From adiabatic transformation to chaos, ergodicity and non equilibrium thermodynamics

Anatoli Polkovnikov, Boston University

“Verily at the first Chaos came to be, but next wide-bosomed Earth, the ever-sure foundations of all the deathless ones who hold the peaks of snowy Olympus, and dim Tartarus in the depth of the wide-pathed Earth, and Eros…” Hesiod, Theogony

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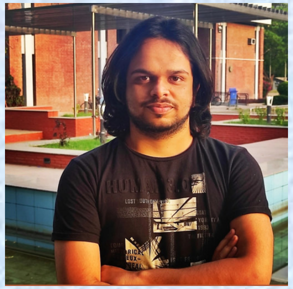
Partial list of recent collaborators

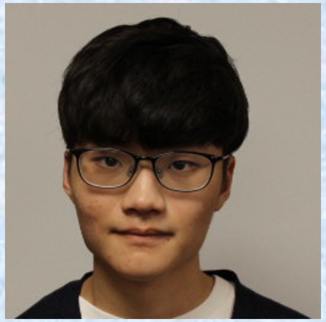
Souvik Bandyopadhyay, BU



Sourav Bhattacharjee, ICFO

Kirill Matirko, RSE

Cedric Lim, Berkeley



Hyeongjin Kim , BU

Mohit Pandey, Menten, AI

Tyler LeBlond Oak Ridge

Michael Flynn, BU

Dries Sels, NYU-Flatiron

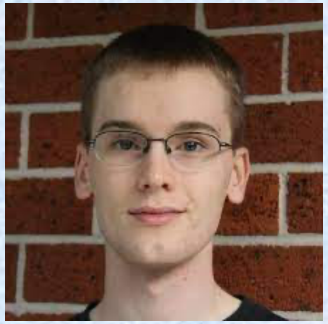
Marcos Rigol, PennState

Łukasz Iwanek, Wroclaw

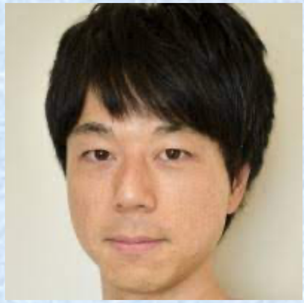
Adam Sajna, Wroclaw

Anushya Chandran , BU

Robin Schäfer, BU



David Long, UMD



Tatsuhiko Ikeda, Riken

Anatoly Dymarsky, UKY

David Campbell, BU

Marcin Mierzejewski, Wroclaw

Pieter Claeys, MPIPKS

J. Wurtz, QUERA

B. Barrera, BU

Plan

1. Adiabatic transformations in quantum and classical systems.
2. Beyond Born-Oppenheimer Approximation. Non-Markovian equilibrium in a moving frame: *A path forward when stochastic thermodynamics does not apply.*
3. Integrability, chaos and ergodicity through adiabatic transformations and low frequency noise: *Universality of emergent chaos and break down of local thermalization close to integrability.*
4. Integrable (non-dissipative) regimes as natural attractors of adiabatic flows and macroscopic dynamics.

Eigenstate adiabatic deformations in quantum systems

$$H(\lambda)|\psi_n(\lambda)\rangle = E_n(\lambda)|\psi_n(\lambda)\rangle$$

Define adiabatic gauge potential (AGP) - generator of these transformations

$$i\hbar\partial_\lambda|\psi_n(\lambda)\rangle \equiv \mathcal{A}_\lambda|\psi_n(\lambda)\rangle \quad \leftrightarrow \quad \langle n|\mathcal{A}_\lambda|m\rangle = i\hbar\frac{\langle n|\partial_\lambda H|m\rangle}{E_m - E_n}, \quad E_n \neq E_m$$

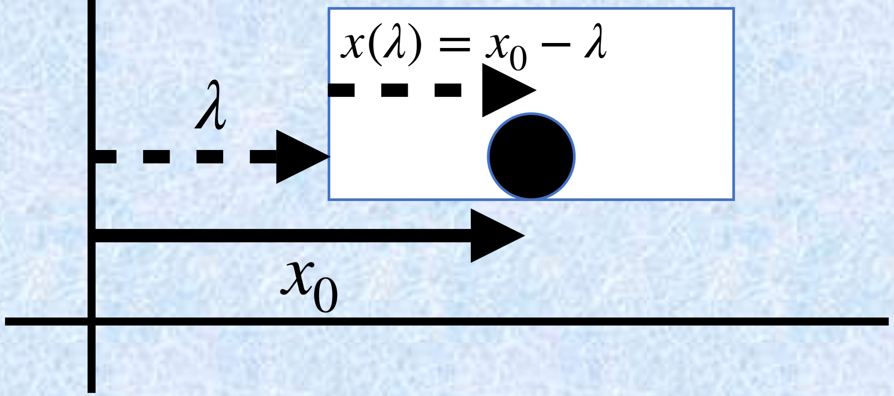
$$|\psi_n(\lambda)\rangle = U|\psi_n(0)\rangle \ \rightarrow \ \mathcal{A}_\lambda = i\hbar(\partial_\lambda U)U^\dagger \qquad \left[G_\lambda, H\right] = 0, \quad G_\lambda = \partial_\lambda H + \frac{i}{\hbar}[\mathcal{A}_\lambda, H]$$

AGP is the “Hamiltonian” generating adiabatic transformations, in particular the Schrieffer-Wolff transformations. Adiabatic transformations are associated with emergent conservation laws and hence integrability. Extension of the Noether theorem.

Classical systems: AGP is the generator of trajectory-preserving canonical transformations:

$H \to H + \partial_\lambda H\,\delta\lambda$, find canonical transformations $x(\lambda), p(\lambda)$, $\quad \dfrac{\partial x}{\partial\lambda} = -\dfrac{\partial\mathcal{A}_\lambda}{\partial p_\lambda}, \dfrac{\partial p}{\partial\lambda} = \dfrac{\partial\mathcal{A}_\lambda}{\partial q_\lambda}$: such that

Example: Translations (Galilean transformation)



$$x(\lambda) = x_0 - \lambda, \quad p(\lambda) = p_0$$

$$\frac{\partial x}{\partial \lambda} = -1 = -\frac{\partial \mathscr{A}_\lambda}{\partial p}, \quad \frac{\partial p}{\partial \lambda} = 0 = \frac{\partial \mathscr{A}_\lambda}{\partial x} \quad \Rightarrow \quad \mathscr{A}_\lambda = p$$

Quantum systems:

$$\psi_n(x;\lambda) = \psi_n(x-\lambda) \;\to\; i\hbar\partial_\lambda \psi_n(x-\lambda) = -i\hbar\partial_x\psi_n(x-\lambda) = \hat{p}\psi_n(x;\lambda) \;\to\; \hat{\mathscr{A}}_\lambda = \hat{p}$$

Motion in a moving frame: expand the state $|\psi(t)\rangle$ in the instantaneous basis:

$$|\psi(t)\rangle = \sum_n a_n(t)|\phi_n(\lambda)\rangle, \quad H(\lambda)|\phi_n(\lambda)\rangle = E_n(\lambda)|\phi_n(\lambda)\rangle$$

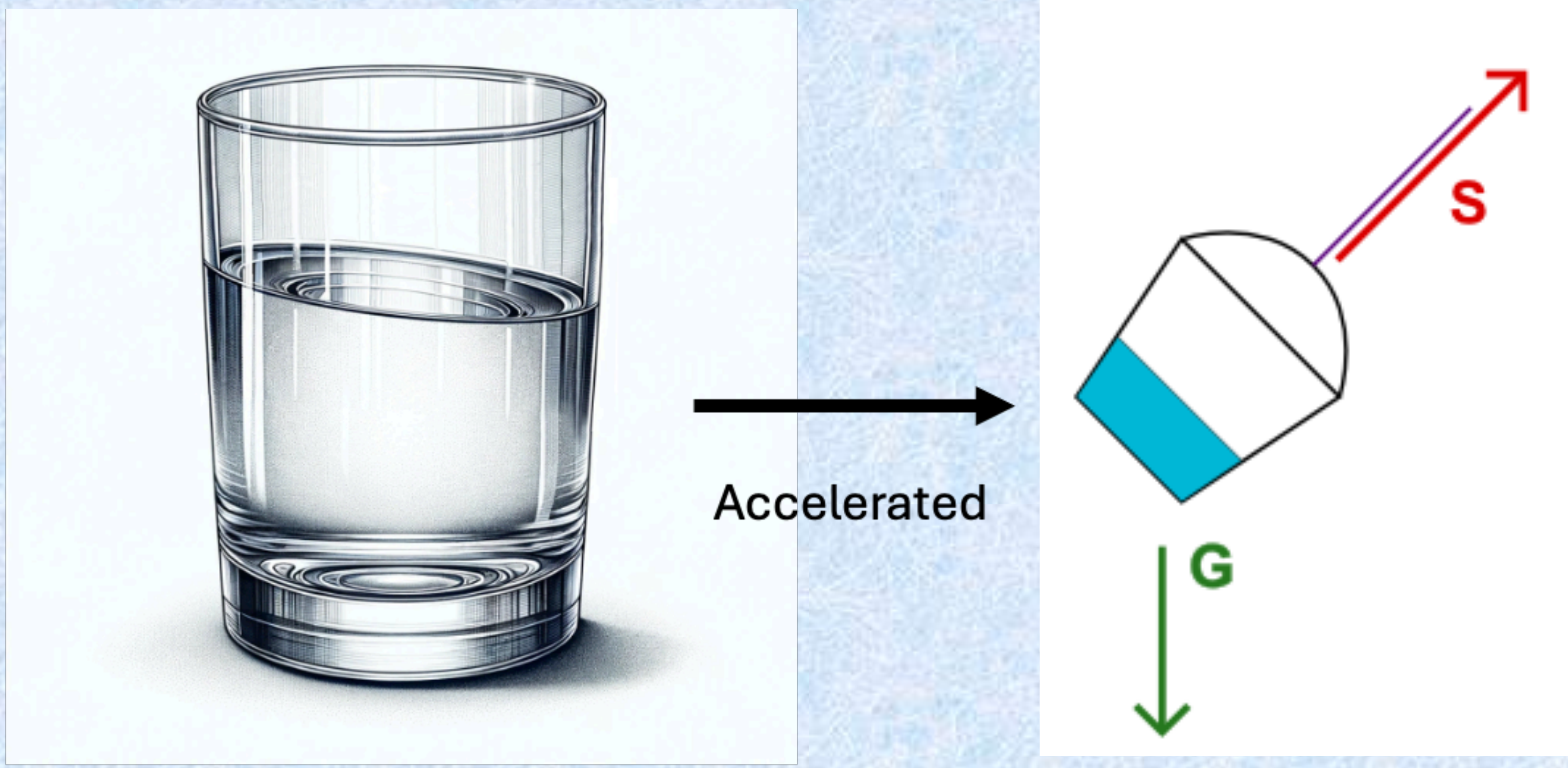
$$i\hbar\partial_t|\psi\rangle = H(\lambda(t))|\psi\rangle \quad \leftrightarrow \quad i\hbar\sum_n \dot{a}_n|\phi_n\rangle + i\hbar\sum_n a_n\dot{\lambda}\partial_\lambda|\phi_n\rangle = \sum_n a_n E_n|\phi_n\rangle$$

Interpret the first term as a moving derivative:

$i\hbar\partial_t^{(m)}|\psi\rangle = (H - \dot{\lambda}\mathscr{A}_\lambda)|\psi\rangle$. Translations: recover Galilean transformation.

Moving frame: non-Markovian forces like inertia lead to equilibration of fast particles in a moving frame.

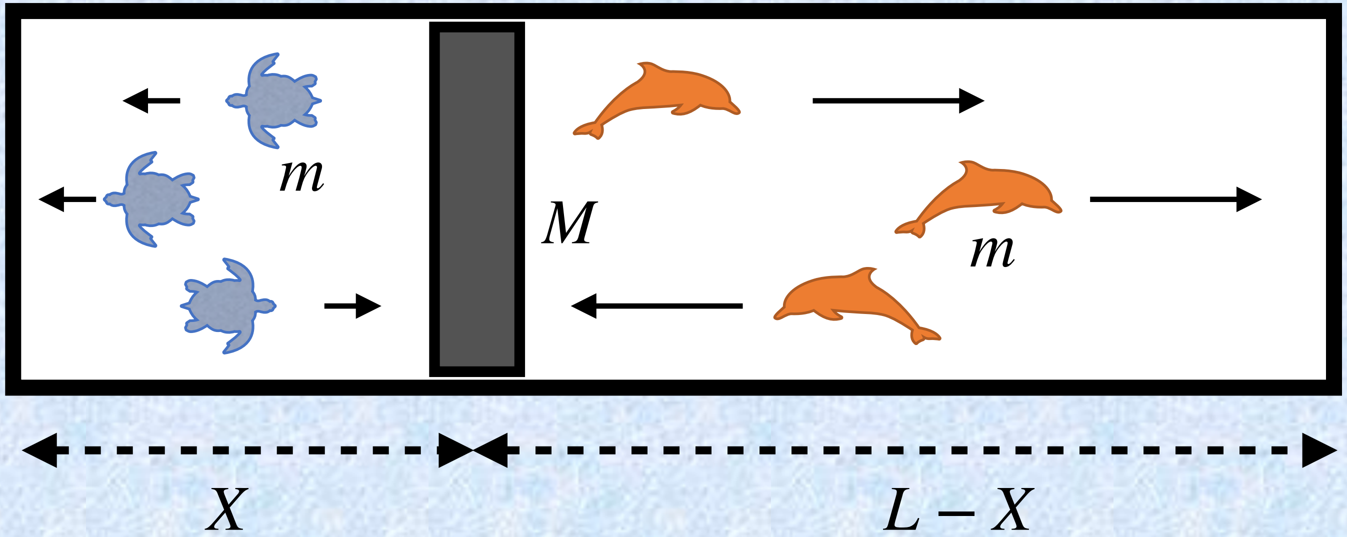
Stationary



$$i\hbar\partial_t^{(m)}|\psi\rangle = (H - \dot{\lambda}\mathscr{A}_\lambda)|\psi\rangle$$

- Locality of AGP implies modified moving frame equilibrium, no entropy production, non-Markovianity, modified equations of motion.
- Nonlocal AGP usually implies dissipation and Markoviantiy to some degree.
- AGP is usually local at low temperatures, close to integrability, in the presence of symmetries.

Adiabatic transformations and (beyond) the Born-Oppenheimer Approximation (BOA)



Old problem. How does this system equilibrate?

How can we describe dynamics of the piston?

$$\hat{H} = \frac{\hat{P}^2}{2M} + \sum_{j\in\text{left}} \frac{\hat{p}_j^2}{2m} + V(x_j, X) + \sum_{j\in\text{right}} \frac{\hat{p}_j^2}{2m} + V(x_j, L - X)$$

Standard approach to such problems BOA (does not lead to equilibration):

1) Treat X, P classically.

2) Find adiabatic equilibrium, e.g. the ground state, of fast particles at fixed X: $|\psi(x_j, X)\rangle$

3) Treat the energy $H_{\text{cl}}(X, P) = \langle\psi(x_j, X)|\hat{H}|\psi(x_j, X)\rangle$ as a classical Hamiltonian for X, P.

4) Solve equations of motion for $X(t), P(t)$.

Beyond BOA: typically add stochastic noise. Fails quantitatively in this system.

Formal approach to BOA

1. Find unitary diagonalizing the Hamiltonian: $U^\dagger(X)\hat{H}U(X) = \mathrm{diag}(E_1(X), E_2(X), \dots)$

2. Neglect transitions between levels and treat $E_n(X)$ as the BO potential.

Note $\mathscr{A}_X = i\hbar(\partial_X U)U^\dagger$ is the generator of adiabatic unitary transformations, i.e. it is the AGP.

Issues with BOA:

- Usually hard to go beyond systematically;
- Classical degrees of freedom become operators due to mixing with quantum.
- Technical problem: semiclassics comes from the saddle point approximation:

$$\sum_n |\psi_n(X_j)\rangle\langle\psi_n(X_j)|\, e^{\frac{i}{\hbar}\hat{H}\delta t} \sum_m |\psi_m(X_{j+1})\rangle\langle\psi_m(X_{j+1})|\, e^{\frac{i}{\hbar}\hat{H}\delta t}\dots$$

Need to deal with overlaps of non-orthogonal states.

Beyond BOA

Bernardo Barrera:

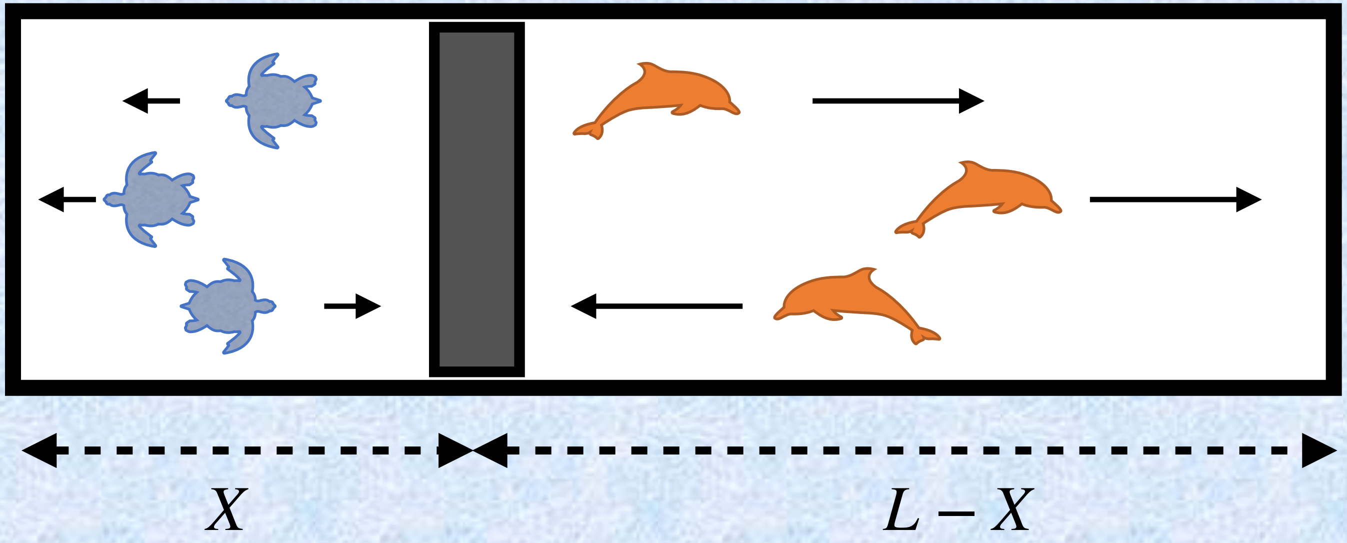
- Define the unitary operator diagonalizing full Hamiltonian:
- $\hat{H}' = \hat{U}^\dagger(\hat{X})\hat{H}\hat{U}(\hat{X}) = \mathrm{diag}(E_n)$. Nothing depends on time!
- Perform the Wigner-Weyl transform for $\hat{X}$ only: $\hat{H}'(X,P) = \hat{U}^\dagger(X) * \hat{H}(X,P) * \hat{U}(X)$

$$\hat{A} * \hat{B} = \hat{A}(X,P) e^{\frac{i\hbar}{2}\Lambda} \hat{B}(X,P), \quad \Lambda = \frac{\overleftarrow{\partial}}{\partial X}\frac{\partial}{\partial P} - \frac{\overleftarrow{\partial}}{\partial P}\frac{\partial}{\partial X}$$

