Thermodynamic tradeoff relations in quantum systems

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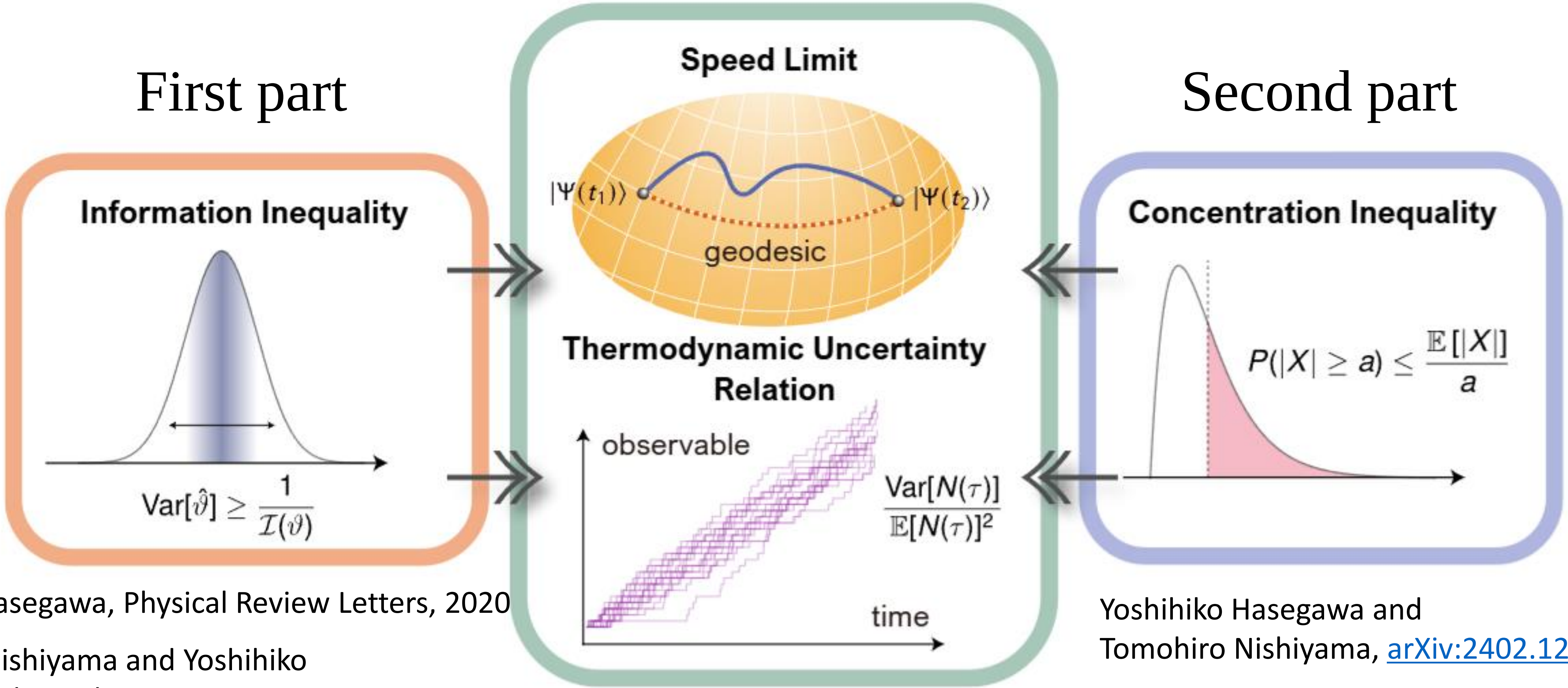
First part

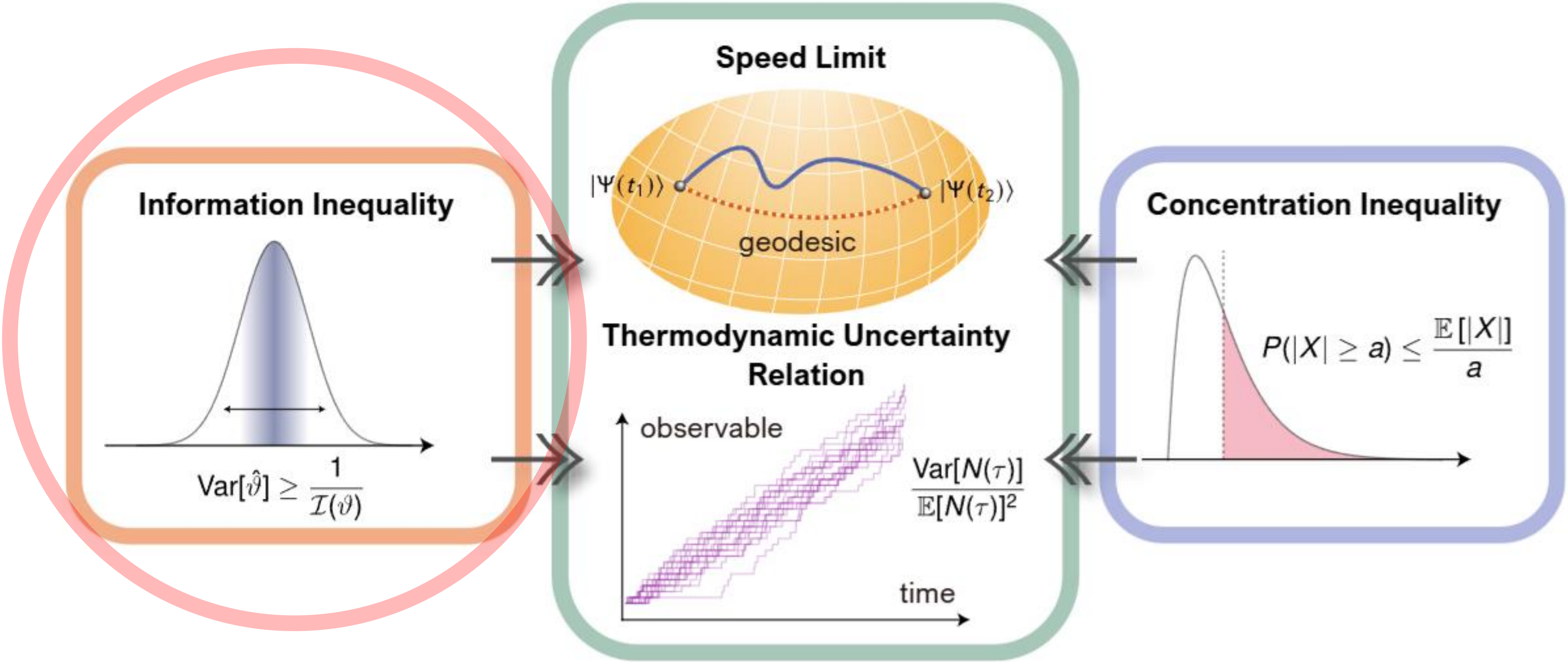
Second part



Yoshihiko Hasegawa, Physical Review Letters, 2020

Tomohiro Nishiyama and Yoshihiko Hasegawa, Physical Review E, 2024

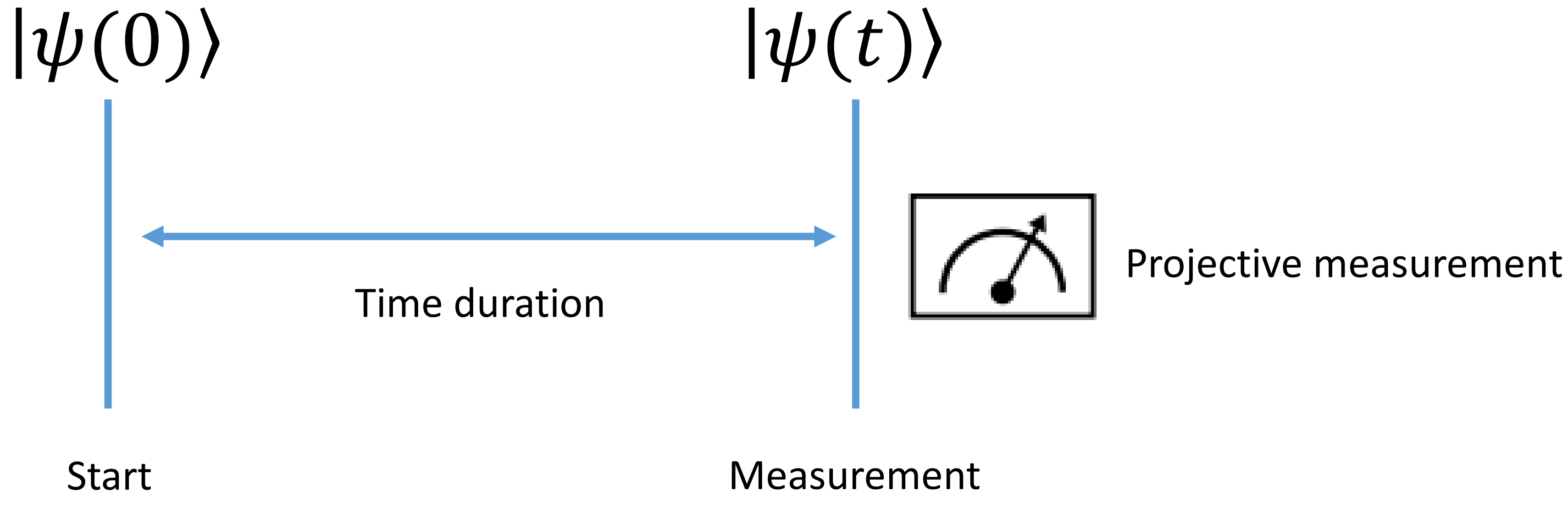
Yoshihiko Hasegawa and Tomohiro Nishiyama, arXiv:2402.12197



