THE THERMODYNAMICS OF THE QUANTUM MPEMBA EFFECT

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Recent work

The thermodynamics of the quantum Mpemba effect

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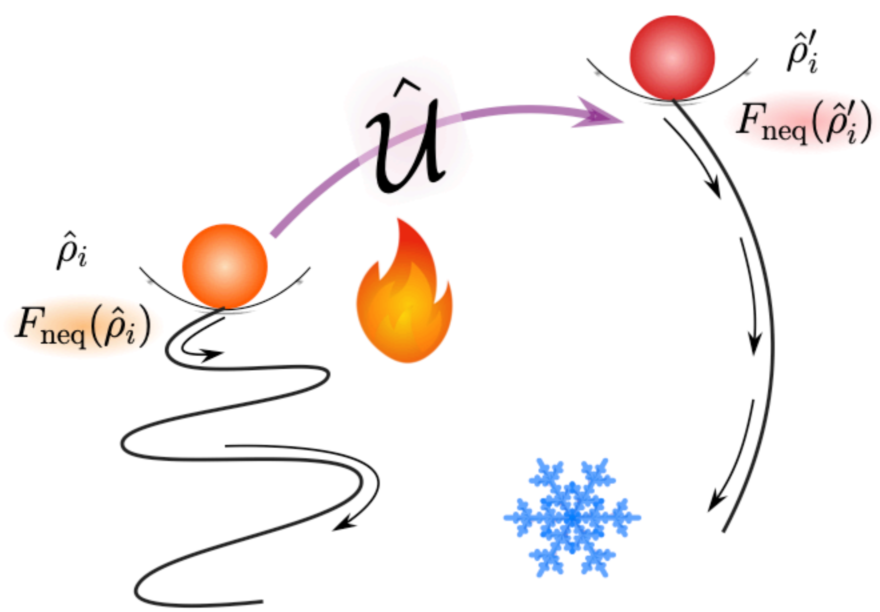

summary

Mattia Moroder
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Oisín Culhane
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John Goold
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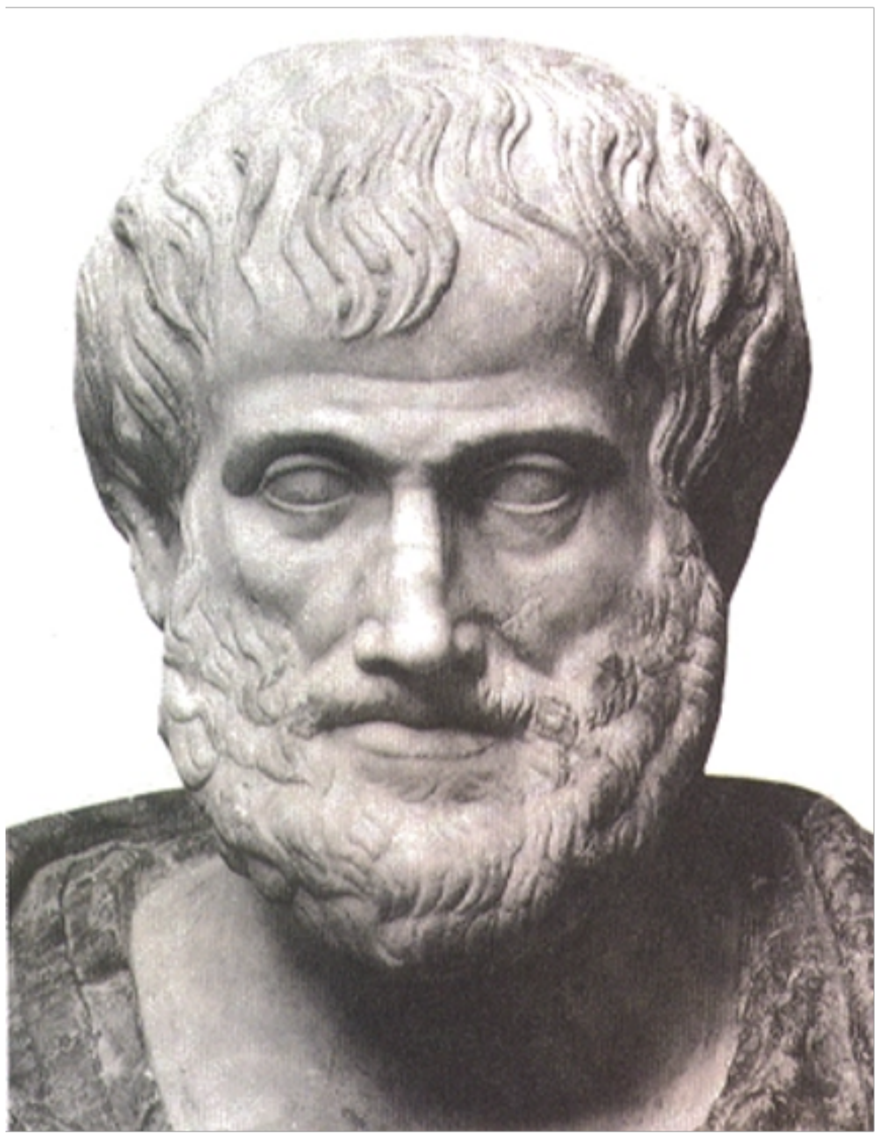
I. History and Background
II. Open quantum systems and thermodynamics
III. Quantum Mpemba effect
IV. Results
V. Something related I am currently working on

Hot water can freeze faster than colder water

History

“The inhabitants of Pontus when they encamp on the ice to fish pour warm water round their reeds that it may freeze the quicker”



Further references to the phenomena have been made by:

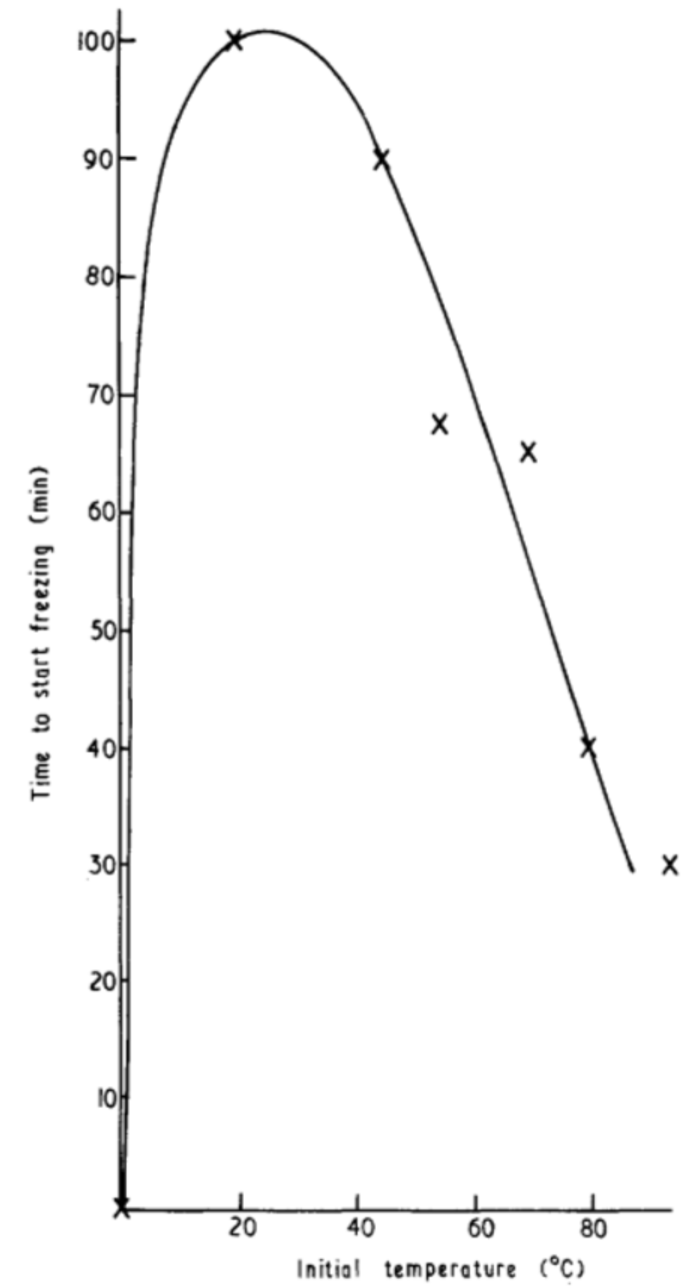
Francis (or Roger?) bacon

“A little warm water will freeze more easily than completely cold”

Rene Descartes

“water that has been kept on a fire for a long time freezes faster than other”

The Mpemba Effect



E. B. Mpemba and D. G. Osborne,
Physics Education 4, 172 (1969).

Interest

Nonequilibrium thermodynamics of the Markovian Mpemba effect and its inverse

Zhiyue Lu and Oren Raz Authors Info & Affiliations

Edited by David A. Weitz, Harvard University, Cambridge, MA, and approved April 4, 2017 (received for review January 23, 2017)

May 1, 2017 | 114 (20) 5083-5088 | https://doi.org/10.1073/pnas.1701264114

The Mpemba effect in spin glasses is a persistent memory effect

Marco Baity-Jesi [1 2], Enrico Calore [3 4], Andres Cruz [2 5], Luis Antonio Fernandez [2 6], José Miguel Gil-Narvión [2], Antonio Gordillo-Guerrero [2 7 8], David Iñiguez [2 9], Antonio Lasanta [10], Andrea Maiorano [2 11], Enzo Marinari [11 12 13], Victor Martin-Mayor [2 6], Javier Moreno-Gordo [2 5], Antonio Muñoz Sudupe [2 6], Denis Navarro [14 15], Giorgio Parisi [16 12 13], Sergio Perez-Gaviro [2 5 17], Federico Ricci-Tersenghi [11 12 13], Juan Jesus Ruiz-Lorenzo [2 8 18], Sebastiano Fabio Schifano [4 19], Beatriz Seoane [2 20 21], Alfonso Tarancón [2 5], Raffaele Tripiccione [3 4], David Yllanes [22 23 23]

