

Main points

Anticipation and Negative Group Delay in a retina

C. K. Chan (陳志強) ckchan@gate.sinica.edu.tw Institute of Physics, Academia Sinica, Nankang, Taipei, Taiwan



• Anticipatory dynamics (AD) is unusual in that causality is not ordered by time: a retina can produce AD

- The phenomenon of Negative Group Delay (NGD) in the propagation of a pulsed signal can give rise to AD.
- Our goal: to understand the mechanism of anticipatory dynamics by using the concept of negative group delay in a retina

Anticipation in Retina



Flash-lag phenomenon

A moving ring should be at the center of the flash but

Model Result

$$H(\omega) = Y(\omega)/X(\omega) \equiv G(\omega)e^{i\Phi(\omega)} : \text{Response function}$$
$$\Phi(\omega) = -\arctan\left[\frac{w(\beta^2 - gk + w^2)}{\beta gk + \alpha(\beta^2 + w^2)}\right]$$



we see it as moving faster. Motion induces anticipation in our perception

https://slideplayer.com/slide/11470660/



Recording from retina experiments by Berry et al. showed that the anticipation occurs at the retinal network. Group delay : $\delta(\omega) \equiv -d\Phi(\omega)/d\omega$



Only slow time varying signal can be anticipated [3]

Voss's NGD Model for prediction

Experimental Result

Delayed negative feed back can produce prediction The response y(t) is now ahead of the input x(t) [2]

$$\dot{y}(t) = -\alpha y(t) + K[x(t) - y(t - \tau)]$$

Storage of y(t) needed



PHYSICAL REVIEW E 93, 030201(R) (2016)

Our NGD model for a Retina



- Retina from bull frogs on multi-electrode array
- Whole field stimulation through an LED

There are two types of responses from retina: ON and OFF The ON response is delayed The OFF response is advanced Only the OFF pathway of a retina is anticipatory

$$\dot{y}(t) = -\alpha y(t) + K(x(t) - z(t))$$
$$\dot{z}(t) = -\beta z(t) + gy(t)$$

Summary

 Anticipatory dynamics in a retina can be understood as a phenomenon of negative group delay
Only slow varying signal can be anticipated
There is no violation of causality

Berry et al, Anticipation of moving stimuli by the retina, Nature, **1999**, 398, 334
Voss, H.U. Anticipating chaotic synchronization. Physical Review E **2000**, 61, 5115.
C. K. Chan et al, Anticipation and negative group delay in a retina, Physical Review E, **2021**, 103, L020401

3. Relaxational entropy production classifies structures of interaction networks

Yoshiaki Horiike^{1, 2}, Shin Fujishiro, and Masaki Sasai

¹Department of Applied Physics, Nagoya University, Nagoya, Japan

²Department of Neuroscience, University of Copenhagen, Copenhagen, Denmark

Models & Methods

Results

3 classes of relaxational entropy production

relaxational entropy production of 7 networks

In short...

- Relaxational entropy production can classify interaction networks
- The classification **reflects structure** of interaction networks

In poster...

- Pathways of flux
- Quantifying network structure

Scaling in Collective Intermittent Wind **Power Production**

Poster 4.

Samy Lakhal¹, Jim Sardonia², Mahesh M. Bandi¹.

Power turbines measure wind speed fluctuations.

Unexpectedly, total farm power is more intermittent than single turbine.

¹ Nonlinear and Non-equilibrium Physics Unit, OIST Graduate University, Onna 904 0495, Japan ² Exus Renewables North America, Pittsburgh, Pennsylvania, USA

An explanation: atmospheric wind is correlated in time and space, and so are power output.

These correlations prevent extreme-events from smoothing out during aggregation.

More on Poster 4

Irreversibility properties of ECGs as indicators of heart conditions

Cesar Maldonado*, Nazul Merino Negrete* & Raúl Salgado-García[†] *IPICyT - Mexico. †CInC-UAEM - Mexico

Healthy (& young) bodies produce more entropy than ill (& elder) ones.

We symbolize ECG signals

We do it for 3, 5, and 7 symbols.