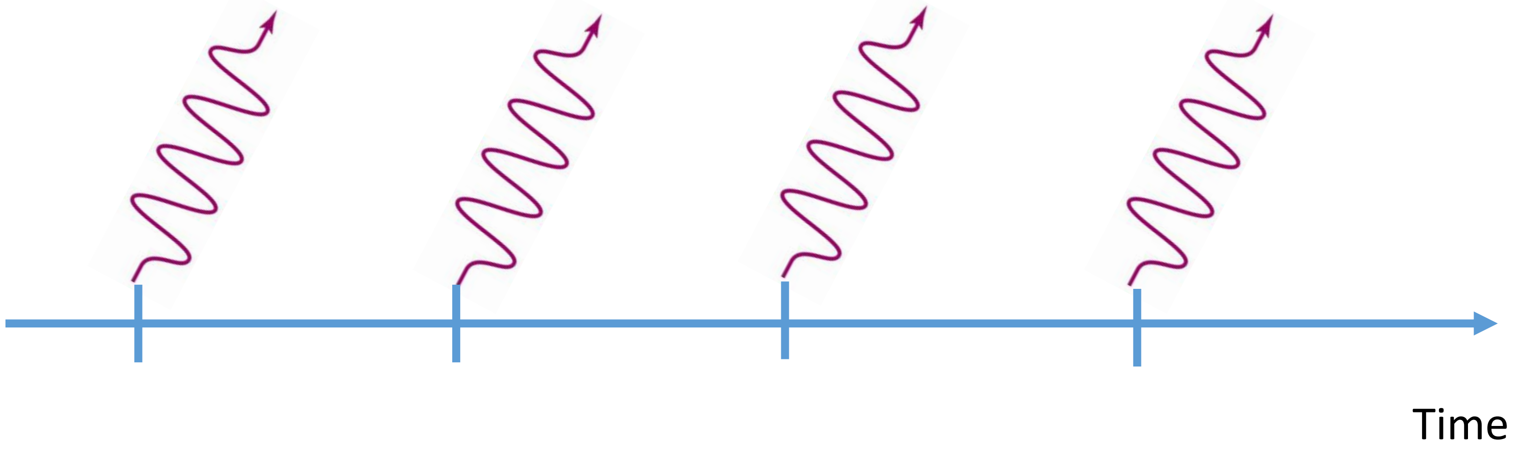
Information Inequality
$\mathrm{Var}[\hat{\vartheta}] \geq \frac{1}{\mathcal{I}(\vartheta)}$
Speed Limit
$|\Psi(t_1)\rangle$
$|\Psi(t_2)\rangle$
geodesic
Thermodynamic Uncertainty Relation
observable
time
$\frac{\mathrm{Var}[N(\tau)]}{\mathbb{E}[N(\tau)]^2}$
Concentration Inequality
$P(|X| \geq a) \leq \frac{\mathbb{E}[|X|]}{a}$

Quantum time-keeping device

Stopwatch



Chime



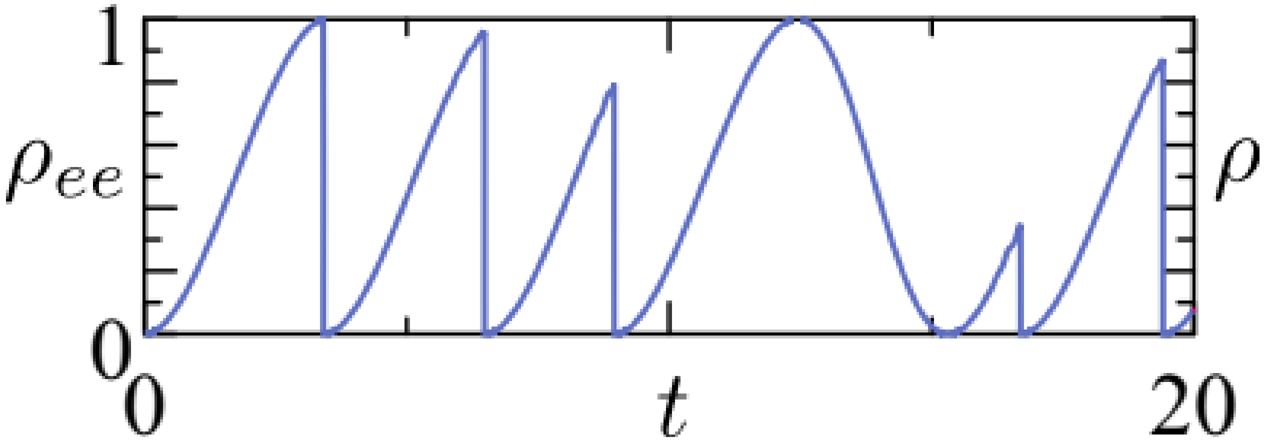
Continuous measurement

- Chime quantum clock inevitably interacts with environment
- Such dynamics can be described by the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) equation

$$\frac{d\rho}{dt} = -i[H, \rho] + \sum_m \left[L_m \rho L_m^\dagger - \frac{1}{2}\{L_m^\dagger L_m \rho + \rho L_m^\dagger L_m\} \right]$$

where L_m is a jump operator and ρ is density operator

- For example, $L = \sqrt{\kappa}|g\rangle\langle e|$



Continuous measurement and matrix product state

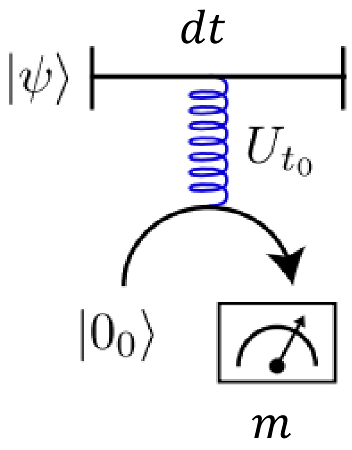
$$\frac{d\rho}{dt} = \mathcal{L}\rho = -i[H,\rho] + \sum_m \left[L_m \rho L_m^\dagger - \frac{1}{2}\left\{ L_m^\dagger L_m \rho + \rho L_m^\dagger L_m \right\} \right]$$

- Continuous measurement is represented by the Kraus operator
- For $[t, t+dt]$

$$\rho(t+dt) = M_0 \rho M_0^\dagger + \sum_{m=1}^{N_C} M_m \rho M_m^\dagger = \sum_{m=0}^{N_C} M_m \rho M_m^\dagger$$

$$M_0 = \mathbb{I}_S - idtH_{\text{eff}} \quad \text{No jump}$$

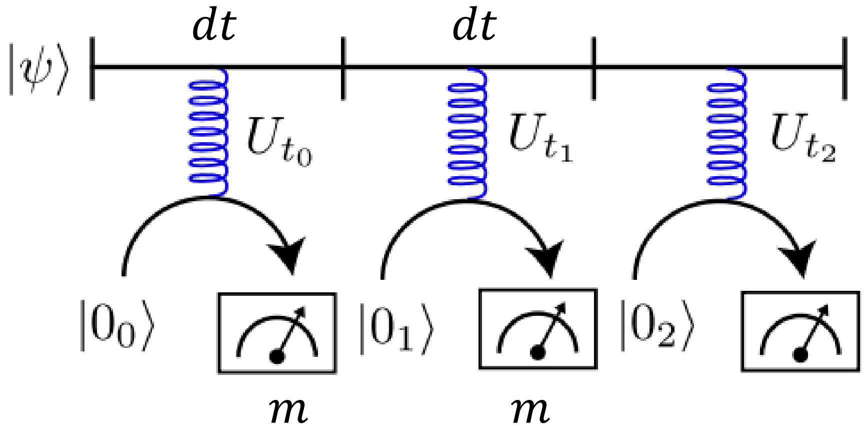
$$M_m = \sqrt{dt} L_m \quad \text{Jump}$$



Continuous measurement and matrix product state

- Applying the Kraus operators repeatedly within $[0, \tau]$

$$\rho(\tau) = \sum_{m_{N_\ell}} \cdots \sum_{m_0} M_{m_{N_\ell - 1}} \cdots M_{m_0} \rho M_{m_0}^\dagger \cdots M_{N_\ell - 1}^\dagger$$

