Random quantum circuit toolbox for the many-body problems

Frontiers in Non-Equilibrium Physics 2024 **09.07.2024**
Yukawa Institute for Theoretical Physics

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Plan of the talk

- Motivation and background
- The random unitary circuit toolbox
- Examples

In collaboration with

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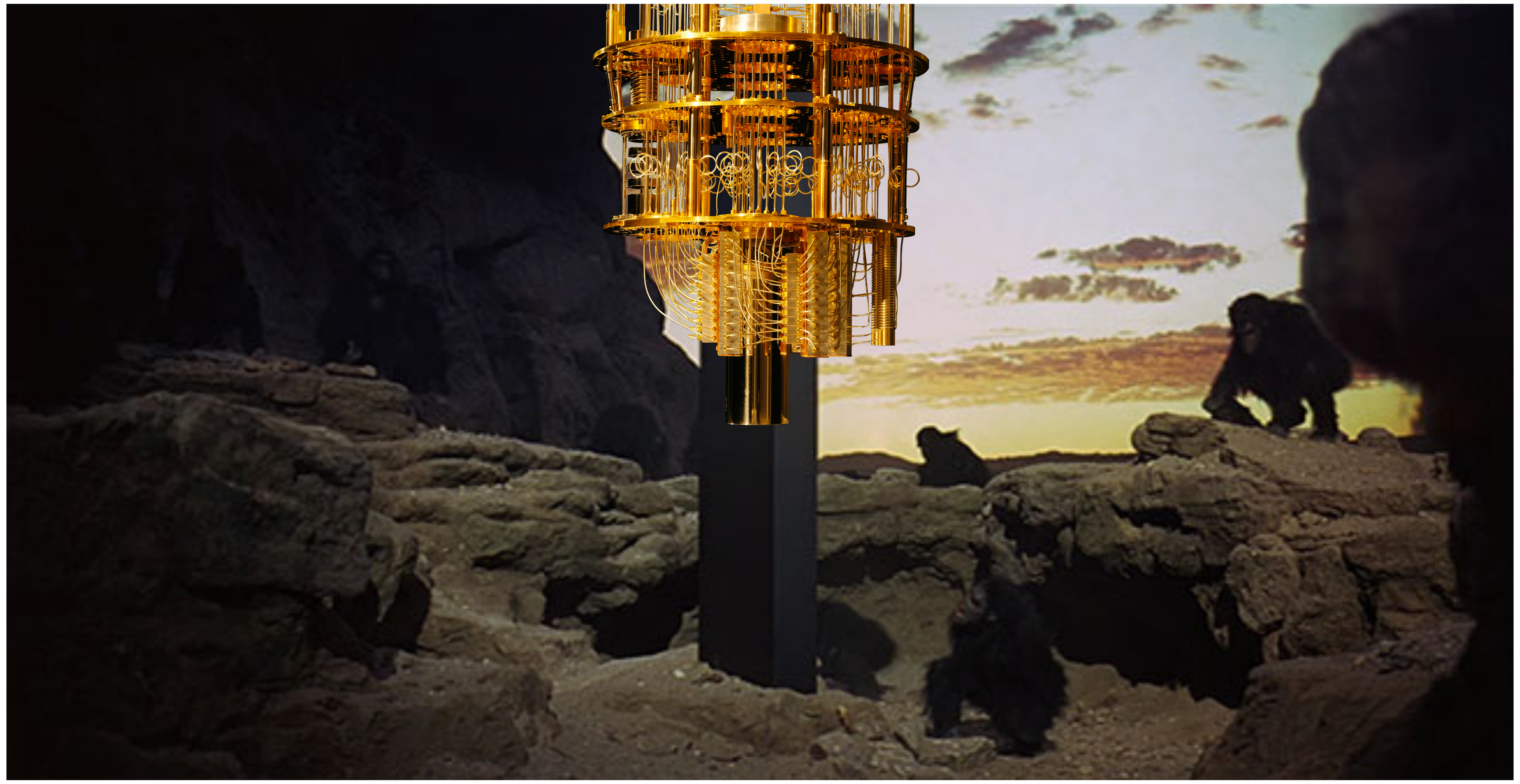
XT, P. Sierant, Phys. Rev. Lett. **132**, 140401 (2024)
XT, P. Sierant, Entropy, **26**, 471 (2024)
XT, A. De Luca, P. Calabrese, arXiv:2405.14514
XT, E. Tirrito, P. Sierant, arXiv: 2407.03929

Noisy intermediate scale quantum devices



Unprecedented control in manipulating **many-body** systems (and measurements)

$$i\hbar\partial_t|\Psi\rangle = H|\Psi\rangle$$

Understanding how quantum resources evolve is a key question

Quantum information dynamics is challenging

Quantum resources are *non-linear* objects in the system state, system sizes are *exponentially large* with the degrees of freedom.

These same resources sometime constrain the numerical efficiency of simulation algorithms.

Example: $S_k^{\mathrm{ent}} = \dfrac{1}{1-k} \log \mathrm{tr}(\rho_A^k)$ $\qquad \rho_A = \mathrm{tr}_B |\Psi\rangle\langle\Psi|$

It limits the computational efficiency of simulating a given state via tensor networks. Bond dimension $\chi \sim \exp(S_1^{\mathrm{ent}})$, and $S_1^{\mathrm{ent}} \propto t$ for generic evolutions.

U. Schollwock, Annals of Physics 326, 96 (2011)

Desiderata

Obtain quantitative, possibly analytical, insights on quantum dynamics for large systems and *any given evolution*.

but

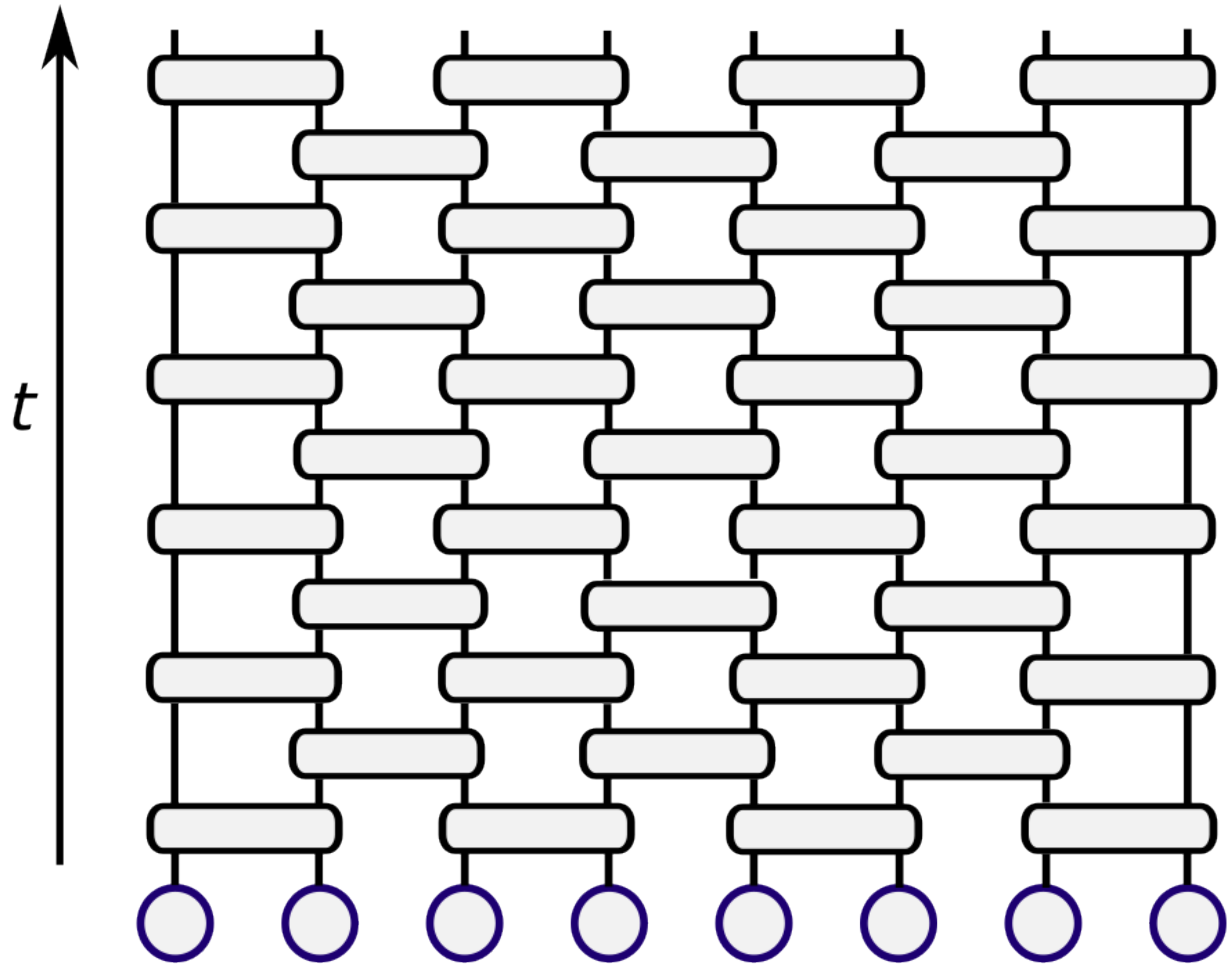
Hamiltonians are complicated. Their fine structure determine the effective degrees of freedom, quasiparticles and their interactions and decays, etc.