- Expand in $\hbar$. Leading order: BOA approximation
- Next leading order (similar to Schrieffer-Wolff but integrate out slow variables):

$$\hat{H}_{\mathrm{eff}} = \frac{P^2}{2M} - \frac{1}{M} P \hat{\mathscr{A}}_X + \frac{1}{2M}\hat{\mathscr{A}}_X^2 + \sum_{j \in \mathrm{left}} \frac{\hat{p}_j^2}{2m} + V(x_j, X) + \sum_{j \in \mathrm{right}} \frac{\hat{p}_j^2}{2m} + V(x_j, L - X)$$

- Planck's constant magically disappeared. Got moving (Galilean) term + interactions without writing any equations of motion!



$$\mathscr{A}_X = \hat{\mathscr{D}}_L - \hat{\mathscr{D}}_R$$

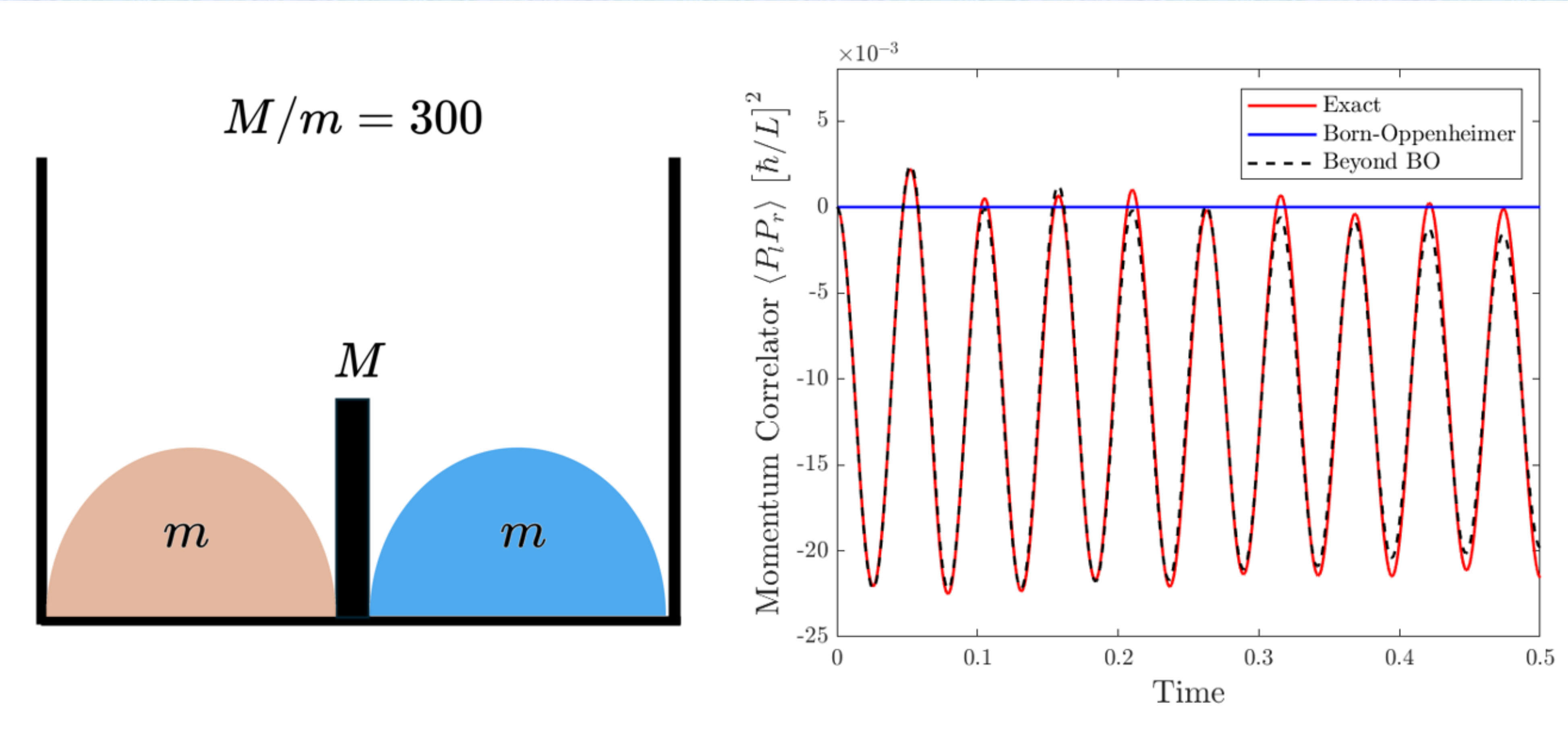
$$\hat{\mathscr{D}}_L = \sum_{j \in \text{left}} \frac{\hat{x}_j \hat{p}_j + \hat{p}_j \hat{x}_j}{2X}$$

$$\hat{H}_{\text{eff}} = \frac{P^2}{2M} - \frac{1}{M} P(\hat{\mathscr{D}}_L - \hat{\mathscr{D}}_R) + \frac{1}{2M}(\hat{\mathscr{D}}_L - \hat{\mathscr{D}}_R)^2 + \sum_{j \in \text{left}} \frac{\hat{p}_j^2}{2m} + V(x_j, X) + \sum_{j \in \text{right}} \frac{\hat{p}_j^2}{2m} + V(x_j, L - X)$$

- Automatically get moving frame (Galilean) transformations (second term)
- Got "electron-phonon" and mediated electron-electron interactions.
- Well defined classical limit (perform a sophisticated canonical transformation classically)
- Can adiabatically (self-consistently) follow the effective Hamiltonian. E.g. for the ground state

$$H_{\text{cl}}(X, P) = \langle \psi_0(x_j, X, P) | \hat{H}_{\text{eff}} | \psi_0(x_j, X, P) \rangle, \quad \frac{dX}{dt} = \frac{\partial H_{\text{cl}}(X, P)}{\partial P}, \quad \frac{dP}{dt} = -\frac{\partial H_{\text{cl}}(X, P)}{\partial X}$$

Work still in progress: emergent momentum correlations beyond BOA



Interesting direction (in progress): moving frame equilibrium:

$$Z(X,P) = \exp[-\beta \hat{H}_{\text{eff}}(X,P)] \qquad \hat{H}_{\text{eff}} = \frac{P^2}{2M} - \frac{1}{M} P \hat{\mathscr{A}}_X + \frac{1}{2M} \hat{\mathscr{A}}_X^2 + \hat{H}_{\text{int}}$$

Motion on a constant entropy curve implies Hamiltonian dynamics for X,P: minimum action principle follows from the maximum entropy principle.

AGP and long time response

$$\langle n|\mathscr{A}_\lambda|m\rangle = i\frac{\langle n|\partial_\lambda H|m\rangle}{\omega_{nm}} \to i\frac{\langle n|\partial_\lambda H|m\rangle\omega_{nm}}{\omega_{nm}^2+\mu^2}$$

$$\mathscr{A}_\lambda = \frac{1}{2}\int dt\,\mathrm{sgn}(t)e^{-\mu|t|}\partial_\lambda H(t), \quad \partial_\lambda H(t) \equiv e^{\frac{i}{\hbar}Ht}\partial_\lambda H e^{-\frac{i}{\hbar}Ht}$$ C. Jarzynski (1995)

Natural measure of complexity of adiabatic transformations: fidelity susceptibility, "quantum" Fisher information, "quantum" geometric tensor.

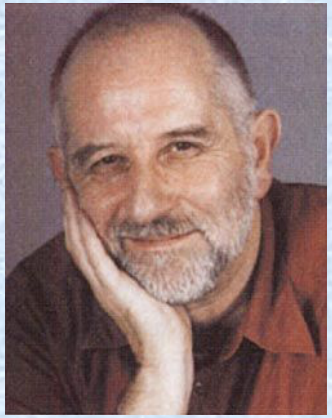
$$\hbar^2\chi_\lambda = \hbar^2\sum_n \rho_n\langle n|\overleftarrow{\partial_\lambda}\partial_\lambda|n\rangle_c \equiv \sum_n \rho_n\langle n|\mathscr{A}_\lambda^2|n\rangle_c \leftrightarrow \int D\vec{x}D\vec{p}\;\rho(E)\mathscr{A}_\lambda^2(\vec{x},\vec{p})$$

$$\hbar^2\chi_\lambda = \sum_{m\neq n}\rho_n\frac{\omega_{nm}^2}{(\omega_{nm}^2+\mu^2)^2}|\langle n|\partial_\lambda H|m\rangle|^2 = \int_{-\infty}^{\infty} d\omega\frac{\omega^2\Phi_\lambda(\omega)}{(\omega^2+\mu^2)^2}$$

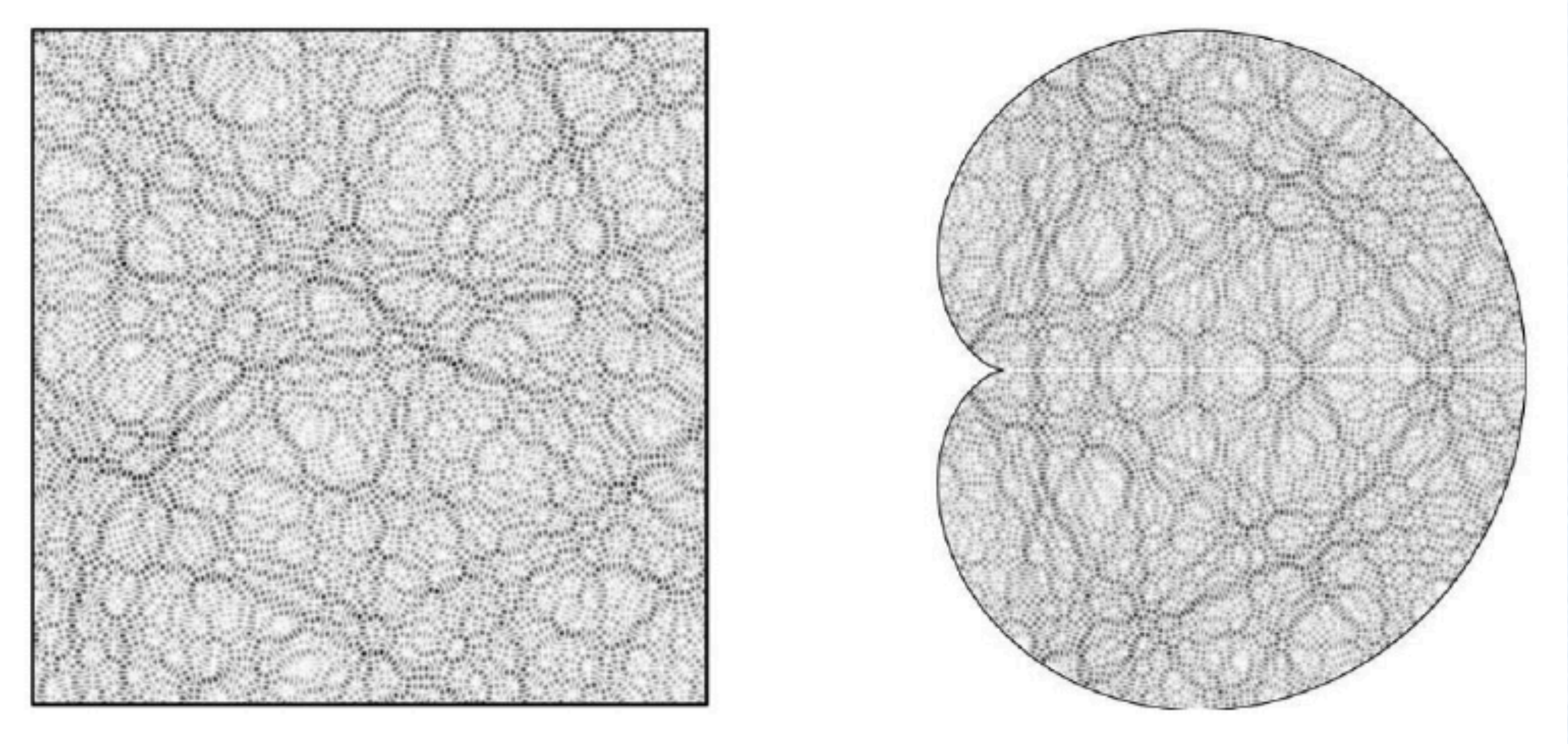
$$\Phi_\lambda(\omega) = \frac{1}{4\pi}\sum_n \rho_n\int dt e^{i\omega t}\langle n|\partial_\lambda H(t)\partial_\lambda H(0)+\partial_\lambda H(0)\partial_\lambda H(t)|n\rangle_c \propto \epsilon''(\omega) \sim \Gamma_{\mathrm{FGR}}(\omega)$$

χ is determined by the low-frequency noise of $\partial_\lambda H(t)$/dissipation for modulation of $\lambda(t)$.

Chaos through adiabatic transformations



Standard approach to quantum chaos. Berry conjecture 1977: Wigner function for quantum eigenstates in the classical limit approaches, the microcanonical distribution. Eigenstates are random superpositions of plain waves with the same energy.



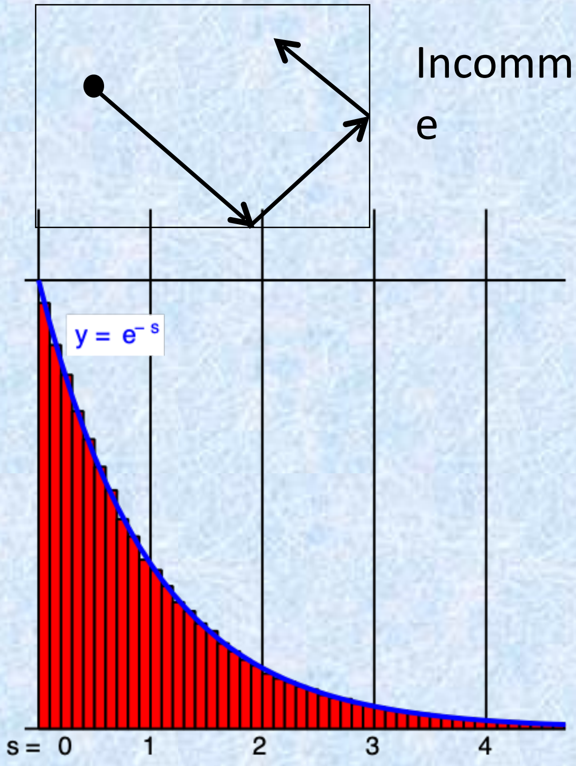
Spatial probability distribution in a high (6000th) energy eigenstate of a cardioid billiard vs. random superposition of plane waves.

Two powerful conjectures about energy levels in single-particle systems

Non-chaotic "generic systems" (energy is some of contributions from extensive number of symmetry sectors: expect Poisson statistics (Berry-Tabor conjecture, 1977)

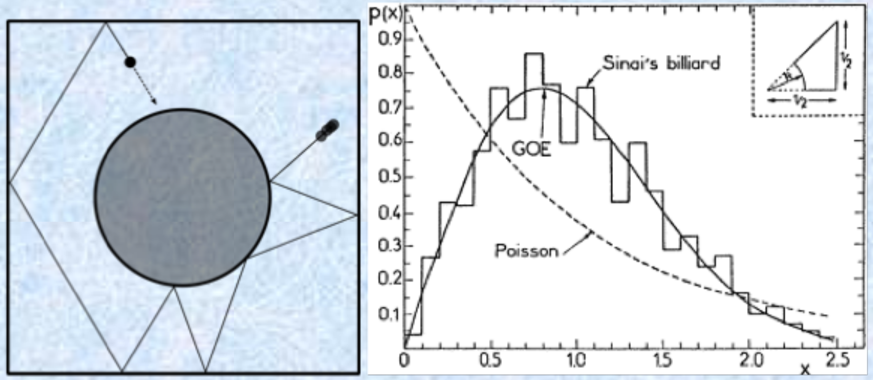
(Bohigas, Giannoni, Schmit) BGS conjecture (1984), must be attributed to M. Berry as well!: random matrix statistics (GOE) or (GUE) in chaotic generic systems

Examples

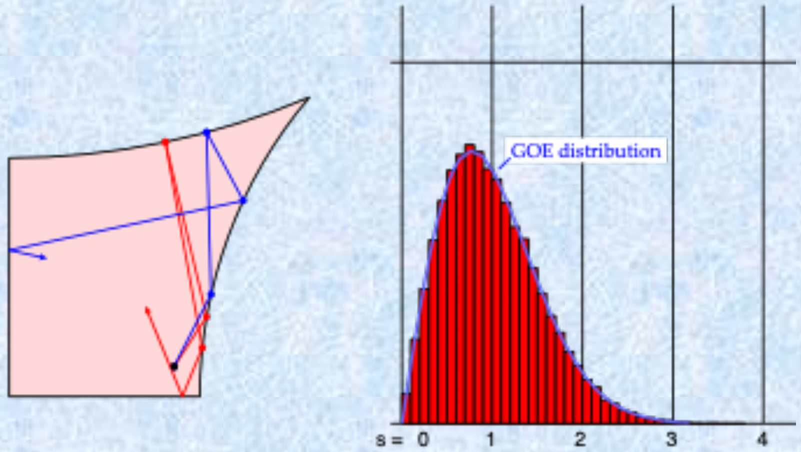


Incommensurat
e

Z. Rudnik, 2008



Sinai Billiard. O. Bohigas et. al. 1984



Another chaotic billiard by Sinai. Z. Rudnik, 2008

Eigenstate Thermalization Hypothesis - extension of Berry's conjecture

J. Deutsch (1992), M. Srednicki (1994, 1996), M. Rigol, V. Dun'ko, M. Olshanii (2008)

M. Srednicki

Expectation values of observables

$$\langle O(t)\rangle = \sum_{nm} O_{mn}\rho_{nm}\exp[-i(E_n - E_m)t] \to_{t\to\infty} \sum_n \rho_{nn}O_{nn}$$

General ETH ansatz (direct generalization of the RMT). Explains emergence of thermodynamics.

$$O_{mn} = O(E)\delta_{mn} + f(E,\omega)e^{-S(E)/2}\sigma_{nm}, \quad E = \frac{E_n + E_m}{2}, \omega = E_n - E_m$$

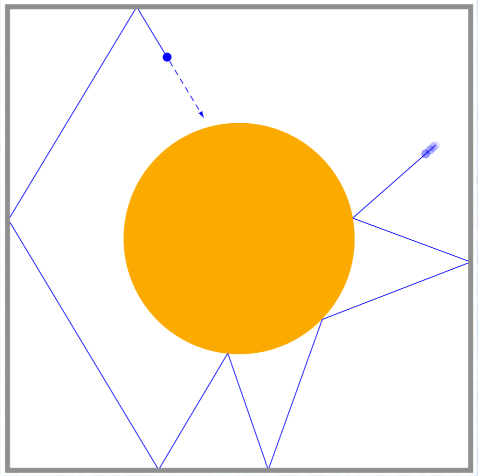
$O(E) \equiv \overline{O}$ — is the micro canonical average,

$$|f^2(E,\omega)| = \cosh\frac{\beta\omega}{2}\Phi_+(\omega) - \sinh\frac{\beta\omega}{2}\Phi_-(\omega), \quad \Phi_\pm(\omega) = \frac{1}{4\pi}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\left(\overline{O(t)O(0) \pm O(0)O(t)}\right)$$

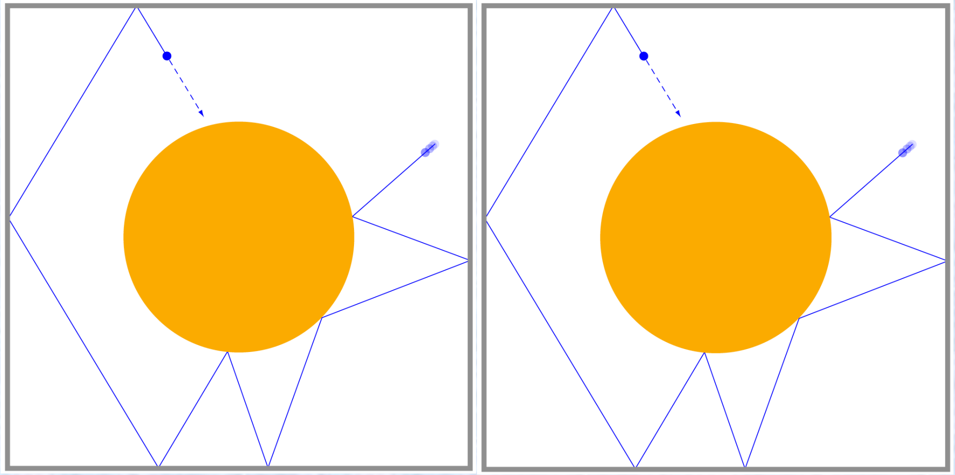
From the ETH ansatz can recover all thermodynamic relations: fluctuation-dissipation, detailed balance, Onsager, Maxwell, … (L. D'Alessio, Y. Kafri, A.P., M. Rigol, ETH review 2016)

Level statistics is a measure of ergodicity, not chaos.