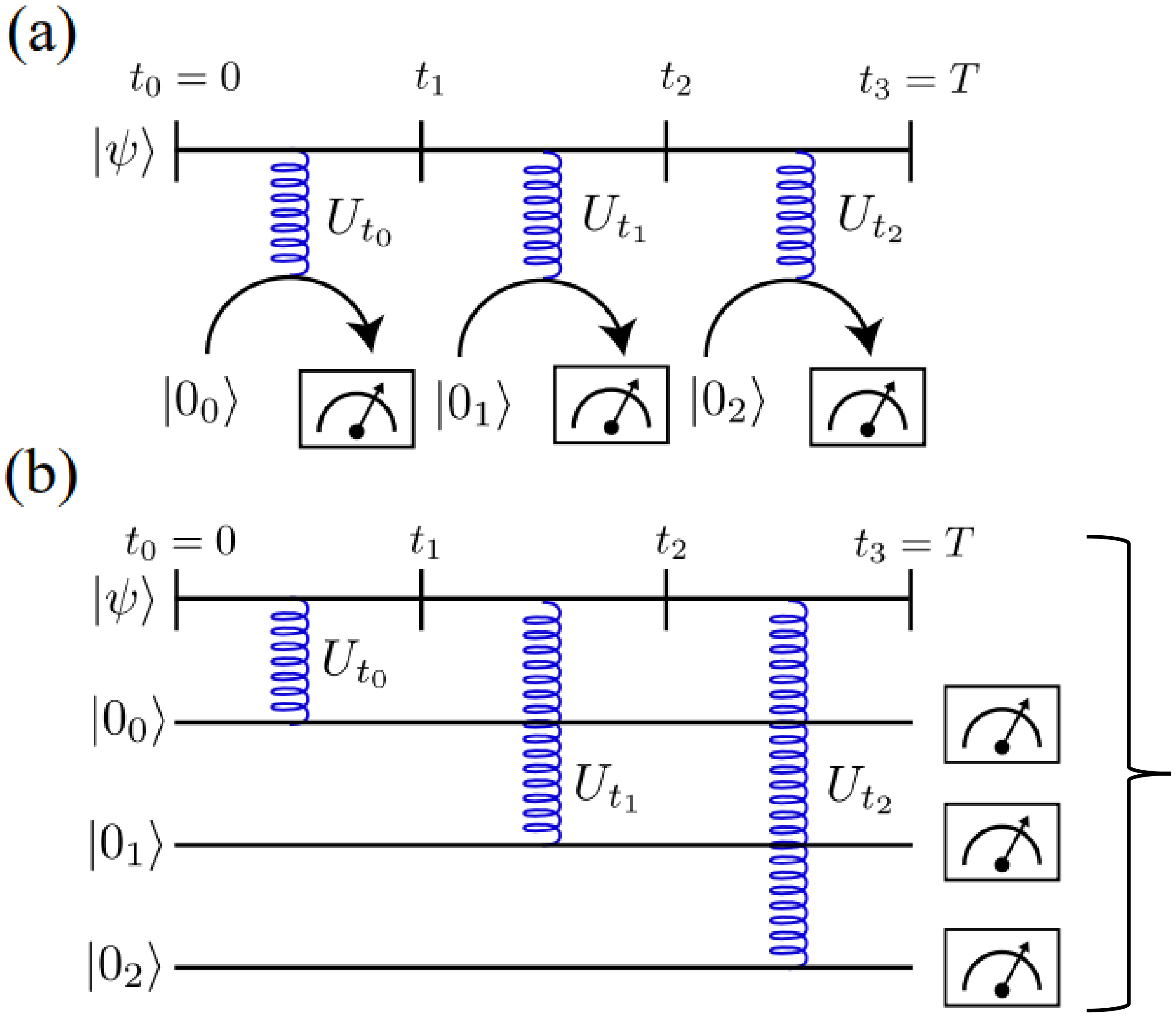


- Continuous measurement can be represented by a matrix product state (MPS)

$$|\Psi(\tau)\rangle = \sum_{m_0, \ldots, m_{N_\ell - 1}} M_{m_{N_\ell - 1}} \ldots M_{m_0} |\psi_S(0)\rangle \otimes |m_{N_\ell - 1}, \ldots m_0\rangle$$

$$\mathrm{Tr}_{\text{field}} \left[|\Psi(\tau)\rangle\langle\Psi(\tau)| \right] = \rho(\tau)$$

Continuous measurement and matrix product state



$|\Psi(\tau)\rangle$

- All the jump information is encoded in MPS
- Measurement of jump information can be performed by Hermitian operator at the final time

$$|\Psi(\tau)\rangle = \sum_{m_0,\ldots,m_{N_\ell-1}} M_{m_{N_\ell-1}} \ldots M_{m_0} |\psi_S(0)\rangle \otimes |m_{N_\ell-1},\ldots m_0\rangle$$

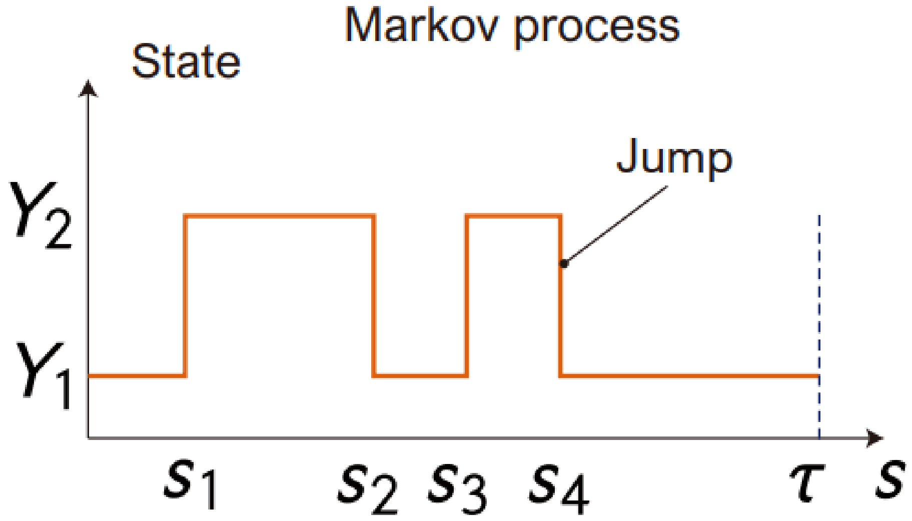
Continuous matrix product state (cMPS)

[Verstraete et al., Phys. Rev. Lett. 2010, Osborne et al., Phys. Rev. Lett., 2010]

- In the continuous limit, MPS becomes continuous MPS (cMPS)
- cMPS encodes classical/quantum stochastic processes into quantum field

$\phi_m^\dagger(s)$: field operator satisfying the commutation

$$\phi_{m_4}^\dagger(s_4)\,\phi_{m_3}^\dagger(s_3)\,\phi_{m_2}^\dagger(s_2)\,\phi_{m_1}^\dagger(s_1)\,|vac\rangle$$



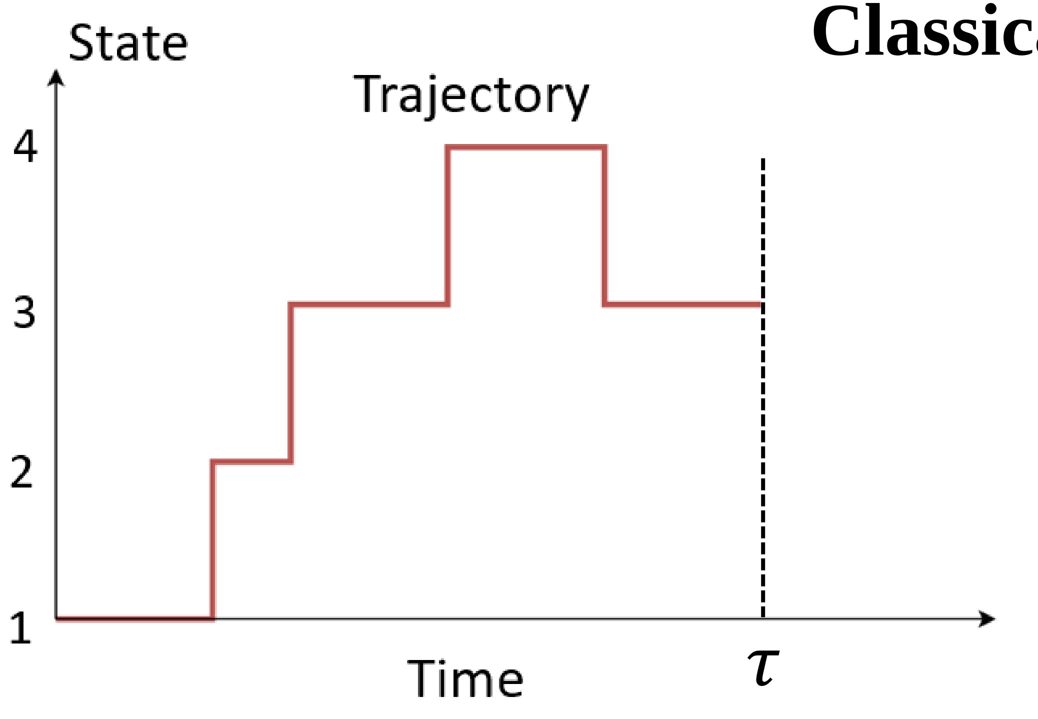
$$|\Psi(t)\rangle = \mathcal{U}(t)\,|\psi_S(0)\rangle \otimes |vac\rangle$$

System state Field state

$$\mathcal{U}(t) \equiv \mathbb{T} e^{-i\int_0^t ds\left[H\otimes\mathbb{I}_{\mathrm{fld}} + \sum_m \left(iL_m\otimes\phi_m^\dagger(s) - iL_m^\dagger\otimes\phi_m(s)\right)\right]}$$