Desiderata

Obtain quantitative, possibly analytical, insights on quantum dynamics for large systems and ~~*any given evolution*~~ for typical quantum evolution, based solely on minimal principles:

1. Locality
2. Unitarity (for noisy dynamics complete-positive-trace-preserving-ness)
3. Symmetry
4. Topology
5. ….

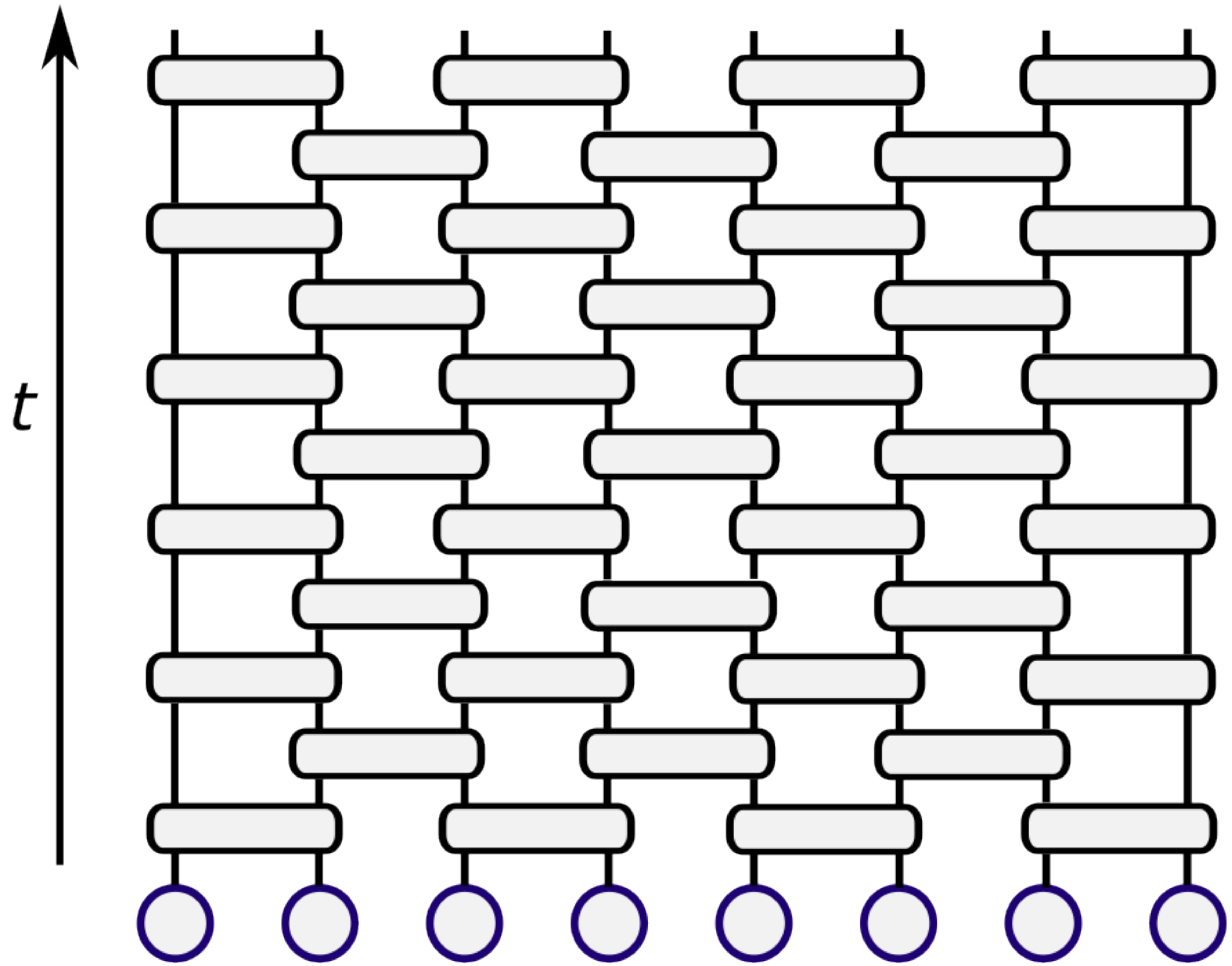
Random quantum circuits



- Typicality: each gate is randomly iid from ensemble $\mathscr{E}$
- Locality: The gates act on a given geometry and architecture
- Unitarity: The ensemble $\mathscr{E}$ is a subspace of $\mathscr{U}$ unitary gates
- Symmetry: The elements U of $\mathscr{E}$ commute with each symmetry generator G, i.e. $[U, G] = 0$

M. Fisher, et al. Annual Review of Condensed Matter Physics 14, 335 (2023)

Random quantum circuits



M. Fisher, et al. Annual Review of Condensed Matter Physics 14, 335 (2023)

- A single realisation of the circuit is **exponentially costy** to simulate (for generic chaotic systems)
- It is **ensemble-averaging** that leads to substantial simplifications: emergent statistical mechanics description and tensor network simulability.

Receipt

We focus on a particular class of operators which are written for $\rho = U\rho_0 U^\dagger$

$$\mathcal{A} = -\log\left[\sum_{W\in\mathcal{W}} \mathrm{tr}\left(WU|\Psi\rangle\langle\Psi|U^\dagger)^m\right)\right] \quad \mathcal{B} = -\log\left[\mathrm{tr}\left([\mathrm{tr}_X(U|\Psi\rangle\langle\Psi|U^\dagger)]^m\right)\right]$$

Up to pre-factor, these formulae include as magic measures and participation entropies, and entanglement entropy, etc.

The general receipt for analytical insights and efficient tensor network representation requires three ingredients

1. The replica trick
2. Self-averaging of the ensemble
3. The Weingarten calculus for $\mathscr{E}$

A. Nahum, et al. Phys. Rev. X **7**, 031016 (2017)
A. Nahum, et al. Phys. Rev. X **8**, 021014 (2018)
XT, P. Sierant, Phys. Rev. Lett. **132**, 140401 (2024)

Receipt: observable and replica-trick

Using properties of the tensor product we find

$$\mathcal{A} = -\log\left[\sum_{W\in\mathcal{W}} \operatorname{tr}\left(W^{\otimes m}(U|\Psi\rangle\langle\Psi|U^\dagger)^{\otimes m})\right)\right]$$

Consider the doubled space $\rho \mapsto |\rho\rangle\rangle$, $U\rho U^\dagger \mapsto U \otimes U^\dagger |\rho\rangle\rangle$, $\operatorname{tr}(A^\dagger B) = \langle\langle A | B \rangle\rangle$. Simple algebra leads to

$$\mathcal{A} = -\log\left[\langle\langle\mathcal{V}|(U\otimes U^*)^{\otimes m})|\Psi\rangle\rangle^{\otimes 2m}\right]$$

Where we have defined the operator

$$\mathcal{V} = \sum_{W\in\mathcal{W}} W^{\otimes m}$$

Receipt: self-averaging

We have $U = \prod_t \prod_{i,i+1} U_{i,i+1,t}$ the time evolution of the circuit, with each $U_{i,i+1,t} \in \mathscr{E}$. Since this is a stochastic space, we can consider multiple proxies

$$\mathcal{A} = -\log\left[\langle\langle\mathcal{V}|(U \otimes U^*)^{\otimes m})|\Psi\rangle\rangle^{\otimes 2m}\right]$$

Quench average $$\bar{\mathcal{A}} = \mathbb{E}_{\mathcal{E}}(\mathcal{A})$$

Annealed average $$\tilde{\mathcal{A}} = -\log \mathbb{E}_{\mathcal{E}}\langle\langle\mathcal{V}|(U \otimes U^*)^{\otimes m}|\Psi\rangle\rangle^{\otimes 2m}$$

In general the results for $\bar{\mathscr{A}}$ and $\tilde{\mathscr{A}}$ can be different. However, if the measure in $\mathscr{E}$ is concentrated, then self-averaging emerge. E.g., when $\mathscr{E} = \mathscr{U}$ the unitary group with Haar measure

Receipt: Weingarten calculus

The annealed average translate into a tensor network by linearity

$$\tilde{\mathcal{A}} = -\log\langle\langle\mathcal{V}|\mathbb{E}_{\mathcal{E}}[(U\otimes U^*)^{\otimes m}]|\Psi\rangle\rangle^{\otimes 2m}$$

This requires the calculation of

$$\mathbb{E}[(U_{i,i+1}\otimes U_{i,i+1})^{\otimes m}] = \sum_{T_1,T_2\in \mathrm{Comm}} \mathrm{Wg}_{T_1,T_2} R(T_1)R(T_2)$$

