\alpha_{d-1}}\tilde{e}_{0a_1\cdots a_{d-1}}\partial_{\alpha_1}y^{a_1}\cdots\partial_{\alpha_{d-1}}y^{a_{d-1}}$$

Geometric current

Here it is easy to check $\nabla_\mu J^\mu = 0$ and $J^\mu\partial_\mu y^a = 0$.

Furthermore, we see $J^\mu \propto \frac{1}{\sqrt{-g}} \longleftarrow \varepsilon^{\mu\alpha_1\cdots\alpha_{d-1}} \propto \frac{1}{\sqrt{-g}}$

3. Effective field theory for perfect fluids

We extend an argument in Dubovsky, Hui, Nicolis & Son 2012 to a curved spacetime.

Dynamical variable

$y^a(x)$: co-moving coordinate of fluids, $\psi(x)$: phase of a conserved quantity.

u^μ : fluid 4-velocity

Symmetry in flat spacetime

Poincare symmetry (= translation + Lorentz) ⟶ general coordinate transformation in a curved spacetime

volume preserving diffeo: $y^a \to f^a(y)$ with $\det(\partial_b f^a) = 1$

(volume = a number of particles in the fluid)

phase transformation: $\psi(x) \to \psi(x) + c$

Curves spacetime

$\eta_{\mu\nu} \to g_{\mu\nu}$

derivative expansion

low energy effective theory

$$S = \int d^d x \sqrt{-g} F(b, z) \qquad b = \sqrt{\det B^{ab}}, \ z := u^\mu \partial_\mu \psi = \frac{J^\mu}{b} \partial_\mu \psi \qquad B^{ab} = g^{\mu\nu} \frac{\partial y^a}{\partial x^\mu} \frac{\partial y^b}{\partial x^\nu}$$

$$J^\mu = b u^\mu \qquad b = \sqrt{\det B^{ab}}$$

Geometric current

$$J^\mu = -\frac{1}{(d-1)!} \frac{1}{\sqrt{-\tilde{g}}} e^{\mu\alpha_1 \cdots \alpha_{d-1}} \tilde{e}_{0a_1 \cdots a_{d-1}} \partial_{\alpha_1} y^{a_1} \cdots \partial_{\alpha_{d-1}} y^{a_{d-1}}$$

Conserved current for $\psi(x) \to \psi(x) + c$:

$$N_1^\mu(x) := \frac{\delta S}{\delta \partial_\mu \psi} = n_1(x) u^\mu(x)$$

ネーターの定理

$$n_1 = F_z := \partial_z F$$

charge density

完全流体

$$T^\mu{}_\nu = \varepsilon u^\mu u_\nu + P(u^\mu u_\nu + \delta^\mu_\nu)$$

4. Expression of entropy current for perfect fluids

EMT

$$T^{\mu\nu}(x) := \frac{2}{\sqrt{-\tilde{g}}}\frac{\partial S}{\partial g_{\mu\nu}(x)} = g^{\mu\nu}F + 2F_z\frac{\partial z}{\partial g_{\mu\nu}} + 2F_b\frac{\partial b}{\partial g_{\mu\nu}}$$

$$z = \frac{J^\mu}{b}\partial_\mu\psi$$

$$= g^{\mu\nu}F + \frac{2F_z}{b}\frac{\partial(J^\mu\partial_\mu\psi)}{\partial g_{\mu\nu}} - 2\frac{F_z z - F_b b}{b}\frac{\partial b}{\partial g_{\mu\nu}}$$

$$b = \sqrt{\det B^{ab}}$$

$$\frac{\partial(J^\mu\partial_\mu\psi)}{\partial g_{\mu\nu}} = -\frac{bz}{2}g^{\mu\nu}$$

$$\frac{\partial b}{\partial g_{\mu\nu}} = -\frac{b}{2}g^{\mu\alpha}g^{\nu\beta}\underline{B_{ab}\partial_\alpha y^a\partial_\beta y^b} = -\frac{b}{2}(g^{\mu\nu} + u^\mu u^\nu)$$

$$= g_{\alpha\beta} + u_\alpha u_\beta$$

$$J^\mu \propto \frac{1}{\sqrt{-g}}$$

$$(B_{ab}\partial_\mu y^a\partial_\nu y^b)u^\nu = B_{ab}\partial_\mu y^a\frac{\partial y^b}{\partial\tau} = 0$$

$$B^{ab} = g^{\mu\nu}\frac{\partial y^a}{\partial x^\mu}\frac{\partial y^b}{\partial x^\nu}$$

$$(B_{ab}\partial_\mu y^a\underline{\partial_\nu y^b)g^{\nu\alpha}\partial_\alpha y^c} = B_{ab}B^{bc}\partial_\mu y^a = \partial_\mu y^c$$

$$T^\mu{}_\nu = (F - F_b b)\delta^\mu_\nu + (F_z z - F_b b)u^\mu u_\nu$$

P $\qquad$ $\varepsilon + P$

$$P = F - F_b b \qquad \varepsilon = F_z z - F$$

+ charge density $\quad n_1 = F_z$

$$P = F - F_b b \qquad \varepsilon = F_z z - F = z n_1 - F \qquad n_1 = F_z$$

$$d\varepsilon = z dn_1 + F_z dz - F_z dz - F_b db = z dn_1 - F_b db$$

compared with thermodynamic relation $d\varepsilon = Tds + \mu_1 dn_1$

$$s = b,\ T = -F_b,\ \mu_1 = z$$

b is an entropy density !