Observable

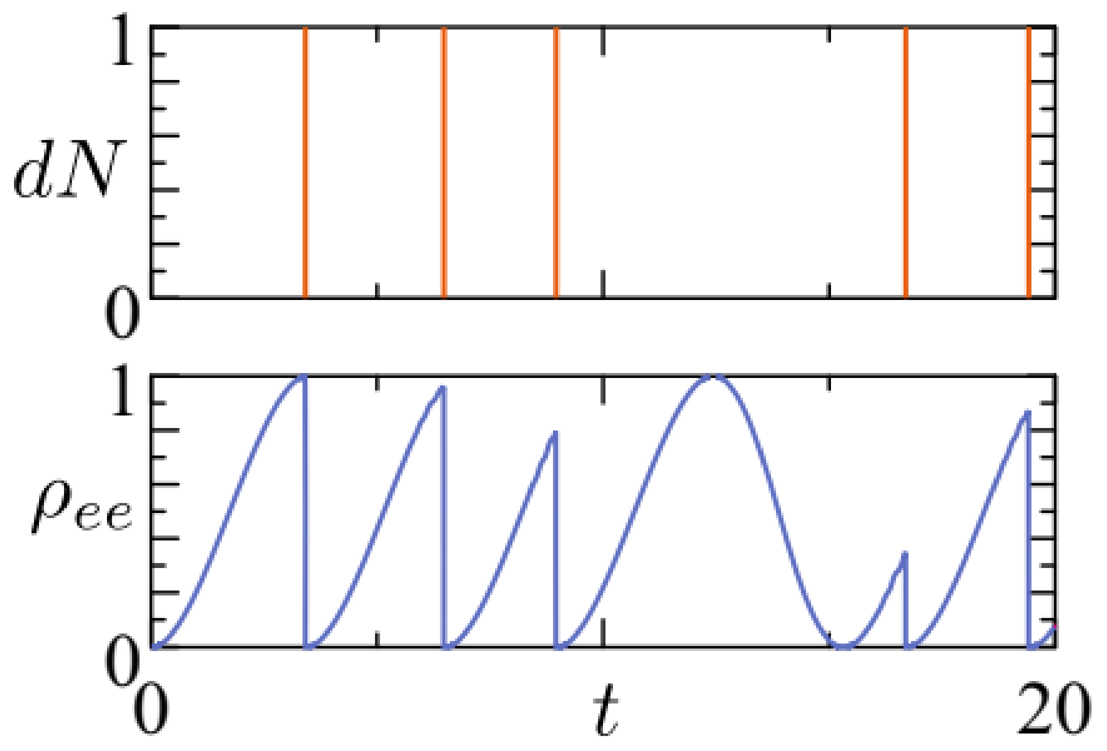
Classical



$$\phi = \sum_{m \neq n} C_{mn} N_{mn}$$

N_{ij} is the number of transitions from ith state to jth state

Quantum



$$N = \sum_{m} C_m N_m$$

N_m is the number of mth jump event

Observable

- The observable in continuous measurement is

$$N = \sum_m C_m N_m$$

- N_m can be calculated by the total number operator

$$\hat{N}_m = \int_0^{\tau} dt\, \phi_m^{\dagger}(t)\, \phi_m(t)$$

- Then, the expectation of N_m becomes

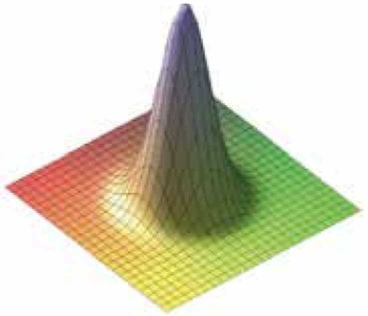
$$\langle \Psi(\tau) | I_S \otimes \hat{N}_m | \Psi(\tau) \rangle$$

cMPS state

Classical Cramér-Rao inequality

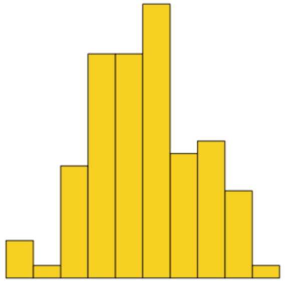
Classical estimation

Prob. dist.



$P(x;\theta)$

Sampling



$\mathcal{D} = \{x_1, x_2, \ldots, x_{N_D}\}$

Estimation

$\Theta(\mathcal{D})$

- Cramér-Rao inequality

$$\mathrm{Var}\left[\hat{\theta}\right] \geq \frac{1}{\mathcal{F}(\theta)}$$

$$\mathcal{F}\,(\theta) = -\left\langle \frac{\partial}{\partial\theta^2} \ln P(x|\theta) \right\rangle$$

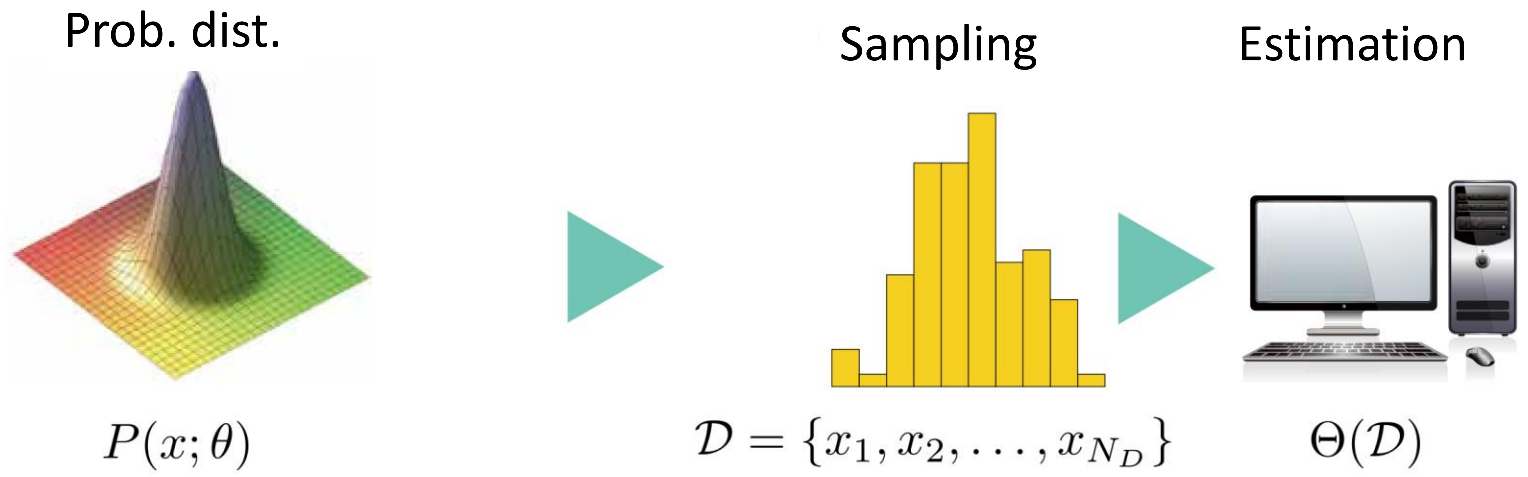
Fisher information

- Generalized Cramér-Rao inequality

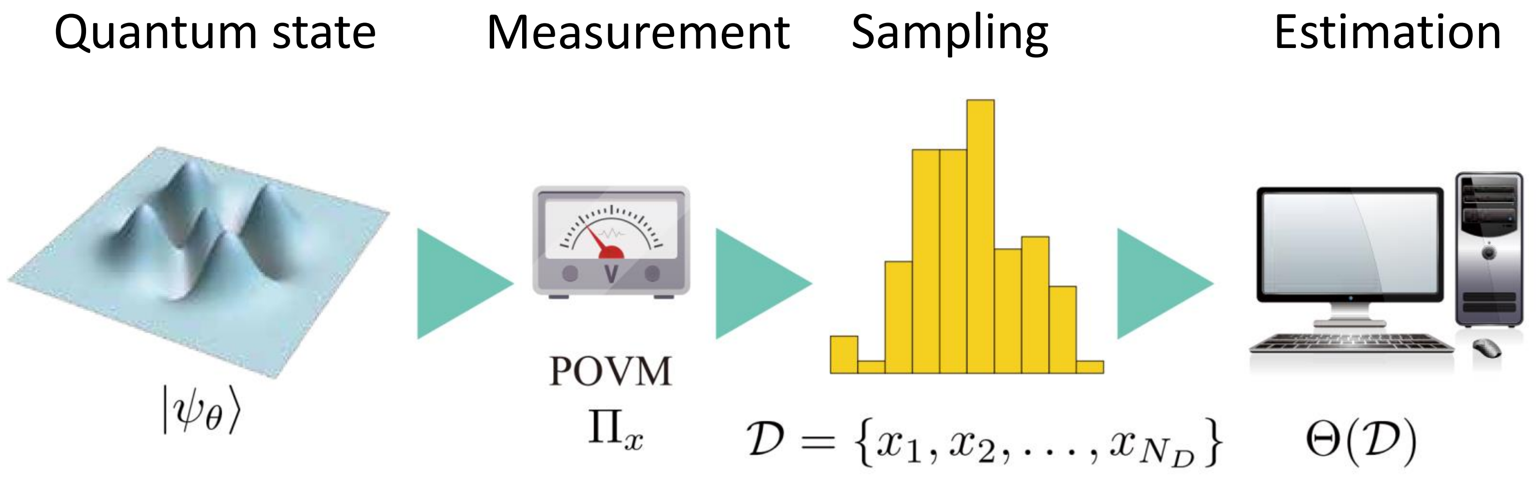
$$\frac{\mathrm{Var}\left[\widehat{\Theta}\,(\theta)\right]}{\left(\partial_\theta \langle \widehat{\Theta} \rangle\right)^2} \geq \frac{1}{\mathcal{F}(\theta)}$$