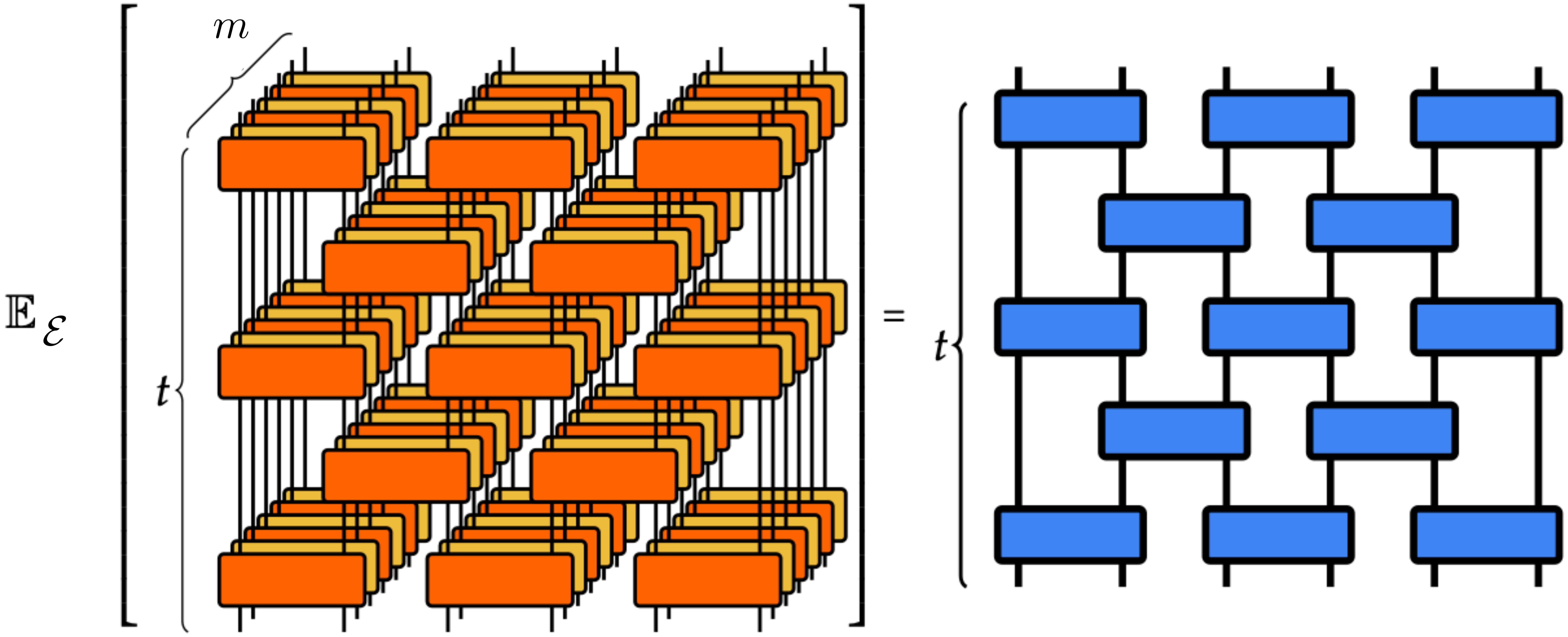
Where Comm is the set of operators O such that $[O, U^{\otimes m}] = 0$ and $R(T_\alpha)$ is a representation of T_α.

D. Gross, S. Nezami, M. Walter, Communications in Mathematical Physics 385 (2021), 1325-1393

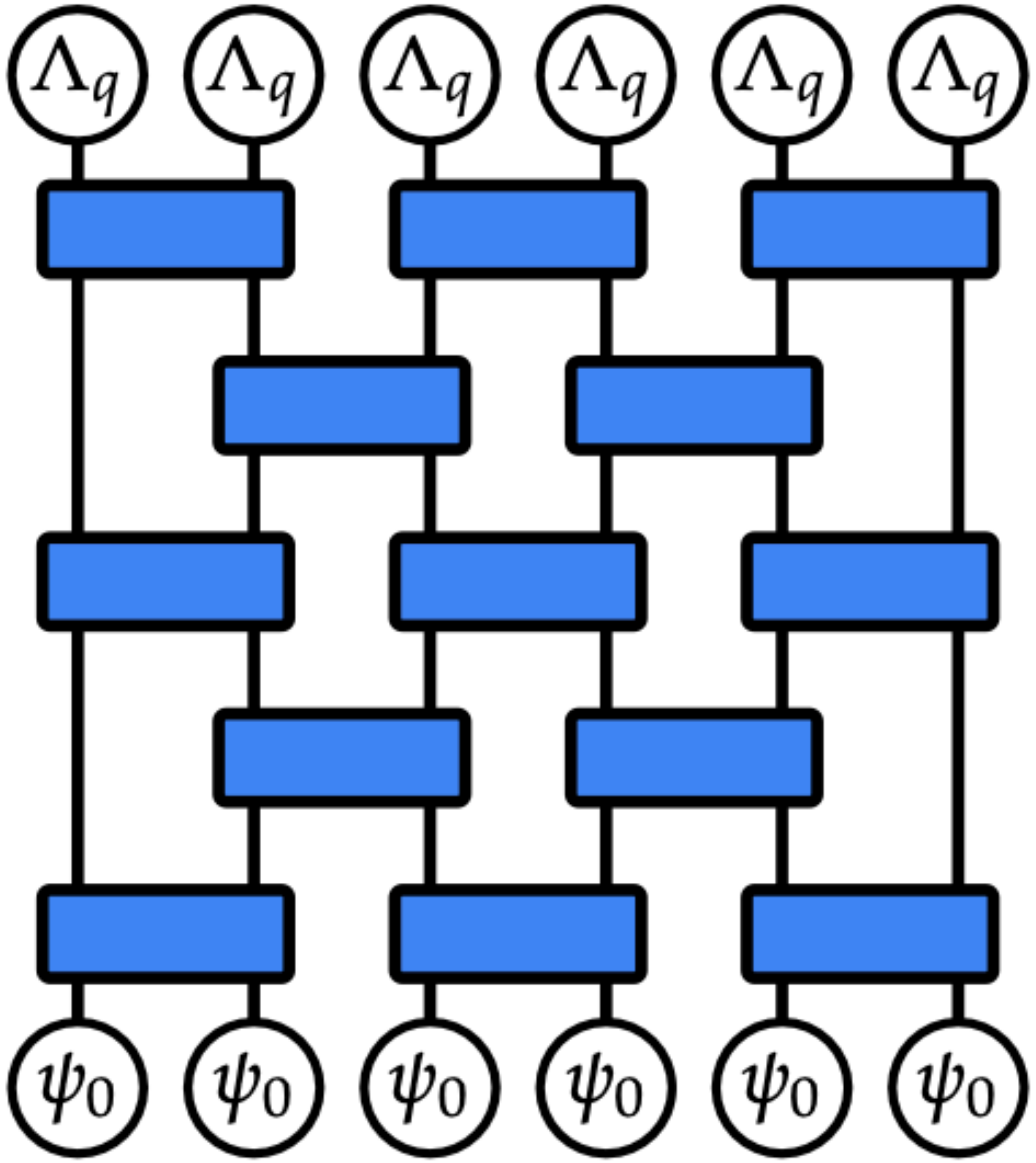
Receipt: Weingarten calculus

The annealed average translate into a tensor network by linearity

$$\tilde{A} = -\log \langle\langle \mathcal{V} | \mathbb{E}_{\mathcal{E}}[(U \otimes U^*)^{\otimes m}] | \Psi \rangle\rangle^{\otimes 2m}$$



Efficient tensor network implementation



- Everything now maps to the problem of computing this partition function/ tensor network contraction.
- Caveat: the qubit dimension is fixed by the dimension of the commutant space. E.g., for random Haar gates this is $m!$
- The choice of operator to study fixes mostly the boundary conditions.

Showcase of examples:

- Entanglement growth
- Anticoncentration aka Hilbert space delocalization
- Magic propagation