We compute irreversibility indices for those symbolized sequences

$$e_{p}(P) = \frac{1}{2} \sum_{i,j \in S} (\pi_{i} \rho_{ij} - \pi_{j} \rho_{ji}) \log \frac{\pi_{i} \rho_{ij}}{\pi_{j} \rho_{ji}}.$$
 (1)

$$D_n\left(\mathbb{P}(X_1^n)|\mathbb{P}(X_n^1)\right) = \sum \mathbb{P}(x_1^n) \log \left(\frac{\mathbb{P}(x_1^n)}{\mathbb{P}(x_n^1)}\right) \cdot$$

We introduce the Lag irreversibility function given by:

$$L(\tau) = D(\mathbb{P}(X_n = s_0; X_{n+\tau} = s_{\tau}) | \mathbb{P}(X_n = s_{\tau}; X_{n+\tau} = s_0))$$

$$= \sum_{s_0} \sum_{s_{\tau}} \mathbb{P}(X_n = s_0; X_{n+\tau} = s_{\tau}) \log \begin{pmatrix} \mathbb{P}(X_n = s_0; X_{n+\tau} = s_{\tau}) \\ \mathbb{P}(X_n = s_{\tau}; X_{n+\tau} = s_0) \end{pmatrix}$$

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Irreversibility properties of ECGs as indicators of heart conditions

We obtain clearly higher values of EPR and KLD in healthy subjects

We test the efficiency of our method of discrimination using ROC

analysis

Joint work with Nazul Merino. Irreversibility indices as indicators of heart conditions from electrocardiographic signals. Physica A. 2024

We compute the 1-step Lag irreversibility function for joint variability

signals

Lag irreversibility as a function of the lag

Joint work with Raúl Salgado & Nazul Merino. Sorting ECGs by lag irreversibilty. Physica D. 2023

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Conductance transition with interacting bosons in an Aharonov-Bohm cage

P. S. Muraev,^{1,2} A. R. Kolovsky, ^{2,3} and S. Flach⁴

1 IRC SQC, Siberian Federal University, 660041 Krasnoyarsk, Russia 2 School of Engineering Physics and Radio Electronics, Siberian

Federal University, 660041 Krasnoyarsk, Russia

3 Kirensky Institute of Physics, Federal Research Center KSC SB RAS, 660036 Krasnoyarsk, Russia

4 Center for Theoretical Physics of Complex Systems, Institute for Basic Science, 34126 Daejeon, Republic of Korea

Introduction

One of the most famous example of the ABF (all bands are flat) lattices is the π -flux diamond chain. In the present work we address the effect of an interparticle interaction on the transport of Bose particles across the diamond lattice from the viewpoint of laboratory experiments where one injects bosons into the first site of the lattice by using an external coherent driving pump and withdraws them from the last site with a sink.

The Model

The considered setup is described by the following master equation for the reduced density matrix $\hat{\mathcal{R}}(t)$ of microwave photons,

$$\frac{\mathrm{d}\hat{\mathcal{R}}}{\mathrm{d}t} = -\frac{i}{\hbar_{\mathrm{eff}}} \left[\hat{\mathcal{H}}, \hat{\mathcal{R}}\right] - \frac{\gamma}{2} \left(\hat{a}_L^{\dagger} \hat{a}_L \hat{\mathcal{R}} - 2\hat{a}_L \hat{\mathcal{R}} \hat{a}_L^{\dagger} + \hat{\mathcal{R}} \hat{a}_L^{\dagger} \hat{a}_L\right)$$

where γ is the rate of photon absorption by a measurement device and the Hamiltonian $\hat{\mathcal{H}}$ has the form

$$\begin{aligned} \widehat{\mathcal{H}} &= \hbar_{\rm eff} \Delta \sum_{l=1}^{L} \widehat{n}_l - \frac{1}{2} \sum_{l,l'=1}^{L} J_{l,l'} \hbar_{\rm eff} (\widehat{a}_l^{\dagger} \widehat{a}_{l'} + {\rm H.c.}) + \\ &+ \frac{g \hbar_{\rm eff}^2}{2} \sum_{l=1}^{L} \widehat{n}_l (\widehat{n}_l - 1) + \frac{\Omega \sqrt{\hbar_{\rm eff}}}{2} (\widehat{a}_1^{\dagger} + \widehat{a}_1). \end{aligned}$$

Results

We demonstrate that in the classical regime the system is insulating up to a critical value of the pump strength in the presence of mean-field interactions, while in the quantum regime there is a particle pair transport and weak conductance below the critical pump strength. Additionally, we will show that there is a swift crossover from the quantum into the classical regime by using a pseudoclassical method upon increasing the number of particles or decreasing the effective Planck constant $\hbar_{\rm eff}$.