Mpemba effect in driven granular Maxwell gases

Apurba Biswas, V. V. Prasad, O. Raz, and R. Rajesh
Phys. Rev. E **102**, 012906 – Published 23 July 2020

When does hot water freeze faster then cold water? A search for the Mpemba effect

James D. Brownridge

Check for updates

+ Author & Article Information

Am. J. Phys. 79, 78–84 (2011)

PERSPECTIVE • FREE ARTICLE

Non-equilibrium memory effects: Granular fluids and beyond

A. Patrón[1], B. Sánchez-Rey[2], C. A. Plata[1] and A. Prados[1]

Published 22 September 2023 • Copyright © 2023 EPLA

Europhysics Letters, Volume 143, Number 6

Citation A. Patrón *et al* 2023 *EPL* **143** 61002

DOI 10.1209/0295-5075/acf7e5

Mpemba-Like Behavior in Carbon Nanotube Resonators

Symposium: Modeling, Simulation, and Theory of Nanomechanical Materials Behavior | Published: 16 August 2011

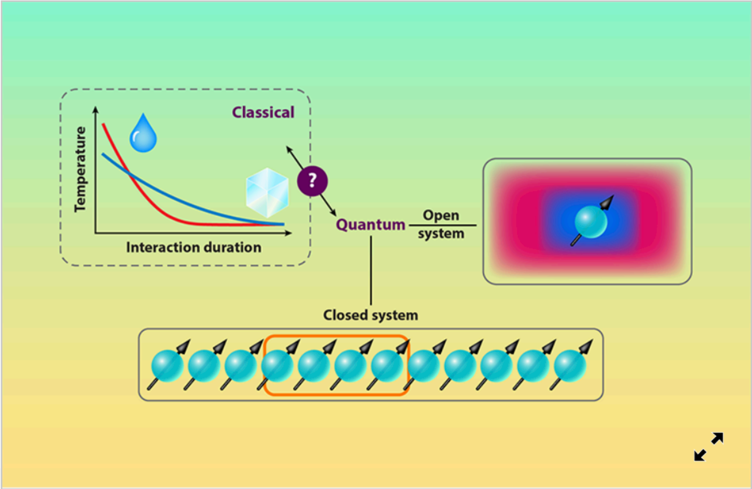
Volume 42, pages 3907–3912, (2011) Cite this article

Recent work

Exploring Quantum Mpemba Effects

Ulrich Warring
Institute of Physics, University of Freiburg, Freiburg, Germany
July 1, 2024 • *Physics* 17, 105

In the Mpemba effect, a warm liquid freezes faster than a cold one. Three studies investigate quantum versions of this effect, challenging our understanding of quantum thermodynamics.



APS/Carin Cain

Featured in Physics | Editors' Suggestion

Inverse Mpemba Effect Demonstrated on a Single Trapped Ion Qubit

Shahaf Aharony Shapira, Yotam Shapira, Jovan Markov, Gianluca Teza, Nitzan Akerman, Oren Raz, and Roee Ozeri
Phys. Rev. Lett. **133**, 010403 – Published 1 July 2024

Physics See Viewpoint: Exploring Quantum Mpemba Effects

Featured in Physics | Editors' Suggestion

Observing the Quantum Mpemba Effect in Quantum Simulations

Lata Kh. Joshi, Johannes Franke, Aniket Rath, Filiberto Ares, Sara Murciano, Florian Kranzl, Rainer Blatt, Peter Zoller, Benoît Vermersch, Pasquale Calabrese, Christian F. Roos, and Manoj K. Joshi
Phys. Rev. Lett. **133**, 010402 – Published 1 July 2024

Physics See Viewpoint: Exploring Quantum Mpemba Effects

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Microscopic Origin of the Quantum Mpemba Effect in Integrable Systems

Colin Rylands, Katja Klobas, Filiberto Ares, Pasquale Calabrese, Sara Murciano, and Bruno Bertini
Phys. Rev. Lett. **133**, 010401 – Published 1 July 2024

Physics See Viewpoint: Exploring Quantum Mpemba Effects

Markovian Mpemba effect

Z. Lu and O. Raz, Non equilibrium thermodynamics of the Markovian Mpemba Effect, PNAS (2017)

$$\frac{\mathrm{d}\vec{p}(t)}{\mathrm{d}t} = \mathscr{L}_{T_b}\vec{p}(t),$$

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = \sum_j R_{ij}(T_b)p_j(t) \quad \text{for } i = 1,2,\cdots,n.$$

$$R_{ij}(T_b) = \begin{cases} \Gamma e^{-\frac{B_{ij}-E_j}{k_B T_b}} & i \neq j \\ -\sum_{k\neq i} R_{ki} & i = j\text{'} \end{cases}$$

Drives system towards a thermal fixed point:

$$\pi_i(T_b) = \frac{e^{-E_i/k_B T_b}}{\sum_i e^{-E_i/k_B T_b}}.$$

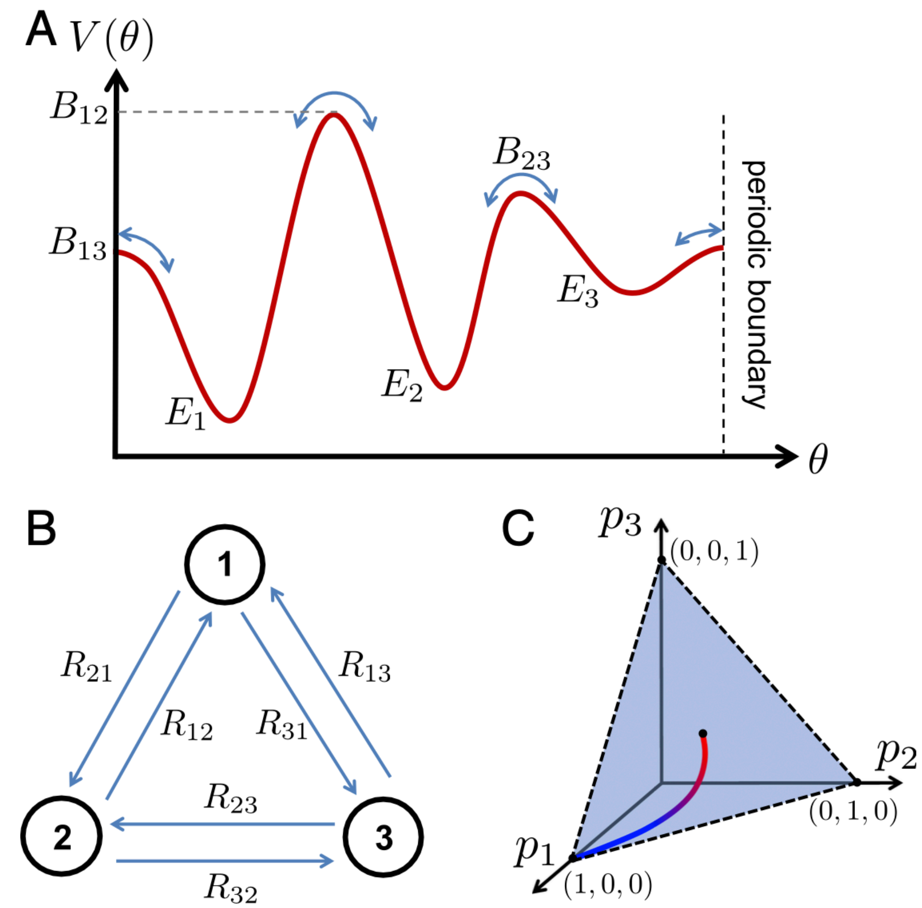


Fig. 1. (*A*) Sketch of an energy landscape with three metastable states. The energy of each state is denoted by E_i, and the barrier between two states is denoted by B_{ij}. (*B*) An effective description of *A* as a three-state system, where R_{ij} denotes transition rate from *j* to *i* and is given by Eq. **3**. (*C*) The probability distribution among the three states can be described by the vector $\vec{p} = (p_1, p_2, p_3)$, and all possible values of $\vec{p}$ form a simplex in (p_1, p_2, p_3) (shaded triangle). The curved line is the quasistatic locus, namely the set of Boltzmann distributions $\vec{\pi}(T)$ corresponding to different temperatures from 0 (blue end) to ∞ (red end).

Markovian Mpemba effect

Z. Lu and O. Raz, Non equilibrium thermodynamics of the Markovian Mpemba Effect, PNAS (2017)

- Distance measure on space of probability distributions $D(\vec{p}(t); \vec{\pi}(T_b))$ (monotonically decreasing in time)
- Temperatures $T_b < T_c < T_h$
- Initially $\overrightarrow{P}^h(0) = \vec{\pi}(T_h)$ and $\overrightarrow{P}^c(0) = \vec{\pi}(T_c)$ relax to same equilibrium at $\vec{\pi}_{T_b}$
- Mpemba effect if ∃ some time t_m such that $D(\vec{p}_h(t); \vec{\pi}(T_b)) < D(\vec{p}_c(t); \vec{\pi}(T_b))$ $\forall t > t_m$
- Markovian system - probabilities evolve according to

$$\overrightarrow{P}(t) = \vec{\pi}_{T_b} + e^{\lambda_2} a_2 \vec{v}_2 + \ldots + e^{\lambda_n} a_n \vec{v}_n$$