} No symmetry
1+1 architecture

- Quantum Mpemba effect

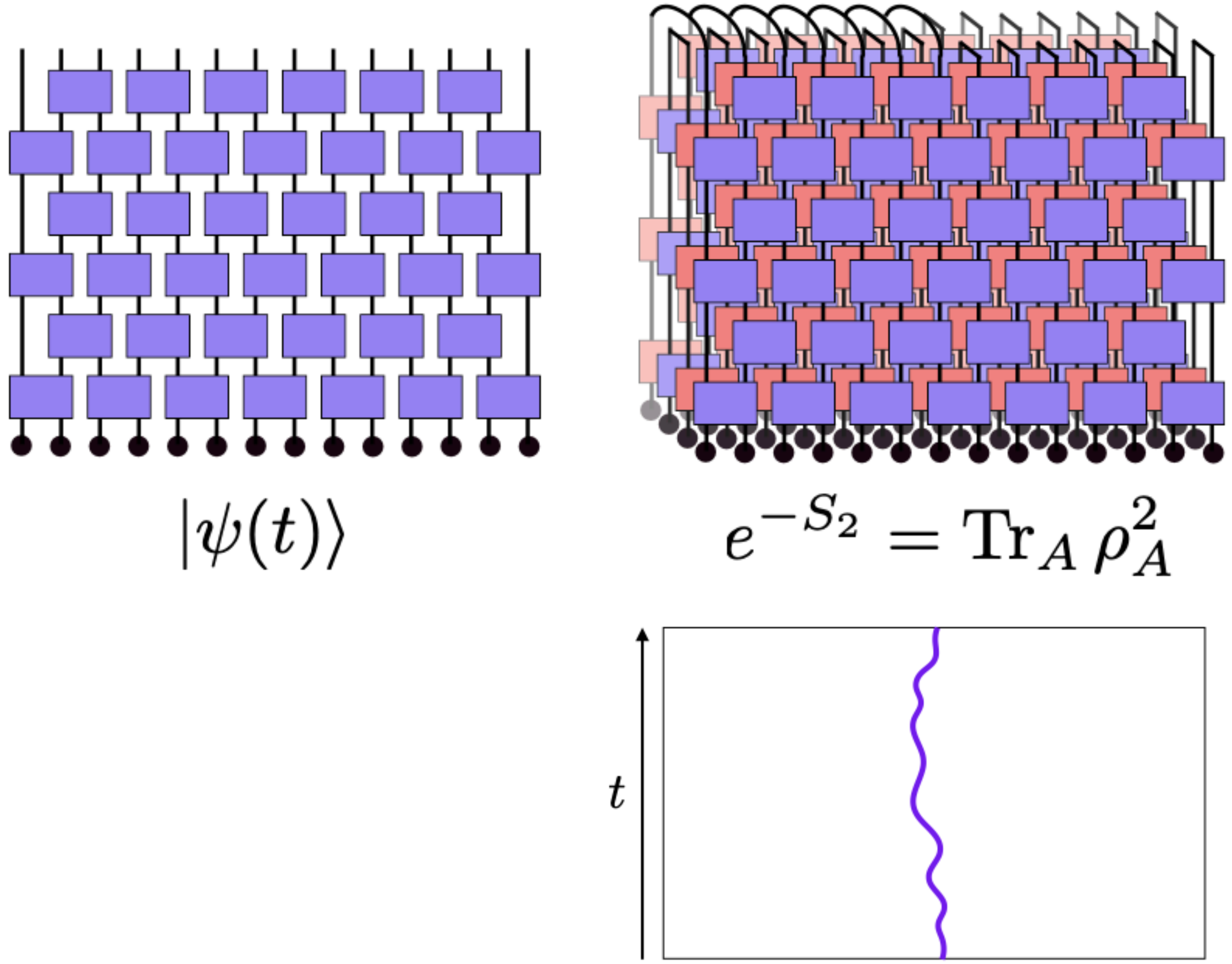
U(1) symmetry
1+1 Architecture

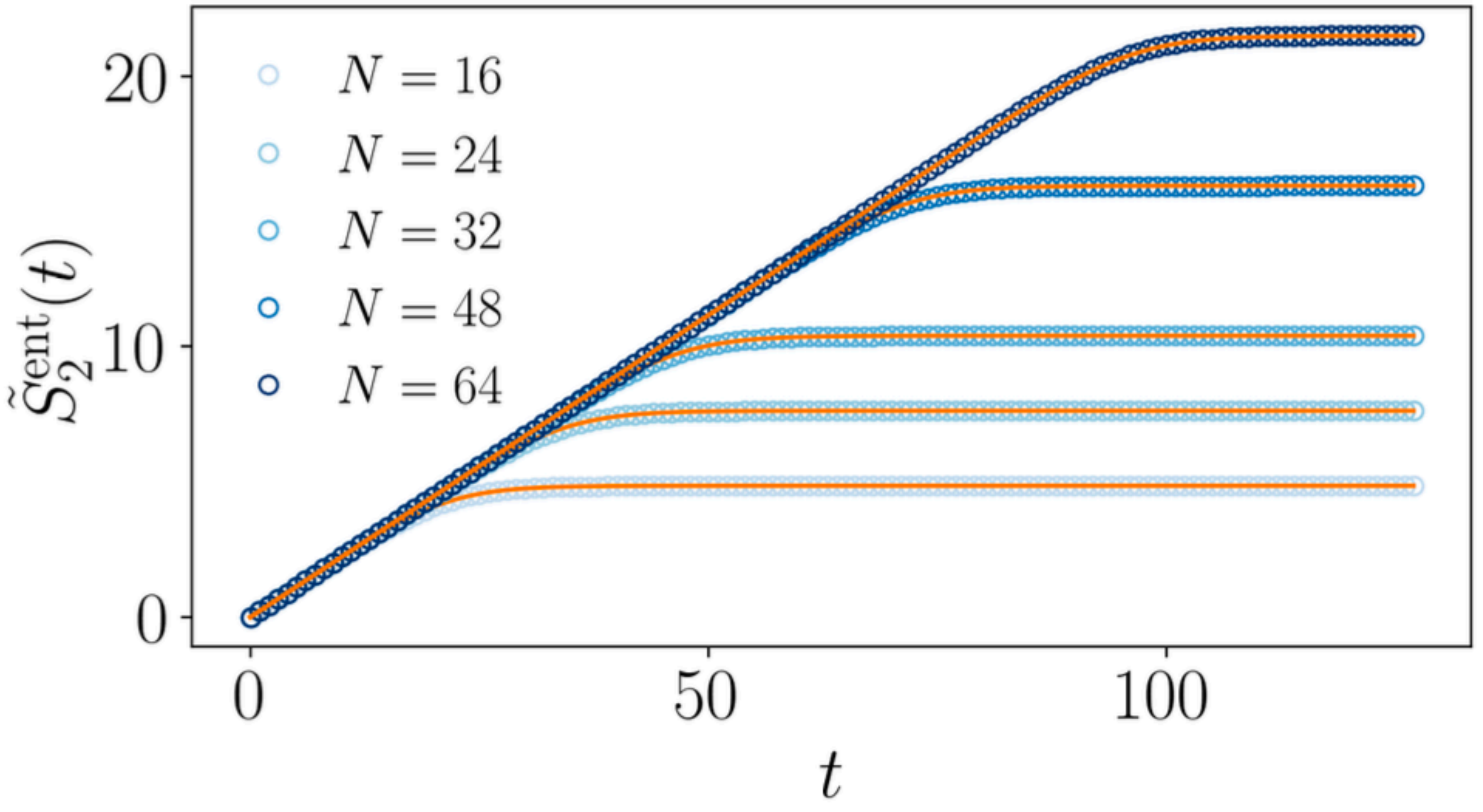
- Error-resilient phase transitions

No symmetry
all-to-all architecture

Renyi-2 entanglement entropy

Requires the calculation of $S_2 = -\log \mathrm{tr}(\rho_A^2)$ for a bipartition of my system $A \cup B$. The problem becomes that of a boundary defect





$$S_2 = -\log\left[(2K_d)^t \sum_{s=t+1}^{\infty} u_{N_A,s} + \sum_{s=0}^{t} (2K_d)^s u_{N_A,s}\right] \qquad K_d = 2d/(d^2+1)$$

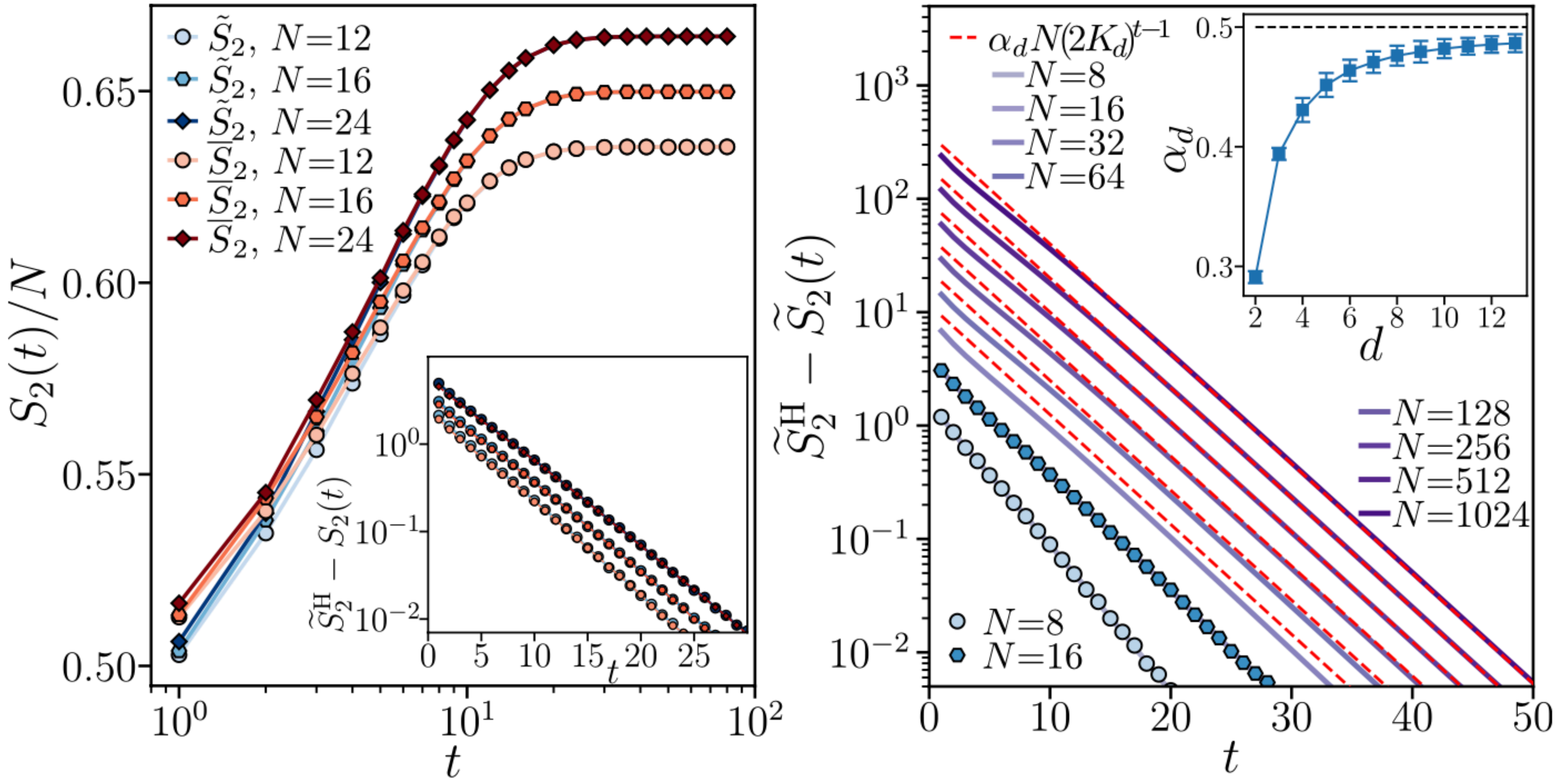
$$u_{z,t} = \sum_{\nu=0}^{N/2-1} \frac{2}{N} \sin\left(\frac{\pi(2\nu+1)}{N}\right) \cos^{t-1}\left(\frac{\pi(2\nu+1)}{N}\right) \sin\left(\frac{\pi(2\nu+1)z}{N}\right)$$

A. Nahum, et al. Phys. Rev. X **7**, 031016 (2017)
XT, P. Sierant, Entropy, **26**, 471 (2024)

Participation entropy

Measures how much the system is localised in the basis $\mathscr{B}$. We consider $\{|n\rangle\}$ the computational basis. Then the inverse participation ration and participation entropy are

$$I_q \equiv \sum_{n\in\mathcal{B}} (\langle n|\rho|n\rangle)^q = \sum_{n\in\mathcal{B}} p_n^q\,, \qquad S_q = \frac{1}{1-q}\ln[I_q]$$