7. Trapping of active Brownian and run-and-tumble particles: A first-passage time approach

E. Q. Z. Moen¹, K. S. Olsen², J. Rønning³ and L. Angheluta¹

- 1. The Njord Center, Department of Physics, University of Oslo
- 2. Institut für Theoretische Physik II Weiche Materie, Heinrich-Heine-Universität Düsseldorf
- 3. Nonlinear and Non-Equilibrium Physics Unit, Okinawa Institute of science and technology

P. 8 Influence of reflecting boundary on fluctuation relation in a lattice random walk model

Exact Work Distribution and Jarzynski's Equality of a Relativistic Particle in an Expanding Piston

Tingzhang Shi Peking University, China July 2nd, 2024

Xianghang Zhang, Tingzhang Shi, H. T. Quan, arXiv:2403.15986

Outline

- Background and motivation
- Setup of the relativistic piston model
- Trajectory of a particle and verification of the Jarzynski's equality
- Relativistic work distribution and its non-relativistic limit
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Background and motivation The Jarzynski's equality

- Second law of thermodynamics: $\langle W \rangle \ge \Delta F$
- Jarzynski's equality:

$$\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}$$

• It is possible to obtain equilibrium thermodynamic

Background and motivation The Jarzynski's equality

• Trajectory work:

 $W(q_0, p_0) = H(q_1, p_1; \lambda_1) - H(q_0, p_0; \lambda_0)$

• The proof of Jarzynski's equality:

$$\begin{split} &\langle e^{\beta W} \rangle \\ &= \int \mathrm{d} p_{i,0} \mathrm{d} q_{i,0} \, \rho(p_{i,0}, q_{i,0}) e^{\beta W(p_{i,0}, q_{i,0}, \tau)} \\ &= \frac{1}{Z_0} \int \mathrm{d} p_{i,0} \mathrm{d} q_{i,0} \, e^{-\beta H_{\lambda(\tau)}(p_{i,\tau}, q_{i,\tau})} \\ &= \frac{1}{Z_0} \int \mathrm{d} p_{i,\tau} \mathrm{d} q_{i,\tau} \, e^{-\beta H_{\lambda(\tau)}(p_{i,\tau}, q_{i,\tau}))} \\ &= \frac{Z_{\tau}}{Z_0}, \end{split}$$

C. Jarzynski, Phys. Rev. Lett. 78, 2690 (1997).

Background and motivation Piston model

• Piston model: Paradigmatic

Background and motivation Piston model

- Piston model: Exact solvable
- Previous work:

Classical Newtonian piston [R. C. Lua and A. Y. Grosberg, 2005] Quantum non-relativistic piston [H. T. Quan and C. Jarzynski, 2012]

Classical relativistic piston (single kick limit) [R. Nolte and A. Engel, 2009]

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Setup of the relativistic piston model

- A single particle inside a one-dimensional cylinder with a moving piston
- The collision with the piston is elastic.
- The gas is of the inverse temperature β .

Setup of the relativistic piston model

- Elastic collision: in the piston frame, the speed of the particle doesn't change during a collision.
- Trajectory work: $W(q_0, p_0) = H(q_1, p_1; \lambda_1) H(q_0, p_0; \lambda_0)$
- Time slides are defined in the laboratory frame.

Setup of the relativistic piston model

F. Jüttner, Annalen der Physik 339, 856 (1911).

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• Trajectory of a particle:

 The key is to obtain the speed after the n-th collision and the time of the n-th collision with the moving piston.