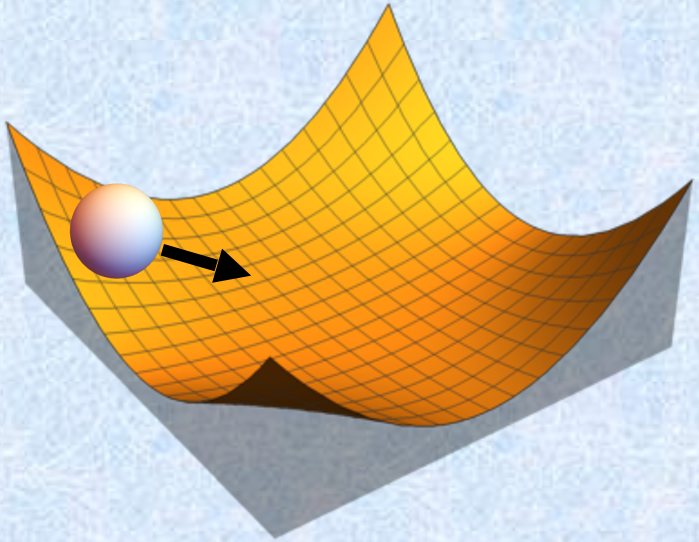
RMT, ETH imply stationary states (long time average) are thermal (J. Deutsch 1992; M. Srednicki 1994; M. Rigol, V. Dunjko, M. Olshanii 2008).



Chaotic ergodic.
GOE level statistics



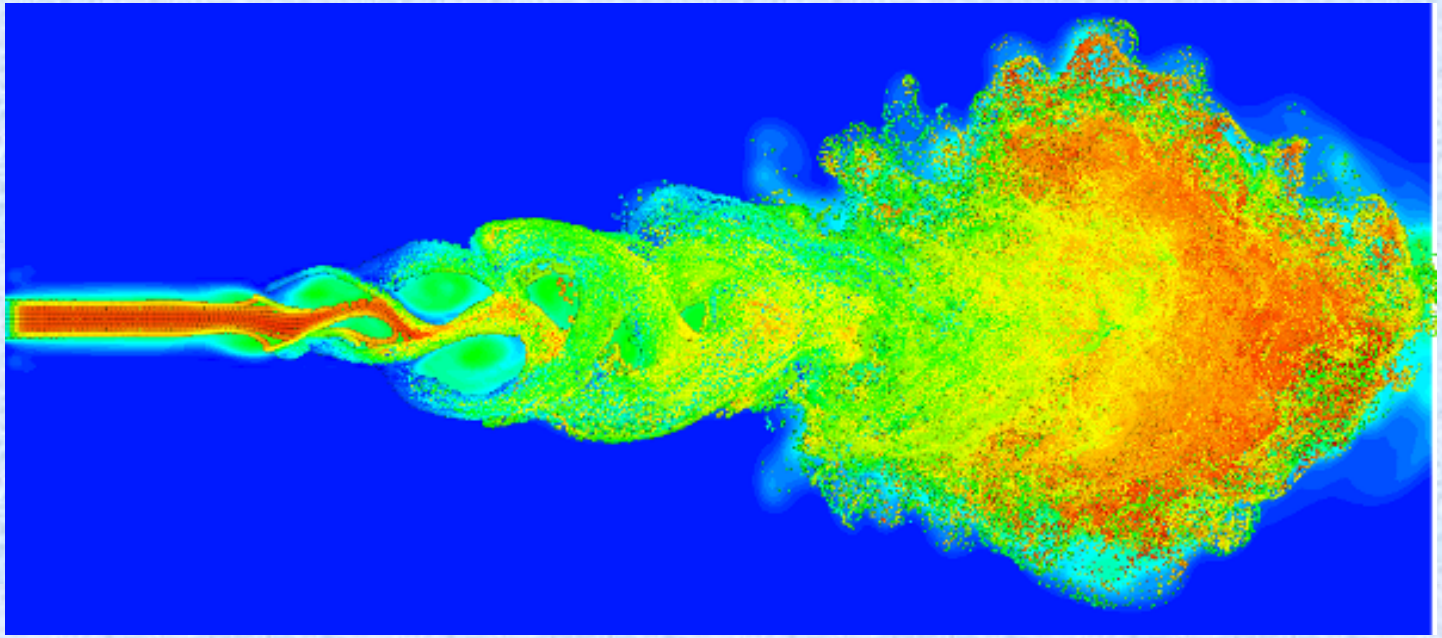
Chaotic, non-ergodic.
Mixed level statistics



Smooth chaotic potentials:
No signatures of RMT even after removing symmetries.

TD limit: usually chaos implies ergodicity, so the measure is fine. However, we need to distinguish the two notions. Many chaotic but non-ergodic models (KAM).

Everyday experience: maximal chaos is ***not*** fastest scrambling or fastest thermalization

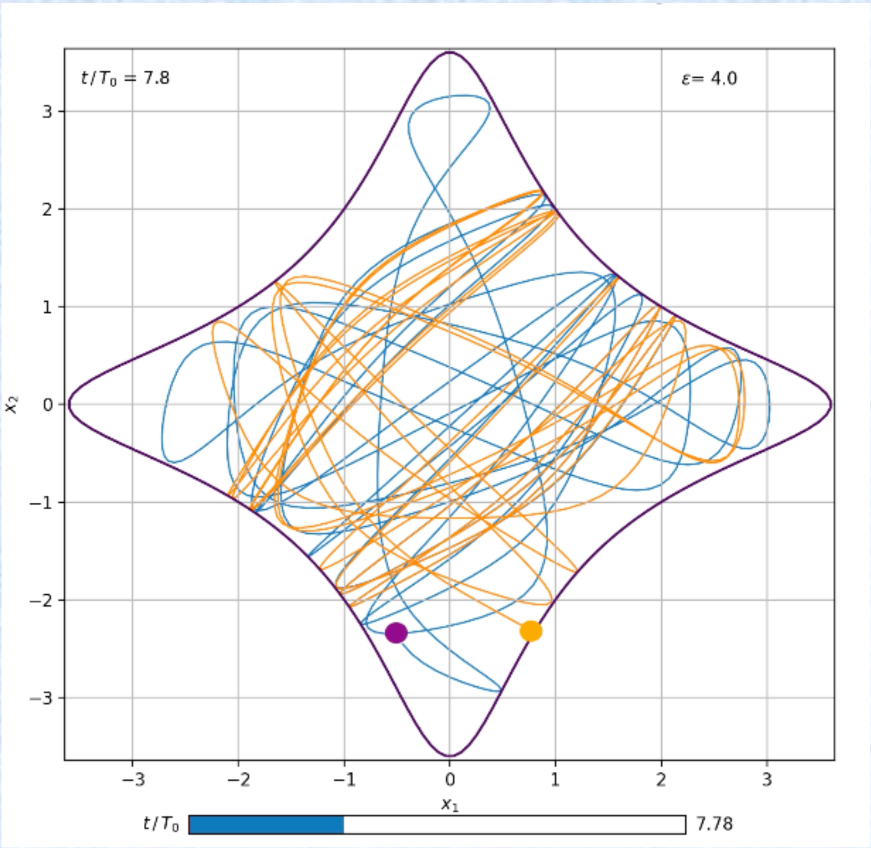


Turbulent flows are more chaotic than laminar and it takes more time to thermalize.

Weakly nonintegrable, nonthermalizing systems are usually more chaotic (less predictable).

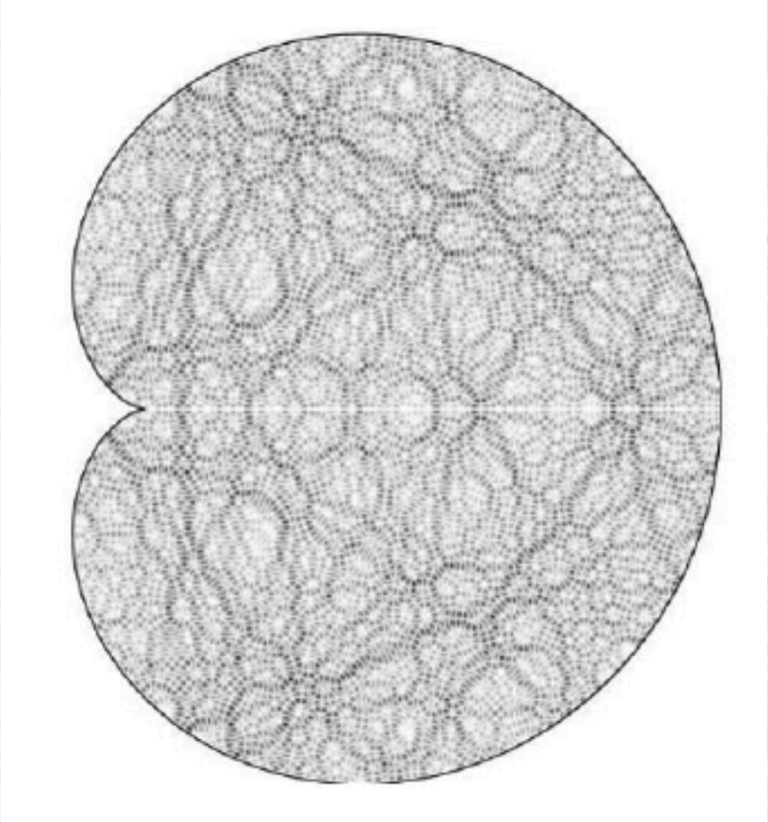
Integrability, Chaos and Thermalization through adiabatic transformations/quantum Fisher information (2020).

Classical systems (standard approach): very fragile trajectories



$$|\delta\vec{x}(t)| \sim |\delta\vec{x}_0| \exp[\lambda t]$$

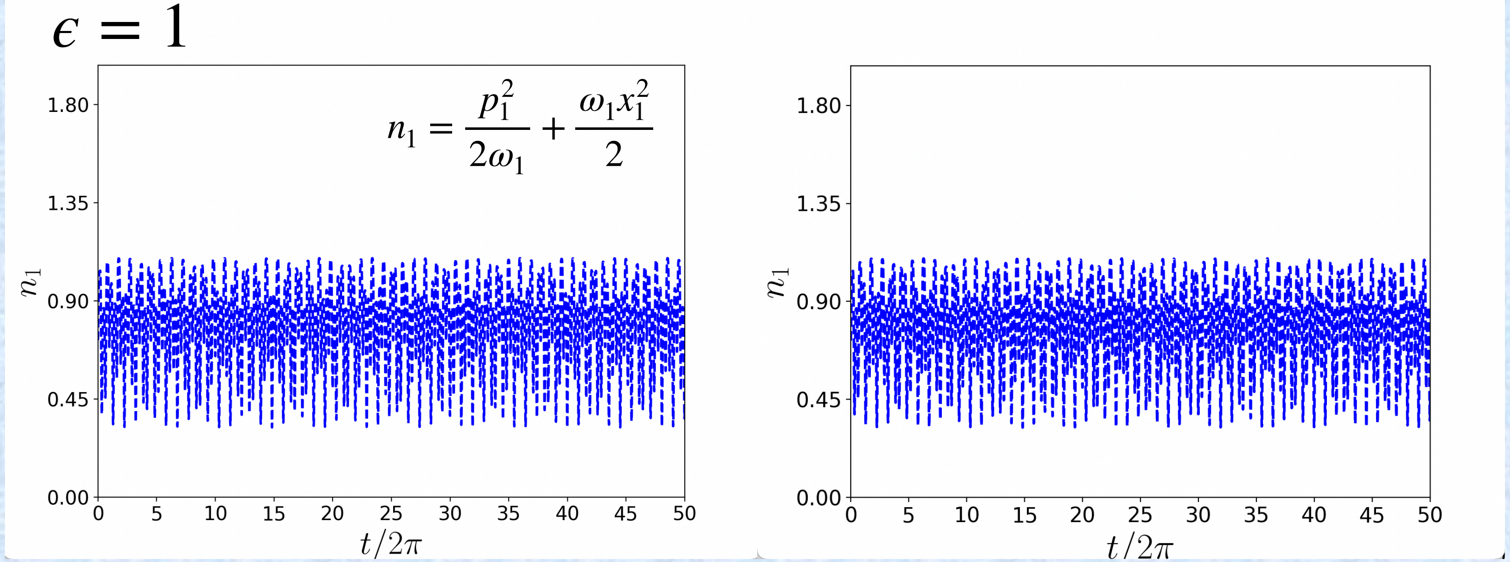
Quantum systems: very fragile eigenstates (large AGP)



$$\chi_\lambda = \sum_n \rho_n \langle n | \overleftarrow{\partial_\lambda} \partial_\lambda | n \rangle_c \equiv \sum_n \rho_n \langle n | \mathcal{A}_\lambda^2 | n \rangle_c$$

Adiabatic complexity and chaos. Intuitive picture. Classical models.

$$\chi_\lambda = \sum_n \rho_n \langle n | \mathscr{A}_\lambda^2 | n \rangle_c \leftrightarrow \int D\vec{x} D\vec{p} \; \rho(E) \mathscr{A}_\lambda^2(\vec{x}, \vec{p})$$

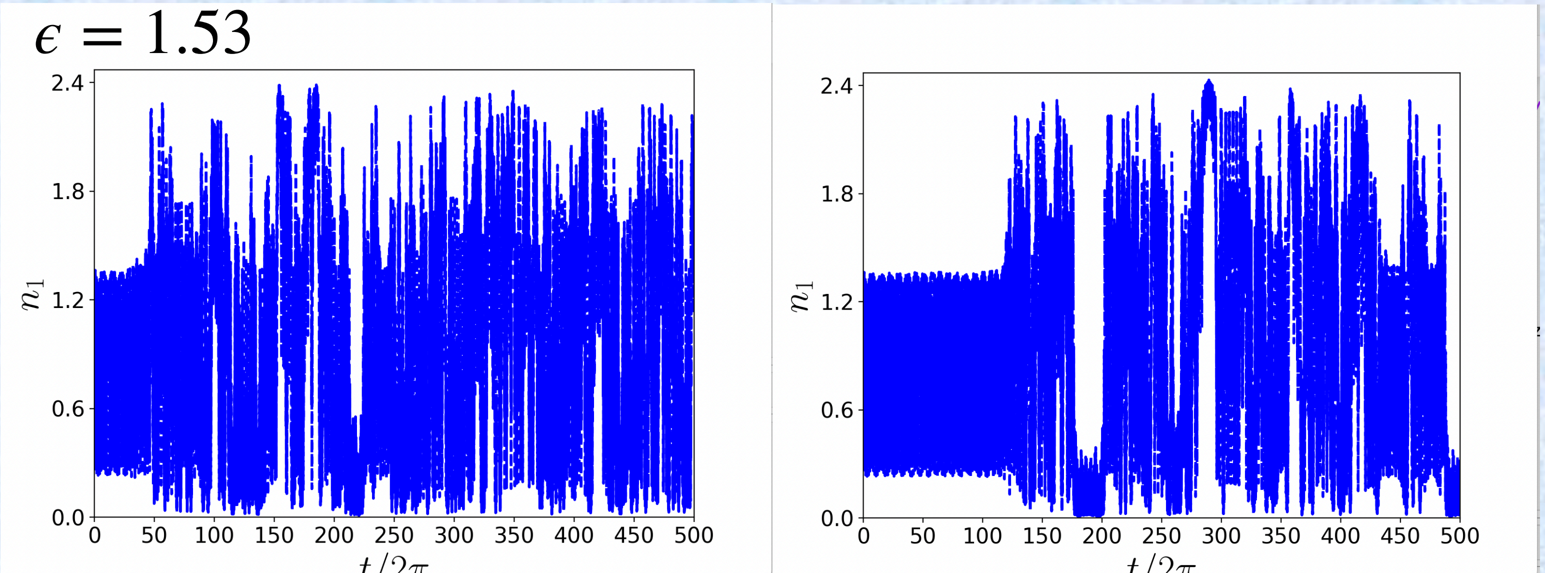


Compare two nearby trajectories: same initial conditions, slightly different Hamiltonians.

Compare long time trajectories at $\omega_1 = \omega_2 = 1$, $\omega_1 = 1.001, \omega_2 = 1$

Small nonlinearity (integrable): trajectories are very similar: must be easy to map them to each other.

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{\omega_1^2 x_1^2}{2} + \frac{\omega_2^2 x_2^2}{2} + \frac{\epsilon}{4} x_1^2 y_1^2$$



Large nonlinearity (chaotic): trajectories are very different.

Chaos as complexity of mapping two trajectories onto each other = complexity of mapping of eigenstates.

Fidelity susceptibility = complexity of adiabatic transformations is defined by the low frequency noise!

$$\hbar^2\chi_\lambda = \int_{-\infty}^{\infty} d\omega \frac{\omega^2\Phi_\lambda(\omega)}{(\omega^2+\mu^2)^2} \sim \frac{\Phi(\mu)}{\mu}$$

More chaotic systems must have longer relaxation times and larger noise at low frequencies.

Integrability is then associated with emergence of spectral gaps.

Chaos defined through adiabatic complexity is tied to dissipation/entropy production (larger $\epsilon''(\omega \to 0)$ means more chaos).