Here $\vec{v}_n$ are the right eigenvectors of the rate matrix R_{ij}

Markovian Mpemba effect

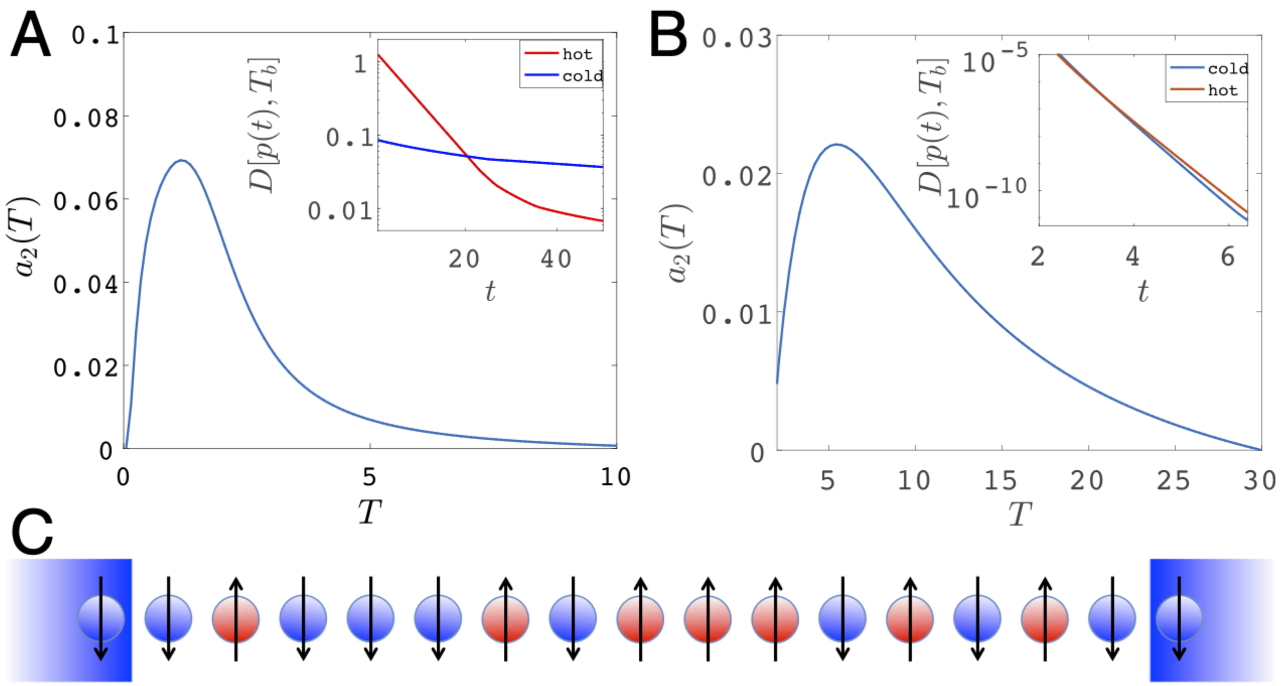
Z. Lu and O. Raz, Non equilibrium thermodynamics of the Markovian Mpemba Effect, PNAS (2017)

$$\vec{P}(t) = \vec{\pi}_{T_b} + e^{\lambda_2} a_2 \vec{v}_2 + \ldots + e^{\lambda_n} a_n \vec{v}_n \qquad \lambda_1 > \lambda_2 \geq \lambda_3 \ldots \geq \lambda_n$$

Long time limit

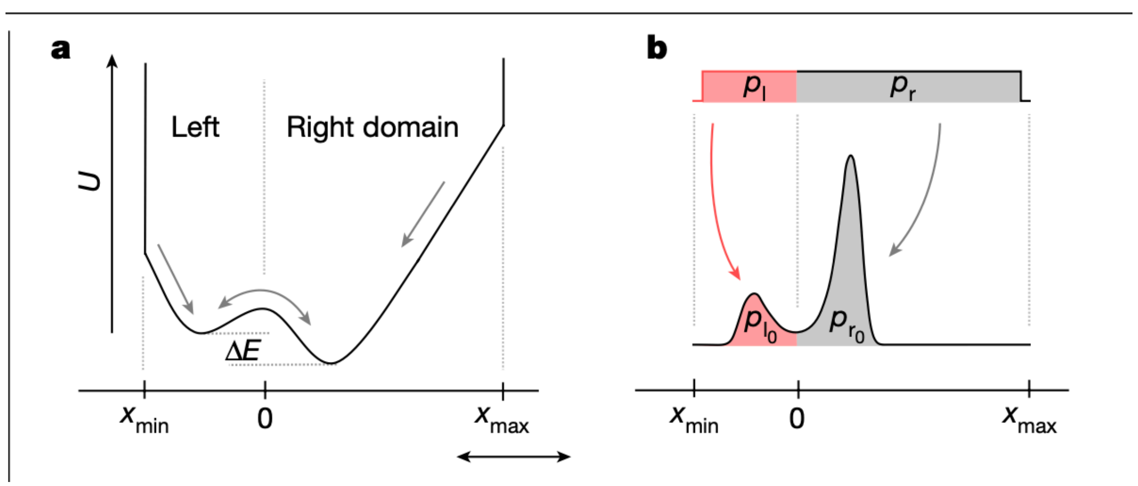
$$\vec{P}^h(t) \approx \vec{\pi}_{T_b} + e^{\lambda_2} a_2^h \vec{v}_2 \qquad \vec{P}^c(t) \approx \vec{\pi}_{T_b} + e^{\lambda_2} a_2^c \vec{v}_2$$

