Jump Trajectories for General Quantum Master Equations

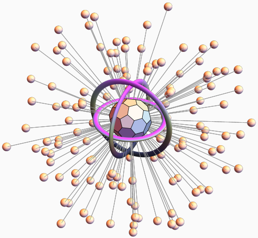
Paul Menczel, Theoretical Physics Laboratory, RIKEN

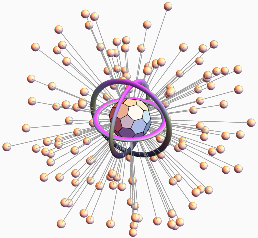
12 July 2024

Frontiers in Non-Equilibrium Physics 2024

Setting

- OQS described by master equation
- Discuss quantum jump trajectories





Setting

- OQS described by master equation
- Discuss quantum jump trajectories

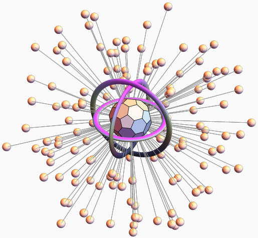
What master equation?

- Lindblad:

$$\partial_t \rho = -\mathrm{i}[H, \rho] + \sum_\alpha \gamma_\alpha \big(L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\}/2 \big)$$

$\rightarrow$ Markovian weak-coupling limit (cf. next talk)

$\rightarrow$ Jump trajectory framework well known



Setting

- OQS described by master equation
- Discuss quantum jump trajectories

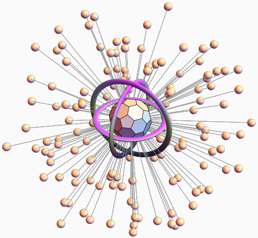
What master equation?

- Lindblad
- Non-Markovian Lindblad (negative rates):

$$\partial_t \rho = -\mathrm{i}[H, \rho] + \sum_\alpha \gamma_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\}/2 \right)$$

$\rightarrow$ TCL master equation, Redfield equation

$\rightarrow$ May be completely positive (not CP-divisible)



Setting

- OQS described by master equation
- Discuss quantum jump trajectories

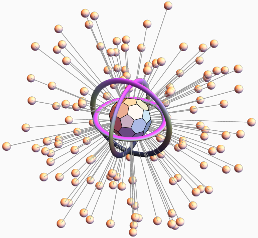
What master equation?

- Lindblad
- Non-Markovian Lindblad (negative rates)
- Non-Physical Lindblad (complex rates, non-Hermitian H):

$$\partial_t \rho = -\mathrm{i}[H, \rho] + \sum_\alpha \gamma_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\}/2 \right)$$

$\rightarrow$ Embedding for physical non-Markovian dynamics (pseudomodes)

$\rightarrow$ Non-Hermitian state ρ



Setting

- OQS described by master equation
- Discuss quantum jump trajectories

What master equation?

- Lindblad
- Non-Markovian Lindblad (negative rates)
- Non-Physical Lindblad (complex rates, non-Hermitian H)
- Generic time-local master equation:

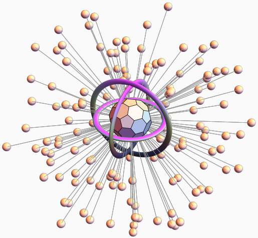
$$\partial_t \rho = A\rho + \rho B^\dagger + \sum_\alpha C_\alpha \rho D_\alpha^\dagger$$

$\rightarrow$ Most general mathematical form

$\rightarrow$ Feedback, counting fields

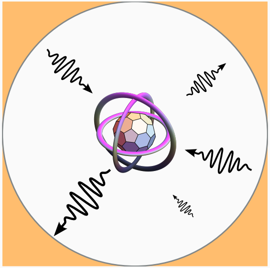
Setting

- OQS described by master equation
- Discuss quantum jump trajectories



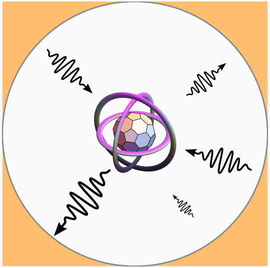
What master equation?

