Thermodynamics of fidelity in quantum gates

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with Tomotaka Kuwahara (RIKEN) and Keiji Saito (Kyoto U)





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Outline

- Background and motivation
- Setup and main results
- Numerical demonstration

Quantum gates



superconducting (wiki)



trapped ions (wiki)



photonic (Madsen+, Nature 2022)

NMR (wiki)



Quantum gates



superconducting (wiki)



trapped ions (wiki)



Quantum gates: Hadamard, CNOT, CZ, Toffoli, etc.



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Quantum gates



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NMR (wiki)

Implementation of fast and high-fidelity quantum gates

trapped-ion qubit: Ballance+, PRL (2016) solid-state spin: Huang+, PRL (2019) two-qubit gate: Hegde+, PRL (2022) CNOT gate: Xie+, PRL (2023)









Bit flip, phase flip (Pauli channels) Amplitude damping, phase damping, etc.

Quantum gates are unavoidably affected by noises





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- Analysis of gate fidelity
 - conservative laws Ozawa, PRL (2002)
 - short operational times Abad+, PRL (2022)
 - imperfect timekeeping Xuereb+, PRL (2023)
 - fluctuations of external control Jiang+, PRA (2023)



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• Motivation: Elucidate thermodynamic effects on fidelity of quantum gates

Quantum gate

-

- Ideal dynamics $\varrho_{\tau} = U_g \varrho_0 U_g^{\dagger}$
 - $\dot{\varrho}_t = -i[H_t, \varrho_t]$

$$U_g = \overrightarrow{\mathbb{T}} \exp\left(-i \int_0^\tau dt H_t\right)$$

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 - Dissipative jump operators \rightarrow local detailed balance $L_c(t) = e^{s_c(t)/2} L_{c'}(t)^{\dagger}$ e.g., energy relaxation

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 - Dissipative jump operators \rightarrow local detailed balance $L_c(t) = e^{s_c(t)/2} L_{c'}(t)^{\dagger}$ e.g., energy relaxation
 - Non-dissipative jump operators \rightarrow self-adjoint $L_c(t) = L_c(t)^{\dagger}$ e.g., phase damping

$$U_g = \overrightarrow{\mathbb{T}} \exp\left(-i \int_0^\tau dt H_t\right)$$

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i.e., $s_c(t) = 0 \& c' = c$

$$\begin{split} |\varphi\rangle\langle\varphi| - |\text{deal} \rightarrow U_g |\varphi\rangle\langle\varphi| U_g^{\dagger} \\ |\varphi\rangle\langle\varphi| - |\text{Reality} \rightarrow \mathscr{E}(\varphi) \end{split}$$

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 $F(U_g | \varphi \rangle \langle \varphi | U_g^{\dagger}, \mathscr{E}(\varphi)) = \langle \varphi | U_g^{\dagger} \mathscr{E}(\varphi) U_g | \varphi \rangle$



Average fidelity

 $\mathcal{F} := \left[d\varphi \left\langle \varphi \, | \, U_g^{\dagger} \mathscr{E}(\varphi) U_g^{} | \, \varphi \right\rangle \le 1 \right]$

Nielsen, Phys. Lett. A (2002)

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on Neumann entropy change

 $Q_t L_c(t)^{\dagger} S_c(t)$: environmental entropy change



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Average dissipation

$$\Sigma := \int d\varphi \, \Sigma_{\varphi}(\tau) \ge 0$$

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Result I

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Relation between fidelity and entropy production



$$\gamma := \int_0^\tau dt \sum_c \left[\Delta L_c(t)\right]^2$$
$$[\Delta L]^2 := \frac{(d \operatorname{tr} \{L^{\dagger}L\} - |\operatorname{tr} \{L\}|^2)}{d(d+1)} \ge$$



Result I





Result |



✓ Valid for arbitrary times and protocols



Result |



✓ Valid for arbitrary times and protocols Thermodynamic upper bound on error

$$1 - \mathscr{F} \le \sqrt{\gamma \Sigma/2}$$



Estimation of entropy production

 $\Sigma \ge \frac{2(1 - \mathcal{F})^2}{\gamma}$

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$$\mathcal{F} = 1 - \gamma + O(\Gamma^2 \tau^2)$$
$$\gamma = \frac{d}{d+1} \Gamma \tau, \quad \Gamma := \sum_c \Gamma_c$$