Classical and quantum estimation

Classical estimation



Quantum estimation



Quantum Fisher information

- In quantum estimation, there is freedom on the measurement operator Π_x (POVM)
- Quantum Cramér-Rao inequality is

$$\mathrm{Var}[\hat{\theta}] \geq \frac{1}{\mathcal{F}_Q(\theta)}$$

where $\mathcal{F}_Q(\theta)$ is the quantum Fisher information (QFI)

- For mixed state and non-unitary dynamics, QFI is difficult to calculate in general
- For pure state $|\psi_\theta\rangle$, $\mathcal{F}_Q(\theta)$ is given by

$$\mathcal{F}_Q(\theta) = 4\left[\langle \partial_\theta \psi_\theta \mid \partial_\theta \psi_\theta \rangle + \left(\langle \partial_\theta \psi_\theta \mid \psi_\theta \rangle\right)^2\right]$$

Quantum TUR for continuous measurement

[Hasegawa, Phys. Rev. Lett., 2020]

- Consider a hypothetical parameter inference in continuous measurement
- Let $\theta \in \mathbb{R}$ be a parameter. Suppose

$$L_m(\theta) = \sqrt{1+\theta} L_m, H(\theta) = (1+\theta) H$$

- From quantum Cramer-Rao inequality

$$\frac{\mathrm{Var}[N]}{(\partial_\theta \langle N \rangle_\theta)^2} \geq \frac{1}{\mathcal{F}_Q(\theta)}$$

$N = \sum_m C_m N_m$

The number of jumps

- We obtain

$$\frac{\mathrm{Var}[N]}{\langle N \rangle^2} \geq \frac{h}{\mathcal{F}_Q(\theta)}$$

h depends on type of continuous measurement

Quantum TUR for continuous measurement

[Hasegawa, Phys. Rev. Lett., 2020]

- Quantum Fisher information for continuous measurement can be calculated via two-sided GKSL equation [Gammelmark & Mølmer, Phys. Rev. Lett., 2014]
- For the jump measurement ($h = 1$)

$$\frac{\mathrm{Var}[N]}{\langle N \rangle^2} \geq \frac{1}{\mathcal{A}(\tau) + \mathcal{B}_q(\tau)}$$ (steady-state condition)

$$\frac{\mathrm{Var}[N]}{\langle N \rangle^2} \geq \frac{1}{\mathcal{A}(\tau)}$$ Classical case

T: time duration

$$\mathcal{A}(\tau) = \tau \sum_m \mathrm{Tr}\left[L_m \rho^{ss} L_m^\dagger\right]$$: frequency of jump

(corresponds to dynamical activity)

$\mathcal{B}_q$: coherent term contribution (difficult to calculate)

$\mathcal{A}(\tau) + \mathcal{B}_q(\tau)$: Quantum dynamical activity

Exact representation of quantum dynamical activity

[Nishiyama & Hasegawa, Phys. Rev. E, 2024]

- In [Hasegawa, Phys. Rev. Lett., 2020], only $\tau \to \infty$ representation was calculated
- In [Nishiyama & Hasegawa, Phys. Rev. E, 2024], we derived its exact representation for arbitrary τ

$$\mathcal{B}(\tau) = \underbrace{\mathcal{A}(\tau)}_{\text{Classical dynamical activity}} + \underbrace{8\int_0^\tau ds_1 \int_0^{s_1} ds_2 \mathrm{Re}\left(\mathrm{Tr}_S\left[H_{\mathrm{eff}}^\dagger \check{H}_S(s_1 - s_2)\rho_S(s_2)\right]\right) - 4\left(\int_0^\tau ds \mathrm{Tr}_S\left[H_S \rho_S(s)\right]\right)^2}_{\text{Coherent dynamics contribution}}$$

Classical dynamical activity

$$\mathcal{A}(\tau) = \tau \sum_m \mathrm{Tr}\left[L_m \rho^{ss} L_m^\dagger\right]$$

Coherent dynamics contribution

Exact representation of quantum dynamical activity

[Nishiyama & Hasegawa, Phys. Rev. E, 2024]

- Upper bound can be derived

$$\mathcal{B}(\tau) \leq \overline{\mathcal{B}}(\tau)$$

$$\overline{\mathcal{B}}(\tau) \equiv \mathcal{A}(\tau) + 8 \int_0^{\tau} ds_1 \sigma_{H_S}(s_1) \int_0^{s_1} ds_2 \sigma_{H_{\mathrm{eff}}}(s_2)$$

$$\sigma_{\mathcal{O}}(s) \equiv \sqrt{\langle (\mathcal{O} - \langle \mathcal{O} \rangle (s))^{\dagger} (\mathcal{O} - \langle \mathcal{O} \rangle (s)) \rangle}$$

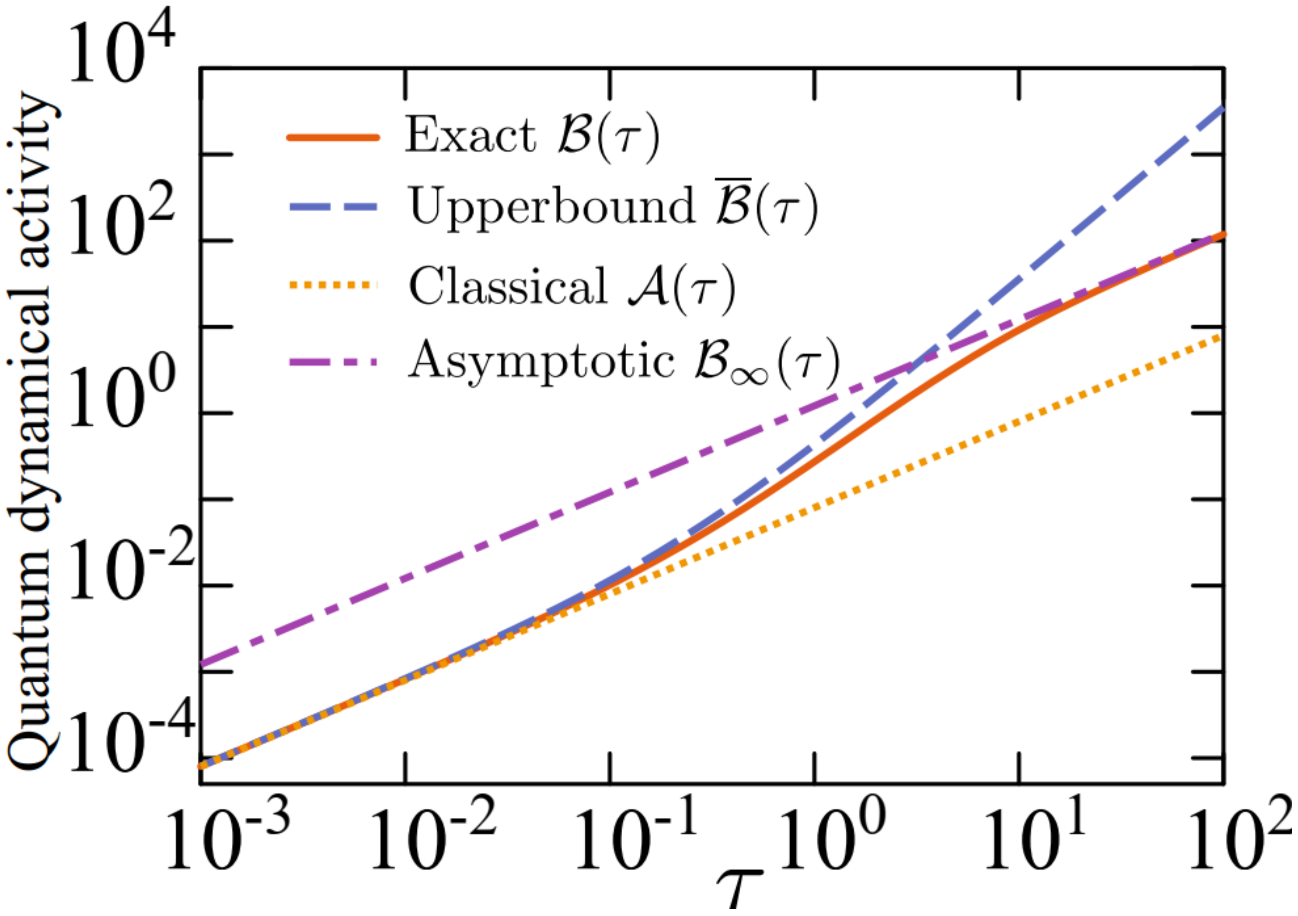
Standard deviation

- The upper bound scales as $O(\tau^2)$

Exact representation of quantum dynamical activity
[Nishiyama & Hasegawa, Phys. Rev. E, 2024]

$$H = \Delta |e\rangle\langle e| + (\Omega/2)\left(|e\rangle\langle g| + |g\rangle\langle e|\right)$$ $|e\rangle$: excited, $|g\rangle$: ground