- Lindblad
- Non-Markovian Lindblad (negative rates)
- Non-Physical Lindblad (complex rates, non-Hermitian H)
- Generic time-local master equation



Lindblad equation:

$\partial_t \rho = -\mathrm{i}[H, \rho] + \sum_\alpha \gamma_\alpha \big(L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\}/2 \big)$

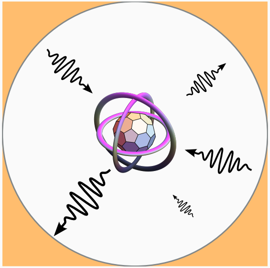


Lindblad equation:

$$\partial_t \rho = -\mathrm{i}[H,\rho] + \sum_\alpha \gamma_\alpha \big(L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\}/2\big)$$

Jump Trajectories:

- Stochastic evolution of pure state $|\psi\rangle$ such that $\rho = \mathbb{E}\{|\psi\rangle\langle\psi|\}$
- $$\partial_t|\psi\rangle = \sum_\alpha \Big[\frac{1}{\langle L_\alpha^\dagger L_\alpha\rangle^{1/2}} L_\alpha|\psi\rangle - |\psi\rangle\Big]\,\mathrm{d}N_\alpha + \Big[-\mathrm{i}H - \sum_\alpha \gamma_\alpha (L_\alpha^\dagger L_\alpha - \langle L_\alpha^\dagger L_\alpha\rangle)\Big]|\psi\rangle\,\mathrm{d}t$$
- $\mathrm{d}N_\alpha$ are Poisson increments, $\mathbb{E}\{\mathrm{d}N_\alpha\} = \gamma_\alpha \langle L_\alpha^\dagger L_\alpha\rangle\,\mathrm{d}t$ $\rightarrow$ only works for positive rates



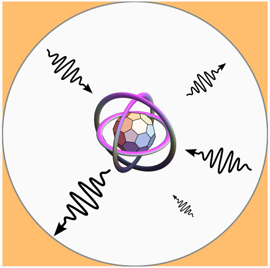
Lindblad equation:

$$\partial_t \rho = -\mathrm{i}[H,\rho] + \sum_\alpha \gamma_\alpha \big(L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\}/2\big)$$

Jump Trajectories:

- Stochastic evolution of pure state $|\psi\rangle$ such that $\rho = \mathbb{E}\{|\psi\rangle\langle\psi|\}$
- $\partial_t |\psi\rangle = \sum_\alpha \Big[\frac{1}{\langle L_\alpha^\dagger L_\alpha\rangle^{1/2}} L_\alpha |\psi\rangle - |\psi\rangle\Big]\,\mathrm{d}N_\alpha + \Big[-\mathrm{i}H - \sum_\alpha \gamma_\alpha (L_\alpha^\dagger L_\alpha - \langle L_\alpha^\dagger L_\alpha\rangle)\Big]|\psi\rangle\,\mathrm{d}t$
- $\mathrm{d}N_\alpha$ are Poisson increments, $\mathbb{E}\{\mathrm{d}N_\alpha\} = \gamma_\alpha \langle L_\alpha^\dagger L_\alpha\rangle\,\mathrm{d}t$
  $\rightarrow$ only works for positive rates

Focus on quantum jump unravellings (Poisson noise)
Quantum diffusion (Gaussian noise) $\rightarrow$ Landi *et al.*, PRXQ (2024)



Advantages:

- Monte-Carlo simulations: $\mathcal{O}(N^2) \rightarrow \mathcal{O}(N)$ in parallel
- Theoretical access to fluctuations

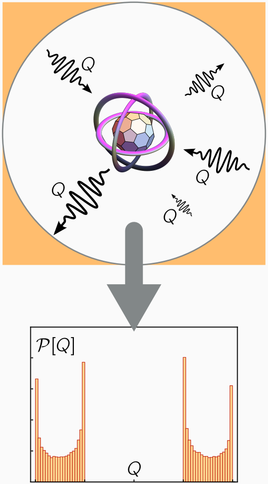
Advantages:

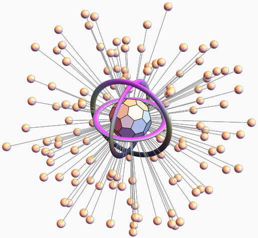
- Monte-Carlo simulations:
  $\mathcal{O}(N^2) \to \mathcal{O}(N)$ in parallel
- Theoretical access to fluctuations

Small advertisement:

- In cyclic operation,
  $$\Delta S \geq 2A\, \lambda_Q \,\mathrm{artanh}\, \lambda_Q$$
  λ_Q characterizes energy frequency dist.
- Leads to power-efficiency trade-off etc.