Ergodic/ETH systems

Extended systems: constant spectral function is expected from diffusion (more generally kinetic-type, Markovian approaches predicting exponential relaxation).

$$\partial_t n = D\frac{d^2 n}{dx^2} \quad \Rightarrow \quad n(x,t) = \sum_k n_k(t) \mathrm{e}^{ikx}, \quad n_k(t) = e^{-Dk^2 t}$$

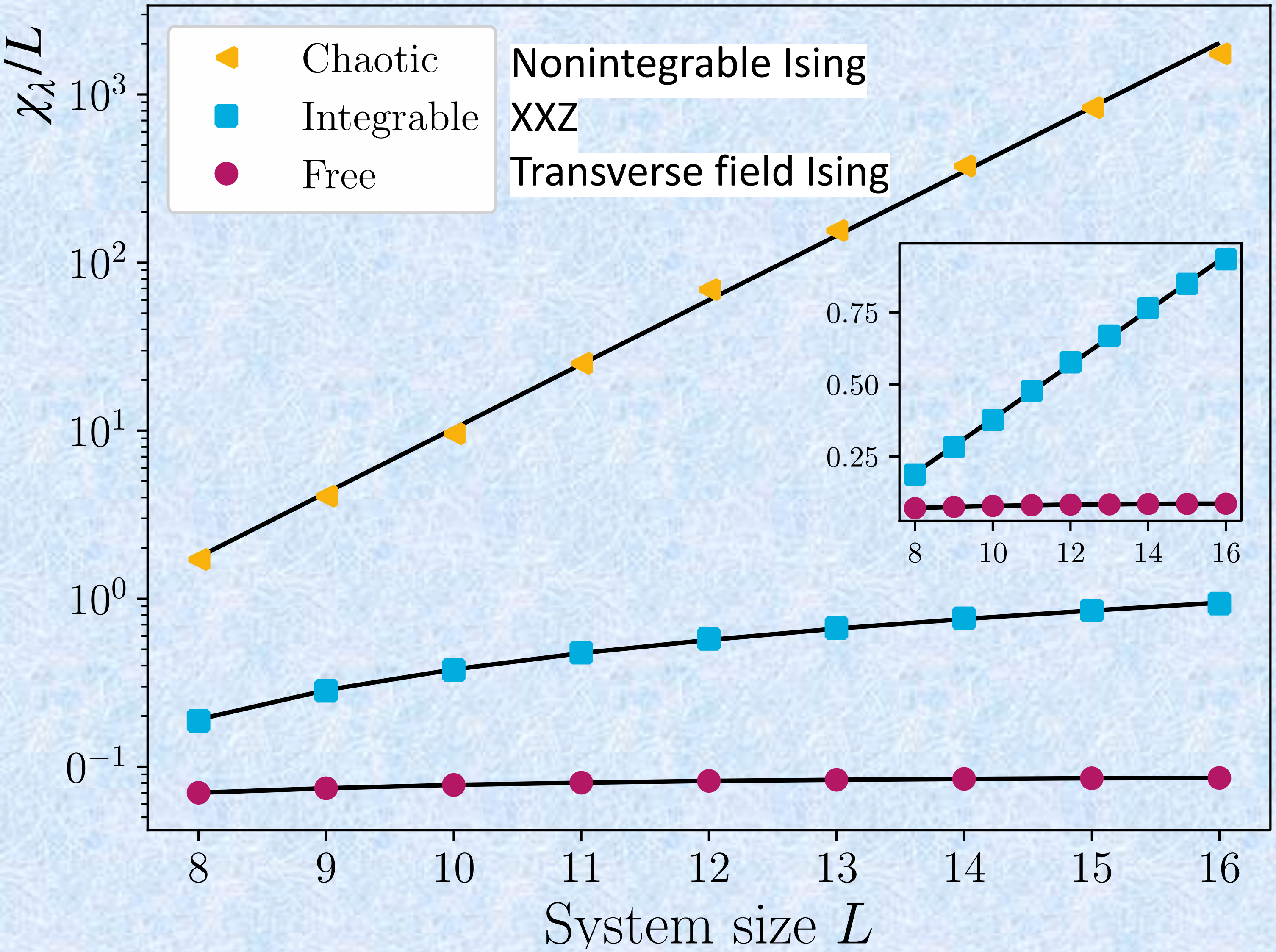
Smallest $k \sim 1/L$ then for $t > t_{\mathrm{Th}}$ $\quad n(t) \propto e^{-t/t_{\mathrm{Th}}}, \quad t_{\mathrm{Th}} = c\frac{L^2}{D} \quad \rightarrow \quad \Phi_\lambda(\omega) \sim C\frac{\omega_{\mathrm{Th}}}{(\omega_{\mathrm{Th}}^2 + \omega^2)} \rightarrow \frac{C}{\omega_{\mathrm{Th}}}, \quad \omega \rightarrow 0$

Alternative explanation from RMT: $|E_n - E_m| \leq \hbar\omega_{\mathrm{Th}}$

ETH: $|\langle n | \partial_\lambda H | m\rangle|^2 \sim C' e^{-S} \quad \Rightarrow \quad \Phi_\lambda(\omega) \approx |\langle n | \partial_\lambda H | m\rangle|^2 \Omega(E) \approx C', \quad C' = \frac{C}{\omega_{\mathrm{Th}}}$

$$\chi_\lambda(\mu) = \frac{\Phi_\lambda(\mu)}{\mu} \sim \frac{C}{\mu\,\omega_{\mathrm{Th}}}$$

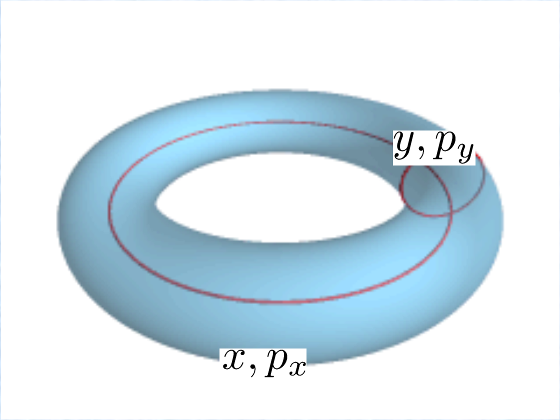
Numerical results



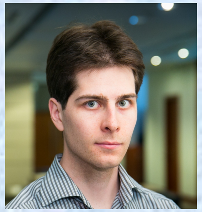
PRX, 2020

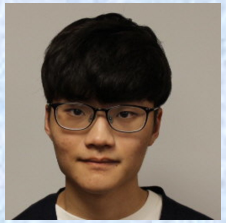
Ergodic model χ_λ agrees with ETH; free (TFI) model $\mathscr{A}_\lambda$ is a local operator (A. del Campo, Ma M. Rams, and W. H. Zurek, 2012); XXZ chain - $\mathscr{A}_\lambda$ is a quasi-local operator (has some long range tails).

Generic Integrable systems: expect finite χ, $\Phi(\omega) \to 0, \quad \omega \to 0 \;\to\; \chi(\mu) = \mathrm{const}(\mu)$



- Integrable motion is a superposition of 1D motions along the Tori.
- Each frequency is generally finite unless we fine-tune to a special saddle point like a separatrix.
- Very small spectral weight at low frequencies.

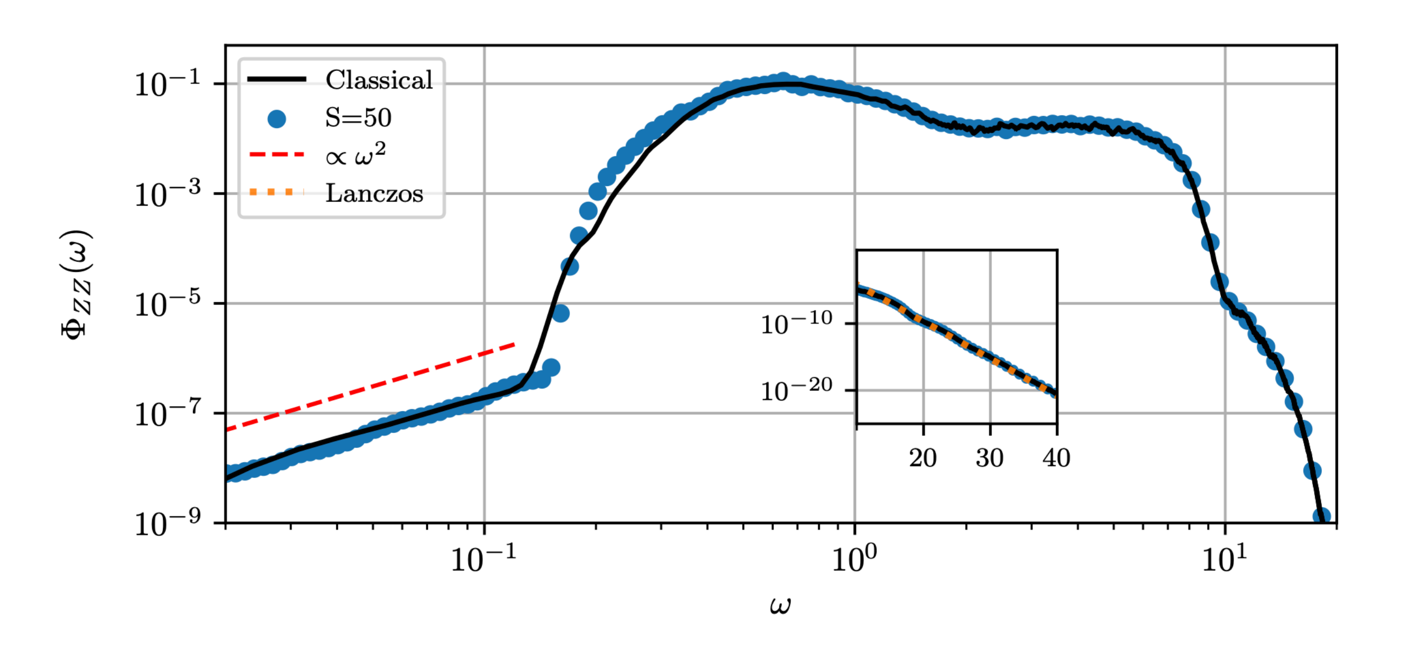


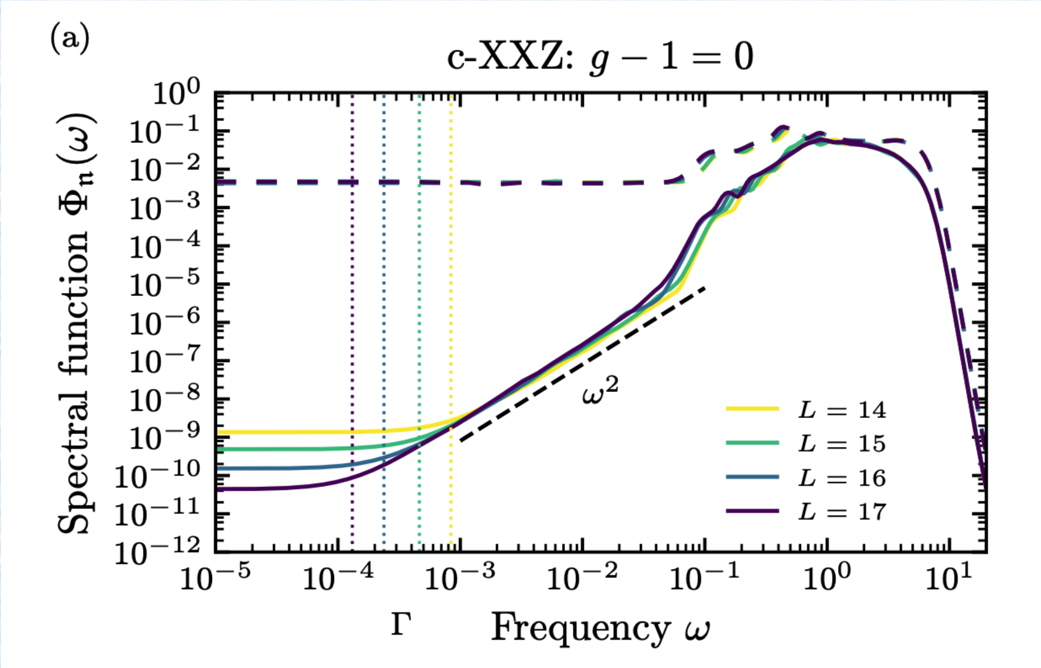


Two-spin integrable classical model:

$$H = -J_x s_1^x s_2^x - J_y s_1^y s_2^y - J_z s_1^z s_2^z$$

XXZ integrable quantum spin chain



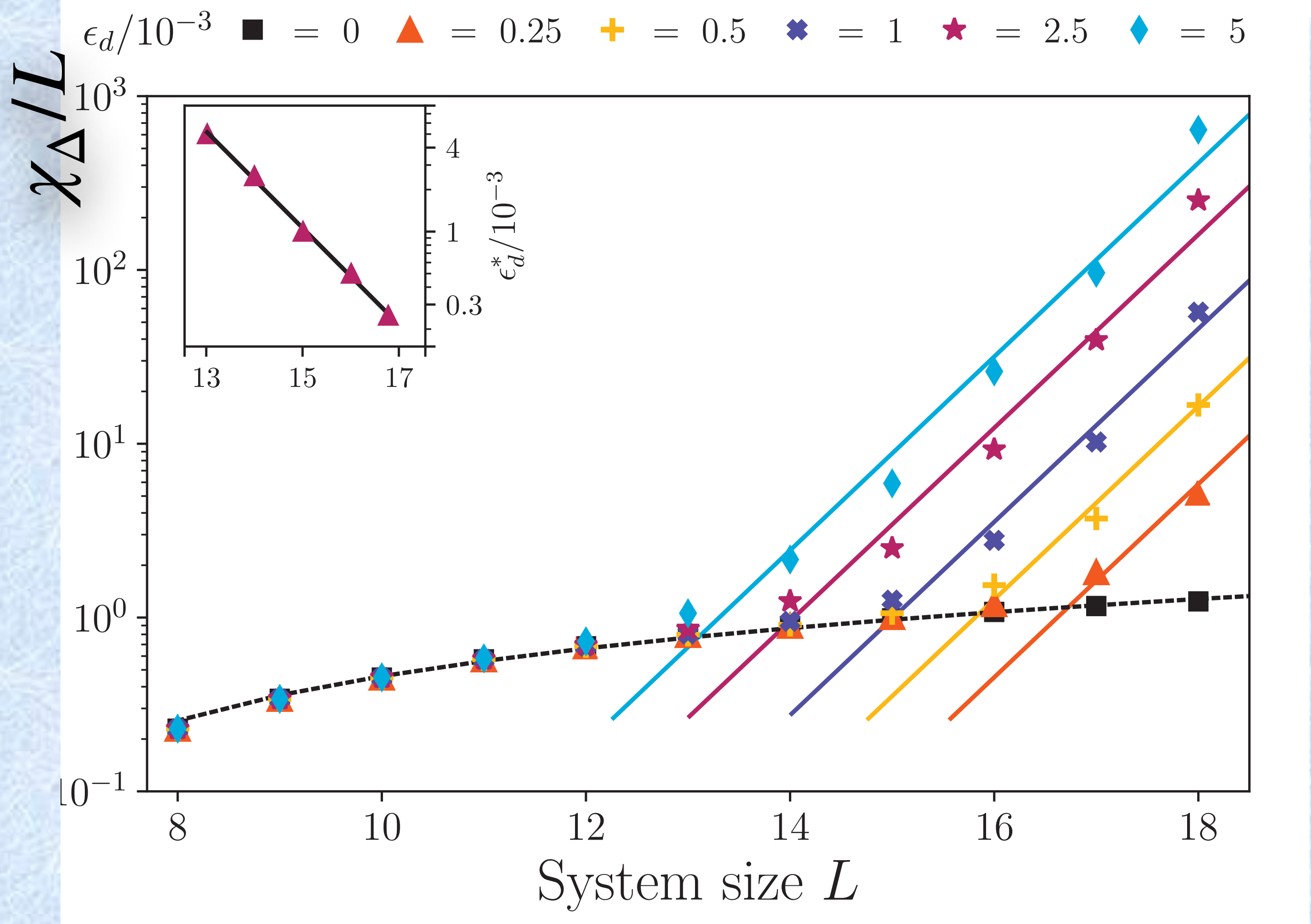


How do chaos and ergodicity emerge at weak integrability breaking?

$$H_{\mathrm{XXZ}} = \sum_{i=1}^{L-1} (\sigma_{i+1}^x \sigma_i^x + \sigma_{i+1}^y \sigma_i^y) + \Delta \sum_{i=1}^{L-1} \sigma_{i+1}^z \sigma_i^z$$

$$H \to H + \epsilon V$$

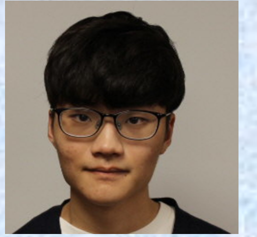
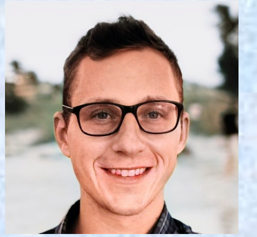
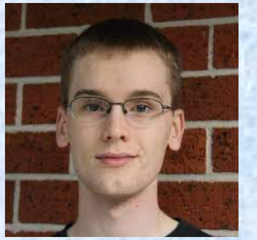
$$V = \sigma^z_{[L+1]/2}, \quad V = \sum_j \sigma_j^z \sigma_{j+2}^z$$



Results:

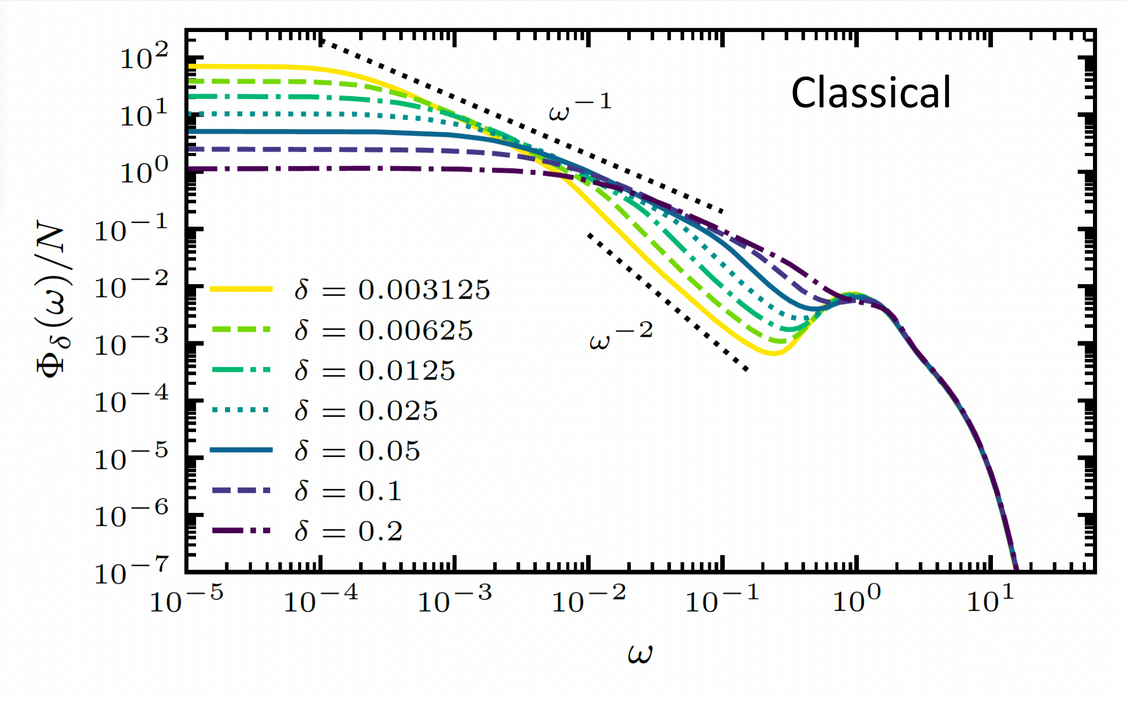
- Very sharp transition from integrable to chaotic behavior.
- χ is a very sensitive probe of chaos.
- Need exponentially (within numerics) small in L perturbation to break integrability
- Instead of expected $\chi \propto e^S$ scaling obtained $\chi \propto e^{2S}$ scaling, saturating the upper bound: $|\langle n|\partial_\lambda H|n+1\rangle| \sim 1 \gg \exp[-S/2]$