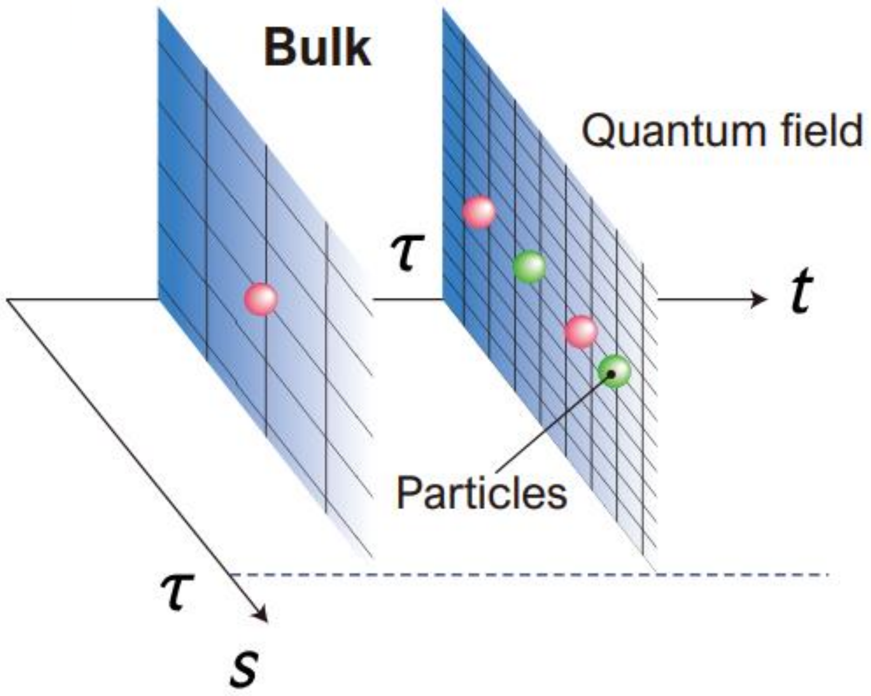
$$L = \sqrt{\kappa}\,|g\rangle\langle e|$$



Heisenberg-Robertson uncertainty relation and TUR

[Hasegawa, Nat. Comm., 2023]

- Consider the Heisenberg-Robertson uncertainty relation in the bulk space



$$[\![\mathcal{X}]\!][\![\mathcal{Y}]\!] \geq \frac{1}{2}|\langle\psi|[\mathcal{X},\mathcal{Y}]|\psi\rangle|$$

- Considering specific $\mathcal{X}$ and $\mathcal{Y}$, it is shown that the uncertainty relation reduces to quantum TUR

Heisenberg-Robertson uncertainty relation and TUR

[Hasegawa, Nat. Comm., 2023]

- Recall that the scaled unitary for the cMPS is

$$\mathcal{U}(t) = \mathbb{T}\exp\left[-i\int_0^\tau ds\left(\frac{t}{\tau}H_{\mathrm{sys}}\otimes\mathbb{I}_{\mathrm{fld}} + \sum_m\left(i\sqrt{\frac{t}{\tau}}L_m\otimes\phi^\dagger(s) - i\sqrt{\frac{t}{\tau}}L_m^\dagger\otimes\phi(s)\right)\right)\right]$$

- Corresponding Hamiltonian can be defined by

$$\mathcal{K}(t) \equiv -i\frac{d\mathcal{U}^\dagger(t)}{dt}\mathcal{U}(t) \qquad \mathcal{U}(t) = \mathbb{T}e^{-i\int_0^t \mathcal{K}(t')dt'}$$

- Let $\mathcal{C}$ be a counting observable. Define its Heisenberg picture:

$$\mathcal{C}(t) = \mathcal{U}^\dagger(t)\mathcal{C}\mathcal{U}(t)$$

Heisenberg-Robertson uncertainty relation and TUR

[Hasegawa, Nat. Comm., 2023]

- Then the Heisenberg-Robertson UR provides a quantum TUR

$$[\![\mathcal{K}(t)]\!][\![\mathcal{C}(t)]\!] \geq \frac{1}{2}|\langle\psi|[\mathcal{K}(t),\mathcal{C}(t)]|\psi\rangle|$$

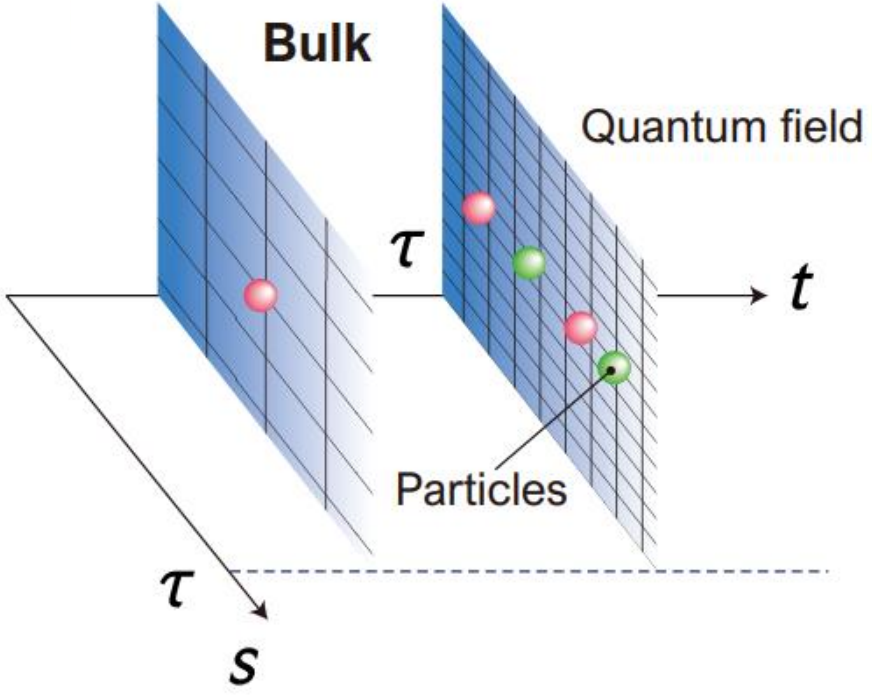
⬇

$$\frac{[\![\mathcal{C}]\!]_\tau^2}{\tau^2\left(\partial_\tau\langle\mathcal{C}\rangle_\tau\right)^2} \geq \frac{1}{\mathcal{B}(\tau)}$$

- It can be seen that the Heisenberg-Robertson uncertainty relation plays an important role not only for QSL but also for TUR.

Application of another uncertainty relation
[Nishiyama & Hasegawa, arXiv:2402.09680]

- Using the scaled cMPS representation, we can identify the continuous measurement as a closed quantum dynamics (i.e., unitary evolution)
- Besides the Heisenberg-Robertson uncertainty relation, we can apply other uncertainty relations to obtain TURs and QSLs in GKSL dynamics



Maccone-Pati uncertainty relation
[Maccone & Pati, Phys. Rev. Lett. 114, 039902 (2015)]

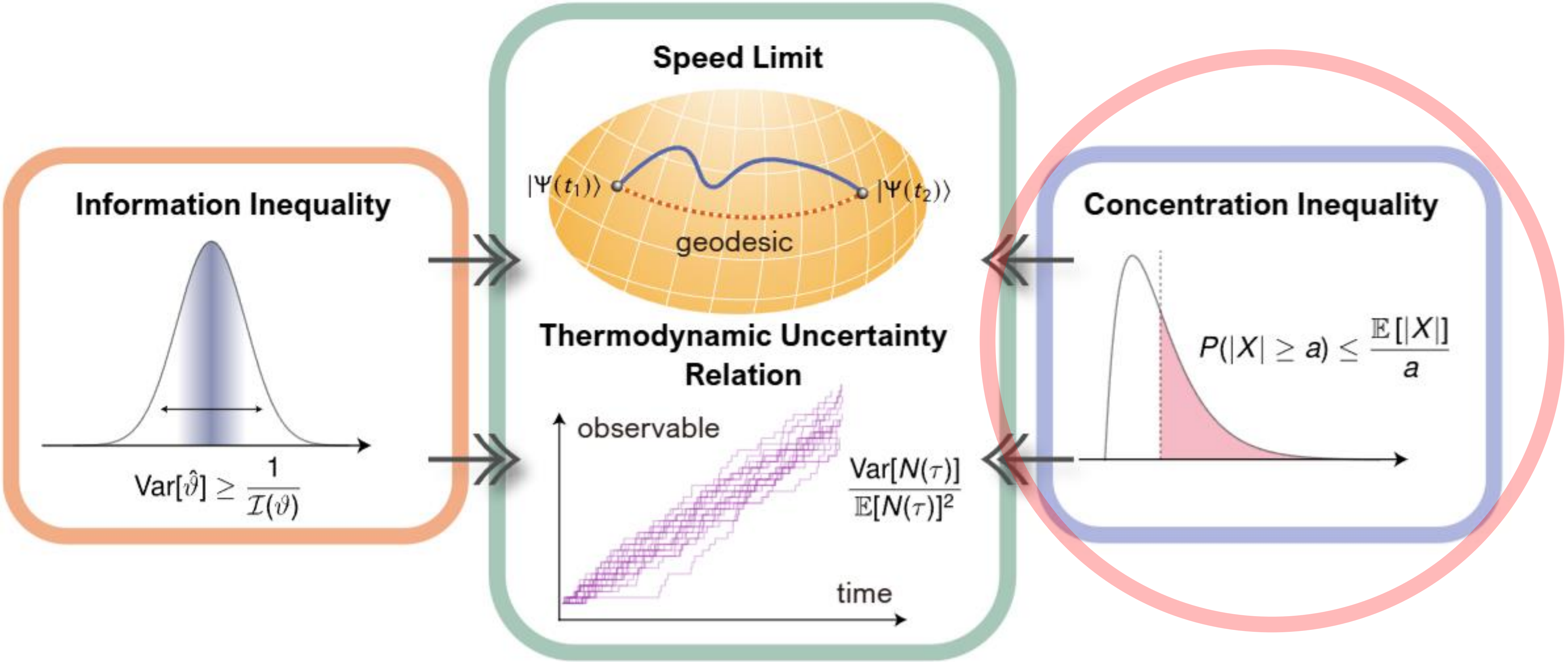
- Maccone and Pati derived an uncertainty relation that is tighter than Heisenberg-Robertson uncertainty relation
- Let A, B be Hermitian operators and $|\overline{\psi}\rangle$ be a state orthogonal to $|\psi\rangle$