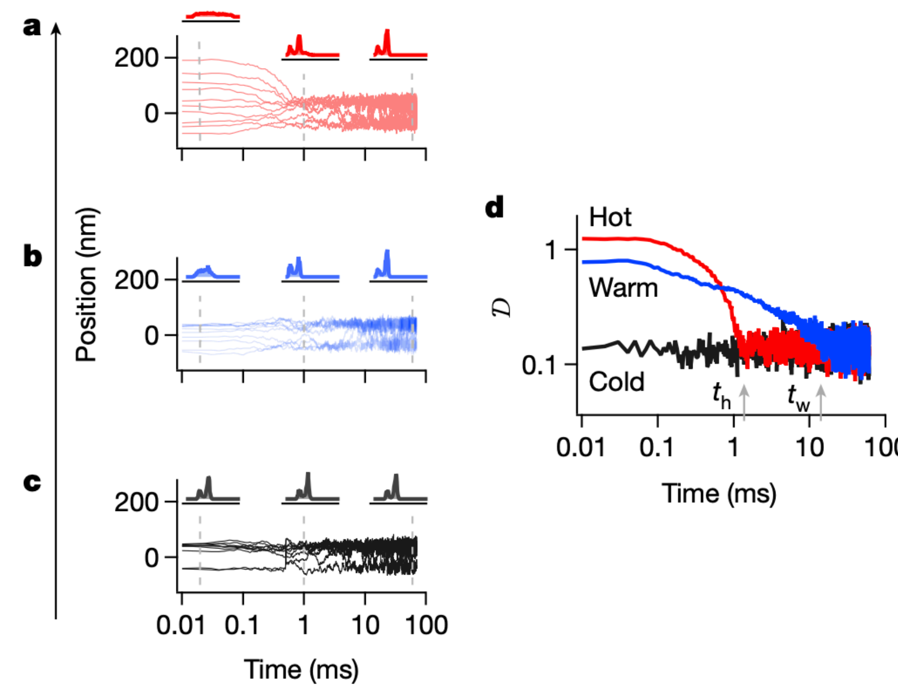
Mpemba effect when: $|a_2^c| > |a_2^h|$

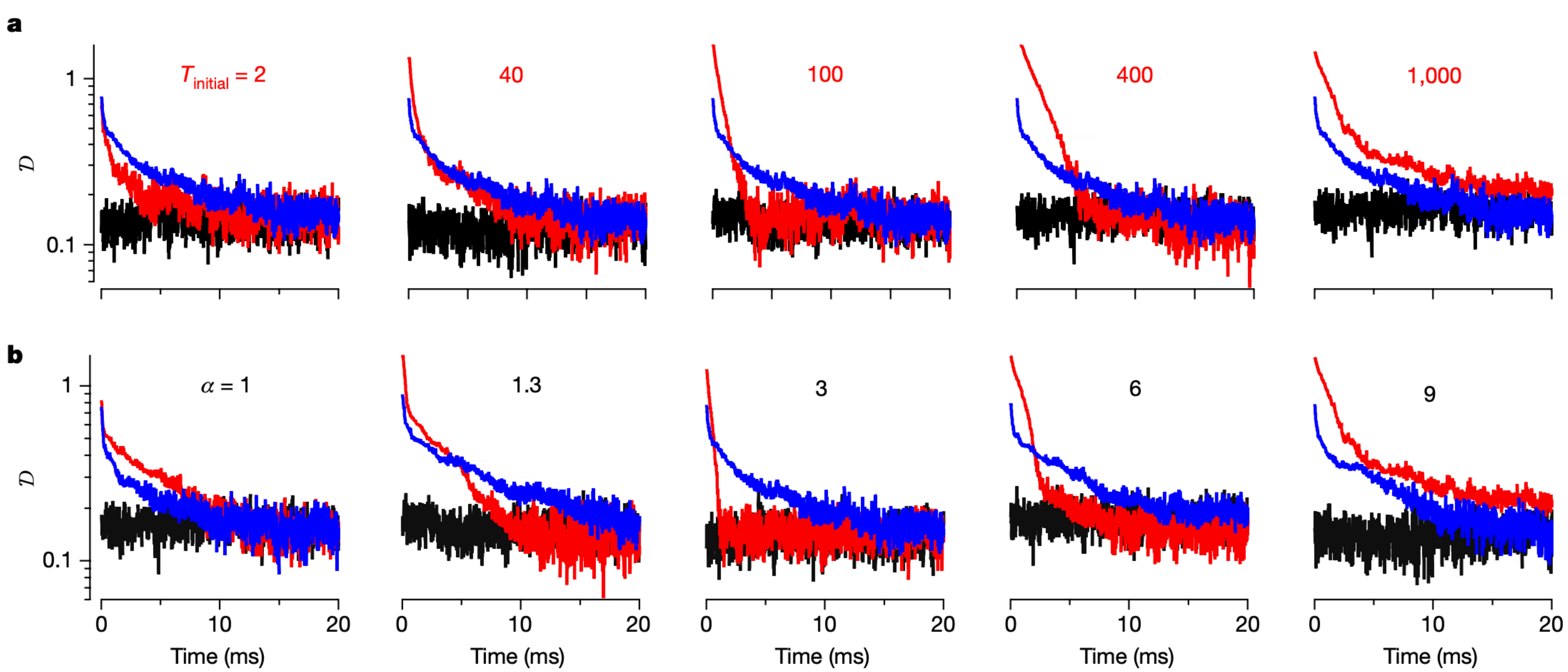


Experiment

A. Kumar and J. Bechhoefer, Exponentially faster cooling in a colloidal system , Nature, 64, (2020)





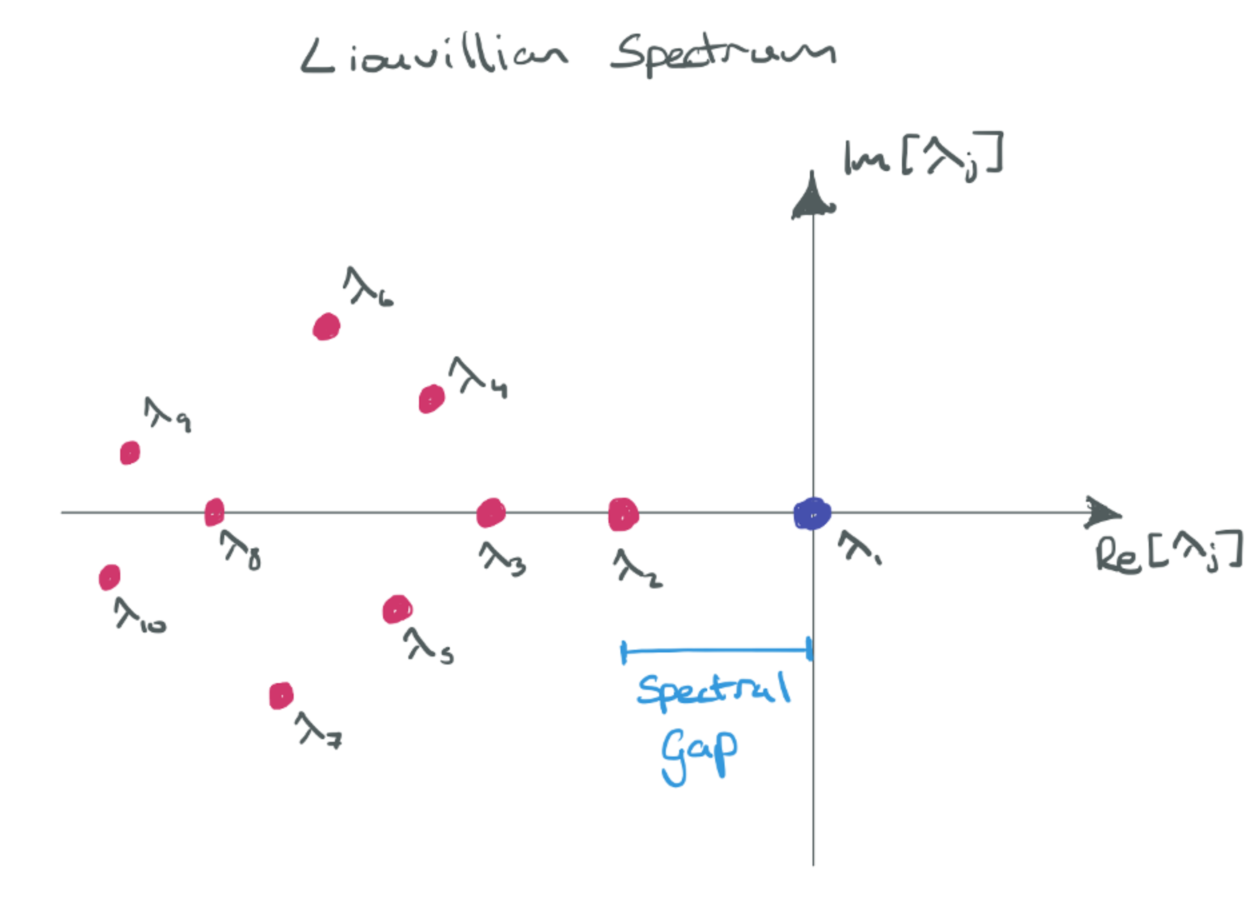


Strong Mpemba effect when: $|a_2^h| = 0$

Exponentially faster cooling !

Quantum Markov Systems

$$\frac{d\hat{\rho}(t)}{dt} = \mathcal{G}\hat{\rho}(t) \qquad \mathcal{G} = i\delta + \mathcal{D}$$



$$\delta(\,\cdot\,) = -\,[\hat{H}, \cdot\,]$$

$$\mathcal{D}(\,\cdot\,) = \sum_l \hat{L}_l(\,\cdot\,)\hat{L}_l^\dagger - \frac{1}{2}\{\hat{L}_l^\dagger\hat{L}_l, (\,\cdot\,)\}$$

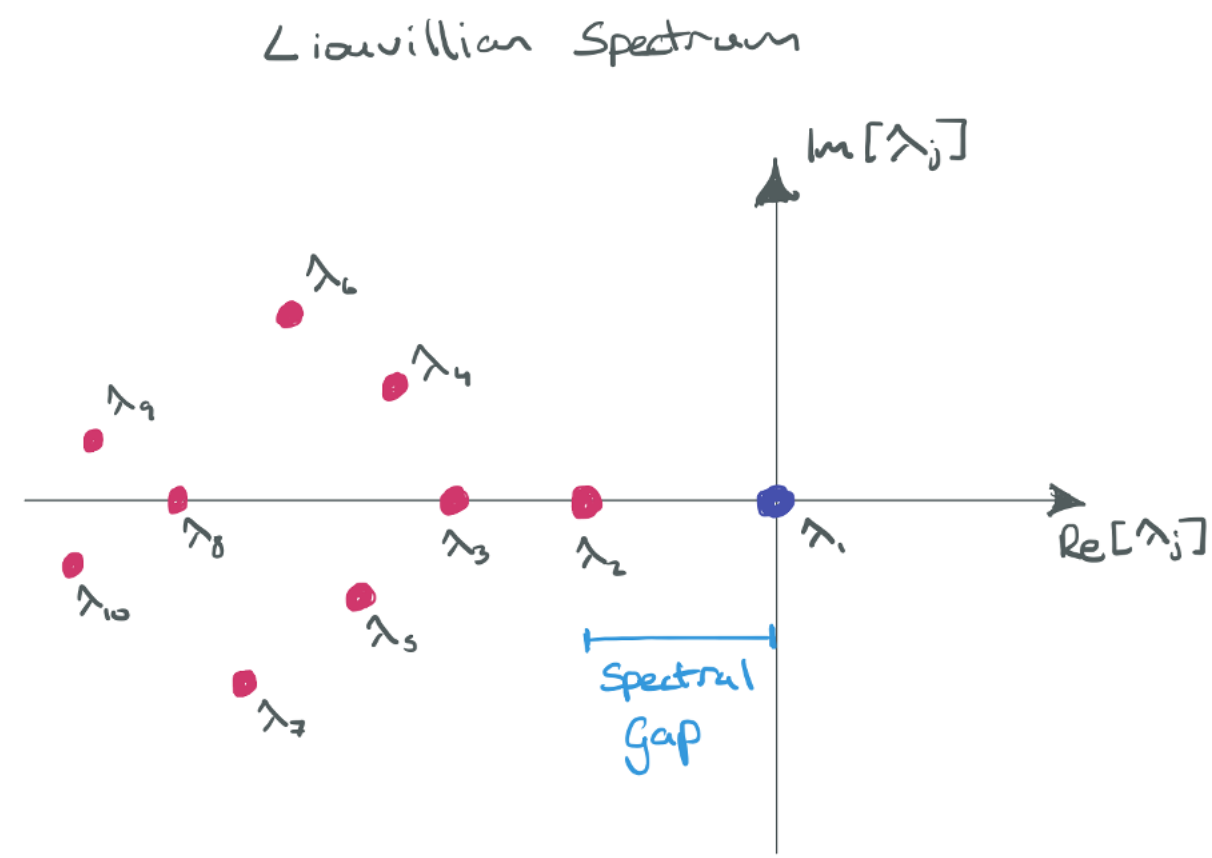
$$\mathcal{G}[\hat{r}_k] = \lambda_k \hat{r}_k \qquad \mathcal{G}^\dagger[\hat{l}_k] = \lambda_k \hat{l}_k$$

Eigenvalues ordered in ascending order according to their real part :

$$0 = \lambda_1 < |\Re(\lambda_2)| \leq |\Re(\lambda_3)| \leq \ldots$$

Quantum Markov Systems

$$\hat{\rho}(t) = e^{\mathcal{G}t}\hat{\rho}_i = \hat{\tau} + \sum_{k=2}^{D^2} Tr[\hat{l}_k^{\dagger}\hat{\rho}_i]\hat{r}_k e^{\lambda_k t}$$



Spectral gap: $|\Re(\lambda_2)|$

Defines longest timescale:

$$|\hat{\rho}(t) - \hat{\tau}| \propto \exp(\Re(\lambda_2)t)$$

timescale: $t = \dfrac{1}{|\lambda_2|}$

Speeding up equilibration

Federico Carollo, Antonio Lasanta, and Igor Lesanovsky Phys. Rev. Lett. **127**, 060401 – (2021)