Other thermodynamics relation automatically follows as

$$\varepsilon + P - \mu_1 n_1 = zF_z - F_b b - zF_z = -F_b b = Ts$$

$$J^\mu = bu^\mu = su^\mu$$

The geometric current is the entropy current !

Ex. Expanding Universe (scalar+radiation)

SA and K. Kawana, "Entropy and its conservation in expanding Universe", International Journal of Modern Physics A38 (2023) 2350072 [arXiv:2210.03323 [hep-th]].

$$\rho = \rho_\phi + \rho_R \qquad P = P_\phi + P_R$$

Radiation $$\dot{\rho}_R + 4H\rho_R = \underline{\Gamma(\rho_\phi + P_\phi)} \qquad P_R = \frac{\rho_R}{3}$$

coupling to scalar field

Scalar field $$\ddot{\phi} + (3H + \underline{\Gamma})\dot{\phi} + \frac{\partial V}{\partial \phi} = 0$$ **EoM**

coupling to radiation

Energy density $$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

Pressure $$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

Potential $$V(\phi) = \frac{m_\phi^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

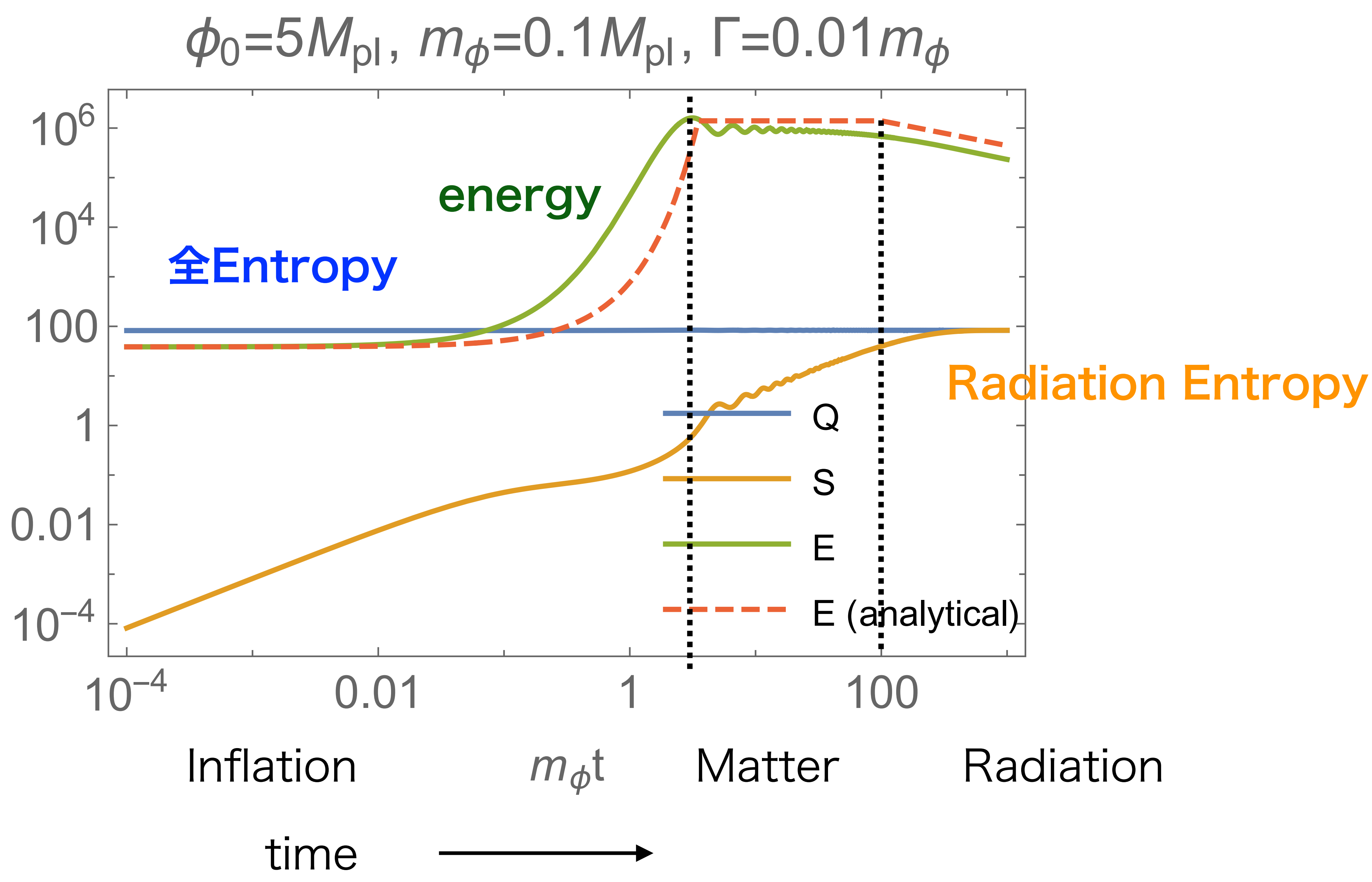
Numerical results

$$\lambda = 10^{-2}, \quad m_\phi = 0.1 M_{\rm Pl}, \quad \Gamma = 10^{-2} m_\phi$$