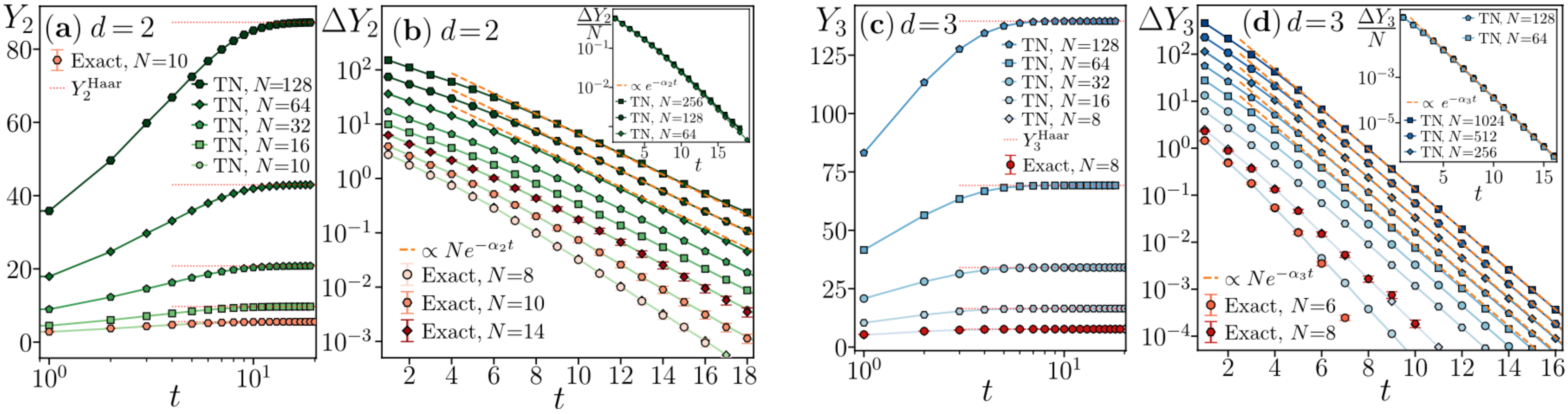


XT, P. Sierant, Entropy, **26**, 471 (2024)

Stabilizer and CSS entropy

Scalable measures of magic (nonstabilizerness). We consider two examples, for qubits (4 replica) and qutrits (3 replica)

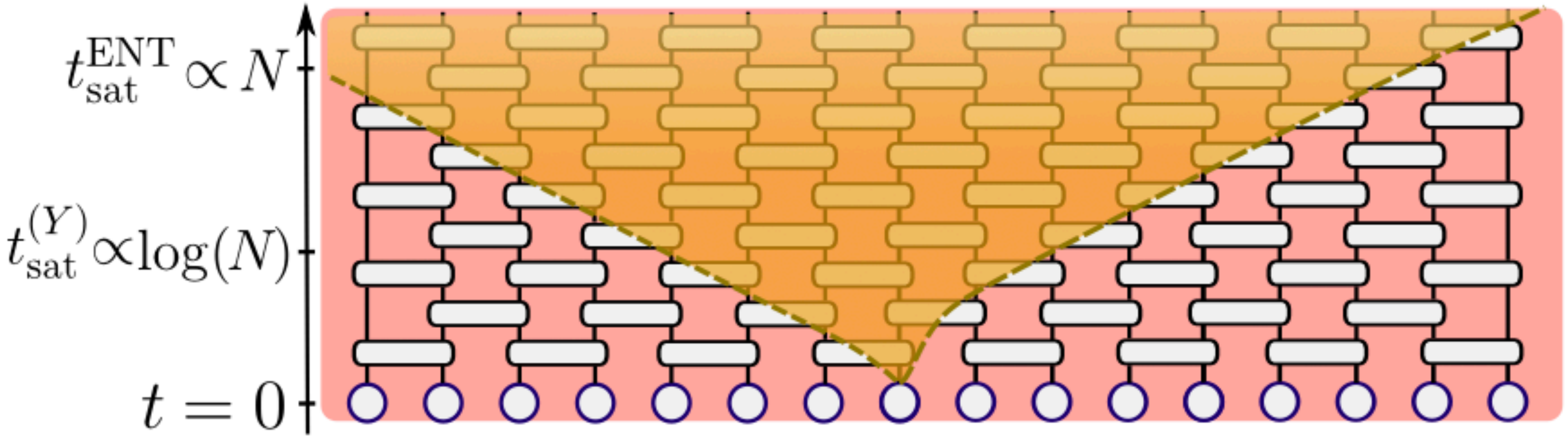
$$Y_2 \equiv -\log \sum_{P\in\mathcal{P}_N} \frac{\langle \Psi_t|P|\Psi_t\rangle^4}{2^N} \qquad Y_3 \equiv -\log \sum_{P\in\mathcal{P}_N} \frac{\langle \Psi_t|P|\Psi_t\rangle^3}{3^N}$$



XT, E. Tirrito, P. Sierant, arXiv: 2407.03929

Lessons

A main result is that the timescale for both Hilbert space delocalisation ("anticoncentration") and magic spreading is logarithmic in system size $\tau \sim \log(N)$, while entanglement requires timescales $\tau \sim N$ to saturate.