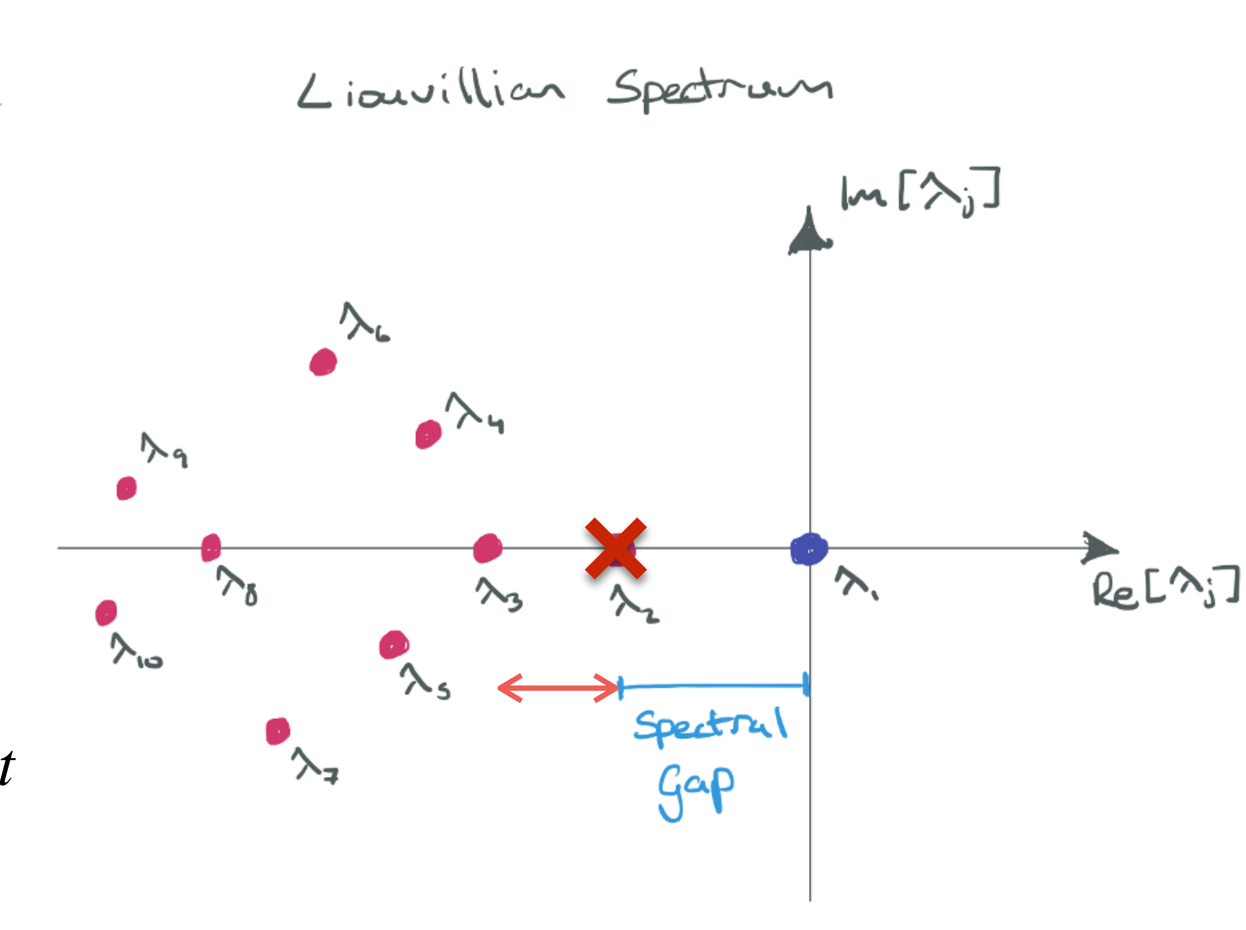
Find U such that:

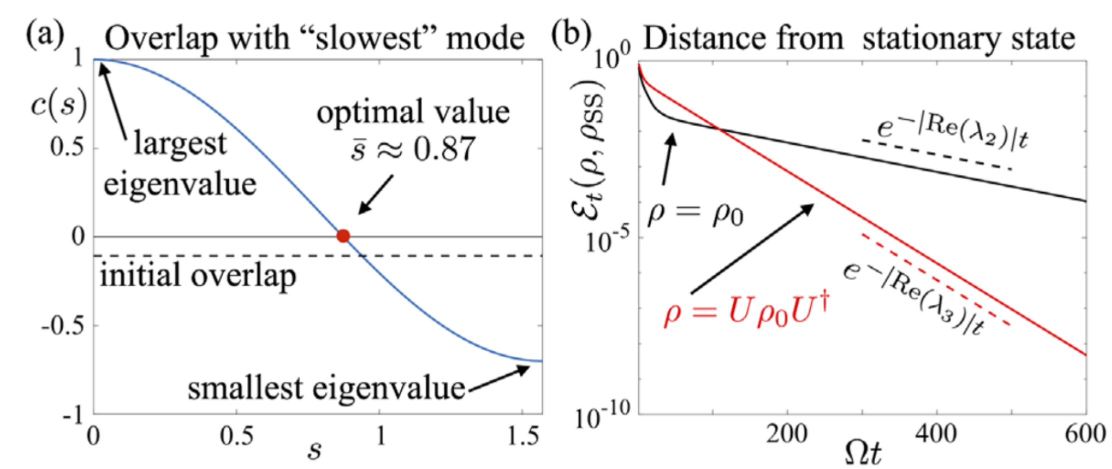
$$Tr(\hat{l}_2 \hat{U} \hat{\rho}_i \hat{U}^\dagger) = 0$$

$$\hat{\rho}(t) = e^{\mathcal{G}t}\hat{\rho}_i = \hat{\tau} + \sum_{k=3}^{D^2} Tr[\hat{l}_k^\dagger \hat{U} \hat{\rho}_i \hat{U}^\dagger]\hat{r}_k e^{\lambda_k t}$$

$$|\hat{\rho}(t) - \hat{\tau}| \propto \exp(\Re(\lambda_3)t)$$

timescale: $t = \dfrac{1}{|\lambda_3|}$





Exponential speed up !

Speeding up equilibration

Federico Carollo, Antonio Lasanta, and Igor Lesanovsky Phys. Rev. Lett. **127**, 060401 – (2021)

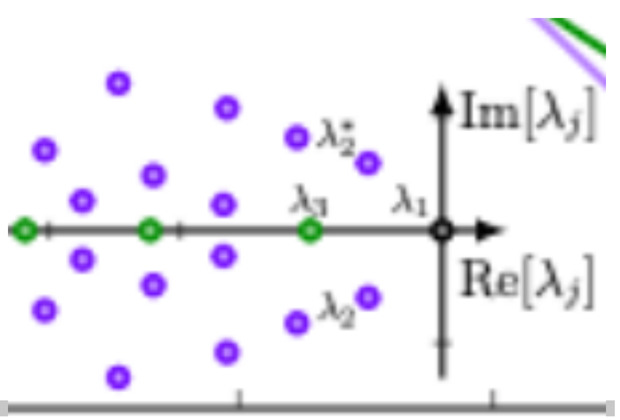
Assumptions on constructing U

Restricted to initial pure states

Spectral gap defined by a real eigenvalue ! Restrictive

Can have a spectral gap defined by a complex conjugate pair

$$\hat{\rho}(t) \propto \hat{\tau} + e^{\Re(\lambda_2)t}\left(\underline{Tr(\hat{l}_2\hat{\rho}_i)}\hat{r}_2 e^{i\Im(\lambda_2)t} + \underline{Tr(\hat{l}_2^\dagger\hat{\rho}_i)}\hat{r}_2^\dagger e^{-i\Im(\lambda_2)t}\right)$$



Thermodynamics

The thermodynamics of the quantum Mpemba effect

Mattia Moroder,[1,*] Oisín Culhane,[2,†] Krissia Zawadzki,[2,3,‡] and John Goold[2,4,§]

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[2] *School of Physics, Trinity College Dublin, Dublin 2, Ireland*

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[4] *Trinity Quantum Alliance, Unit 16, Trinity Technology and Enterprise Centre, Pearse Street, Dublin 2, D02YN67*

Interested in thermal fixed points

Quantum detailed balance wrt to thermal state

$$\frac{d\hat{\rho}(t)}{dt} = \mathscr{G}\hat{\rho}(t) \qquad \mathscr{G} = i\delta + \mathscr{D}$$

Davies maps:

E. Davies, J. Funct. Anal. 34, 421 (1979).

Davies Generator

$$\frac{d\hat{\rho}(t)}{dt} = \mathcal{G}\hat{\rho}(t) \qquad \mathcal{G} = i\delta + \mathcal{D}$$

Quantum detailed balance wrt to thermal state: $\hat{\tau}_\beta = e^{-\beta\hat{H}}/Z$

Mathematically this means :

R. Alicki, Rep. Math. Phys. 10, 249 (1976).

$$\langle \hat{A}, \mathcal{D}^\dagger(\hat{B})\rangle_{\hat{\tau}_\beta} = \langle \mathcal{D}^\dagger(\hat{A}), \hat{B}\rangle_{\hat{\tau}_\beta}$$

With: $\langle \hat{A}, \hat{B}\rangle_{\hat{\tau}_\beta} = Tr(\hat{\tau}_\beta \hat{A}^\dagger \hat{B})$