$$[\![A]\!]^2 + [\![B]\!]^2 \geq \pm i\langle\psi|[A,B]|\psi\rangle + |\langle\psi|(A \pm iB)|\bar{\psi}\rangle|^2$$

$$[\![A]\!][\![B]\!] \geq \pm \frac{\frac{i}{2}\langle\psi|[A,B]|\psi\rangle}{1 - \frac{1}{2}\left|\left\langle\psi\left|\frac{A}{[\![A]\!]} \pm i\frac{B}{[\![B]\!]}\right|\bar{\psi}\right\rangle\right|^2}$$

$[\![A]\!]$: Standard deviation of A

- Then we derived quantum TURs and QSLs for open quantum dynamics using the Maccone-Pati uncertainty relation



Yoshihiko Hasegawa and Tomohiro Nishiyama, arXiv:2402.12197

Concentration inequality

- Many TURs take advantage of information inequalities such as Cramer-Rao inequality

$$\mathrm{Var}[\hat{\vartheta}] \geq \frac{1}{\mathcal{I}(\vartheta)}$$

- Concentration inequalities constitute another pivotal class of statistical tools.

$$P(|X| \geq a) \leq \frac{\mathbb{E}[|X|]}{a} \qquad P(Z > \theta\mathbb{E}[X]) \geq (1-\theta)^2 \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}$$

- We derived the *thermodynamic concentration inequalities* (TCI) that provide lower bounds for the probability distribution of observables.

Dynamics

- Again, we consider continuous measurement in GKSL equation

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_m \left[L_m \rho L_m^\dagger - \frac{1}{2}\{L_m^\dagger L_m \rho + \rho L_m^\dagger L_m\} \right]$$

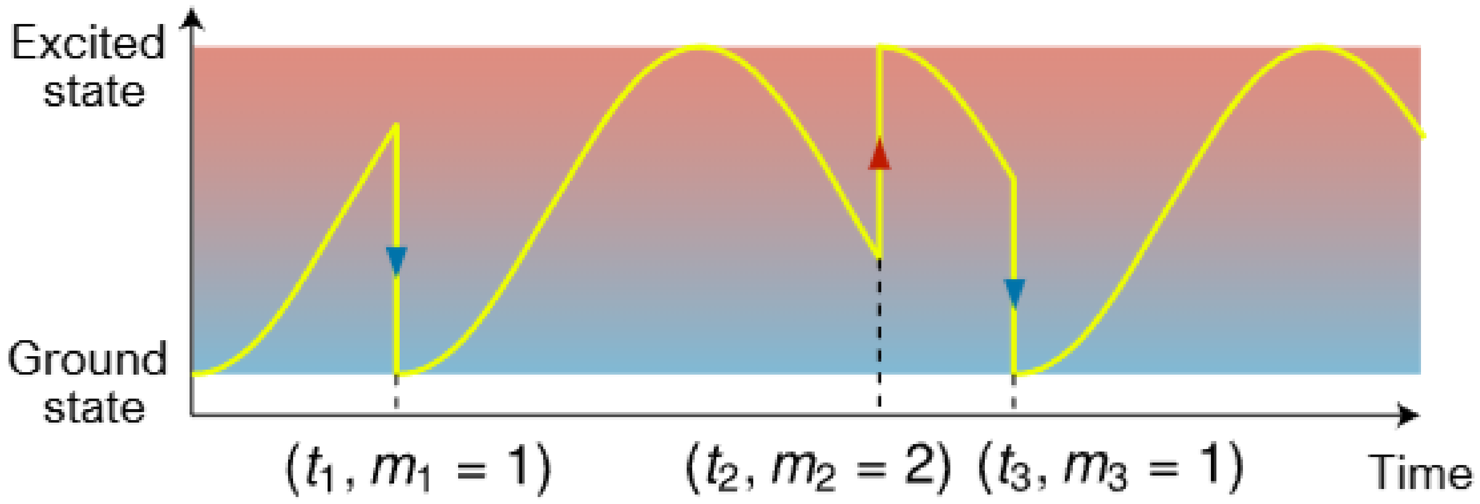
- GKSL equation can recover classical Markov process as a particular case

$$\frac{d}{dt}\mathbf{P}(t) = \mathbf{W}\mathbf{P}(t)$$

where $\mathbf{P}(t)$ is probability distribution and $\mathbf{W}$ is transition rate.

Observable with no-jump condition

- So far, we have considered the counting observable that counts the number of jumps within time interval
- Here, we consider an observable that satisfies "no-jump condition"
- Let ζ be a trajectory of continuous measurement



Trajectory $\zeta_\tau = [(t_1, m_1 = 1), (t_2, m_2 = 2), (t_3, m_3 = 1)]$

Observable with no-jump condition

- Let $N(\zeta)$ be a function of a trajectory ζ
- $N(\zeta)$ can be arbitrary as long as the no-jump condition is met
- The "no-jump condition" is given by

$$N(\zeta_\emptyset) = 0$$

where $\zeta_\emptyset$ is a trajectory with no-jump

- Apparently, this condition is met by the counting observable that counts the number of jump events

Thermodynamic concentration inequality
[Hasegawa & Nishiyama, arXiv:2402.19293]

■ For the observable with the no-jump condition, the following relation holds

$$\cos\left[\frac{1}{2}\int_0^\tau \frac{\sqrt{\mathcal{B}(t)}}{t}dt\right]^2 \leq P(N(\tau)=0)$$ Quantum case

$$e^{-\mathcal{A}(\tau)} \leq P(N(\tau)=0)$$ Classical case

$\mathcal{B}(\tau)$: Quantum dynamical activity
$\mathcal{A}(\tau)$: Classical dynamical activity

Thermodynamic concentration inequality

[Hasegawa & Nishiyama, arXiv:2402.19293]

- Dynamical activities $\mathcal{A}(\tau)$ and $\mathcal{B}(\tau)$ quantify the activity of the system
  - Larger $\mathcal{A}(\tau)$ and $\mathcal{B}(\tau)$ = more jumps, more intense coherent dynamics
- As the dynamical activity increases, the probability $P(N(\tau) = 0)$ decreases.
- By using the thermodynamic concentration inequality, several trade-off relations can be derived

Sketch of derivation

- From MPS representation

$$|\Phi(\tau)\rangle = \sum_{m_{K-1},\cdots,m_0} V_{m_{K-1}} \cdots V_{m_0} |\psi_S(0)\rangle \otimes |m_{K-1},\cdots,m_0\rangle$$

$$= \sum_{\boldsymbol{m}} \mathcal{V}_{\boldsymbol{m}} |\psi_S(0)\rangle \otimes |\boldsymbol{m}\rangle$$

$\mathbf{m} = \mathbf{0}$ is associated with no-jump

- Then the probability of no-jump is

$$\mathfrak{p}(\tau) = \left\langle \psi_S(0) \middle| \mathcal{V}_0^\dagger \mathcal{V}_{\mathbf{0}} \middle| \psi_S(0) \right\rangle$$

$$\begin{aligned} |\langle \Psi(0) \mid \Psi(\tau)\rangle|^2 &= |\langle \psi_S(0)|\mathcal{V}_{\mathbf{0}}|\psi_S(0)\rangle|^2 \\ &\leq \left| \left\langle \psi_S(0) \middle| \mathcal{V}_{\mathbf{0}}^\dagger \mathcal{V}_{\mathbf{0}} \middle| \psi_S(0) \right\rangle \right| \\ &= \mathfrak{p}(\tau). \end{aligned}$$