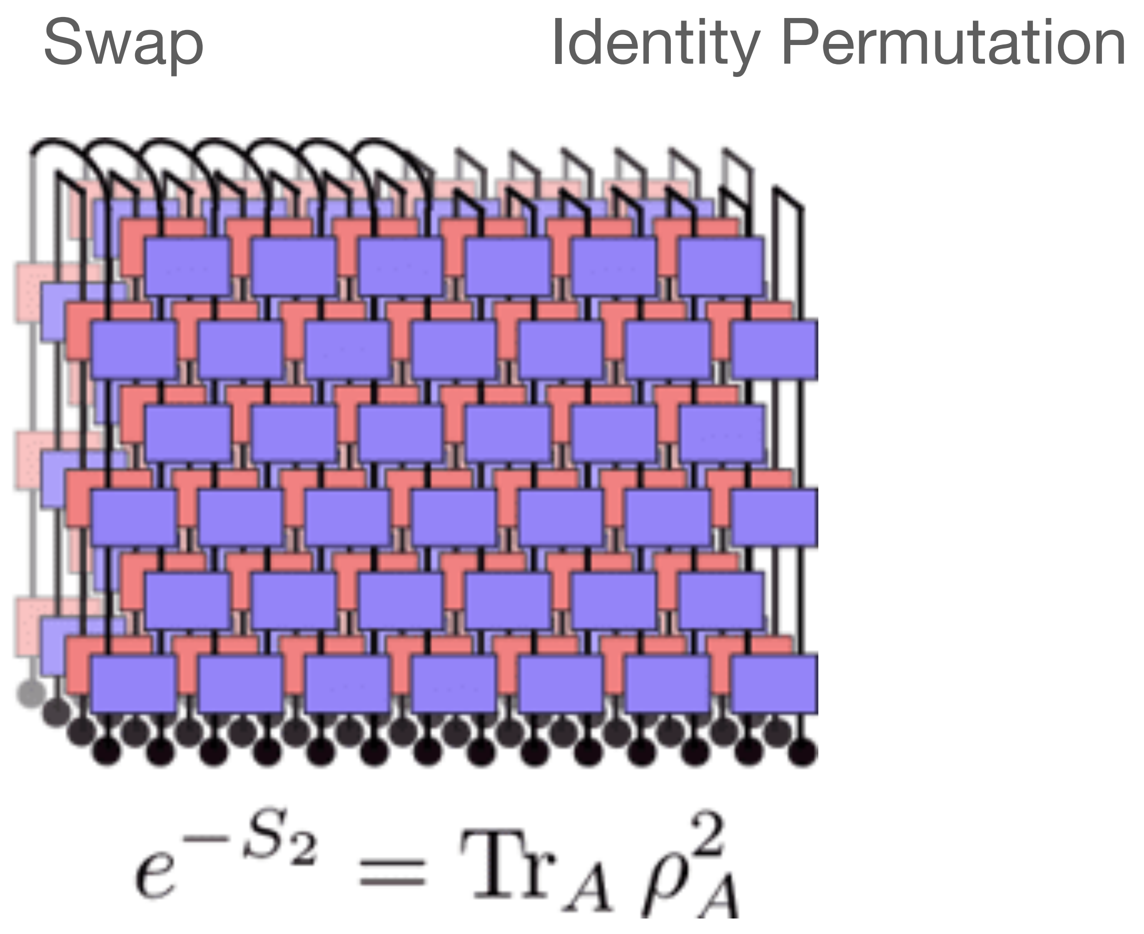


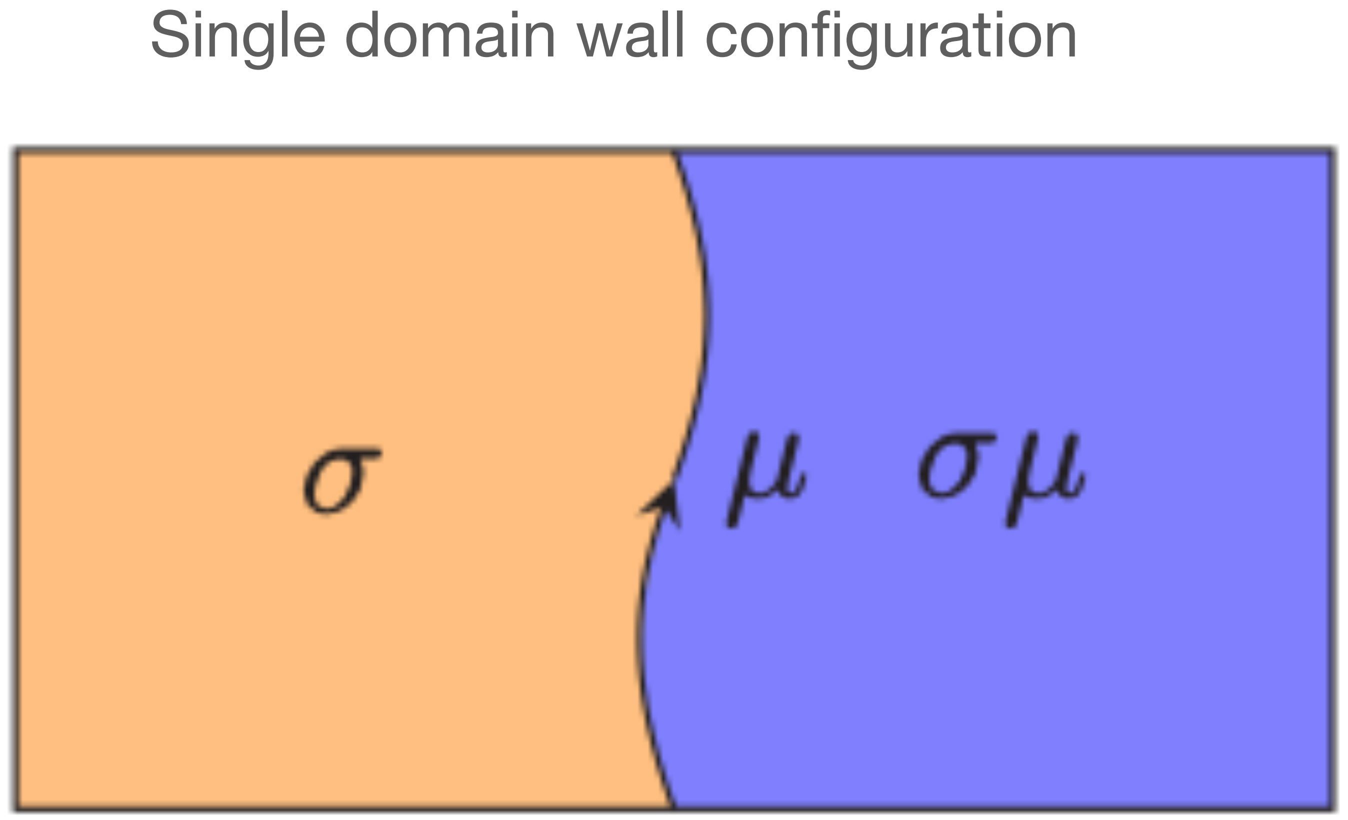
XT, P. Sierant, Entropy, **26**, 471 (2024)

XT, E. Tirrito, P. Sierant, arXiv: 2407.03929

Qualitative explanation

Entanglement





Localization features correspond instead of free (quasi-free) boundary conditions. The sum over all possible configurations of domain walls among all $d!$ permutations.
Random strings in $\mathbb{Z}_d \to$ Domain walls annihilate faster $\to$ In large d this gives an exp. decay.

T. Zhou, A. Nahum, Phys Rev B **99**, 174205 (2019)
XT, P. Sierant, Entropy, **26**, 471 (2024)

Quantum Mpemba effect

A non trivial example is when symmetry is present. Symmetry increase the dimension of the commutant space.

A case of study is the Quantum Mpemba effect. Roughly this states that the farther from the equilibrium, the fastest I equilibrate. While non-universal, its appearance is ubiquitous.

For isolated quantum systems, a version of Mpemba effect has been recently discussed in the framework of asymmetry.

Quantum circuits allow to unveil the origin of the Mpemba physics for generic chaotic (isolated) systems.

E B Mpemba and D G Osborne 1969 Phys. Educ. **4** 172
M. Moroder, O. Culhane, K. Zawadzki, J. Goold, arXiv: 2403.16959
F. Ares, S. Murciano, P. Calabrese Nat Commun **14**, 2036 (2023)

Quantum Mpemba effect in random circuits

The evolution $U_t = \prod_s \prod_{i,i+1} U_{i,i+1,t}$ preserves the global $U(1)$ symmetry generated by the operator $Q = \sum_i Z_i$. Consider a state that breaks the $U(1)$ symmetry, like the tilted ferromagnet $|\Psi(\theta)\rangle = \bigotimes_{i=1}^{N} e^{-i\theta Y_i}|0\rangle$.

For any bipartition, $\rho_A \mapsto_{t\to\infty} \mathbb{I}/2^{N_A}$ because of simple thermodynamic principles. In particular, it restores the symmetry. How fast is this restoration obtained?

S. Liu, H.-K. Zhang, S. Yin, S.-X. Zhang, arXiv:2403.08459
XT, A. De Luca, P. Calabrese, arXiv:2405.14514

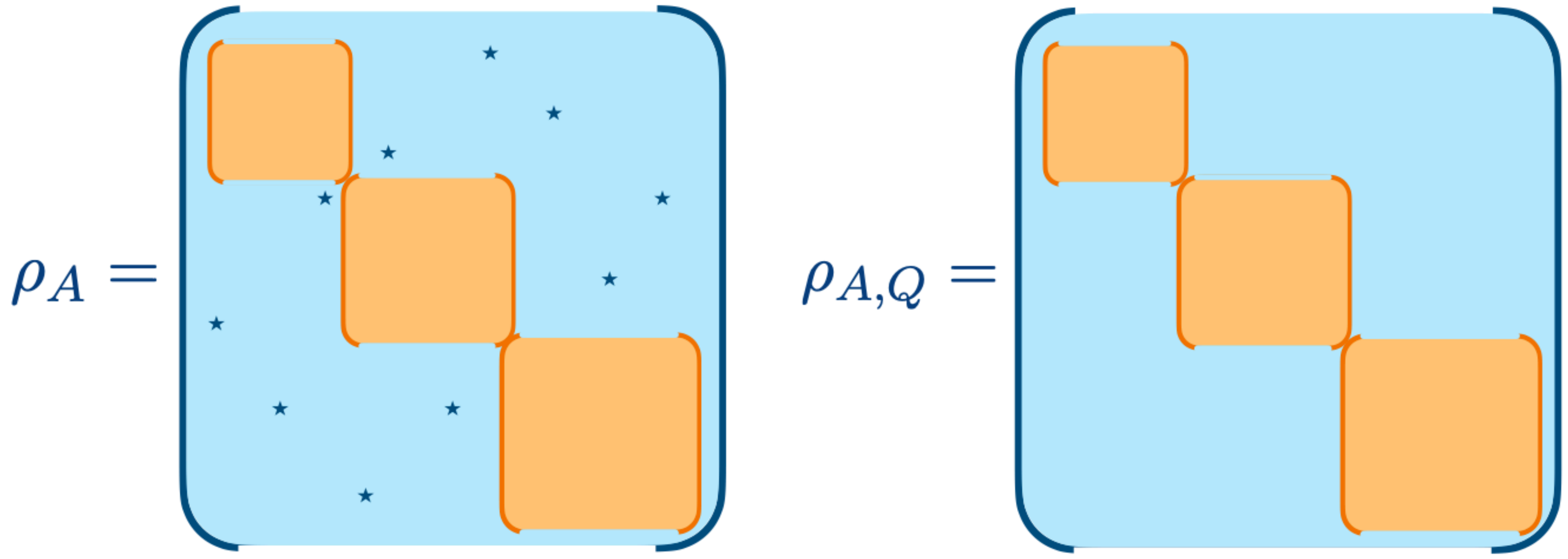
Quantum Mpemba effect in random circuits

Entanglement asymmetry

$$\Delta S_A^{(n)}(\rho_A) = S_n(\rho_{A,Q}) - S_n(\rho_A)$$

$$\rho_A = \mathrm{tr}_B U_t |\Psi(\theta)\rangle\langle\Psi(\theta)| U_t^\dagger$$