KMS inner product:

$$[\hat{H}, \hat{\tau}_\beta] = 0$$

Importantly δ and $\mathcal{D}$ share a common eigenspace: $|n\rangle\langle m|$

$$\omega_{nm} = E_n - E_m$$

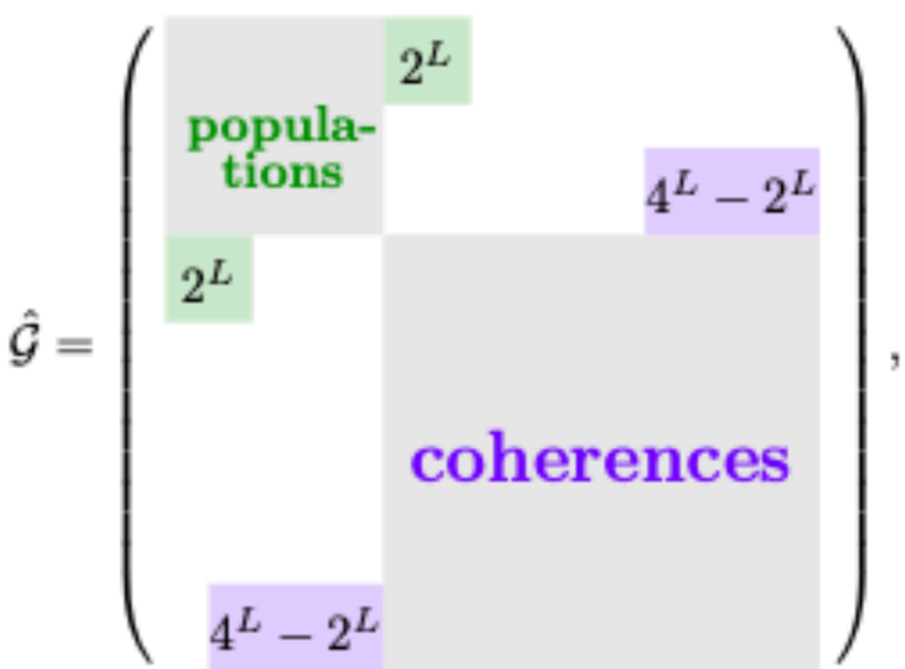
Davies Generator

$$\frac{d\hat{\rho}(t)}{dt} = \mathscr{G}\hat{\rho}(t) \qquad \mathscr{G} = i\delta + \mathscr{D}$$

$$\mathscr{G} = \mathscr{G}_P \oplus \mathscr{G}_C$$

$\mathscr{G}_P = \mathscr{D}_P$ Populations sub-block

$\mathscr{G}_{\mathscr{C}} = i\delta + \mathscr{D}_C$ Coherences sub-block



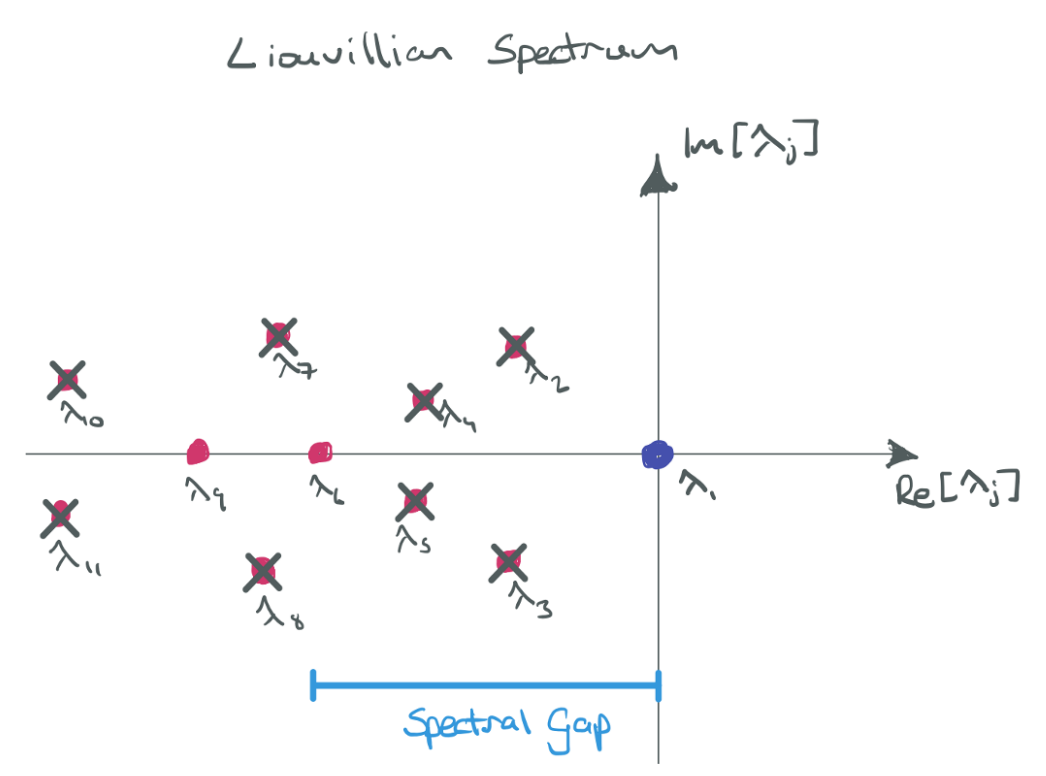
An essential point :

Left eigenmatrices, $\hat{l}_k$, corresponding to **populations** are **diagonal** in energy eigenbasis with purely real eigenvalues λ_k

Left eigenmatrices, $\hat{l}_k$, corresponding to **coherences** are pure **off diagonal** in energy eigenbasis with eigenvalues λ_k in complex conjugate pair

Deleting coherent overlaps

Left eigenmatrices, $\hat{l}_k$, corresponding to **coherences** are pure **off diagonal** in energy eigenbasis with eigenvalues λ_k in complex conjugate pair



Any transformation that brings the initial state to be diagonal in the energy eigenbasis will kill **all** overlaps with coherent modes !

New thermalisation rate now defined by the first real eigenvalue !

Exponential speed up ! $\hat{U}_1 \hat{\rho}_i \hat{U}_1^\dagger = \hat{\Lambda}$

But is this a Mpemba effect ?

Non equilibrium free energy

Consider internal energy of some arbitrary state

$$U(\hat{\rho}(t)) = -\beta^{-1}Tr[\hat{\rho}(t)\log e^{-\beta\hat{H}}]$$

$$= \beta^{-1}Tr[\hat{\rho}(t)\log \frac{e^{-\beta\hat{H}}}{Z}] - \beta^{-1}Tr(\hat{\rho}(t)\log Z)$$

$$= \beta^{-1}Tr[\hat{\rho}(t)\log \hat{\tau}_\beta] - \beta^{-1}S(\hat{\rho}(t)) + \beta^{-1}S(\hat{\rho}(t)) + F_{eq}$$

$$= \beta^{-1}D(\hat{\rho}(t)\,||\,\hat{\tau}_\beta) + \beta^{-1}S(\hat{\rho}(t)) + F_{eq}$$

$$F_{\text{neq}}(\hat{\rho}(t)) = \beta^{-1}D(\hat{\rho}(t)\,||\,\hat{\tau}_\beta) + F_{\text{eq}}$$

Relative entropy

$$F_{\text{neq}}(\hat{\rho}(t)) = \beta^{-1} D(\hat{\rho}(t) \,||\, \hat{\tau}_\beta) + F_{\text{eq}}$$

Relative entropy $D(\hat{\rho} \,||\, \hat{\sigma}) = Tr\hat{\rho}(\log \hat{\rho} - \log \hat{\sigma})$

Klein's inequality $D(\hat{\rho} \,||\, \hat{\sigma}) \geq 0$

Pinksers inequality: $D(\hat{\rho} \,||\, \hat{\sigma}) \geq |\hat{\rho} - \hat{\sigma}|_1^2/2$

Monoticity: $D(\Gamma[\hat{\rho}] \,||\, \Gamma[\hat{\sigma}]) \leq D(\hat{\rho} \,||\, \hat{\sigma})$

$$F_{neq}(\hat{\rho}(t)) \geq F_{eq} \forall t$$

Entropy production in Davies maps

$$F_{\text{neq}}(\hat{\rho}(t)) = \beta^{-1} D(\hat{\rho}(t) \,||\, \hat{\tau}_\beta) + F_{\text{eq}}$$

$$\frac{dS_s}{dt} = \dot{\Sigma} - \dot{\Phi}$$

$$\dot{\Phi} = \beta \dot{Q}$$

Entropic heat flow

$$\dot{\Sigma} = -\beta \frac{dF_{neq}(\hat{\rho}(t))}{dt}$$

$$\dot{\Sigma} \geq 0$$

Relative entropy

$$\dot{\Sigma} = 0 \quad \hat{\rho}(t) \to \hat{\tau}_\beta$$

Nice division: $D(\hat{\rho}(t) \,||\, \hat{\tau}_\beta) = \mathscr{P}(p(t) \,||\, \hat{\tau}_\beta) + \mathscr{C}(\hat{\rho}(t))$

Francica, Goold, Plastina PRE ,99 (2019)

Santo,Celeri, Landi Paternostro NPJ Quantum info ,23 (2019)

Entropy production in Davies maps

$$F_{\text{neq}}(\hat{\rho}(t)) = \beta^{-1} D(\hat{\rho}(t)\,||\,\hat{\tau}_\beta) + F_{\text{eq}}$$

$$D(\hat{\rho}(t)\,||\,\hat{\tau}_\beta) = \mathscr{P}(p(t)\,||\,\hat{\tau}_\beta) + \mathscr{C}(\hat{\rho}(t))$$

$$\mathscr{P}(p(t)\,||\,t_\beta) = \sum_n p_n(t) \ln \frac{p_n(t)}{t_n^\beta}$$

$$\mathscr{C}(\hat{\rho}(t)) = S(\Delta(\hat{\rho}(t))) - S(\hat{\rho}(t))$$

Division allows us to study the coherent and incoherent contributions to entropy production

Quantum Mpemba effect

Three states: $\hat{\rho}(t_0)$ $\hat{\rho}'(t_0) = \hat{U}\hat{\rho}(t_0)\hat{U}^\dagger$ $\hat{\tau}_\beta = e^{-\beta\hat{H}}/Z$

Non equilibrium free energies : $F_{\text{neq}}(\hat{\rho}'(t_0)) > F_{\text{neq}}(\hat{\rho}(t_0)) > F_{\text{eq}}$

Mpemba effect if : $\exists t_m$ Such that $F_{\text{neq}}(\hat{\rho}'(t)) < F_{\text{neq}}(\hat{\rho}(t)) \forall t > t_m$

Key finding of our work:

If the generator has a spectral gap defined by a complex eigenpair then a unitary can always be found such that an exponential speed up and a genuine Mpemba effect occurs