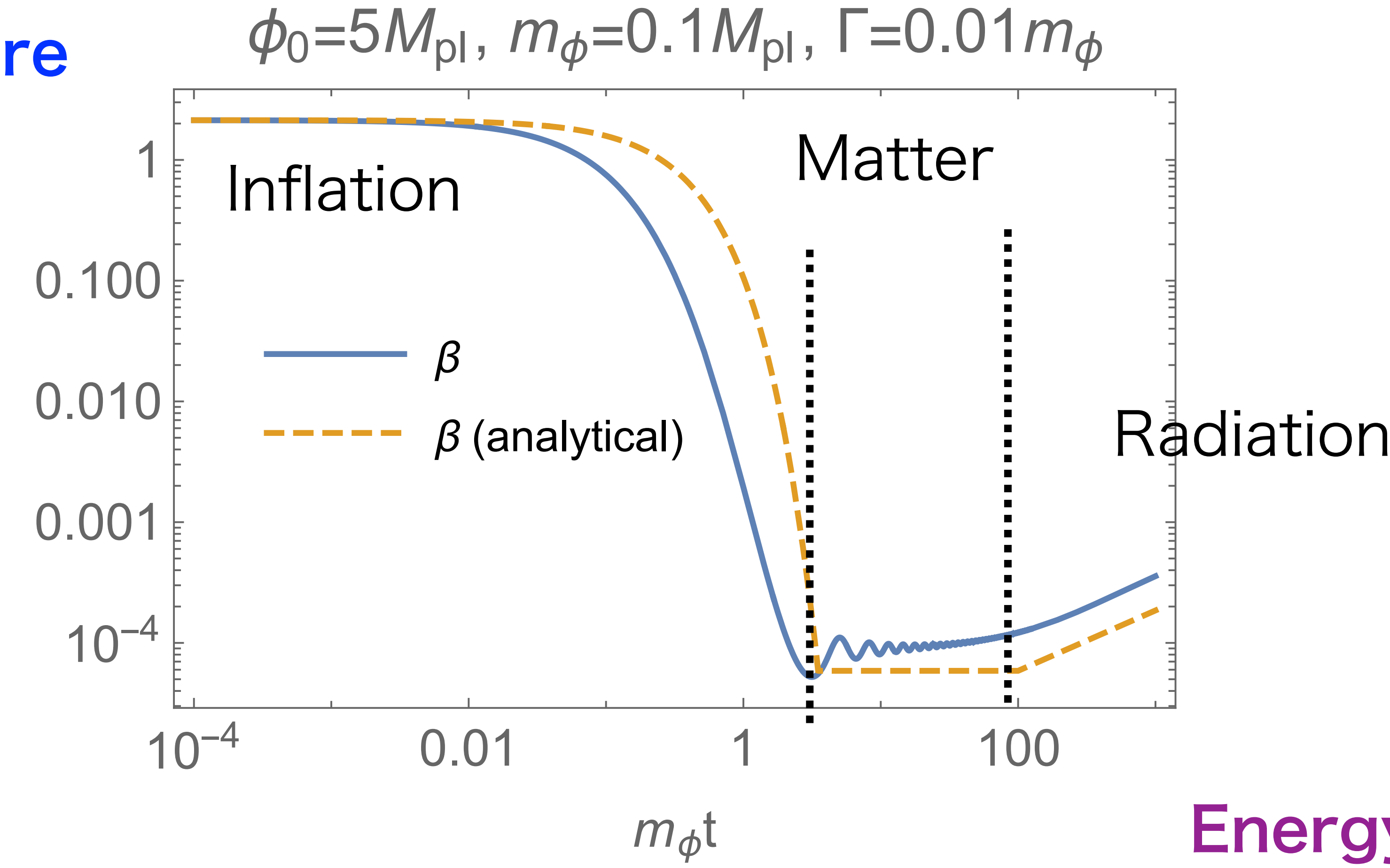
$$M_{\rm Pl} := \frac{1}{\sqrt{G_N}}$$ Planck mass

$$\phi(0) = 5 M_{\rm Pl}, \quad \dot{\phi}(0) = 0, \quad \beta(0) = \frac{2\pi}{H(0)}$$