$$\rho_{A,Q} = \sum_{N_A} \Pi_{N_A} \rho_A \Pi_{N_A}$$

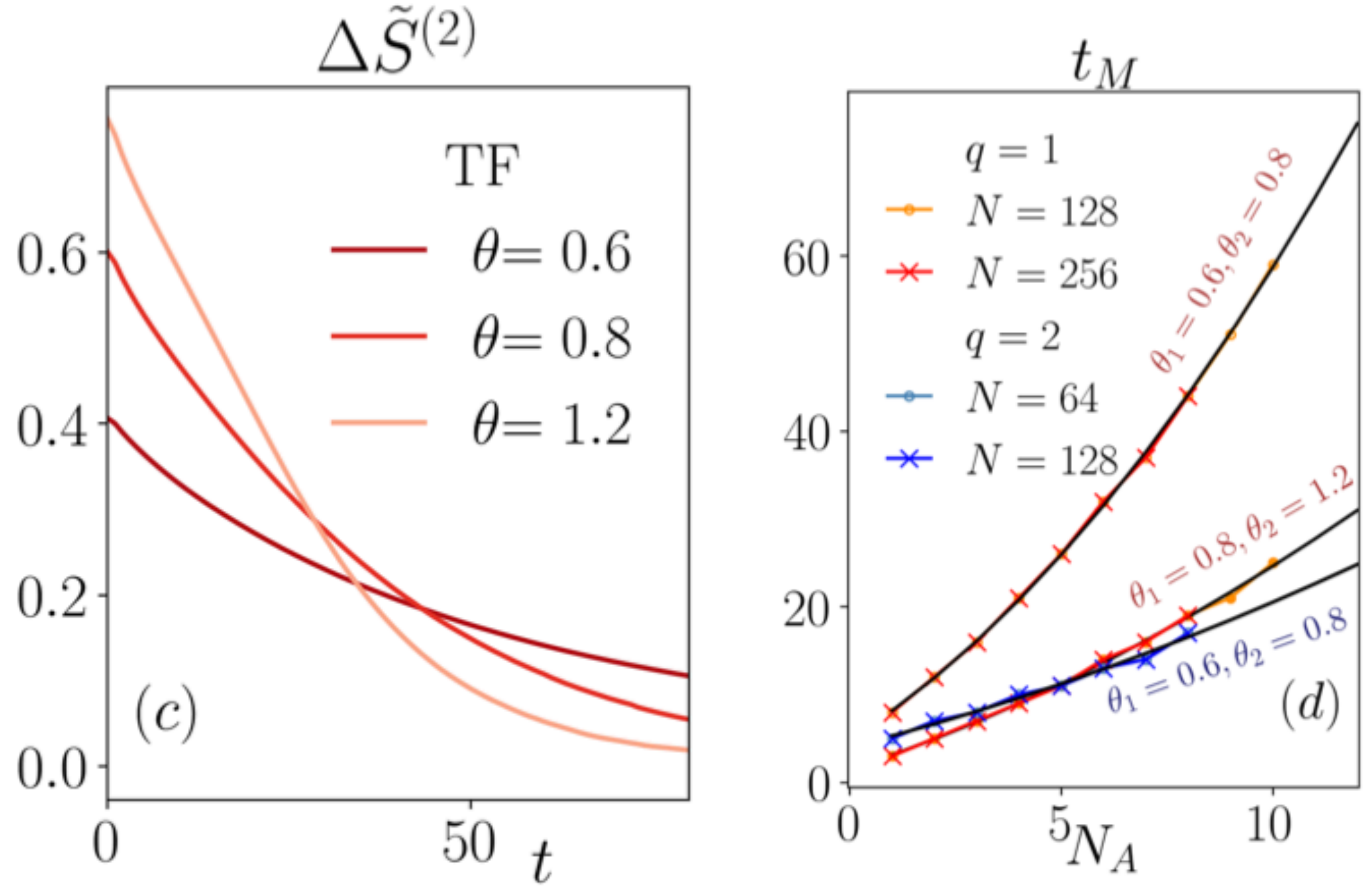


F. Ares, S. Murciano, P. Calabrese Nat Commun **14**, 2036 (2023)

Quantum Mpemba effect in random circuits

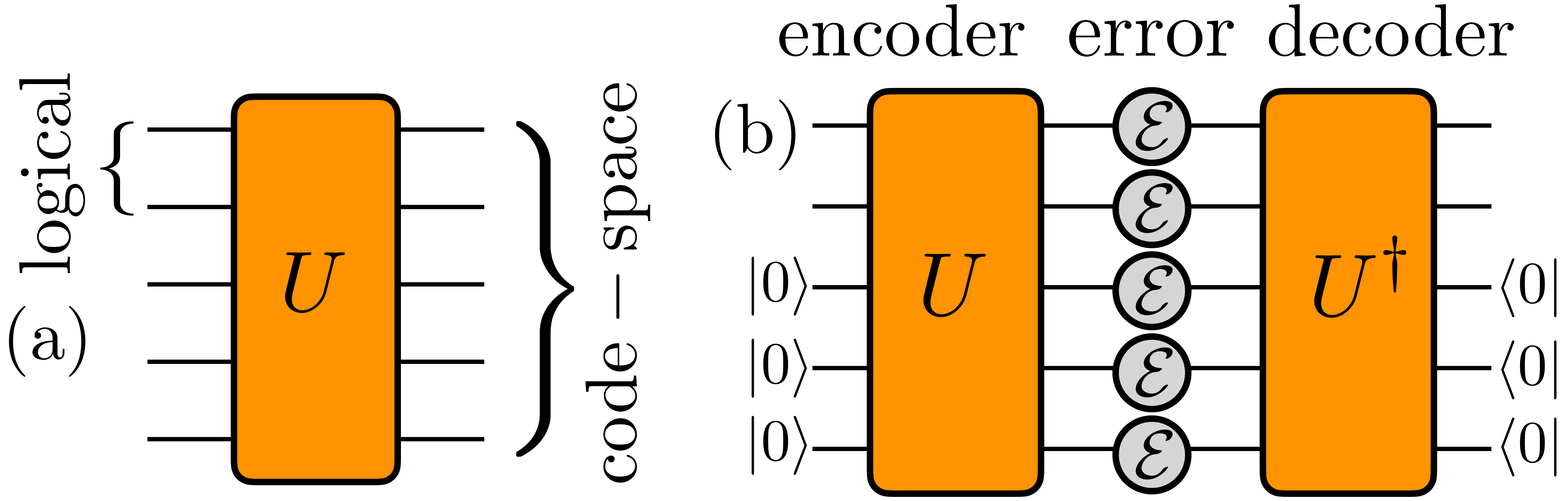
- Mpemba physics visible from the crossing of the entanglement asymmetry.

- Microscopic understanding is also possible. Quite technical: macroscopic fluctuation theory at large qudit dimension + operator spreading arguments.



XT, A. De Luca, P. Calabrese, arXiv:2405.14514

Encoding-decoding circuits with quantum noise



$K = |X|$ Logical qubits

$N = |X \cup \bar{X}|$ Total qubits

$r = K/N$ Code rate

$$\mathcal{E}(\circ) = \sum_\mu K_\mu \circ K_\mu^\dagger$$ Error model

Initial state

$$\rho_{0,X} = |\psi_X\rangle\langle\psi_X|$$

$$\rho_0 = \rho_{0,X} \otimes |0_{\bar{X}}\rangle\langle 0_{\bar{X}}|$$

Decoded state

$$\rho = U^\dagger \mathcal{E}(U\rho_0 U^\dagger)U$$

$$\rho_X = \frac{\langle 0_{\bar{X}}|\rho|0_{\bar{X}}\rangle}{\mathrm{tr}(\langle 0_{\bar{X}}|\rho|0_{\bar{X}}\rangle)}$$

We expect that $\rho_X \neq \rho_{0,X}$

XT, P. Sierant, Phys. Rev. Lett. **132**, 140401 (2024)

Error models

We consider coherent errors or incoherent errors (specifically depolarization)

$$\mathcal{E}_\alpha(\circ) = \prod_{i=1}^{N} e^{-i\alpha Z_i/2} \circ e^{+i\alpha Z_i/2}$$

$$\mathcal{E}_\lambda(\circ) = \prod_{i=1}^{N} \left[\left(1 - \frac{3}{4}\lambda\right) \circ + \frac{1}{4}\lambda \sum_{P=X,Y,Z} P_i \circ P_i \right]$$

XT, P. Sierant, Phys. Rev. Lett. **132**, 140401 (2024)

Results for the fidelity

Coherent errors

$$\tilde{F} = \frac{\left(2^N - 1\right)\left(2^N \cos^{2N}\left(\frac{\alpha}{2}\right) + 1\right)}{2^N\left(2^N - 2^k\right)\cos^{2N}\left(\frac{\alpha}{2}\right) + 2^{k+N} - 1}$$

$$\lim_{k,N\to\infty,r=\text{const}} \tilde{F} = 1 - \Theta(\alpha > 2\arccos(2^{-r/2}))$$

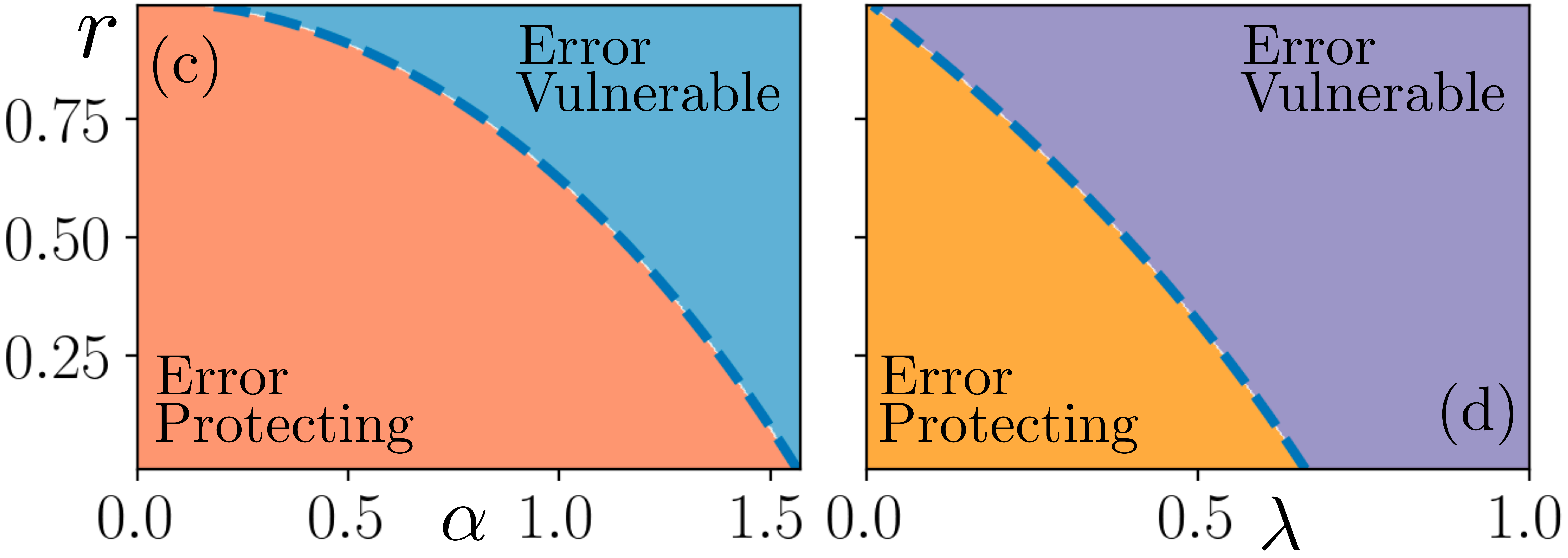
Incoherent errors

$$\tilde{F} = \frac{\left(2^N - 1\right)\left((4-3\lambda)^N + 2^N\right)}{-2^k(4-3\lambda)^N + 2^{k+2N} + (8-6\lambda)^N - 2^N}$$

$$\lim_{k,N\to\infty,r=\text{const}} \tilde{F} = 1 - \Theta(\lambda < \lambda_c(r)) \qquad \lambda_c(r) = \frac{4}{3}\left(1 - 2^{r-1}\right)$$