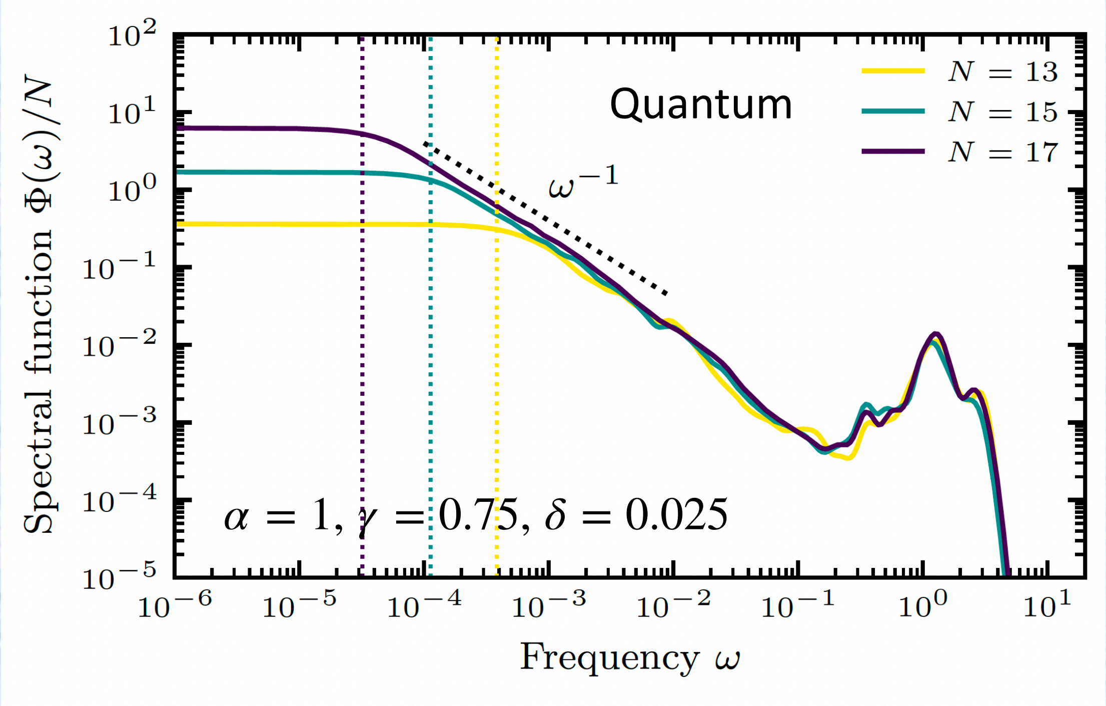
Physical reason for large susceptibility: slow dynamics/prethermalization


Central Spin Model:

$$H = \delta \sum_j h_j s_j^z + \sum_j (1 + \gamma g_j)(s_0^x s_j^x + s_0^y s_j^y + \alpha s_0^z s_j^z), \quad h_j, g_j \in [-1,1]$$

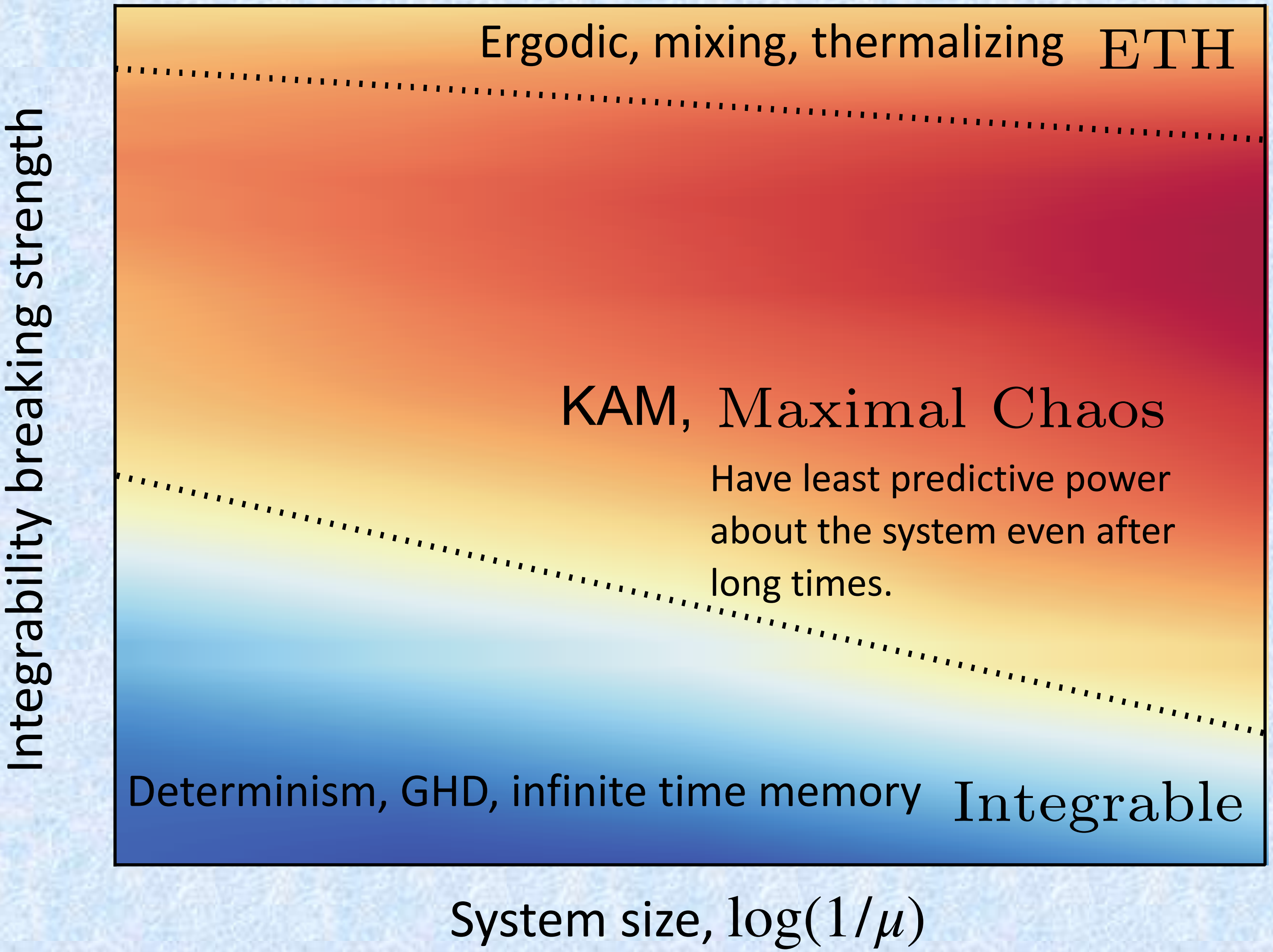




Universal long-time response near integrability. Emerges both in classical and quantum systems. Can have various power laws. More in poster by H. Kim

Emergence of chaos/mixing in many different models

Adiabatic complexity chaos diagram

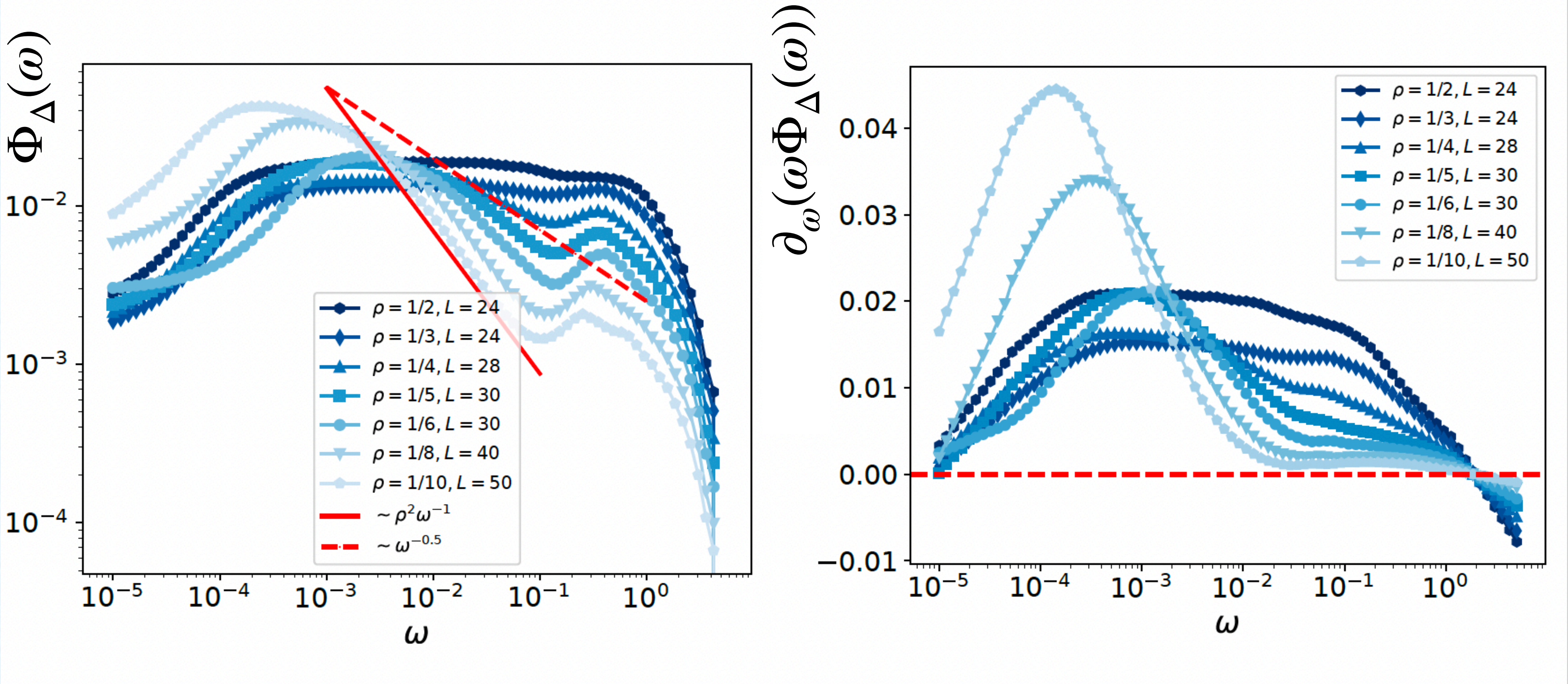


KAM regime: universal (slow) relaxation of observables in time.

Maximal chaos should be distinguished from fastest thermalization (T. Prosen et. al. circuits), SYK, black holes (J. Maldacena, A. Kitaev,…).

Everyday experience, weakly nonintegrable systems are very unstable and are hardest to predict, e.g. turbulence, weather, …

Take ergodic system, go closer to the (integrable) ground state: finite spectral gap, finite AGP, quasiparticles. $H = (J/2)\sum_i (S_i^+ S_{i+1}^- + h.c.) + \Delta S_i^z S_{i+1}^z + \Delta' S_i^z S_{i+2}^z \quad J = \sqrt{2},\ \Delta = (\sqrt{5}+1)/4,\ \Delta' = 1$

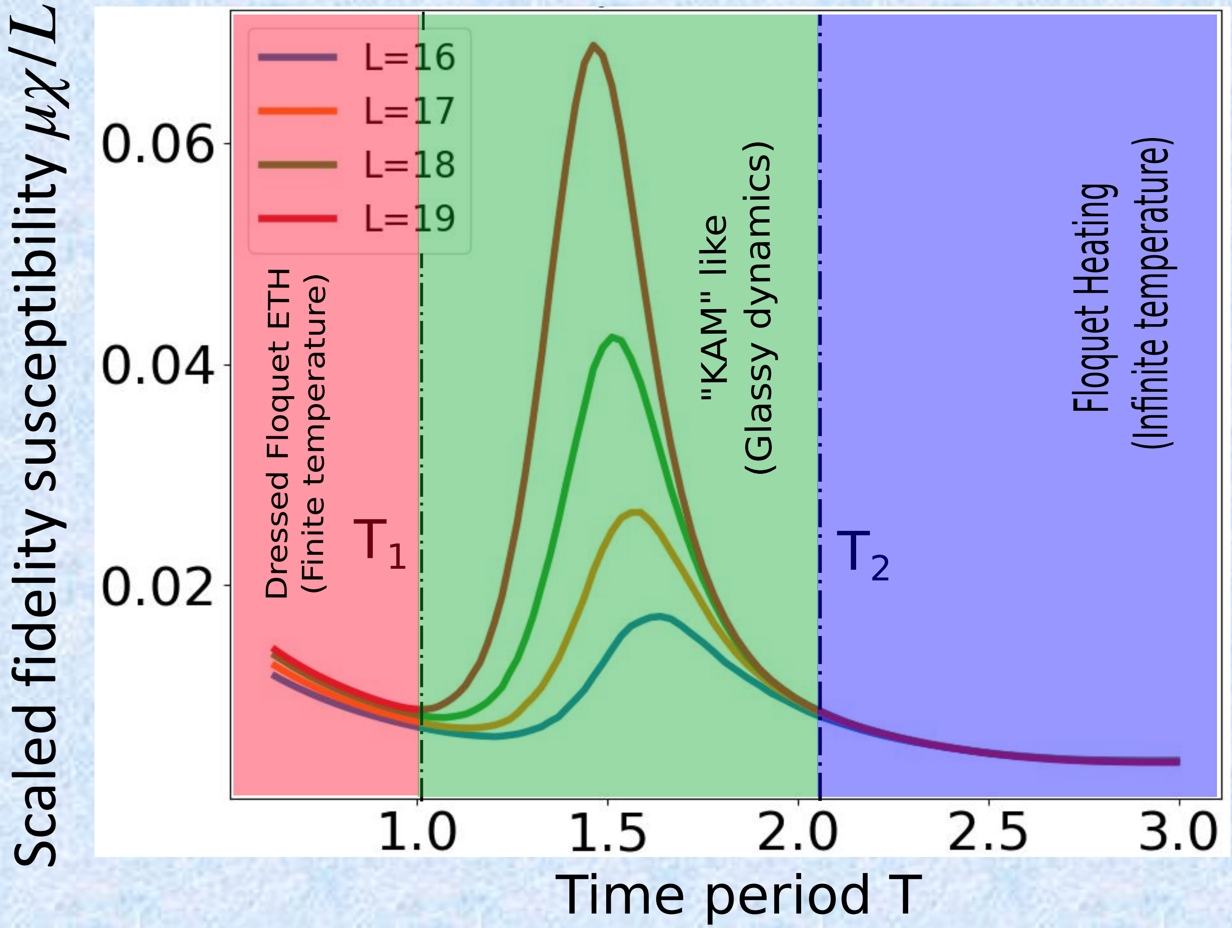


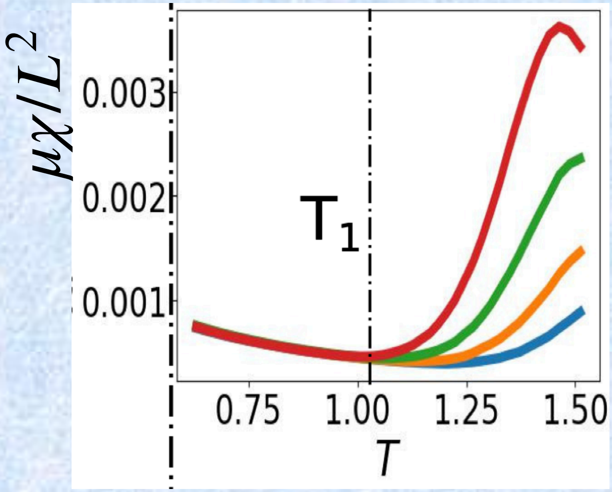
KAM chaotic buffer region precedes integrable quasiparticle phase. Now study transport, connect to chaos.

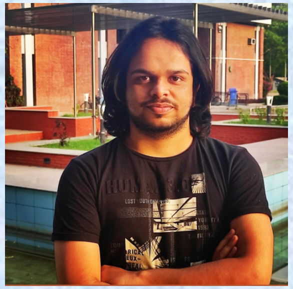
Driven Floquet System

$U_F = \mathrm{e}^{-iH_B\frac{T}{2}}\mathrm{e}^{-iH_A\frac{T}{2}}$ Short period driving: $U_F \approx \mathrm{e}^{-iH_\mathrm{F}T}$, $H_F \approx \dfrac{H_A + H_B}{2}$

Long period driving - expect energy drift, infinite heating, FGR







$T < T_1$ — local H_F

$T > T_2$ — Infinite heating

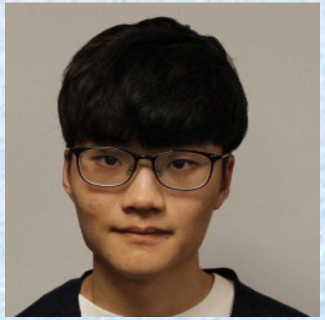
$T_1 < T < T_2$ — no Floquet Hamiltonian —

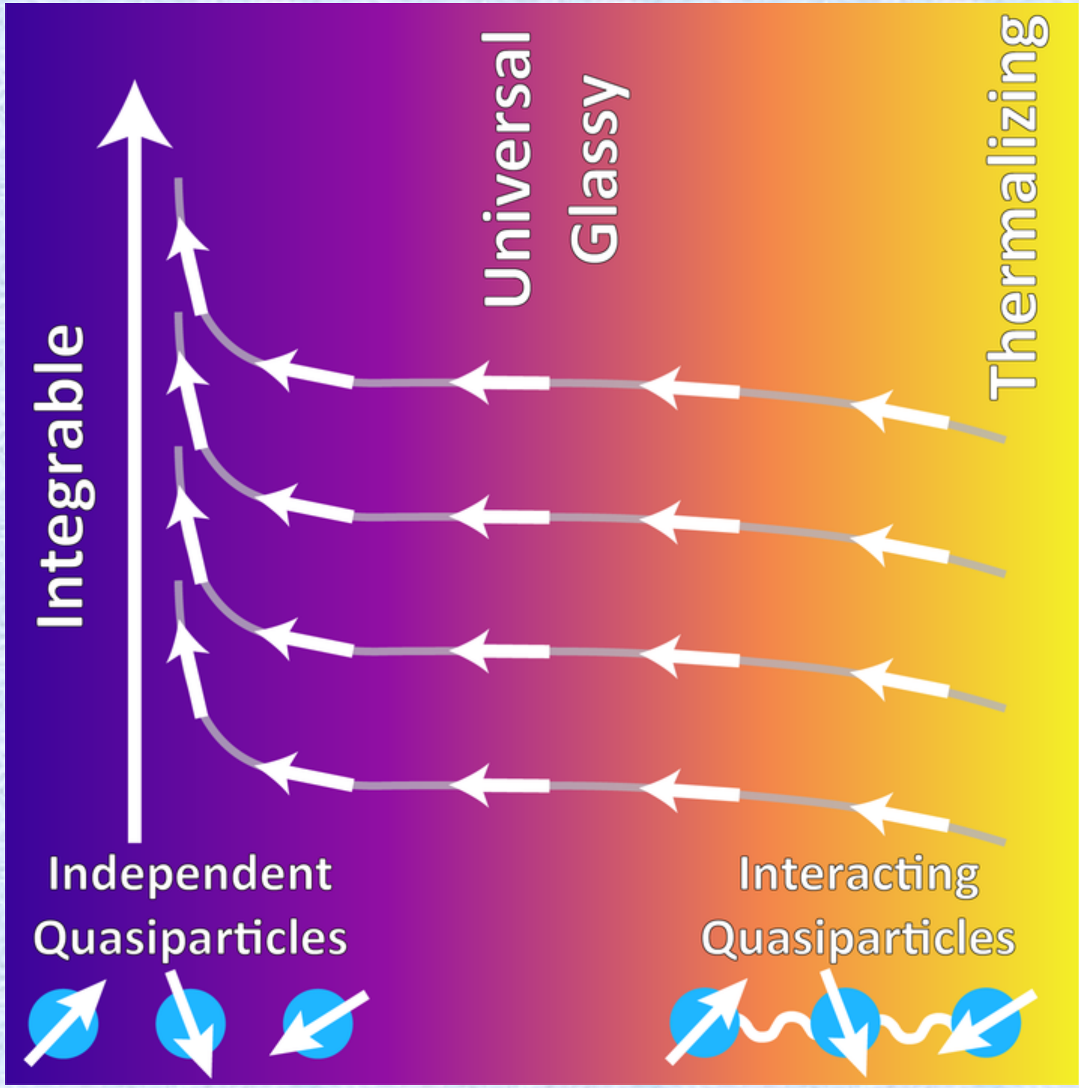
KAM regime, strong chaos, no infinite heating.