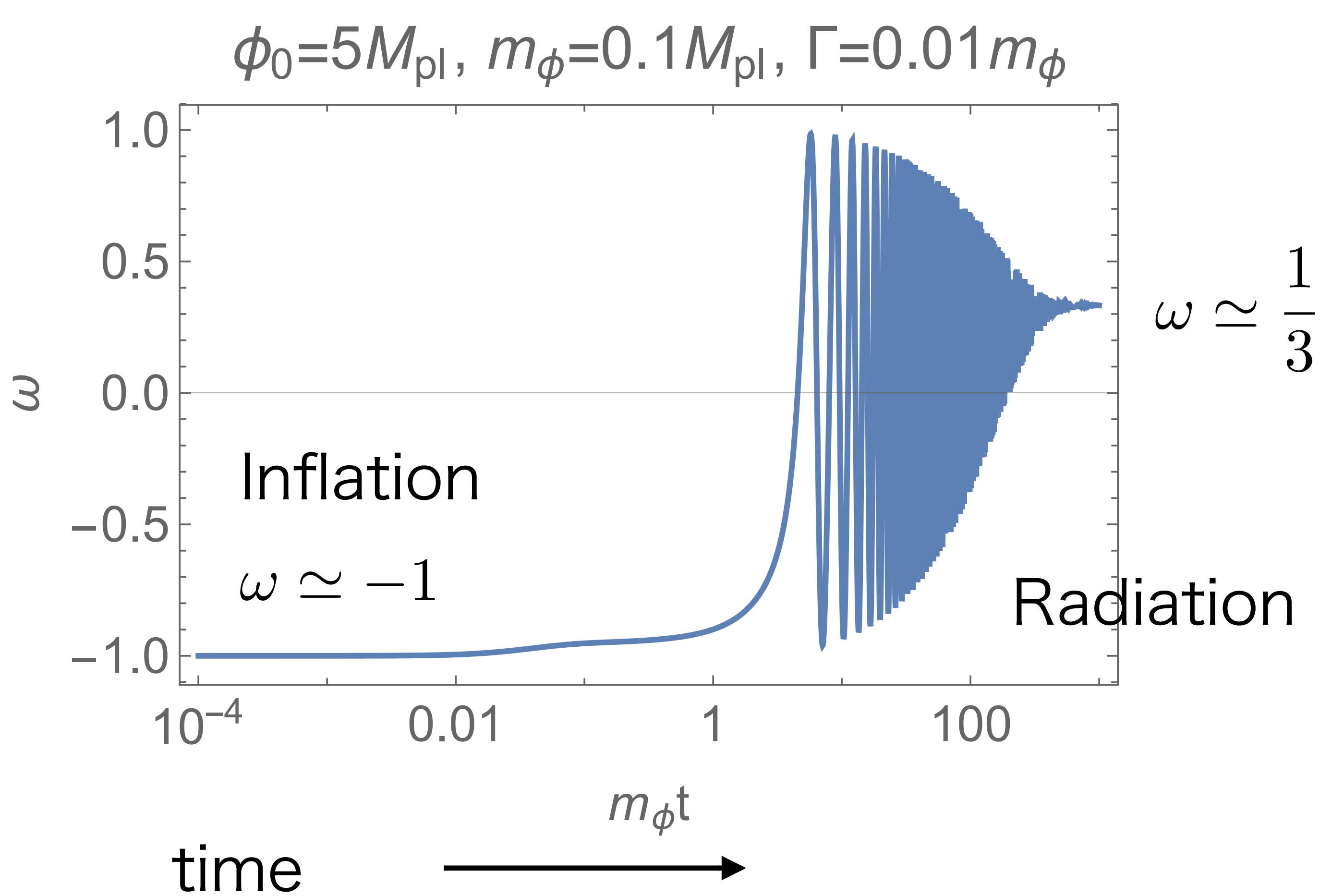
slow roll   de Sitter temperature



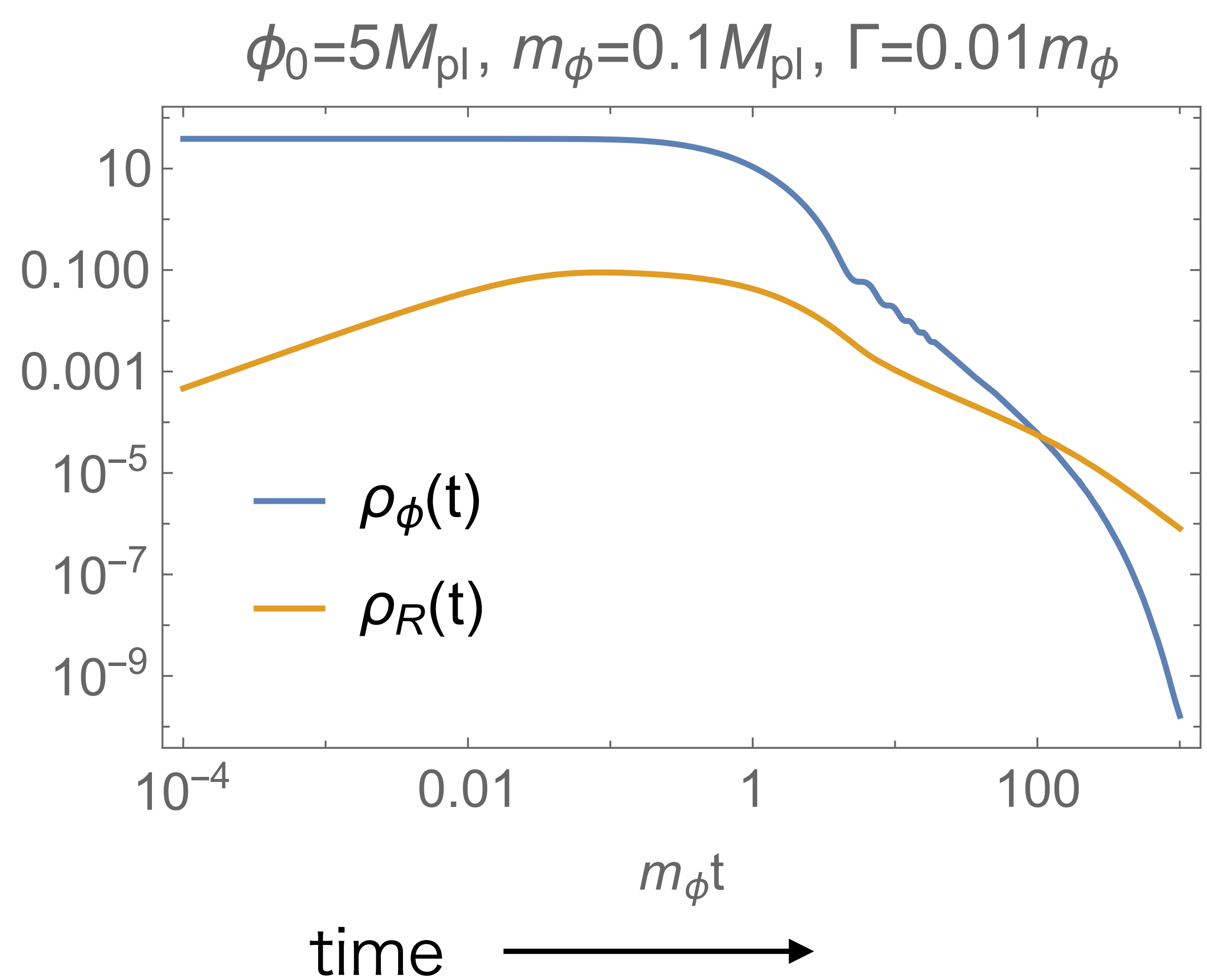
Inverse temperature



EoS $P = w\rho$



Energy density



V. Conclusion

Conclusion

1. Geometric conservation always holds in a curved spacetime.

conserved current $$J^\mu(\tau,y) = \frac{-u^\mu(\tau,y)}{n(\tau,y)\cdot u(\tau,y)\sqrt{h(\tau,y)}}$$

gravitational charge $$Q = \int_{H_{d-1}} d^{d-1}y$$

2. The geometric conserved charge is entropy for perfect fluids.

Interpretation