PM, C. Flindt, and K. Brandner, PRR (2020)





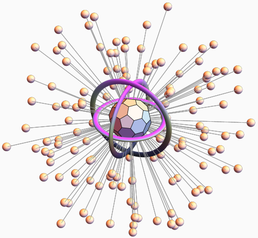
Negative rates (at some or all times):

$\partial_t \rho = -\mathrm{i}[H,\rho] + \sum_\alpha \gamma_\alpha \big(L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\}/2 \big)$

→ Time-convolutionless (TCL) master equation

→ Redfield equation

→ Classical noise models

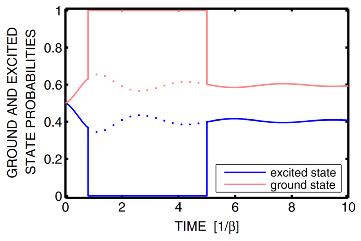


Negative rates (at some or all times):

$$\partial_t \rho = -\mathrm{i}[H,\rho] + \sum_\alpha \gamma_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\}/2\right)$$

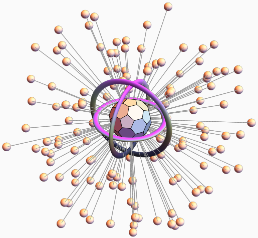
→ Time-convolutionless (TCL) master equation
→ Redfield equation
→ Classical noise models

Approach 1: "backward jumps"



J. Piilo *et al.*, PRL (2008)

Non-Markovian Lindblad



Negative rates (at some or all times):

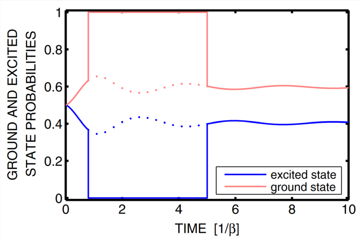
$$\partial_t \rho = -\mathrm{i}[H,\rho] + \sum_\alpha \gamma_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\}/2\right)$$

→ Time-convolutionless (TCL) master equation

→ Redfield equation

→ Classical noise models

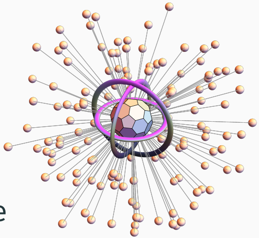
Approach 1: "backward jumps"



- Only works for completely positive dynamics
- Evolution of different trajectories is coupled

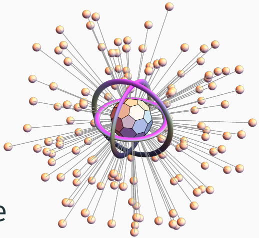
J. Piilo *et al.*, PRL (2008)

Approach 2: "influence martingale"

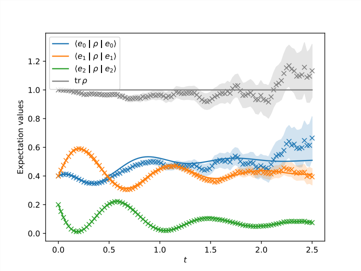


- Introduce scalar μ with $\rho = \mathbb{E}\{\mu|\psi\rangle\langle\psi|\}$
- $|\psi\rangle$ and μ satisfy coupled stochastic differential equations
- Method is part of QuTiP v5 as `nm_mcsolve`

Approach 2: "influence martingale"



- Introduce scalar μ with $\rho = \mathbb{E}\{\mu|\psi\rangle\langle\psi|\}$
- $|\psi\rangle$ and μ satisfy coupled stochastic differential equations
- Method is part of QuTiP v5 as `nm_mcsolve`



- Example: Redfield equation for two qubits coupled to same reservoir (10k trajectories)
- Instability at longer times (μ grows exponentially)

B. Donvil *et al.*, Nat Commun (2022)

Generic time-local master equation:

$$\partial_t \rho = A\rho + \rho B^\dagger + \sum_\alpha C_\alpha \rho D_\alpha^\dagger$$

Hermiticity not preserved, $\rho = \mathbb{E}\{|\psi\rangle\langle\psi|\}$ clearly impossible

- Trajectories in double Hilbert space: $\rho = \mathbb{E}(|\psi_1\rangle\langle\psi_2|)$
  $\rightarrow$ coupled SDEs for $|\psi_1\rangle$ and $|\psi_2\rangle$
  $\rightarrow$ same instability at longer times
  Breuer *et al.*, PRA (1999)
- Embedded in Lindblad equation for $\bar{\rho}$ on double space:
  $$\rho = \mathrm{e}^{\int_0^t \mathrm{d}\tau\, \alpha(\tau)}\, \mathrm{tr}_\mathrm{a}\left[X_\mathrm{a}\bar{\rho}\right]$$
  $\rightarrow$ same instability: $\alpha(\tau) \geq 0$ (zero iff Lindblad)
  Hush *et al.*, PRA (2015)