• Recursion relation of the speed :

$$v_{n+1} = \frac{(c^2 + v_p^2)v_n - 2v_pc^2}{c^2 + v_p^2 - 2v_pv_n}.$$

• Solution to the recursion relation:

$$v_n = \frac{(c+v)\alpha_p^{2n} - c + v}{(c+v)\alpha_p^{2n} + c - v} \cdot c, \quad \text{with} \qquad \alpha_p = \frac{c-v_p}{c+v_p}.$$

- Recursion relation of the time of the n-th collision with the piston: $t_{n+1} = \frac{2L + (v_p + v_n)t_n}{v_n - v_n}$,
- Solution to the recursion relation:

$$t_n = \left[(-\alpha_p^{2n} - \alpha_p + \alpha_p^{n+1} + \alpha_p^n) \frac{v}{c} + (-\alpha_p^{n+1} + \alpha_p^n) \frac{x}{L} + (\alpha_p - \alpha_p^{2n}) \right] \\ \left[-\alpha_p + \alpha_p^{2n} + (\alpha_p + \alpha_p^{2n}) \frac{v}{c} \right]^{-1} \cdot \frac{L(1 + \alpha_p)}{c(1 - \alpha_p)},$$

• Solution to the recursion relation:

$$t_{n} = \left[(-\alpha_{p}^{2n} - \alpha_{p} + \alpha_{p}^{n+1} + \alpha_{p}^{n}) \frac{v}{c} + (-\alpha_{p}^{n+1} + \alpha_{p}^{n}) \frac{x}{L} + (\alpha_{p} - \alpha_{p}^{2n}) \right] \\ \left[-\alpha_{p} + \alpha_{p}^{2n} + (\alpha_{p} + \alpha_{p}^{2n}) \frac{v}{c} \right]^{-1} \cdot \frac{L(1 + \alpha_{p})}{c(1 - \alpha_{p})},$$

• Final collision number: $n = \max\{N | t_N \le \tau\}$

• Final position and momentum:

 $(x_{\tau}, p_{\tau}) = (|L - v_n \tau + (v_n + v_p)t_n|, p_n)$ with p_{τ}

$$_{n} = \frac{mv_{n}}{\sqrt{1 - \left(\frac{v_{n}}{c}\right)^{2}}}.$$

• Liouville's theorem can be proved:

$$\left|\frac{\partial(x_{\tau}, p_{\tau})}{\partial(x, p)}\right| = 1,$$

which verifies the Jarzynski's equality.

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Relativistic work distribution and its non-relativistic limit

- Non-dimenssionalize: m, L, k_B, c=1.
- Definition of the work distribution:

$$P(W) = \int_{-1}^{1} \mathrm{d}x \int_{0}^{1} \mathrm{d}v \frac{e^{-\frac{\beta}{\sqrt{1-v^{2}}}} \delta(W - W_{\tau}(x, v))}{2K_{1}(\beta)(1-v^{2})^{\frac{3}{2}}},$$

• After some tedious calculations, the distribution function of W can be analytically expressed as $P(W) = P_0 \delta(W) + \frac{1}{2K_1(\beta)} \sum_{n=1}^{N} \varphi_n(v_n(W)] \times \frac{e^{-\beta/\sqrt{1-v_n(W)^2}}}{(\alpha_p^{-n}-1) \left[1+\alpha_p^n-v_n(W)(1-\alpha_p^n)\right]}.$

Relativistic work distribution and its non-relativistic limit

• The work distribution P(W) is:

$$P(W) = P_0 \delta(W) + \frac{1}{2K_1(\beta)} \sum_{n=1}^N \varphi_n(v_n(W)) \times \frac{e^{-\beta/\sqrt{1-v_n(W)^2}}}{(\alpha_p^{-n}-1)\left[1+\alpha_p^n-v_n(W)(1-\alpha_p^n)\right]}.$$