Quantum Mpemba effect

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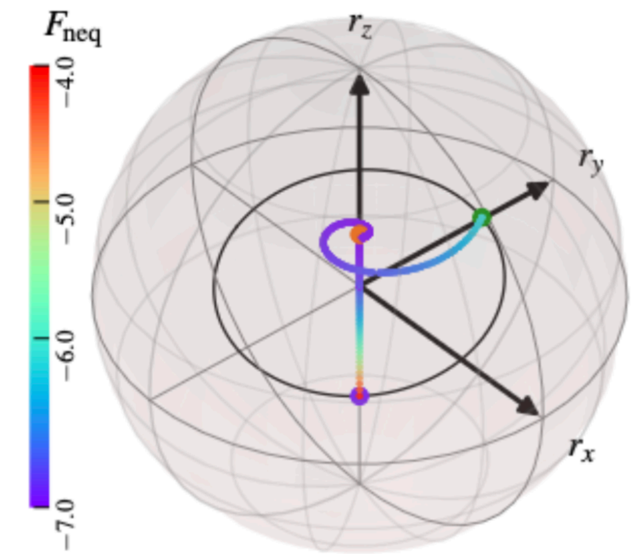
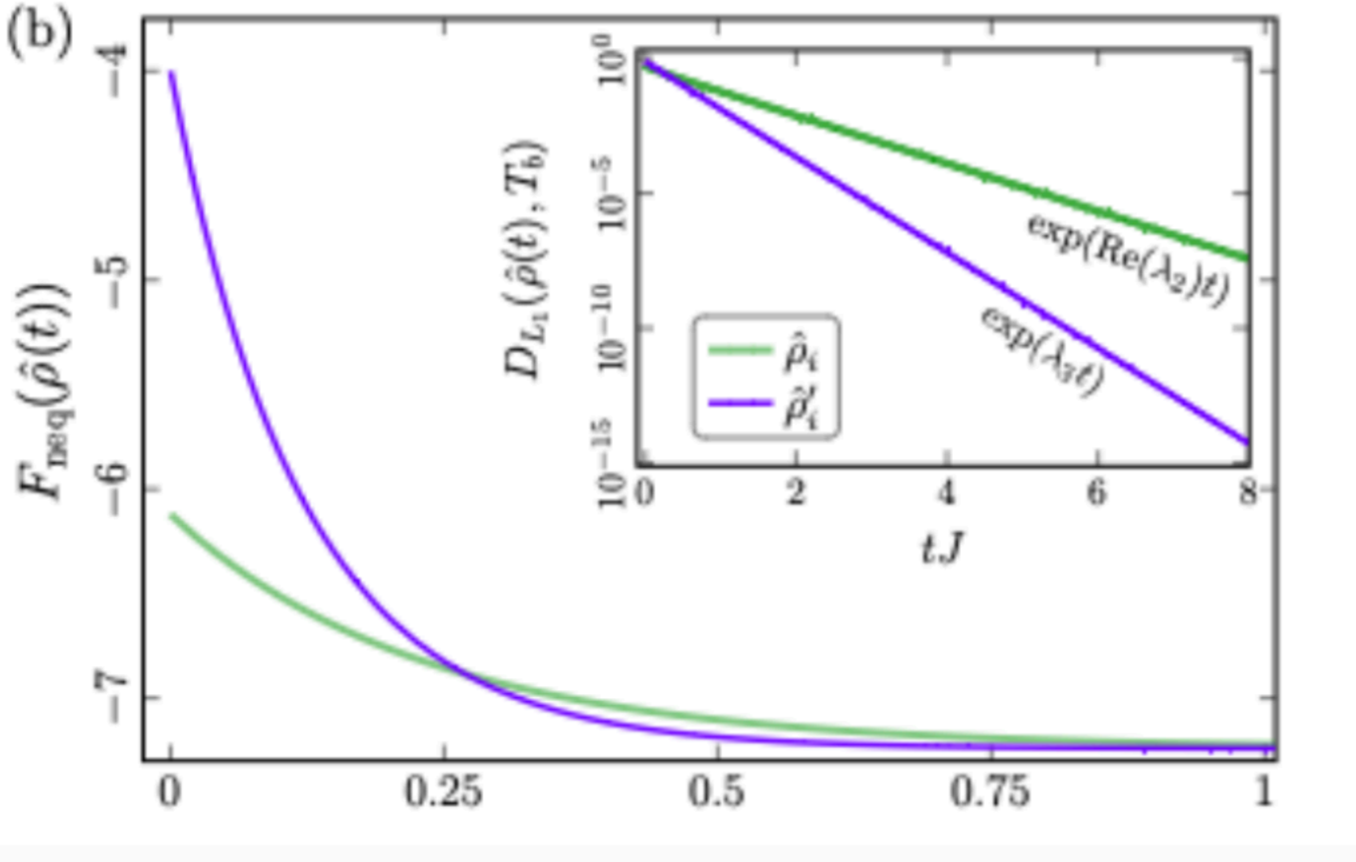
Key finding of our work:

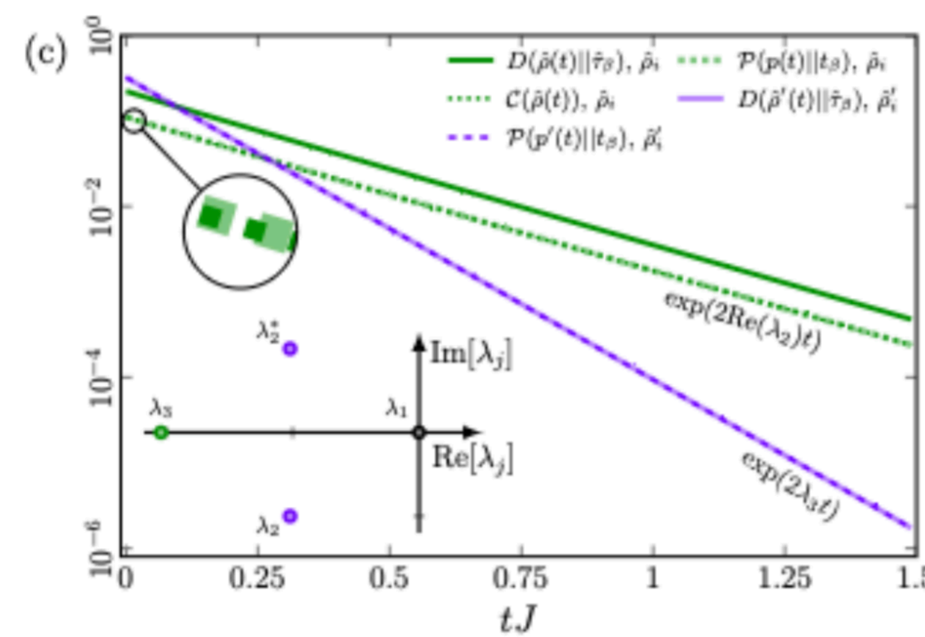
If the generator has a spectral gap defined by a complex eigenpair then a unitary can always be found such that an exponential speed up and a genuine Mpemba effect occurs

Single qubit example

$$\hat{\rho}_i = \frac{1}{2}(\hat{1} + \boldsymbol{r} \cdot \hat{\boldsymbol{\sigma}})$$

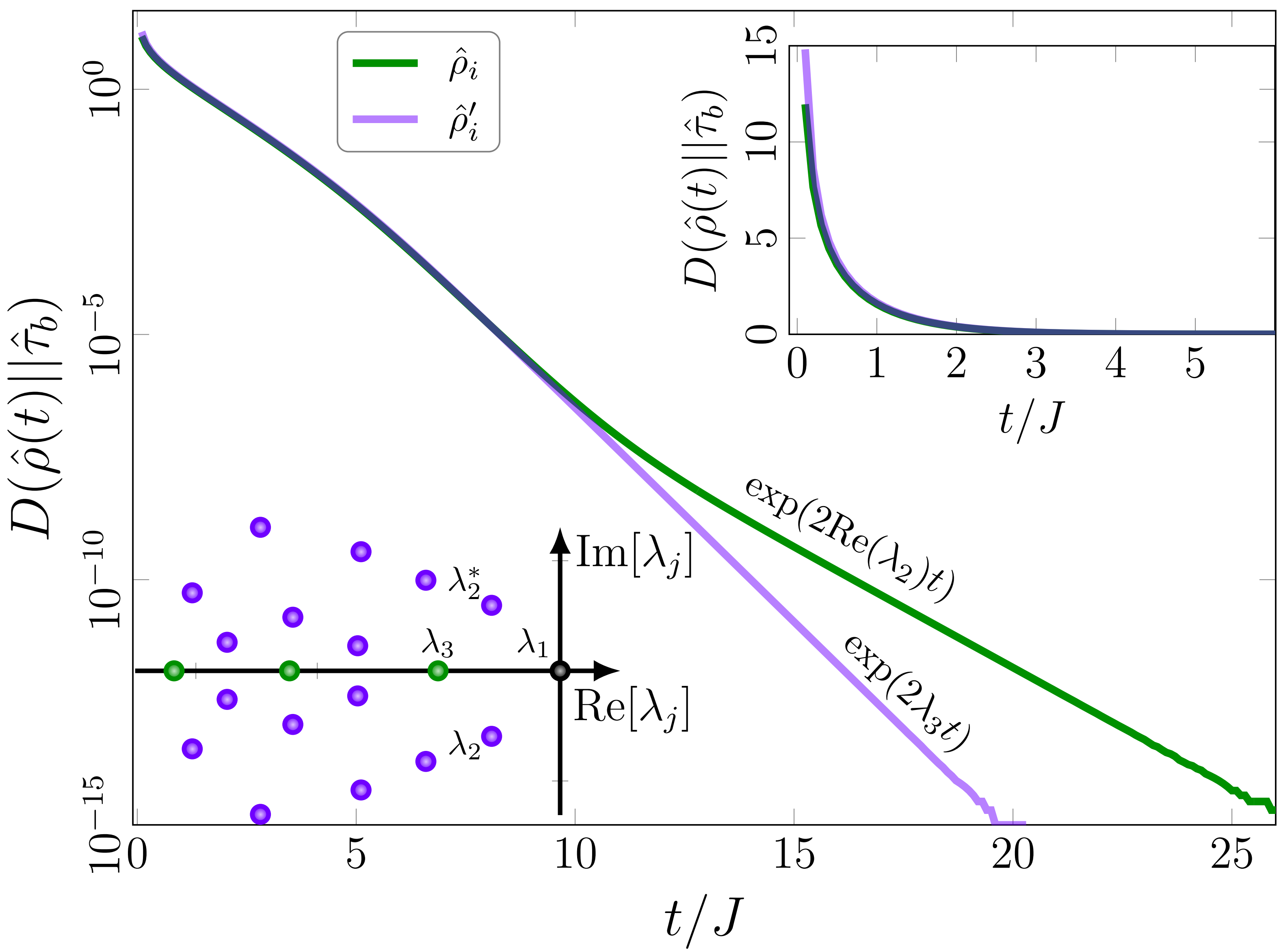






Multi qubit example

$$\hat{H} = -J\sum_{j=1}^{L-1}\hat{\sigma}_j^z\hat{\sigma}_{j+1}^z + h\sum_{j=1}^{L}\hat{\sigma}_j^x$$