Error-resilience transitions

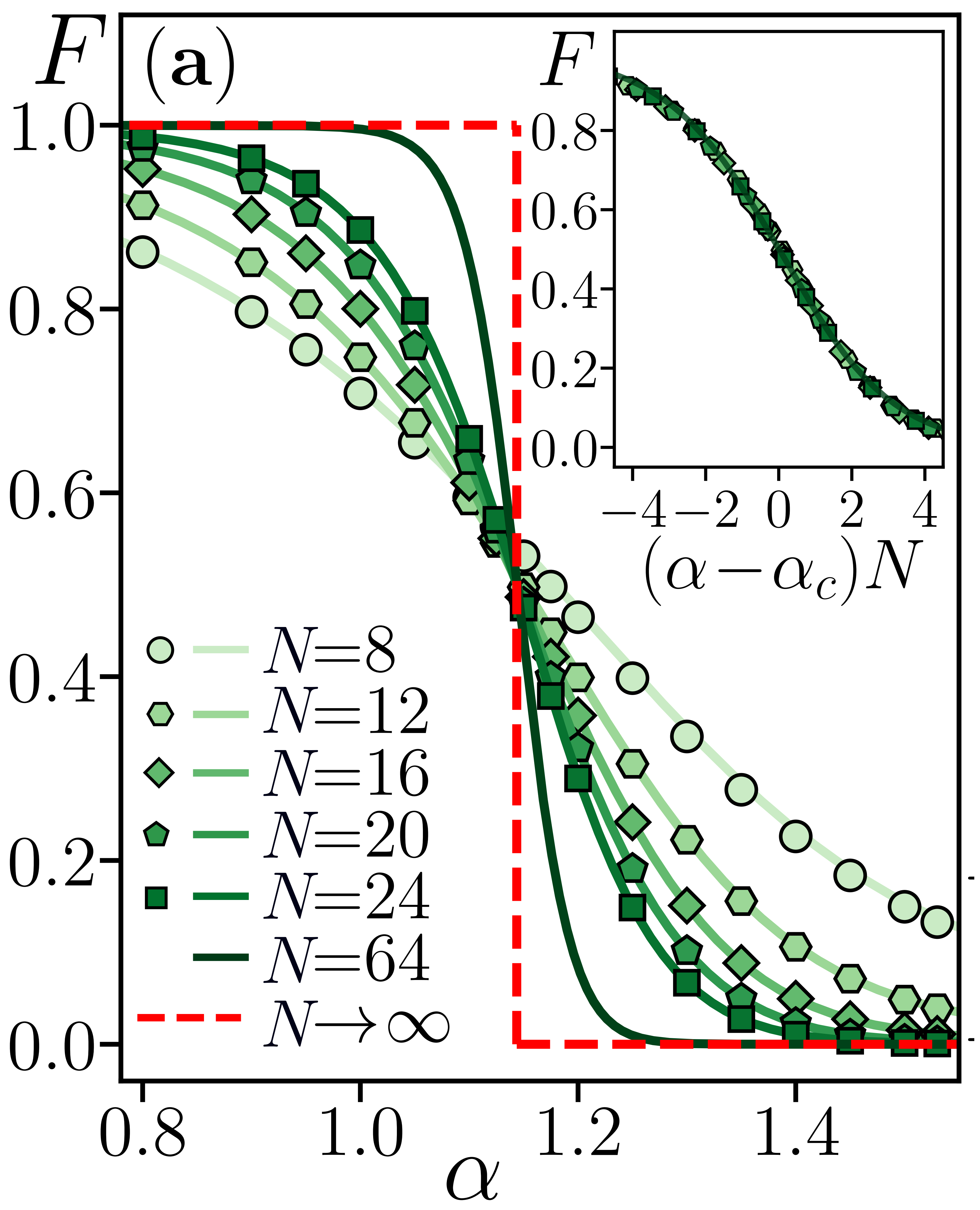
In the scaling limit $K, N \to \infty, r = \text{const}$, non-trivial phase diagram varying α or λ and r



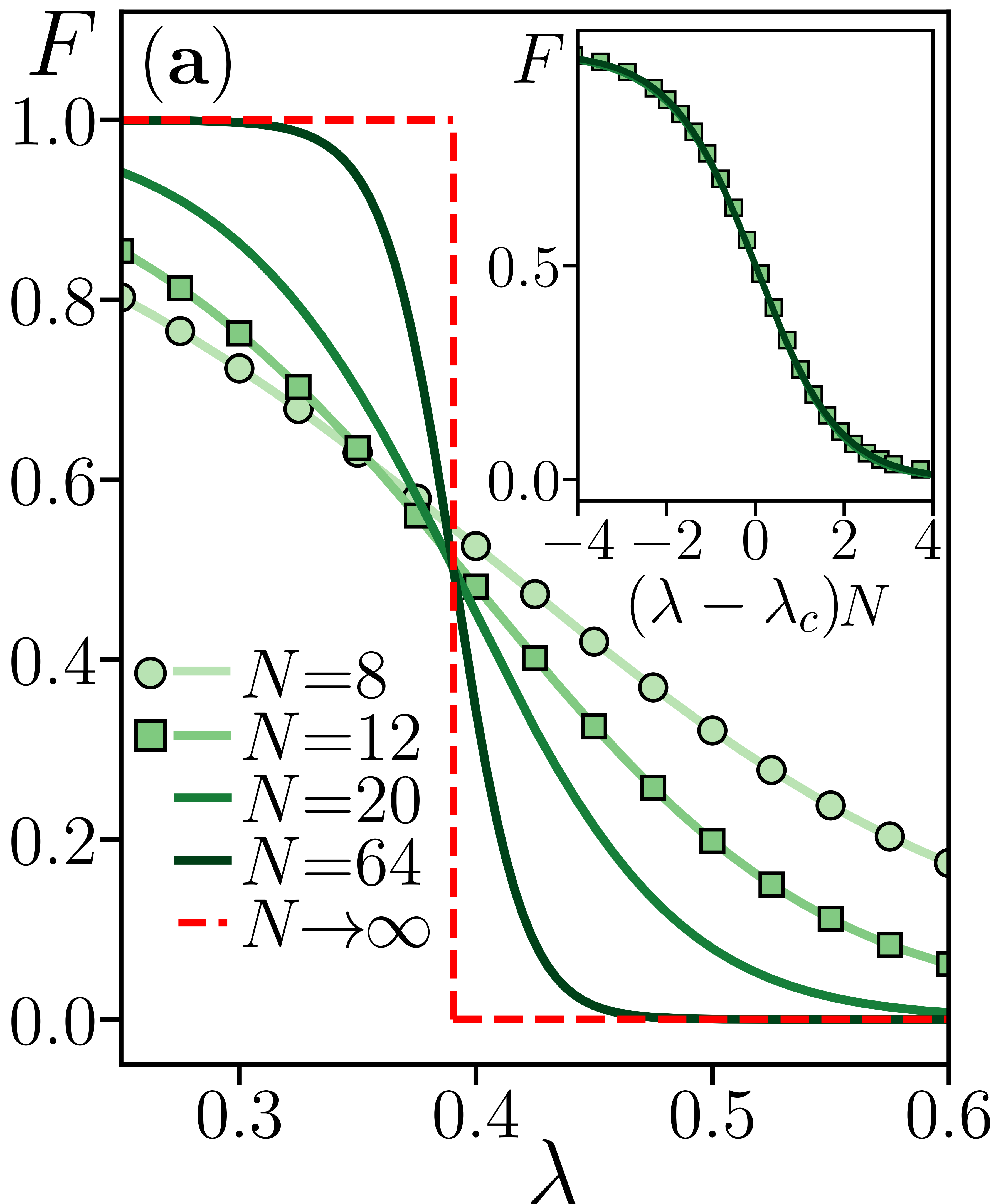
Furthermore, both error-protecting and error-vulnerable phases (EPP and EVP) have multi-fractal features

Results for the fidelity

Coherent errors



Incoherent errors



Further applications of random circuits

- (Classical) Shadow Tomography
- Randomised benchmarking
- Random generators
- Verification and validation
- Quantum error correction

Conclusion

- Random quantum circuits are versatile tools to understand many-body quantum dynamics without the detailed knowledge of tailored Hamiltonian or specific gates.
- They allow for both analytical insights and efficient numerical methods, provided self-averaging is present. (Example where this is not the case: measurement-induced transitions)
- The Tensor Network representation is efficient because the gates are non-unitary after averaging. Bond dimension does not grow extensively, and remains bounded to poly(effective qudit dimension).
- Outlook: extension to noisy quantum dynamics (unital and non-unital noise). State becomes mixed, but self-averaging still holds.

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Quantum noise