• With

$$v_n(W) = \frac{(1 - \alpha_p^n)^3 (1 + \alpha_p^n) + 4W \sqrt{\alpha_p^{3n} \left((1 - \alpha_p^n)^2 + \alpha_p^n W^2\right)}}{(1 - \alpha_p^{2n})^2 + 4\alpha_p^{2n} W^2},$$

$$\int \frac{1 - \xi_n(v), \quad \frac{X_n - 1}{T_n} < v_n(W) \le \frac{X_n + 1}{T_n}}{T_n},$$

$$\varphi_n(v) = \begin{cases} 2, & \frac{X_n + 1}{T_n} < v_n(W) \le \frac{X_{n+1} - 1}{T_{n+1}} \\ 1 + \xi_{n+1}(v), & \frac{X_{n+1} - 1}{T_{n+1}} < v_n(W) \le \frac{X_{n+1} + 1}{T_{n+1}} \end{cases},$$

• Overlap factor:

Relativistic work distribution and its non-relativistic limit

 We may recover the dimension of the expressions and let *c* → ∞, then we have the non-relativistic limit of the work distribution.

$$\begin{split} P(W) = & P_0 \delta(W) + \frac{\sqrt{\beta}}{\sqrt{2\pi} n v_p} e^{-\frac{\beta}{2} \left(\frac{W}{2nv_p} + nv_p\right)^2} f(W), \\ f(W) = \begin{cases} -(n-1)(1+\frac{v_p}{2}) + \frac{W}{4nv_p}, & (n-1)(v_p+2) < \frac{W}{2nv_p} \le (n-1)(v_p+2) + 2 \\ 1, & (n-1)(v_p+2) + 2 < \frac{W}{2nv_p} \le (n-1)(v_p+2) + 2 + 2v_p \\ (n+1)(1+\frac{v_p}{2}) - \frac{W}{4nv_p}, & (n-1)(v_p+2) + 2 + 2v_p < \frac{W}{2nv_p} \le (n+1)(v_p+2) \end{cases} \end{split}$$

R. C. Lua and A. Y. Grosberg, J. Phys. Chem. B 109, 6805 (2005)

The comparison between relativistic work distribution and its non-relativistic limit

$$P(W) = P_0 \delta(W) + \frac{1}{2K_1(\beta)} \sum_{n=1}^N \varphi_n(v_n(W)) \times \frac{e^{-\beta/\sqrt{1 - v_n(W)^2}}}{(\alpha_p^{-n} - 1) \left[1 + \alpha_p^n - v_n(W)(1 - \alpha_p^n)\right]}$$

$$\begin{split} v_n(W) &= \quad \frac{(1-\alpha_p^n)^3(1+\alpha_p^n)+4W\sqrt{\alpha_p^{3n}\left((1-\alpha_p^n)^2+\alpha_p^nW^2\right)}}{(1-\alpha_p^{2n})^2+4\alpha_p^{2n}W^2},\\ \varphi_n(v) &= \begin{cases} 1-\xi_n(v), & \frac{X_n-1}{T_n} < v_n(W) \leq \frac{X_n+1}{T_n}\\ 2, & \frac{X_n+1}{T_n} < v_n(W) \leq \frac{X_{n+1}-1}{T_{n+1}},\\ 1+\xi_{n+1}(v), & \frac{X_{n+1}-1}{T_{n+1}} < v_n(W) \leq \frac{X_{n+1}+1}{T_{n+1}}, \end{cases} \end{split}$$

$$\begin{split} P(W) = P_0 \delta(W) + & \frac{\sqrt{\beta}}{\sqrt{2\pi} n v_p} e^{-\frac{\beta}{2} \left(\frac{W}{2nv_p} + nv_p\right)^2} f(W), \\ f(W) = \begin{cases} -(n-1)(1+\frac{v_p}{2}) + \frac{W}{4nv_p}, & (n-1)(v_p+2) < \frac{W}{2nv_p} \le (n-1)(v_p+2) + 2 \\ 1, & (n-1)(v_p+2) + 2 < \frac{W}{2nv_p} \le (n-1)(v_p+2) + 2 + 2v_p \\ (n+1)(1+\frac{v_p}{2}) - \frac{W}{4nv_p}, & (n-1)(v_p+2) + 2 + 2v_p < \frac{W}{2nv_p} \le (n+1)(v_p+2) \end{cases} \end{split}$$

The comparison between relativistic work distribution and its non-relativistic limit

Relativistic and non-relativistic work distribution with different parameters. The initial length is 1*cm*. The protocol is: $(a)\tau = 0.3ns, v_p = 3 \times 10^7 m/s, T = 3 \times 10^{12} K;$ $(b)\tau = 1ns, v_p = 1 \times 10^7 m/s, T = 3 \times 10^{11} K;$ $(c)\tau = 3ns, v_p = 3 \times 10^6 m/s, T = 3 \times 10^{10} K.$ Relativistic work distribution has no zeros, which might be detected.