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 $\Sigma \ge \frac{2d}{d+1}\Gamma\tau + O(\Gamma^2\tau^2)$

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✓ dephasing rate Γ is estimable in experiments
 Harper+, Nat. Phys. (2020)
 Flammia+, Quantum (2021)

Ideal Hamiltonian

 H_t



H_t

systematic errors





 H_t

systematic errors



Realistic Hamiltonian

 $\hat{H}_t \neq H_t$

Further generalization

 $\mathcal{F} + \sqrt{\gamma \Sigma/2} \ge \frac{|\operatorname{tr}\{\hat{U}_{\tau}^{\dagger}U_{g}\}|^{2} + d}{d(d+1)}$



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$$\mathcal{F} + \sqrt{\gamma \Sigma/2} \ge \frac{|\operatorname{tr}\{\hat{U}_{\tau}^{\dagger}U_{g}\}|^{2} + d}{d(d+1)}$$

✓ Recover original relation for perfect implementation of Hamiltonian

 $\hat{H}_t = H_t \to \mathrm{tr}\{\hat{U}_\tau^\dagger U_g\} = d \to \mathscr{F} +$



$$\sqrt{\gamma \Sigma/2} \ge 1$$

For time-independent protocols

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- time-independent Hamiltonian H
- · jumps between energy eigenstates

$$[H, L_k] = \omega_k L_k$$

 ω_k : energy change

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✓ Hold for arbitrary times

For time-independent protocols



Thermodynamic upper bound on error

$$1 - \mathscr{F} \le 1 - e^{-\Sigma}$$

For time-independent protocols



- Thermodynamic upper bound on error
- ✓ Simple estimation for dissipation

$$\Sigma \ge -\ln \mathscr{F}$$

$$1 - \mathscr{F} \le 1 - e^{-\Sigma}$$

Generic quantum gate coupled to arbitrary reservoirs





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 Q_E : initial state of reservoirs



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$$Q := \int d\varphi \operatorname{tr} \{ H(\varrho_{\tau} - \varrho_{0}) \}$$

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Energy dissipation

$$Q := \int d\varphi \operatorname{tr} \{ H(\varrho_{\tau} - \varrho_{0}) \}$$

• U conserves total energy $\rightarrow Q$ equals heat dissipation to environment

- ✓ generalizable to time-dependent cases



Fidelity-dissipation trade-off relation

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$$\kappa := \frac{d}{(d+1)g}$$

$$g : \text{energy band width of } H$$

Fidelity-dissipation trade-off relation





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Fidelity-dissipation trade-off relation



Hold for arbitrary times and interactions





Fidelity-dissipation trade-off relation



✓ Hold for arbitrary times and interactions Thermodynamic lower bound on error

$$1 - \mathcal{F} \ge \kappa | \mathcal{Q} |$$





Demonstration

Demonstration



 $H_t = \omega_t (\sigma_x \cos \phi_t + \sigma_y \sin \phi_t)/2$ $L_1 \propto \sigma_-, L_2 \propto \sigma_+, L_3 \propto \sigma_z$

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• Fidelity-dissipation relations in quantum gates





Fidelity-dissipation relations in quantum gates



✓ Quantitative characterization for thermodynamic effects on fidelity

Fidelity-dissipation relations in quantum gates



✓ Quantitative characterization for thermodynamic effects on fidelity ✓ Probe dissipation through fidelity

Fidelity-dissipation relations in quantum gates



- ✓ Quantitative characterization for thermodynamic effects on fidelity
- ✓ Probe dissipation through fidelity
- Future works

✓ Role of thermodynamics in dissipative quantum computation Verstraete+, Nat. Phys. (2009)



Fidelity-dissipation relations in quantum gates



- ✓ Quantitative characterization for thermodynamic effects on fidelity ✓ Probe dissipation through fidelity
- Future works

 - Thermodynamics of quantum error correction
 - Landi+, PRA (2020)
 - Danageozian+, PRX Quantum (2022)

✓ Role of thermodynamics in dissipative quantum computation Verstraete+, Nat. Phys. (2009)





Thank you for your attention!