Information geometry considerations

Recent interest in connection between information geometry and stochastic thermodyamics

(Ito, Dechant, Nicholson, Hasegawa and others)

Basic idea: random variable X depending on $\theta = \{\theta_1, \theta_2, \ldots \theta_n\}$
with probability density $p(X|\theta)$

Fisher information matrix: $$I_{ij}(\theta) = \mathbb{E}\left[\frac{\partial \log p(X|\theta)}{\partial \theta_i} \frac{\partial \log p(X|\theta)}{\partial \theta_j}\right]$$

Diagonal elements represent amount of information that observable random variable provides about parameter

Information geometry considerations

The Fisher information defines a Riemannian metric on parameter space

$$ds^2 = \sum_{i,j} g_{ij}\, d\theta_i\, d\theta_j$$

Gives statistical measure of distance over a manifold of probability distributions

$$D_{KL}(p(x;\theta) \parallel p(x;\theta + d\theta)) = \int p(x;\theta) \log \frac{p(x;\theta)}{p(x;\theta + d\theta)}\, dx$$

$$ds^2 = D_{KL}(p(x;\theta) \parallel p(x;\theta + d\theta)) \;\approx \frac{1}{2} I(\theta) d\theta^2$$

Statistical length in parameter space:
$$\mathscr{L} = \int_\gamma ds = \int_\gamma \frac{ds}{d\theta}\, d\theta = \int_{\theta_i}^{\theta_f} \sqrt{I_F(\theta)}\, d\theta$$

Quantum considerations

Working in progress with Laetitia Bettmann (TCD)

Classically the fisher information is
uniquely contractive metric on space of probability distributions
under stochastic maps
(Chestnov's theorem)

Quantum mechanically there is a "garden" of Fisher information metrics that are contractive under CPTP trace preserving maps (Petz 1997)

Consider density matrix:

$$\hat{\rho}(\theta) = \sum_x p(\theta)_x \left| x(\theta) \right\rangle \left\langle x(\theta) \right|$$

$$\mathrm{d}s^2 = \sum_x \frac{|dp_x|^2}{p_x} + \sum_{x,y} \frac{|\partial_\theta \rho_{xy}|^2}{p_x f(p_y/p_x)}$$

$f(x) = x f(1/x)$ and $f(1) = 1$

Quantum considerations

$$\mathrm{d}s^2 = \sum_x \frac{|dp_x|^2}{p_x} + \sum_{x,y} \frac{|\partial_\theta \rho_{xy}|^2}{p_x f(p_y/p_x)}$$

$$\hat{\rho}(\theta) = \sum_x p(\theta)_x \left| x(\theta) \right\rangle \left\langle x(\theta) \right|$$

Eg. $f_{SLD}(x) = \frac{x+1}{2}$ $f_{KMB}(x) = \frac{x-1}{\log(x)}$ $f_{WY}(x) = \frac{1}{4}(\sqrt{x}+1)^2$

$$\mathscr{L} = \int_\gamma ds = \int_\gamma \frac{ds}{d\theta}\, d\theta = \int_{\theta_i}^{\theta_f} \sqrt{I_F(\theta)}\, d\theta = \int_{\theta_i}^{\theta_f} \sqrt{I_F^I(\theta)}\, d\theta + \int_{\theta_i}^{\theta_f} \sqrt{I_F^C(\theta)}\, d\theta$$

Time as a parameter:

$$\theta = t$$

$$\mathscr{L}_0^t = \int_\gamma ds = \int_\gamma \frac{ds}{dt}\, dt = \int_0^t v(t)\, dt$$

Instantaneous velocity:

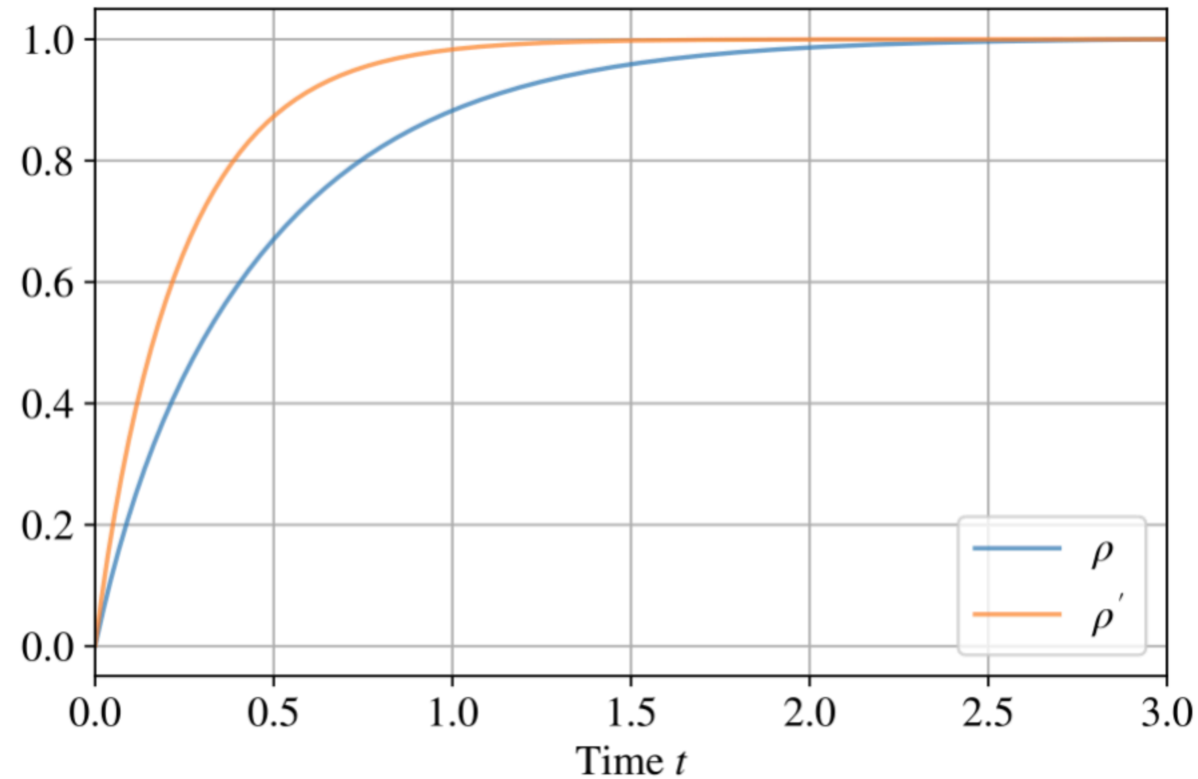
$$v(t) = \sqrt{I(t)}$$

Quantum considerations

$$\mathscr{L}_0^t = \int_\gamma ds = \int_\gamma \frac{ds}{dt}\, dt = \int_0^t v(t)\, dt$$

Degree of completion: $$\phi(s) = \frac{\mathscr{L}_0^s}{\mathscr{L}_0^{t_f}}$$

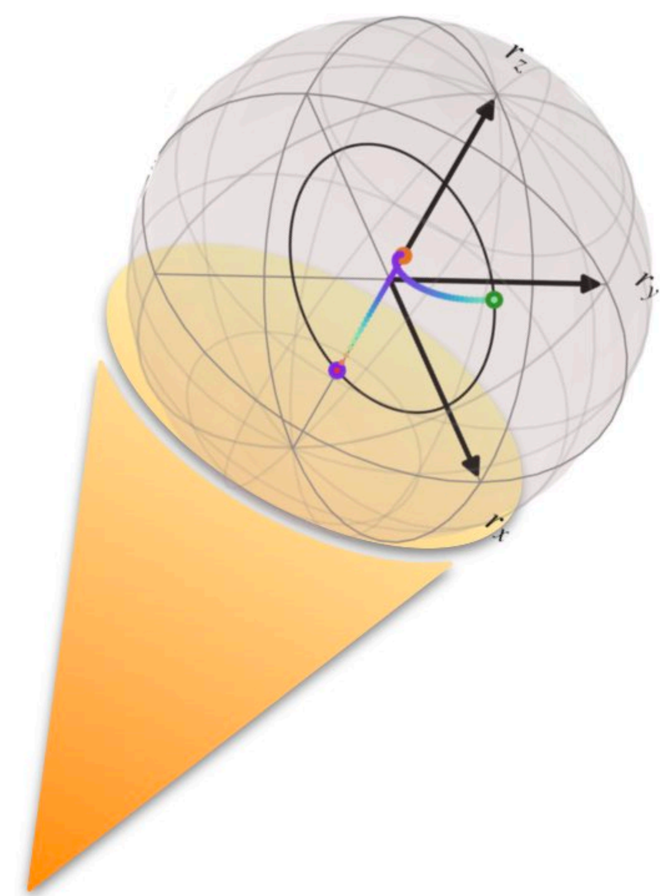
Mpemba:



Some conclusions

If the generator has a spectral gap defined by a complex eigenpair then a unitary can always be found such that an exponential speed up and a genuine Mpemba effect occurs

Closest quantum study to Lu and Raz o



Novel cooling schemes

Dissipative quantum computir

Information geometry a nice framework for unifying the mpemba effects ? ?

thanks

thanks for listening