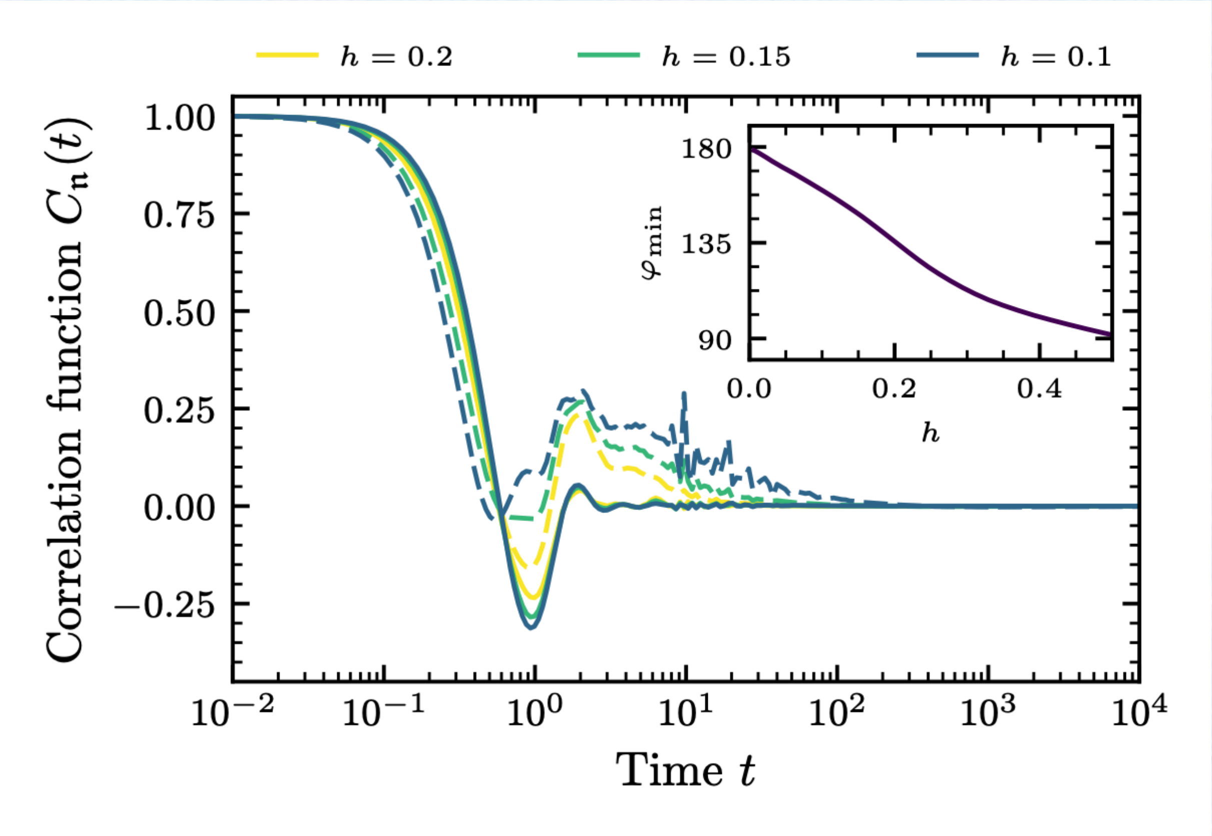
Disordered systems — same story chaos and thermalization are not equivalent. Failure to realize that

→ conceptual mistakes in canonical “MBL” story: RG, LIOMs, Avalanche mechanism, the role of QM…

Universal adiabatic flows towards integrability (integrability is attractive)







More in poster by H. Kim

Fast relaxation along the flows: good for adiabatic state preparation, CD driving, SW transformations, …

Summary

- Emergent non-Markovian dynamics beyond BOA.

- Classical and quantum chaos/ergodicity can be understood through complexity of trajectory/state preserving adiabatic transformations.

- Glassy KAM regime (maximally sensitive/maximally chaotic) generically separates integrable and ergodic phases. Transient in TD limit, stable at all times in small classical systems. Possibly equivalent to Hilbert space fragmentation.

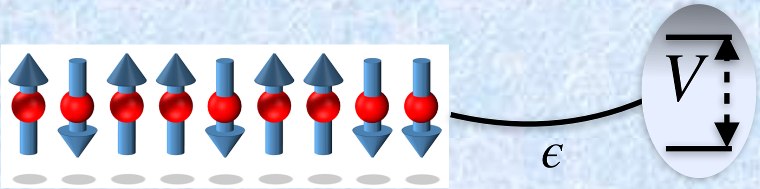
- Universal dynamics near integrability. Temperature plays a similar role in dynamics as integrability breaking parameter.

- Integrable points are generic attractors of adiabatic flows and possibly time evolution in autonomous systems.

Chaos (breakdown of LIOMs) and operator spreading.

$$H = VS_z^0 + \epsilon H_{\text{int}} + H_{\text{bath}}$$

Idea: find recursively LIOM: $[Q, H] = 0$



Probe spin

Use perturbation theory (Birkhoff normal form). First order in ϵ and all orders in $1/V$. Work in the TD limit. Make no assumptions about eigenstates, gaps,…

$$Q = S_z^0 + \frac{1}{V} q_1(\epsilon) + \frac{1}{V^2} q_2(\epsilon) + \ldots, \qquad Q \equiv G_V = S_Z + i[\mathcal{A}_V, H]$$

First order $\quad [q_1, S_z^0] + [S_z^0, \epsilon H_{\text{int}}] = 0, \quad \to \quad q_1 = \epsilon H_{\text{int}}$

Can solve analytically in the linear order in ϵ

$$Q = S_0^z + \frac{\epsilon}{V} H_{\text{int}} + \frac{\epsilon}{V^2} \sigma_0^z [H_{\text{bath}}, H_{\text{int}}] + \frac{\epsilon}{V^3} [H_{\text{bath}}, [H_{\text{bath}}, H_{\text{int}}]] + \ldots$$

This is an expansion of the conserved charge (and the AGP) in the Krylov space.

Stop at N-th order: $Q_N = S_z^0 + \epsilon \sum_{n=0}^{N} \frac{(\sigma_z^0)^n}{V^n} \mathscr{L}^n H_{\text{int}}$

$[Q_{2N}, H] = \frac{\epsilon}{V^{2N+1}} \mathscr{L}^{2N+1} H_{\text{int}} \qquad \Gamma_{2N}^2 = \|i[Q_{2N}, H]\|^2 \approx \epsilon^2 \frac{\|\mathscr{L}^{2N+1} H_{\text{int}}\|^2}{V^{4N+2}}$

This is a convergent procedure (at large V) for any finite-dimensional matrices, free models, Cayley trees,..

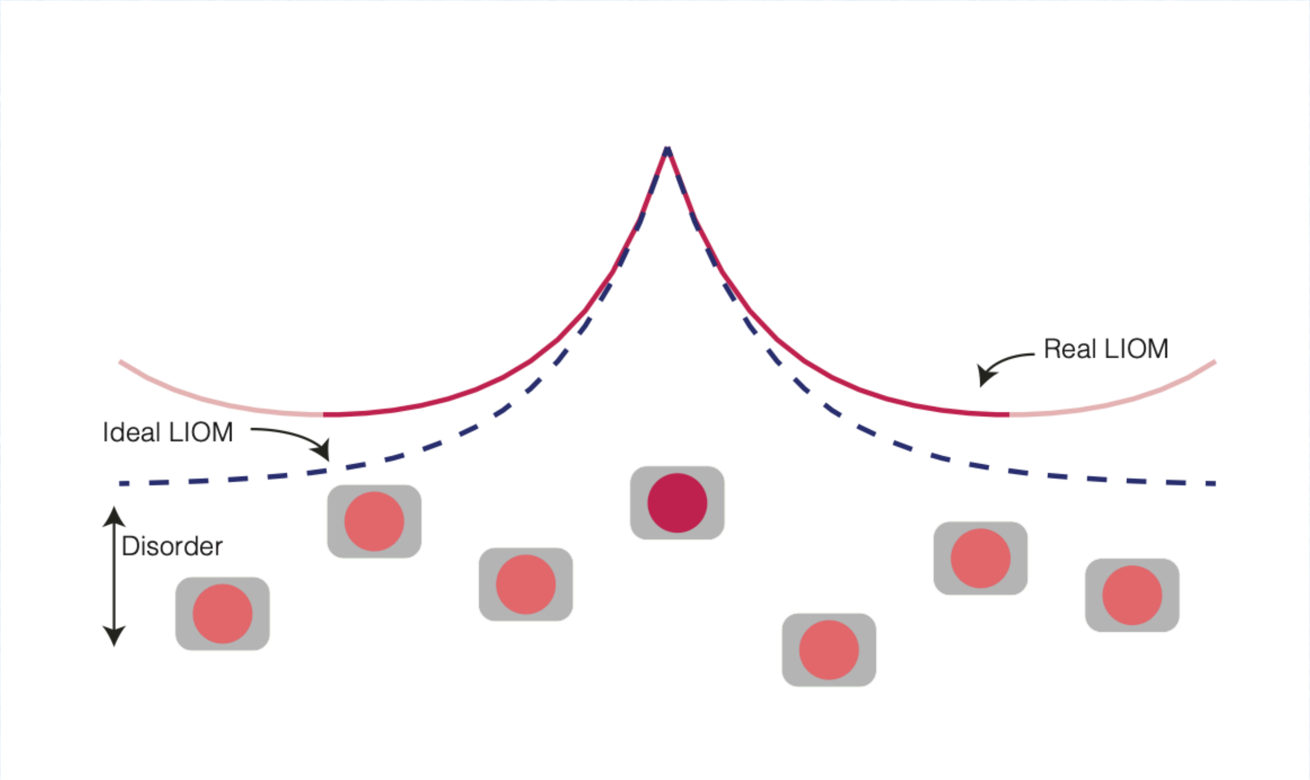
Generic chaotic models (no selection rules): D. E. Parker et. al. 2019; A. Avdoshkin, A. Dymarsky 2020; X. Cao 2021, … Disorder plays no role!

$$\|\mathscr{L}^k O\|^2 \approx \left(\frac{2k}{e\tau}\right)^{2k}, \quad \Rightarrow \quad b_k \sim k$$

Operator growth (short time expansion) sets long time decay rate.

In interacting systems perturbation theory/SW transformation is intrinsically unstable due to virtual UV processes!

LIOM correlation length flows with the distance, no exponential tails!

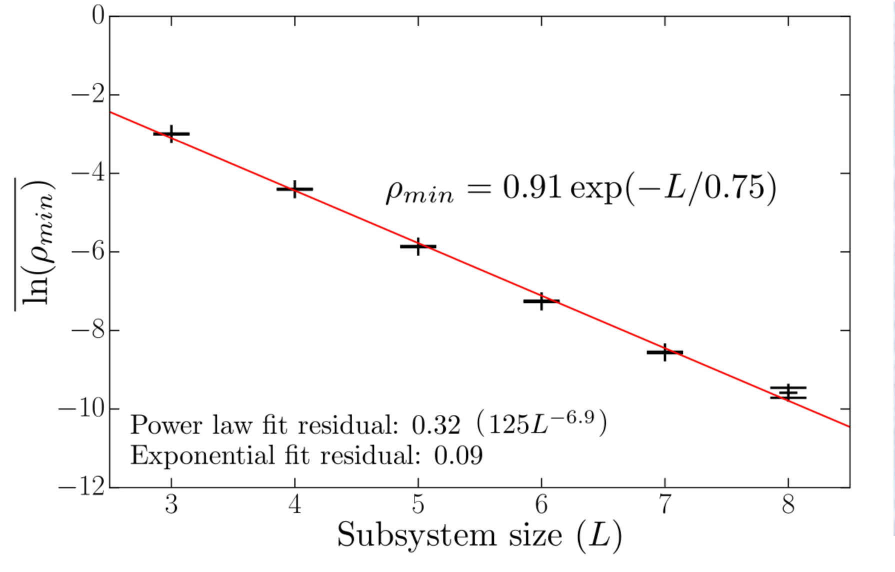


Combine with avalanche instability argument by W. De Roeck and F. Huveneers (2017) - localization is unstable in all dimensions.

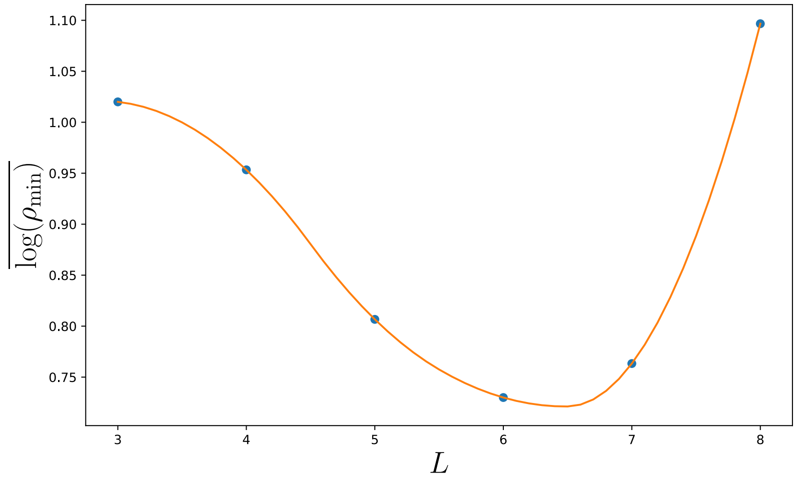
Finite systems. MBL - chaotic (glassy) but non-ergodic (non-mixing) phase.

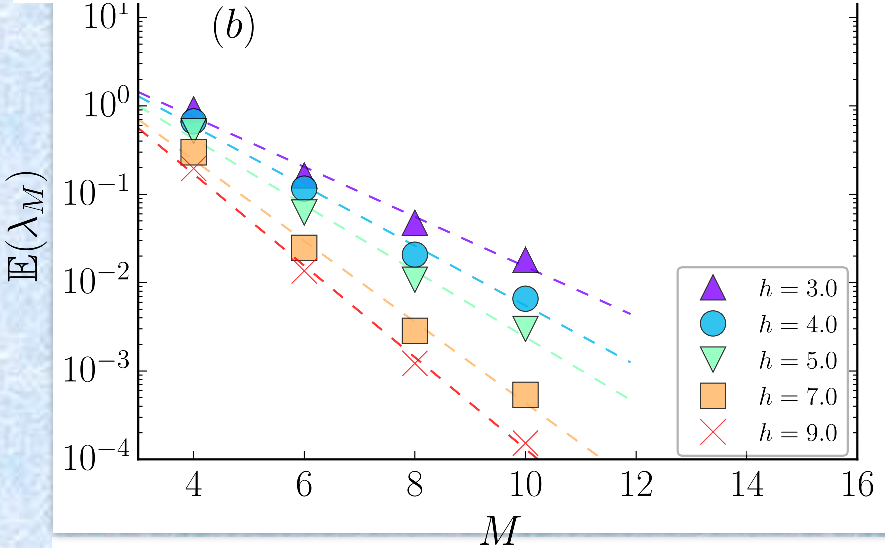
Earlier studies showing exponential L-bits

"Exponential scaling" of the slowest operator in the MBL phase (T O'Brien, D Abanin, G. Vidal, and Z. Papic, PRB, 2016)

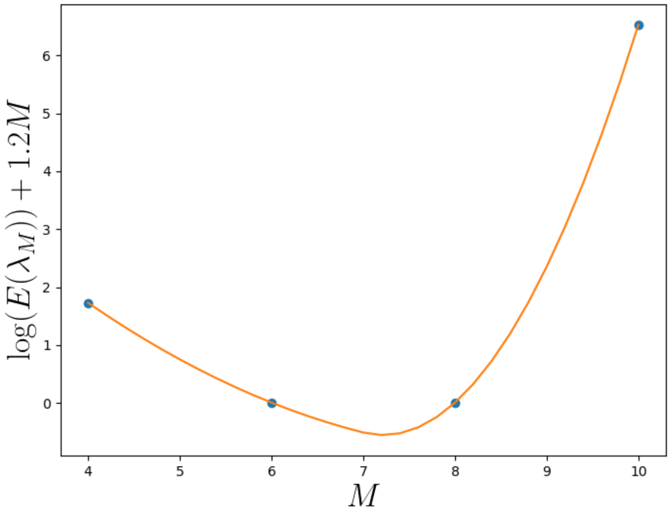


Digitized data with subtracted slope. Strong drift of slope. MBL is unstable using arguments by W. De Roeck, F. Huveneers (2015)