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Summary

- 1D classical relativistic piston model is an exact solvable model. We analytical solve the trajectory and the work distributuion of the relativistic piston model, and verify the Jarzynski's equality.
- In the non-relativistic limit, our results recover the non-relativistic results [Lua, & Grosberg, J. Phys. Chem. B, 109, 6805(2005)].
- We also find that, unlike the non-relativistic case, the maximun number of collisions in this relativistic gas model is finite, and the relativistic work distribution no longer has zeros.
- It is difficult to detect the relativistic effects of the work distribution of the ideal gas in a piston system with the current experimental techniques.

Geometric Thermodynamic Uncertainty Relation for Cyclic Heat Engines

Guo-Hua Xu, Keiji Saito, and Gentaro Watanabe

Motivation

Providing a TUR of cyclic heat engines for general currents including work and heat

Setup: Flywheel as a Degree of Freedom

$$\dot{x} = a(x, y) + \xi_t,$$

$$\langle \xi_t \xi_s \rangle = 2D\delta(t - s)$$

$$\dot{y} = \omega,$$

General current

$$O = \int_0^\tau \mathrm{d}t \Lambda_x(x, y) \circ \mathrm{d}x + \Lambda_y(x, y) \circ \mathrm{d}y$$

Main result I: TUR for cyclic heat engines

Using Cramer-Rao inequality

$$\frac{\operatorname{Var}[O]}{\langle O \rangle^2} \ge \frac{2}{\mathcal{T}\sigma_Q}$$

T. Koyuk, U. Seifert, P. Pietzonka, J. Phys. A: Math. Theor. **52**, 02LT02 (2018)

Geometric bound in the linear response regime

$$\sigma_Q \le \frac{\pi}{2} \mathcal{L}^2$$

$$\sigma_Q$$

Effective entropy production (EP) rate: EP rate for arbitrary distribution Q(x)

$$Q(x) = P_{eq}(\lambda_0, D_0)$$

 σ_Q is EP in the fast driving limit

Main result II: geometric TUR

$$\frac{\operatorname{Var}[O]}{\langle O \rangle^2} \ge \frac{4}{\pi \mathcal{L}^2 \tau}$$

Seeking Postdoctoral Position:

If you know of any opportunities or have suggestions, please feel free to reach out to me. Thank you!

Non-equilibrium Design Principles — Intelligent Molecule for Temporal Pattern Recognition and Computation

No. 12, 2nd week

Zhongmin Zhang and Zhiyue Lu, Department of Chemistry, University of North Carolina at Chapel Hill

Discrepancy design rule: thermally stable states are kineticly slow to reach.

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Full Non-equilibrium Controllability

Universal Fluctuation-Response Relations of Non-equilibrium Dynamics: A Trajectory Information Geometry Framework

> Jiming Zheng^{*} and Zhuyue Lu⁺ University of North Carolina at Chapel Hill Poster Number: #13, 2nd week

arXiv: 2403.10952

System and Setup

Master equation

$$\frac{\boldsymbol{p}(t)}{\partial t} = R \cdot \boldsymbol{p}(t)$$

 ∂

Only Requirement

R is irreducible

Control Parameter

 $R = R(\boldsymbol{\xi})$

Modeling Response

$$\partial_{\xi} \langle Q \rangle \equiv \frac{\partial \langle Q \rangle}{\partial \xi}$$

Sensitivity

$$\Delta \langle Q \rangle = |\langle Q \rangle' - \langle Q \rangle|$$

Non-perturbative Response