1. A source of gravity is "entropy", as the electric charge is the source of EM interaction.

   c.f. "Gravity is entropic force". T. Jacobson 1995, E.P. Verlinde 2011.

2. Through Einstein's equation $G_{\mu\nu} + \Lambda g_{\mu\nu} = 2\kappa T_{\mu\nu}$, the geometric conservation holds in spacetime . What is its mathematical/geometric meaning ?

Future studies

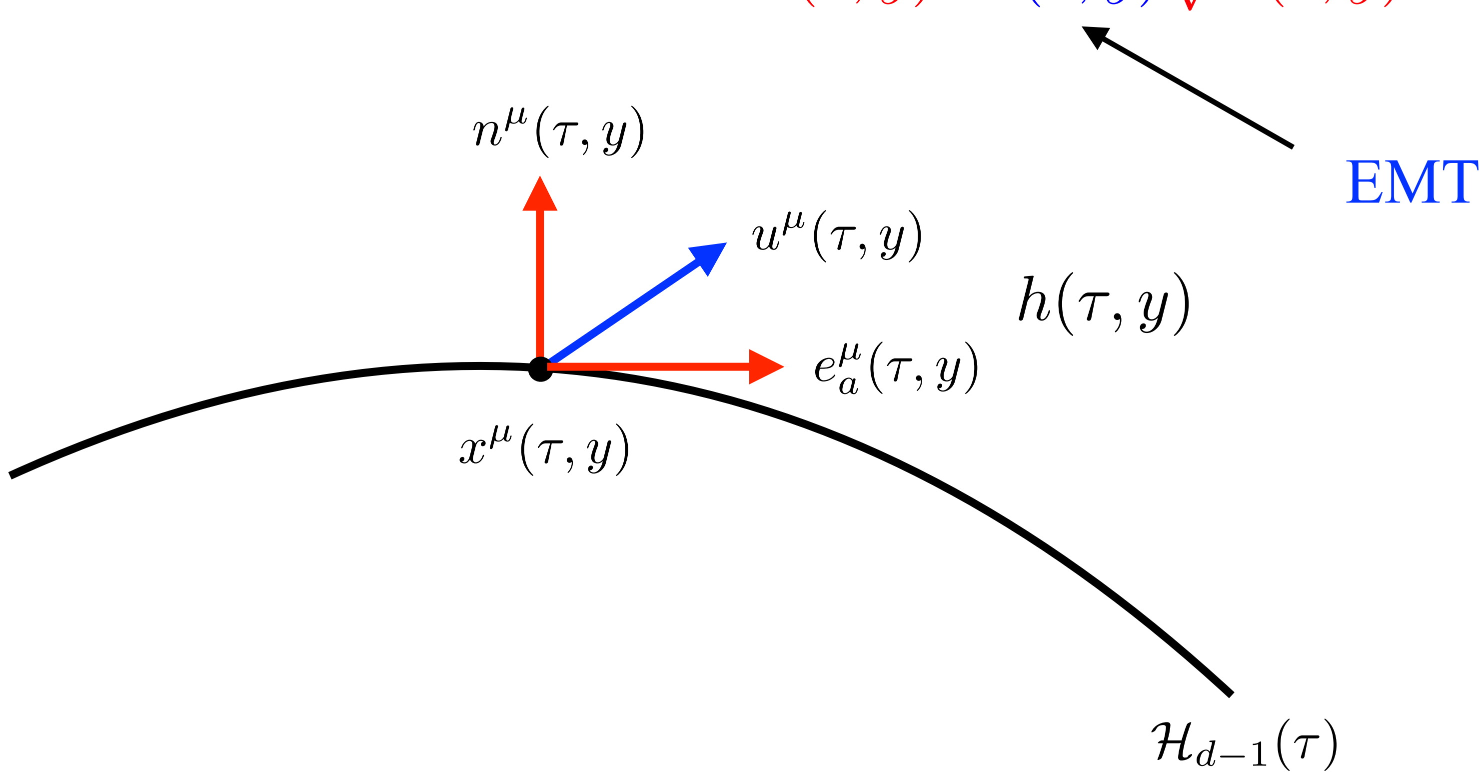
1. What is a physical interpretation of the geometric conservation for dissipative fluids ?