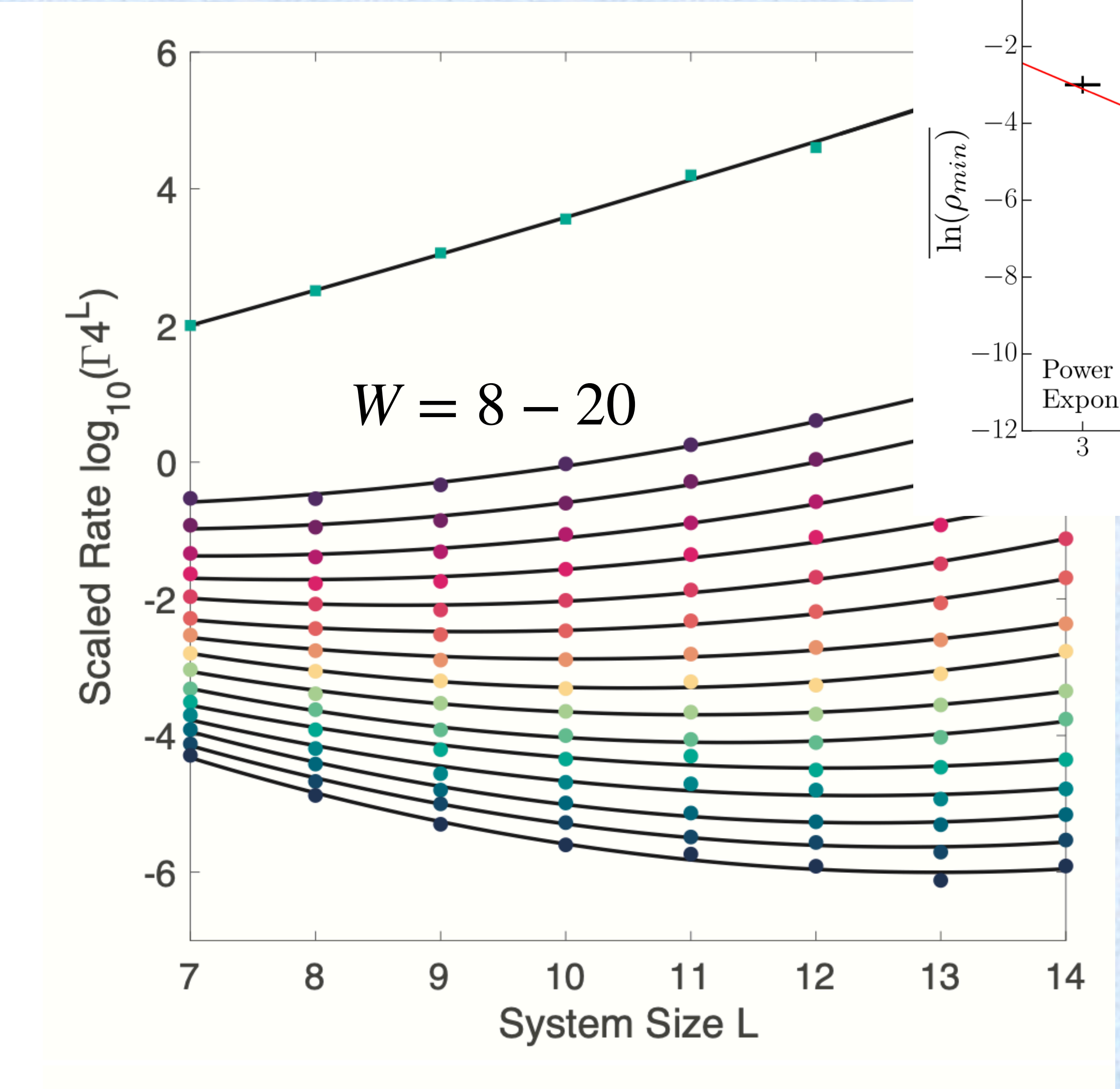


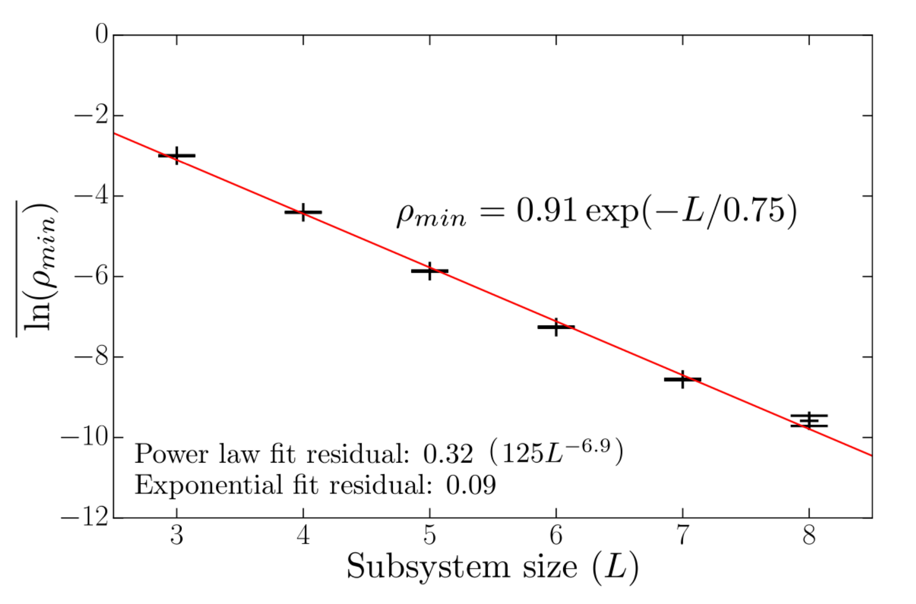
N. Pancotti, M. Knap, D. A. Huse, I. Cirac, and M. Banuls, PRB 2018. "For strong disorder, the decay of λ_M is compatible with an exponential form, $e^{-M/\xi}$"



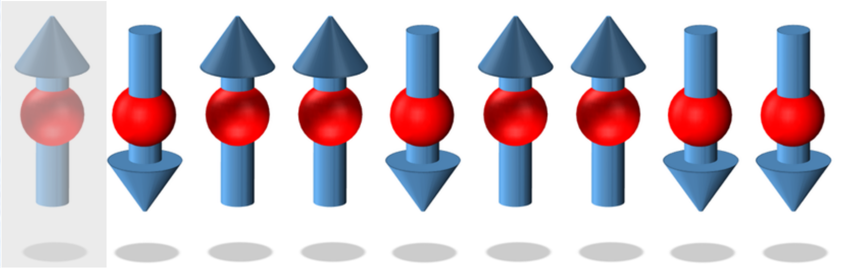
The digitized data for $h = 9$ with a subtracted mean slope. True statement: "For strong disorder, the decay of λ_M is incompatible with an exponential form, $e^{-M/\xi}$."

A. Morningstar, L. Colmenarez, V. Khemani, D. J. Luitz, and D. A. Huse, 2021,D. Sels 2021, no signatures of localization for $W \lesssim 20$.





$$\frac{\|[O,\sigma_0^\alpha]^2\|}{\|O^2\|} = \min, \quad O \sim \tau_L^z$$



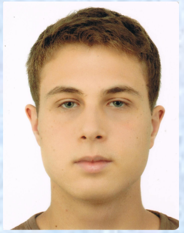
Disorder only changes the mean slope (bare ξ) but not the shape. $t_{\mathrm{Th}} > \exp[70]$.

Conclusions. Outlook.

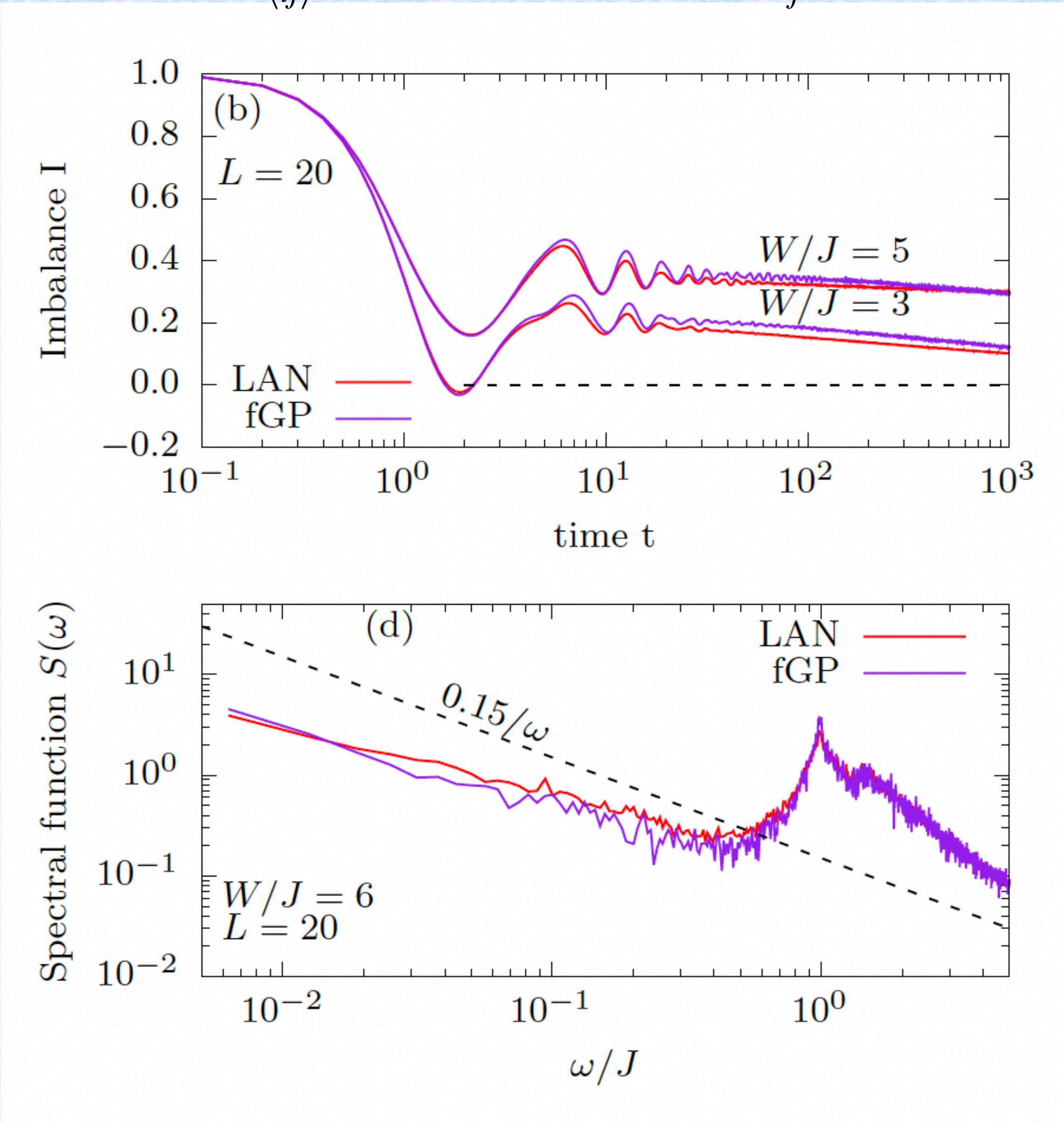
- Can define chaos in quantum/classical Hamiltonian systems via adiabatic complexity.
- No direct transition from Integrability to ergodicity in generic systems. Intermediate (maximally chaotic) KAM phase in all cases. KAM regime is usually transient (prethermalization) in extended systems but parametrically long in time close to integrability in the TD limit.
- Three distinct regimes for time evolution (at weak integrability): integrable, chaotic/glassy/KAM, local equilibrium. Generally a similar situation if weakly break symmetries instead of integrability (Floquet).
- Universal dynamics in the KAM regime. RG possible? Integrable regions are like critical points. Glassy dynamics are like critical slowing down.
- Emergence of $1/\omega$ spectral function ($1/f$-noise) close to integrability in many cases; $\log(t)$ relaxation in time in many systems: quantum and classical, few-body and many-body.
- Disordered systems are not special. “MBL” looks like a transient (in local models) glassy KAM regime: exists in quantum and classical systems, 1D, 2D interacting models share similar universal properties.
- Chaos is possibly delocalization in the Krylov space. Ergodicity - delocalization in phase space (Fock space).

Disordered Interacting Fermions

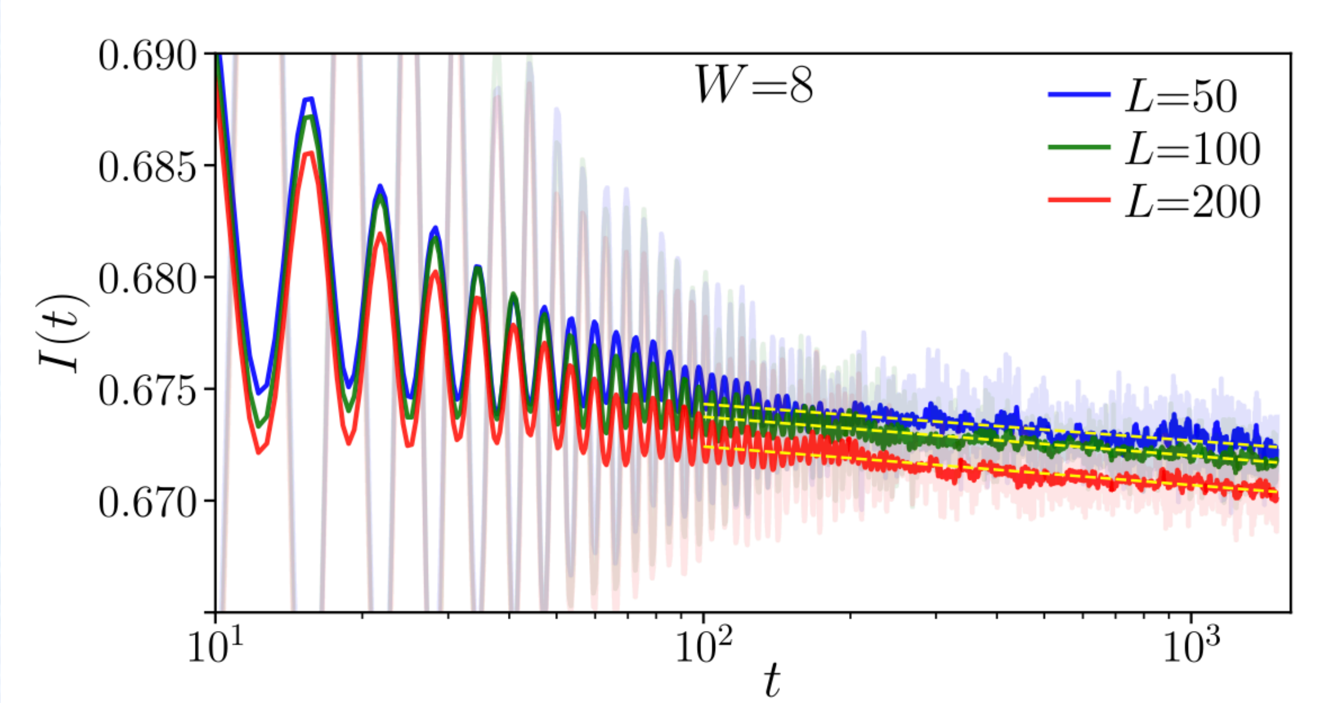
$$H = -\frac{J}{2}\sum_{\langle ij \rangle}(c_i^\dagger c_j + c_j^\dagger c_i) + V n_i n_j + \sum_j \epsilon_j n_j, \quad V = 0.5,\ J = 1$$



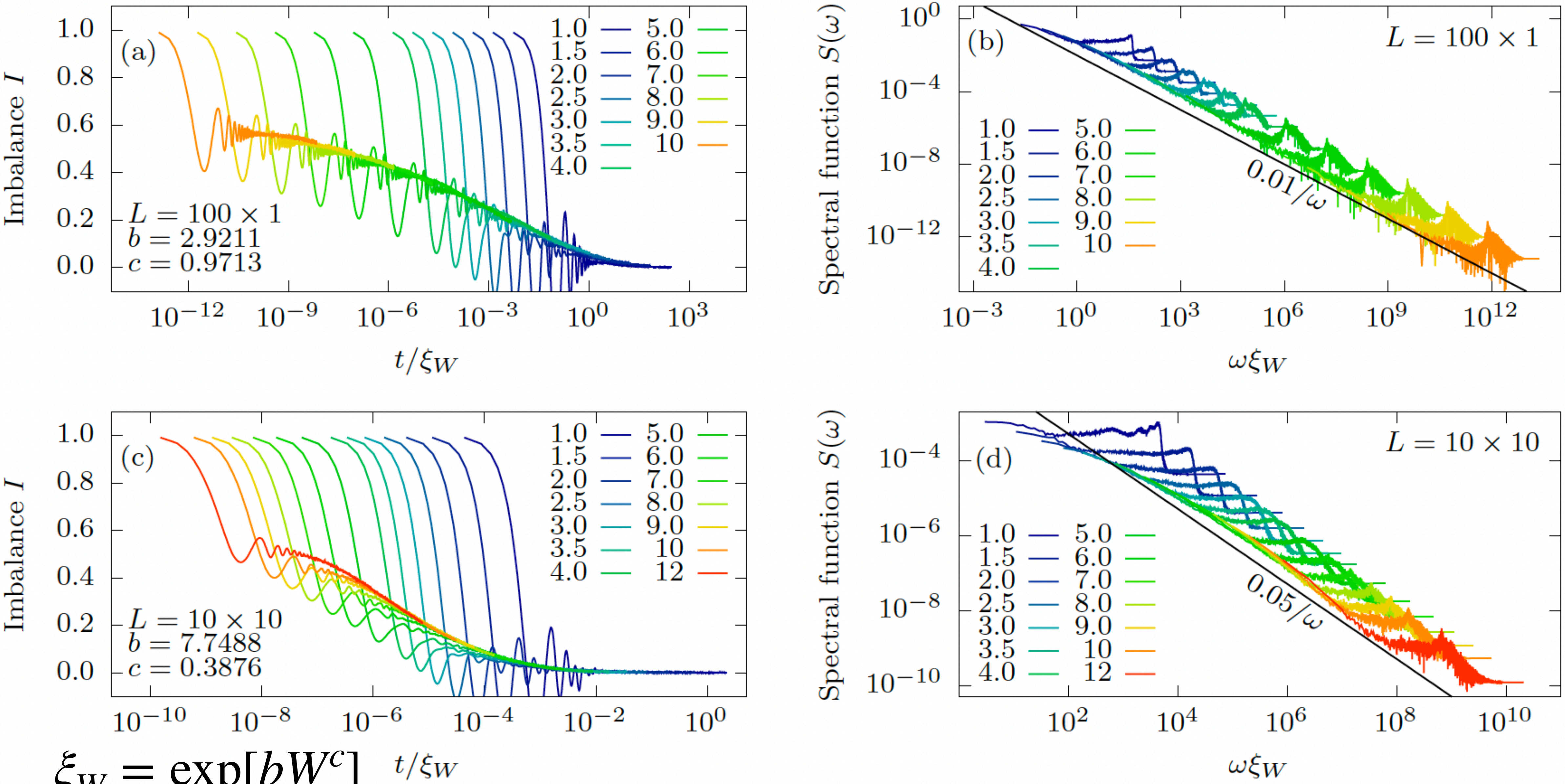

Poster: by Lucasz Iwanek



Disordered quantum XXZ chain "MBL"(P. Sierant, and J. Zakrzewski, 2021)



Universal glassy dynamics in 1D and 2D. Similar observation (PhD. Thesis of J. Wurtz, 2020)



$$\xi_W = \exp[bW^c]$$

MBL: localization in disordered systems are stable to short range interactions.D. Basko, I. Aleiner, B. Altshuler 2006, I. Gornyi, A. Mirlin, D. Polyakov, 2005; V. Oganesyan, D. Huse, 2007, … Onsager prize 2022.

Loosely speaking MBL=Fock space localization.