Non-Physical Lindblad

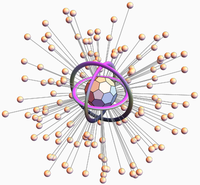
Complex rates and non-Hermitian Hamiltonian:
$\partial_t \rho = -\mathrm{i}[H, \rho] + \sum_\alpha \gamma_\alpha \big(L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\}/2 \big)$

Complex rates and non-Hermitian Hamiltonian:

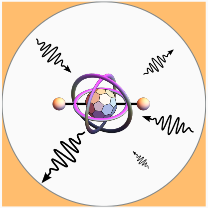
$\partial_t \rho = -\mathrm{i}[H, \rho] + \sum_\alpha \gamma_\alpha \big(L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\}/2\big)$

Why?

Pseudomode framework provides embedding of non-Markovian dynamics (Gaussian bath) in such equation: $\rho_{\mathrm{sys}} = \mathrm{tr}_{\mathrm{pm}}\, \rho$ exactly equals correct system state



$=$



PM, K. Funo, M. Cirio, N. Lambert, and F. Nori, arXiv:2401.11830

Non-Physical Lindblad: Unraveling

Trajectories in double Hilbert space: $\rho = \mathbb{E}(|\psi_1\rangle\langle\psi_2|)$

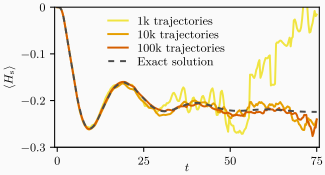
$$\mathrm{d}|\psi_1\rangle = \left[-\mathrm{i}H - \frac{1}{2}\sum_\alpha \left(\gamma_\alpha L_\alpha^\dagger L_\alpha - r_\alpha\right)\right]|\psi_1\rangle\,\mathrm{d}t + \sum_\alpha \left[\sqrt{\frac{\gamma_\alpha}{r_\alpha}} L_\alpha|\psi_1\rangle - |\psi_1\rangle\right]\mathrm{d}N_\alpha,$$

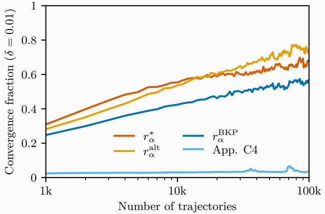
$$\mathrm{d}|\psi_2\rangle = \left[-\mathrm{i}H^\dagger - \frac{1}{2}\sum_\alpha \left(\gamma_\alpha^* L_\alpha^\dagger L_\alpha - r_\alpha\right)\right]|\psi_2\rangle\,\mathrm{d}t + \sum_\alpha \left[\sqrt{\frac{\gamma_\alpha}{r_\alpha}}^{\,*} L_\alpha|\psi_2\rangle - |\psi_2\rangle\right]\mathrm{d}N_\alpha$$

$\rightarrow$ jump rates $r_\alpha > 0$ can be chosen freely
$\rightarrow$ choose them to minimize fluctuations of $|\langle\psi_2 \mid \psi_1\rangle|$
$\rightarrow$ exponential growth of fluctuations unavoidable

PM, K. Funo, M. Cirio, N. Lambert, and F. Nori, arXiv:2401.11830

- Spin-Boson Model: $H_s = \frac{\Delta}{2}\sigma_x$, coupling $Q = \sigma_z$
- Underdamped bath, finite temperature, neglect Matsubara
- Exponential growth of required trajectories
- Other stochastic methods have similar problems
  - → Stockburger, EPL (2016)
  - → Luo *et al.*, PRXQ (2023)





PM, K. Funo, M. Cirio, N. Lambert, and F. Nori, arXiv:2401.11830

Summary

- Quantum jump trajectories are a useful tool.
- For general master equations, one can do jump unravelings in the double Hilbert space.
- They appear to be unavoidably unstable at long times.
- We determined the optimal rates to minimize instability.

- Outlook:
  - Can we use unravelings like this to study fluctuations of physical quantities? (If not, why not?)
  - Is there a similar issue in classical stochastic dynamics?
  - Can the instability be used to characterize completely positive maps?
  - Does it tell us anything about Markovian embeddings?

Jump Trajectories for General Quantum Master Equations

Paul Menczel, Theoretical Physics Laboratory, RIKEN

12 July 2024

Frontiers in Non-Equilibrium Physics 2024