2. Applications of the geometric conservation.

A magic (universal) formula for "entropy" density

$$s(x(\tau,y)) = \frac{-1}{n(\tau,y)\cdot u(\tau,y)\sqrt{h(\tau,y)}}$$

Please calculate "entropy" density in your favorite spacetime.



Thank you for your attention.

Backup

Backup: Noether's 2nd theorem

E. Noether, Gott. Nacho. **1918**(1918)235-257 [arXiv:physics/0503066[physics]]

Local (gauge) symmetry ➡ Conservation laws as constraints or identities

conserved current for an **arbitrary** vector ξ^μ without using EoM $\partial_\mu J^\mu[\xi] = 0$

$$J^\mu[\xi] = \frac{1}{4\kappa}\sqrt{-g}\nabla_\nu\left[\nabla^{[\mu}\xi^{\nu]}\right] = A^\mu{}_\nu\xi^\nu + B^\mu{}_\nu{}^\alpha\xi^\nu{}_{,\alpha} + C^\mu{}_\nu{}^{\alpha\beta}\xi^\nu{}_{,\alpha\beta}$$

$\partial_\mu A^\mu{}_\nu = 0$ non-covariant conserved current

$A^\mu{}_\nu$ ⟶ pseudo-tensor

$J^\mu[\xi]$ ⟶ quasi-local energy (by Stokes theorem)

Trivial conservation due to Noether's 2nd theorem

3a. Entropy current conservation for perfect fluids

Perfect fluid $T^{\mu}{}_{\nu} = \varepsilon u^{\mu} u_{\nu} + P(u^{\mu} u_{\nu} + \delta^{\mu}_{\nu})$

conservation $u^{\nu} \nabla_{\mu} T^{\mu}{}_{\nu} = -\partial_{\tau}\varepsilon - (\varepsilon + P)K = 0 \longrightarrow \partial_{\tau}\varepsilon = -(\varepsilon + P)K$

An other conserved current $N_1^{\mu} = n_1 u^{\mu}$ $\nabla_{\mu} N_1^{\mu} = \partial_{\tau} n_1 + n_1 K = 0$

(We here consider one conserved current, but an extension to more is straightforward.)

Entropy current $s^{\mu} = s u^{\mu}$

Thermodynamics relations

$$ds = d\varepsilon - \mu_1 dn_1$$

chemical potential μ_1

$$Ts = \varepsilon + P - \mu_1 n_1$$

temperature T

$$\nabla_{\mu} s^{\mu} = \partial_{\tau} s + sK = \frac{1}{T}(\partial_{\tau}\varepsilon - \mu_1 \partial_{\tau} n_1 + sTK) = \frac{K}{T}(-\varepsilon - P + \mu_1 n_1 + sT) = 0$$

Entropy current is conserved.

Expanding Universe

(A simple example)

1. Homogeneous and isotropic expanding Universe

$$ds^2 = -(dx^0)^2 + a^2(x^0)\tilde{g}_{ij}dx^i dx^j$$ **FLRW metric**

EMT (perfect fluid) $T^0{}_0 = -\varepsilon(x^0),\ T^i{}_j = P(x^0)\delta^i_j,\ T^0{}_j = T^i{}_0 = 0$

covariant conservation $\nabla_\mu T^\mu{}_\nu = 0$ $\longrightarrow$ $\dot{\varepsilon} + (d-1)(\varepsilon + P)\frac{\dot{a}}{a} = 0$

energy $E(x^0) := -\int d^{d-1}x\,\sqrt{-g}\,T^0{}_0 = V_{d-1}a^{d-1}\varepsilon, \quad V_{d-1} := \int d^{d-1}x\,\sqrt{\tilde{g}}.$

$$\longrightarrow \quad \frac{\dot{E}}{E} = -(d-1)\frac{\dot{a}}{a}\frac{\varepsilon + P}{\varepsilon} + (d-1)\frac{\dot{a}}{a} = -(d-1)\frac{\dot{a}}{a}\frac{P}{\varepsilon} \neq 0$$

The energy is indeed not conserved in expanding Universe.