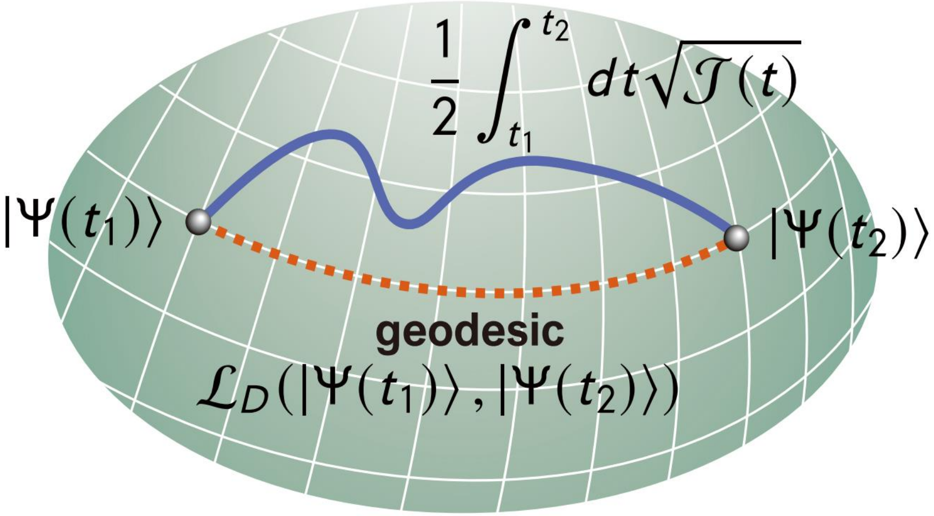
Sketch of derivation

- Next, we obtain a lower bound of the inner product
- From geometric QSL, the inner product and the quantum Fisher information is related via

$$\frac{1}{2}\int_0^\tau \frac{\sqrt{\mathcal{B}(t)}}{t}dt = \frac{1}{2}\int_0^\tau \sqrt{\mathcal{J}(t)}dt \geq \mathbf{arccos}\,|\langle\Psi(\tau)\mid\Psi(0)\rangle|$$

Quantum dynamical activity

Quantum Fisher information



Application: Petrov inequality case

- From the thermodynamic concentration inequality, several trade-off relations can be derived
- Consider the Petrov inequality [V. V. Petrov, J.Stat. Plann. Inference (2007)]

$$P(|X| > b) \geq \frac{(\mathbb{E}[|X|^r] - b^r)^{s/(s-r)}}{\mathbb{E}[|X|^s]^{r/(s-r)}}$$

where $s > r > 0$ and $b > 0$

- We combine the TCI with the Petrov inequality with $b = 0$

Application: Petrov inequality case

- Combining the Petrov inequality with TCI, the following relation holds

$$\frac{\mathbb{E}[|N(\tau)|^s]^{r/(s-r)}}{\mathbb{E}[|N(\tau)|^r]^{s/(s-r)}} \geq \sin\left[\frac{1}{2}\int_0^\tau \frac{\sqrt{\mathcal{B}(t)}}{t}dt\right]^{-2}$$

Quantum case

$$\frac{\mathbb{E}[|N(\tau)|^s]^{r/(s-r)}}{\mathbb{E}[|N(\tau)|^r]^{s/(s-r)}} \geq \frac{1}{1-e^{-\mathcal{A}(\tau)}}$$

Classical case

where $N(\tau)$ is the observable satisfying the no-jump condition.

Application: Petrov inequality case

- For $r = 1$ and $s = 2$

$$\frac{\mathrm{Var}[|N(\tau)|]}{\mathbb{E}[|N(\tau)|]^2} \geq \tan\left[\frac{1}{2}\int_0^\tau \frac{\sqrt{\mathcal{B}(t)}}{t}dt\right]^{-2}$$

- This bound is identical to that derived in [Hasegawa, *Nat. Comm.*, 2023]
- For classical case, the bound becomes

$$\frac{\mathrm{Var}[|N(\tau)|]}{\mathbb{E}[|N(\tau)|]^2} \geq \frac{1}{e^{\mathcal{A}(\tau)} - 1}$$

Application: Markov inequality case

- The reverse Markov inequality states

$$P(X \leq a) \leq \frac{\mathbb{E}[X_{\max} - X]}{X_{\max} - a}$$

where X_{max} is the maximum of X.

- Substituting the bound in the reverse Markov inequality, we have

$$\mathbb{E}[|N(\tau)|] \leq N_{\max} \sin \left[\frac{1}{2} \int_0^\tau \frac{\sqrt{\mathcal{B}(t)}}{t} dt \right]^2$$

$$\mathbb{E}[|N(\tau)|] \leq N_{\max} \left(1 - e^{-\mathcal{A}(\tau)}\right)$$

- This provides upper bound on the expectation

Integral probability metric

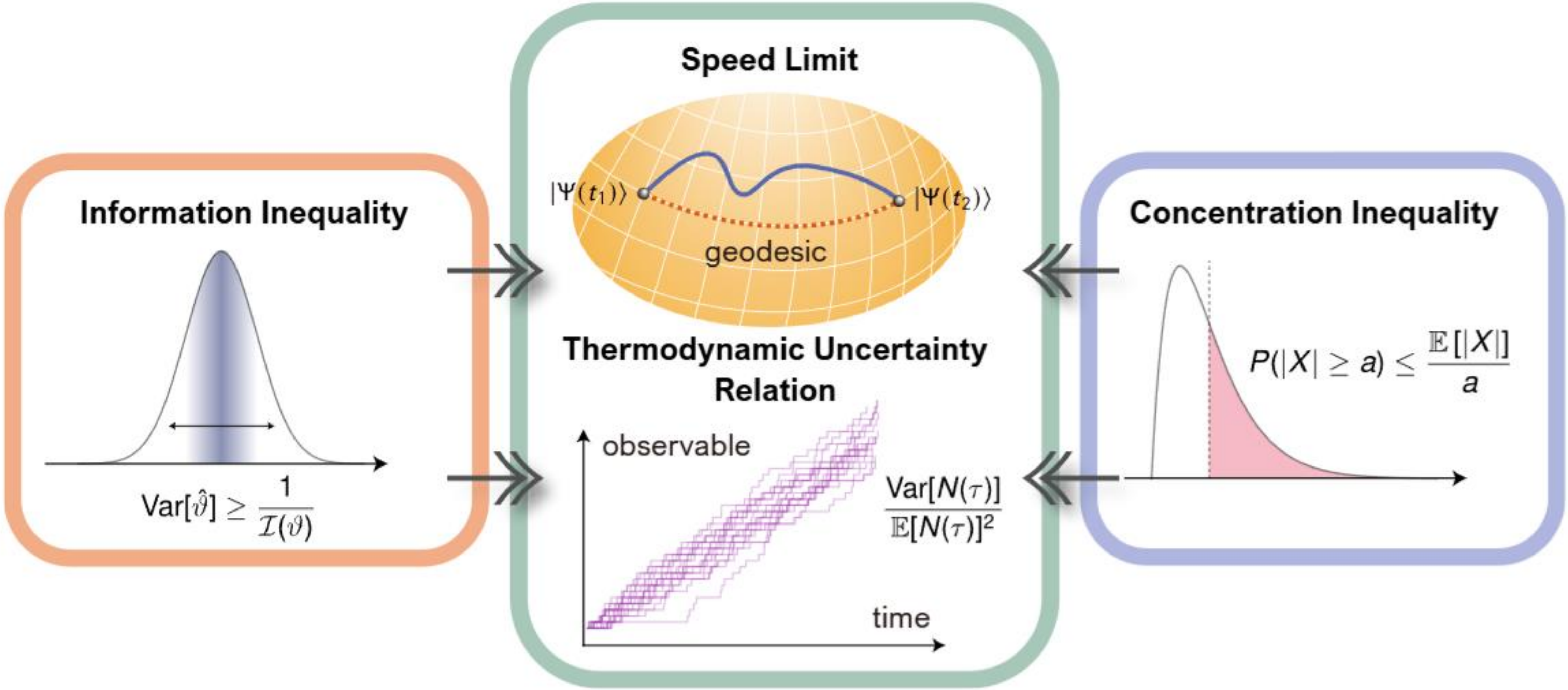
- Integral probability metric (IPM) is defined by

$$D_{\mathcal{F}}(\mathfrak{P}, \mathfrak{Q}) \equiv \max_{f \in \mathcal{F}} |\mathbb{E}_{\mathfrak{P}}[f(X)] - \mathbb{E}_{\mathfrak{Q}}[f(Y)]|$$

- IPM becomes total variation distance or Wasserstein-1 distance for particular set $\mathcal{F}$
- IPM is recently used in trade-off relations [Kwon et al. arXiv:2311.01098 (2023)]
- Combining the IPM with the thermodynamic concentration inequality, we have

$$D_{\mathcal{F}}(\mathbf{P}(\tau), \mathbf{P}(0)) \leq F_{\max}\left(1 - e^{-\mathcal{A}(\tau)}\right)$$

Conclusion



Information Inequality
$\mathrm{Var}[\hat{\vartheta}] \geq \frac{1}{\mathcal{I}(\vartheta)}$
Speed Limit
$|\Psi(t_1)\rangle$
$|\Psi(t_2)\rangle$
geodesic
Thermodynamic Uncertainty Relation
observable
time
$\frac{\mathrm{Var}[N(\tau)]}{\mathbb{E}[N(\tau)]^2}$
Concentration Inequality
$P(|X| \geq a) \leq \frac{\mathbb{E}[|X|]}{a}$