General relativity should have no generic conserved energy.

entropy current $s^\mu(x^0) = -\frac{1}{(n\cdot u)\sqrt{h}}u^\mu = \frac{c_0\delta^\mu_0}{a^{d-1}(x^0)\sqrt{\tilde{g}}}$ c_0: constant, $\sqrt{h} = a^{(d-1)}\sqrt{\tilde{g}}$

$$\nabla_\mu s^\mu = \dot{s}^0 + \Gamma^\mu_{\mu 0}s^0 = -(d-1)\frac{\dot{a}}{a}s^0 + (d-1)\frac{\dot{a}}{a}s^0 = 0 \qquad \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j$$

2. Constant equation of state (EOS)

flat space case

Einstein equation

$$\frac{(d-2)(d-1)}{2}H^2 = 2\kappa\varepsilon \qquad (d-2)\left[\dot{H} + \frac{d-1}{2}H^2\right] = -2\kappa P \qquad H := \frac{\dot{a}}{a}$$

constant EOS $P(x^0) = w\varepsilon(x^0)$

$$a(x^0) = (1 + C_0 H_0 x^0)^{1/C_0} \qquad a(x^0 = 0) = 1 \qquad C_0 := \frac{(d-1)(1+w)}{2}$$

$$\varepsilon(x^0) = \varepsilon_0 \left(\frac{1}{a(x^0)}\right)^{(d-1)(1+w)}$$

internal energy $U(x^0) = V_{d-1}\dfrac{\varepsilon_0}{a^{(d-1)w}(x^0)}$ $\qquad V_{d-1} = \displaystyle\int d^{d-1}x, \quad \sqrt{\tilde{g}} = 1$

volume $V(x^0) = V_{d-1}a^{(d-1)}(x^0)$

entropy current $s^\mu(x^0) = \dfrac{c_0\delta^\mu_0}{a^{d-1}(x^0)}$

entropy $S(x^0) = V_{d-1}c_0$

Thermodynamic entropy

$S = S(U, V, N)$ N: conserved charge

property 1. $S(\lambda U, \lambda V, \lambda N) = \lambda S(U, V, N)$

property 2. 2nd derivative with respect to each variable is negative semi-definite.

凸 for each variable

$$\dot{H} + \frac{(d-1)(1+\omega)}{2} H^2 = 0 \longrightarrow H(x^0) = \frac{H_0}{1 + C_0 H_0 x^0} \longrightarrow \log \frac{a(x^0)}{a_0} = \frac{1}{C_0} \log(1 + C_0 H_0 x^0)$$

$$\varepsilon(x^0) = \frac{H_0^2}{2\kappa} (1 + C_0 H_0 x^0)^{-2} = \frac{\varepsilon_0}{a(x^0)^{2C_0}}$$

2-1. Radiation era

$$\omega = \frac{1}{d-1}$$

$$P(x^0) = \frac{1}{d-1}\varepsilon(x^0) \longrightarrow a(x^0) = \left(1 + \frac{d}{2}H_0 x^0\right)^{2/d}$$

$$a(x^0) = (1 + C_0 H_0 x^0)^{1/C_0}$$

$$C_0 = \frac{d}{2}$$

$$\longrightarrow \quad \varepsilon(x^0) = \frac{\varepsilon_0}{a^d(x^0)}, \quad \sqrt{-g} = a^{d-1}(x^0)$$

$$\longrightarrow \quad U = V_{d-1}\frac{\varepsilon_0}{a(x^0)}, \quad V = V_{d-1}a^{d-1}(x^0), \quad S = V_{d-1}c_0$$

fundamental relation for radiation $S = UG(U/V) = cU\left(\frac{U}{V}\right)^{\alpha}$ no N_i for radiation

$S \sim 1$ by conservation

$$U \sim a^{-1}(x^0), \quad U/V \sim a^{-d}(x^0) \longrightarrow \alpha = -\frac{1}{d}$$

$$\longrightarrow \quad S = cV_{d-1}\rho_0^{\frac{d-1}{d}} \longrightarrow c_0 = c\rho_0^{\frac{d-1}{d}}$$

$$\longrightarrow \quad \boxed{S(U,V) = cU^{1-\frac{1}{d}}V^{\frac{1}{d}}}$$

fundamental relation is determined.

concave conditions are satisfied.

various thermodynamic quantities

(Inverse) temperature
$$\frac{1}{T(x^0)} := \frac{\partial S}{\partial U} = c\frac{d-1}{d}\left(\frac{V}{U}\right)^{\frac{1}{d}} = \frac{d-1}{d}\frac{c}{\varepsilon_0^{\frac{1}{d}}}a(x^0) \qquad \frac{1}{T} = c\frac{d-1}{d}\left(\frac{V}{U}\right)$$

Pressure
$$\frac{P}{T} := \frac{\partial S}{\partial V} = \frac{c}{d}\left(\frac{U}{V}\right)^{1-\frac{1}{d}} = \frac{1}{d-1}\frac{\varepsilon}{T} \longrightarrow \boxed{P(x^0) = \frac{1}{d-1}\varepsilon(x^0)}$$

consistency

Entropy density
$$s := \frac{S}{V} = c\left(\frac{U}{V}\right)^{1-\frac{1}{d}} = \frac{d}{d-1}\frac{\varepsilon}{T} = \frac{\varepsilon+P}{T}$$

thermodynamic relation

Stefan-Boltzmann
$$\varepsilon(x^0) = \left(\frac{d-1}{d}cT(x^0)\right)^d = \sigma_d T^d(x^0) \qquad \sigma_d := \left(\frac{d-1}{d}c\right)^d$$

various thermodynamic quantities are correctly reproduced from the fundamental relation

$$S(U,V) = c\,U^{1-\frac{1}{d}}V^{\frac{1}{d}}$$

$$\varepsilon = \frac{U}{V}$$

2-2. Dark energy (Inflation)

$$P(x^0) = -\varepsilon(x^0) \longrightarrow a(x^0) = e^{H_0 x^0}, \quad \varepsilon(x^0) = \varepsilon_0 = \frac{(d-1)(d-2)}{16\pi^2} H_I^2$$

exponential expansion = inflation

The metric is equivalent to (static) de Sitter spacetime

$$ds^2 = -(dx^0)^2 + e^{2H_I x^0}(dR^2 + R^2 d\Omega_{d-2}^2)$$

Hubble constant $H_I := H_0$

$$x^0 = t + \frac{1}{2H_I}\log(1 - H_I^2 r^2), \quad R = \frac{r e^{-H_I t}}{\sqrt{1 - H_I^2 r^2}}.$$

$$ds^2 = -(1 - H_I^2 r^2)dt^2 + \frac{dr^2}{1 - H_I^2 r^2} + r^2 d\Omega_{d-2}^2$$

cosmological constant $\Lambda = \frac{(d-1)(d-2)}{2} H_I^2 := \frac{(d-1)(d-2)}{2R_H^2}$

$R_H = \frac{1}{H_I}$ radius of de Sitter horizon =radius of Hubble horizon

unform matter with $w = -1$ (dark energy) $\longleftrightarrow$ de Sitter spacetime

$$P(x^0) = -\varepsilon(x^0) \longrightarrow U = V_{d-1}\varepsilon_0 a^{d-1}(x^0), \quad V = V_{d-1}a^{d-1}(x^0), \quad S = c_0\, V_{d-1}$$

fundamental relation $S(U,V,N) = UG(U/V, N/V) = c\,U\left(\frac{N}{V}\right)^{\beta}\left(\frac{U}{V}\right)^{\alpha}$

$$\longrightarrow \quad \frac{1}{T} = c\left(\frac{N}{V}\right)^{\beta}(\alpha+1)\left(\frac{U}{V}\right)^{\alpha}, \qquad \frac{P}{T} = -c\,\varepsilon\left(\frac{N}{V}\right)^{\beta}(\alpha+\beta)\left(\frac{U}{V}\right)^{\alpha}, \quad \longrightarrow \quad P = -\epsilon\frac{\alpha+\beta}{\alpha+1}$$

consistency $P = -\varepsilon \longrightarrow \beta = 1$

$$N := V_{d-1}n_0 a^{\gamma}(x^0) \longrightarrow S \sim a^{d-1}(x^0)\frac{a^{\gamma}(x^0)}{a^{d-1}(x^0)} \sim 1 \longrightarrow \gamma = 0$$

$$\frac{U}{V} = \varepsilon_0$$

concave condition $\longrightarrow \alpha = -1$

$$\boxed{S = c\,N = c\,V_{d-1}n_0}$$

fundamental relation $\longrightarrow c_0 = c\,n_0$

conserved charge $S(x^0) = V_{d-1} a_0^{d-1} \varepsilon_0 \zeta_0$ $c_0 = a_0^{d-1}\epsilon_0\zeta_0$

co-moving volume $V_{d-1} = \int d^{d-1}x\sqrt{\tilde{g}} = \Omega_{d-2}\int_0^{r_{\max}} r^{d-2}dr = \frac{r_{\max}^{d-1}}{d-1}\Omega_{d-2}$

$r_{\max}$: a maximal distance a light can travel from $x^0 = 0$ to $x^0 = \infty$.

light $\frac{dr}{dx^0} = \frac{1}{a(x^0)} = \frac{e^{-H_I x^0}}{a_0} \longrightarrow r_{\max} = \int_0^\infty \frac{dr}{dx^0}dx^0 = \frac{1}{a_0 H_I}$

$$S = \frac{A_H}{4G_N}\frac{d-2}{2}\frac{H_I}{2\pi}\zeta_0$$

$A_H = R_H^{d-2}\Omega_{d-2}$ area of de Sitter horizon

an initial condition $\frac{d-2}{2}\zeta_0 = \frac{2\pi}{H_I} := \frac{1}{T_H}$ de Sitter temperature

$$S = \frac{A_H}{4G_N}$$

Bekenstein-Hawking entropy cf. Gibbons-Hawking 1977

An alternative derivation of entropy for de Sitter spacetime

Entropy, carried by dark energy, is uniformly distributed inside the de Sitter horizon, but NOT only on the horizon.