Sites are $1001010110,\ 0101010110,\ldots$

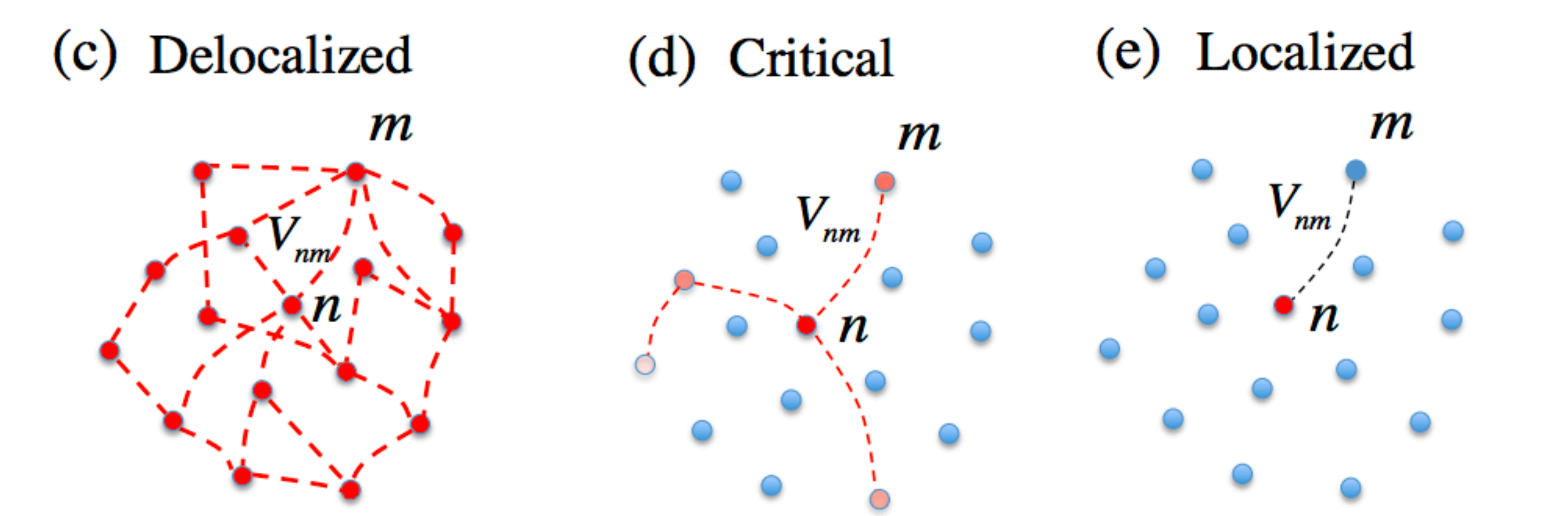
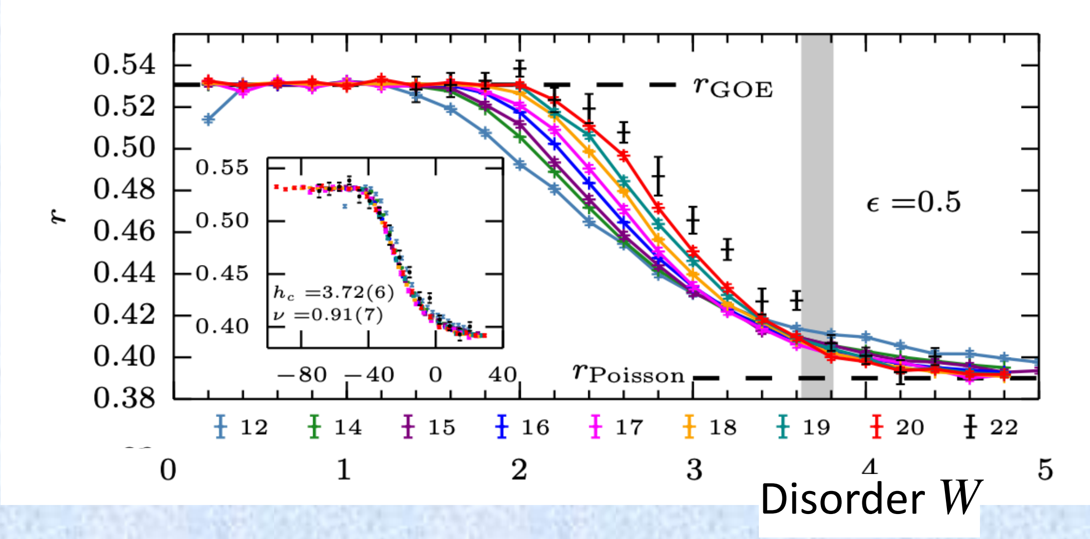


Image taken from Z. Papic talk

Competition between growing density of states and matrix elements. Claim: at strong disorder matrix elements decay faster than density of states grows.

Standard Model for MBL amenable to numerics: disordered Heisenberg (XXZ) chain:

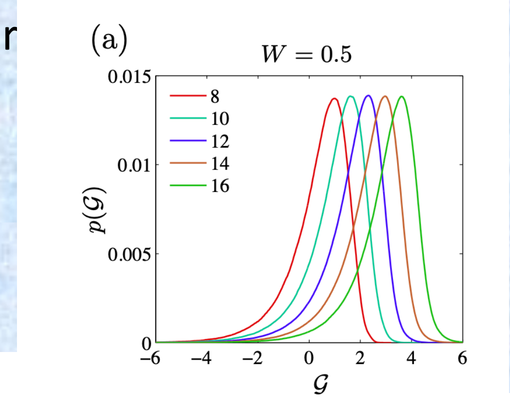
$$H = \sum \mathbf{S}_j \mathbf{S}_{j+1} - h_j S_j^z, \quad h_j \in [-W, W]$$

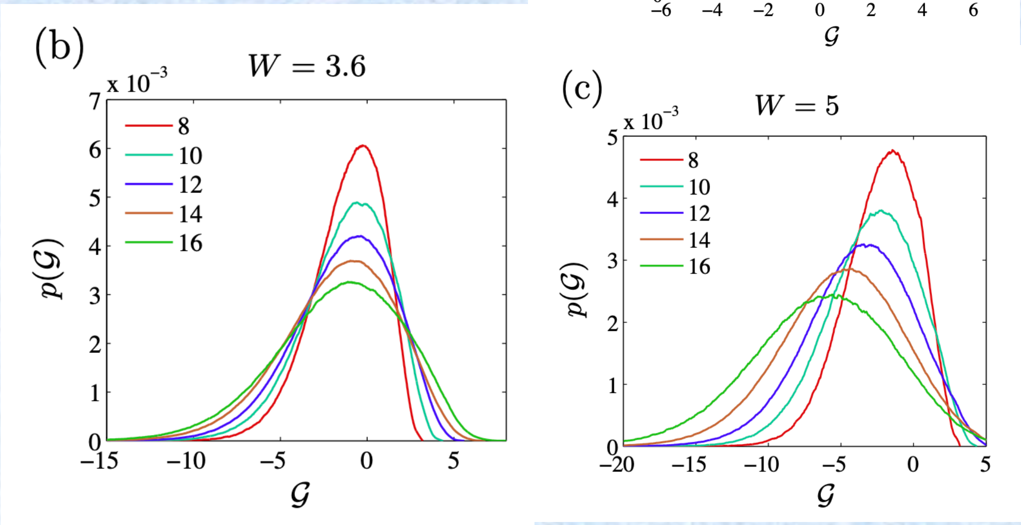


D. J. Luitz, N, Laflorencie, and F. Alet, PRB 2015.
Critical disorder $W_c \approx 3.72$ from level statistics

A different measure similar to χ :

$$G = \log\left(\frac{|V_{n,n+1}|}{E_{n+1} - E_n}\right)$$





M. Serbyn, Z. Papic, and D. Abanin, PRX 2015. Transition at $W_c \approx 3.6$

Many other papers "confirming" MBL transition near $W_c \approx 3.6$.

Analytic proof? of stability of MBL phase by J. Imbrie with few extra assumptions for stability of MBL phase (2016).

Experiments by I. Bloch's group (2015). RG (Vosk, Altman, Huse). Avalanche instability (W. De Roeck, F. Huveneers 2015),….

Standard phenomenology of MBL: existence of local integrals of motion (LIOMs, Lbits). M. Serbyn, Z. Papi´c, D. A. Abanin (2013); V. Oganesyan and D. Huse (2013)

Exponential scaling of the slowest operator with the system size (vs. expected diffusive in ergodic systems)

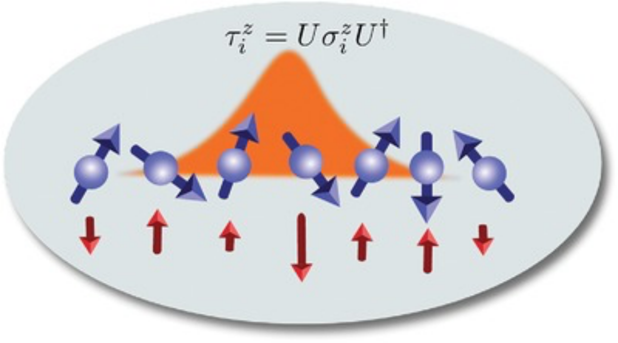
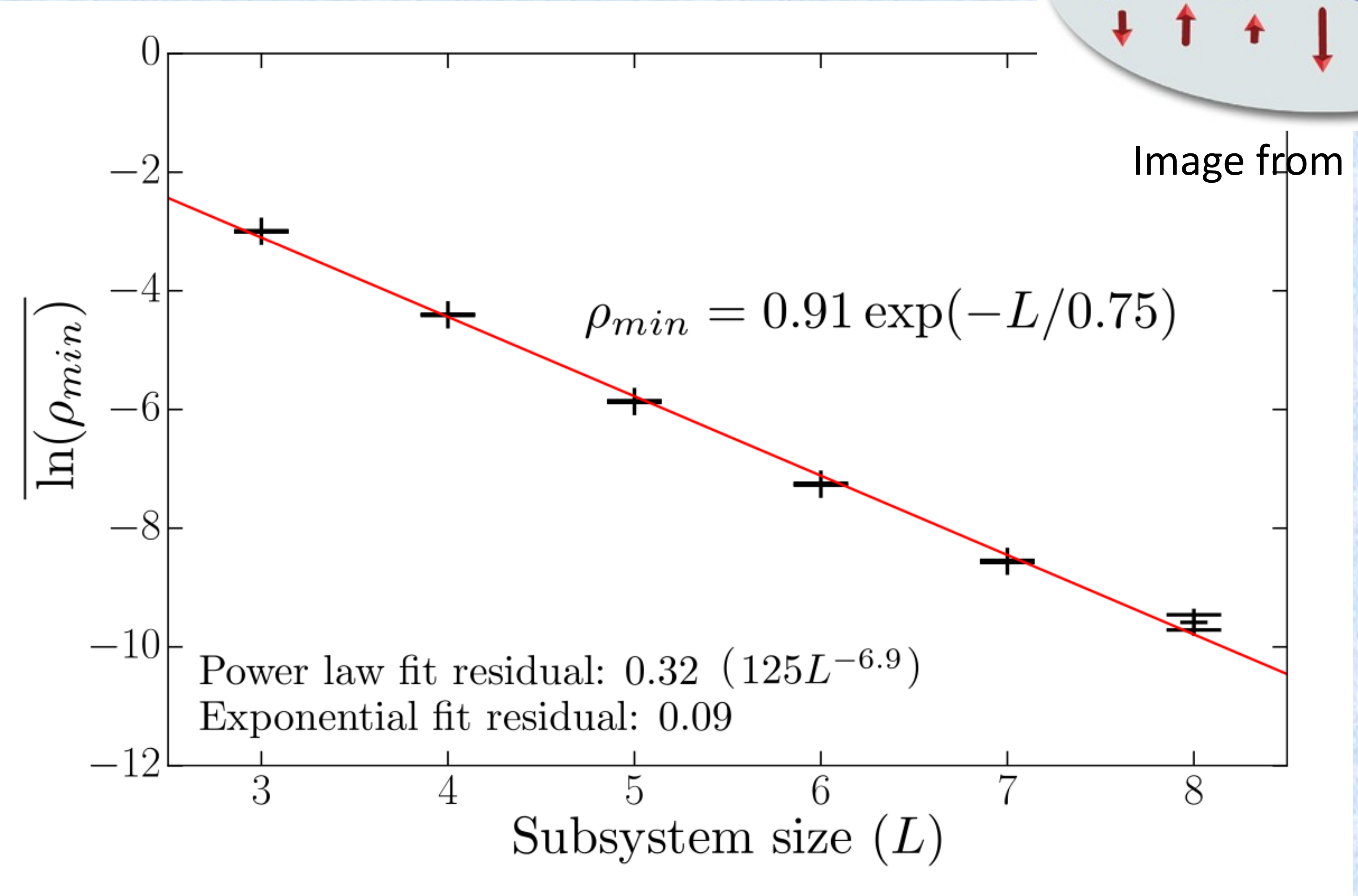
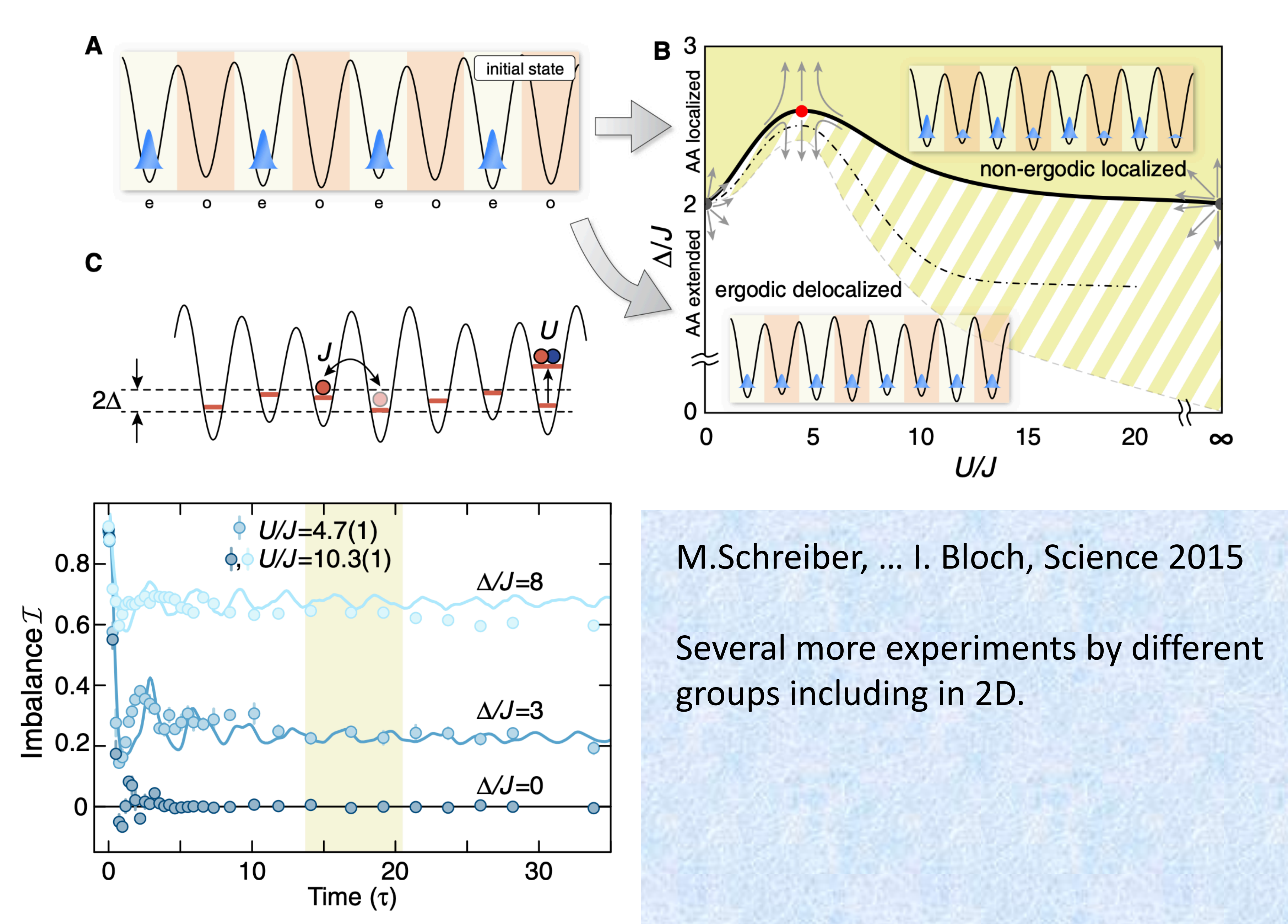


Image from D. Abanin talk



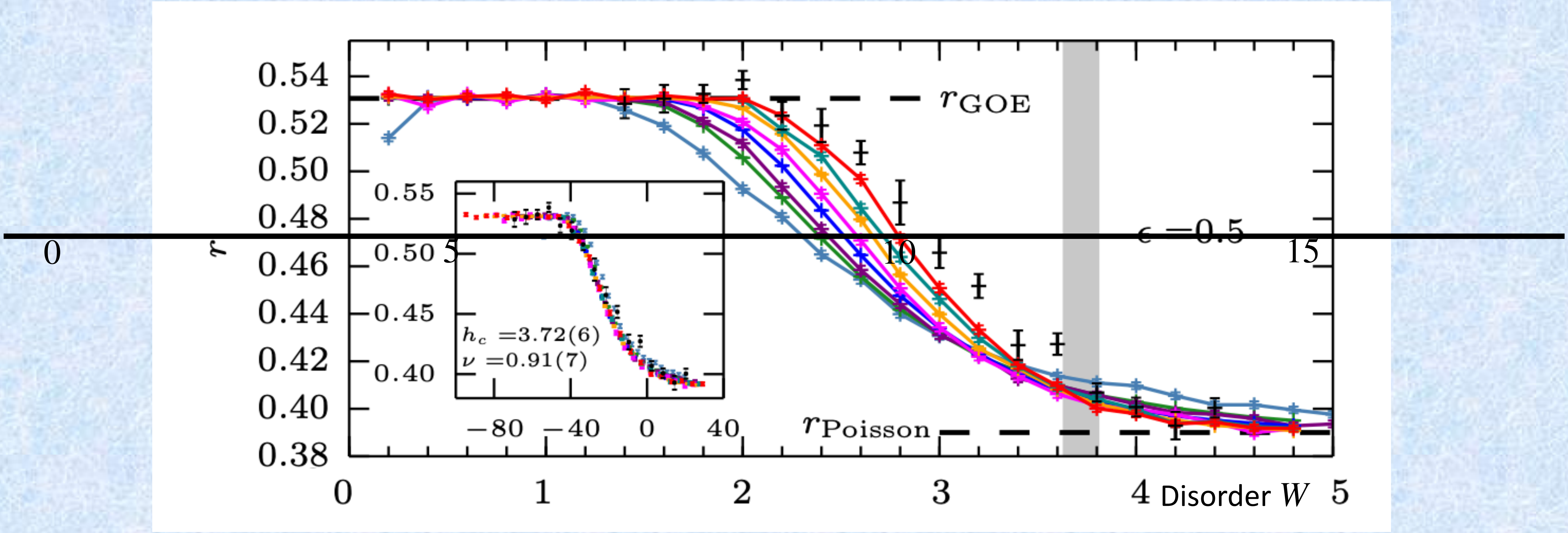
"Exponential scaling" of the slowest operator in the MBL phase (T O'Brien, D Abanin, G. Vidal, and Z. Papic, PRB, 2016)

Beautiful experimental confirmation in cold atoms: interacting fermions with quasi-periodic incommensurate potential



M.Schreiber, … I. Bloch, Science 2015

Several more experiments by different groups including in 2D.



D. J. Luitz, N, Laflorencie, and F. Alet, PRB 2015. Critical disorder $W_c \approx 3.72$ from level statistics. True transition at $W_c > 20$?

Numerical progress in MBL disorder/time scales

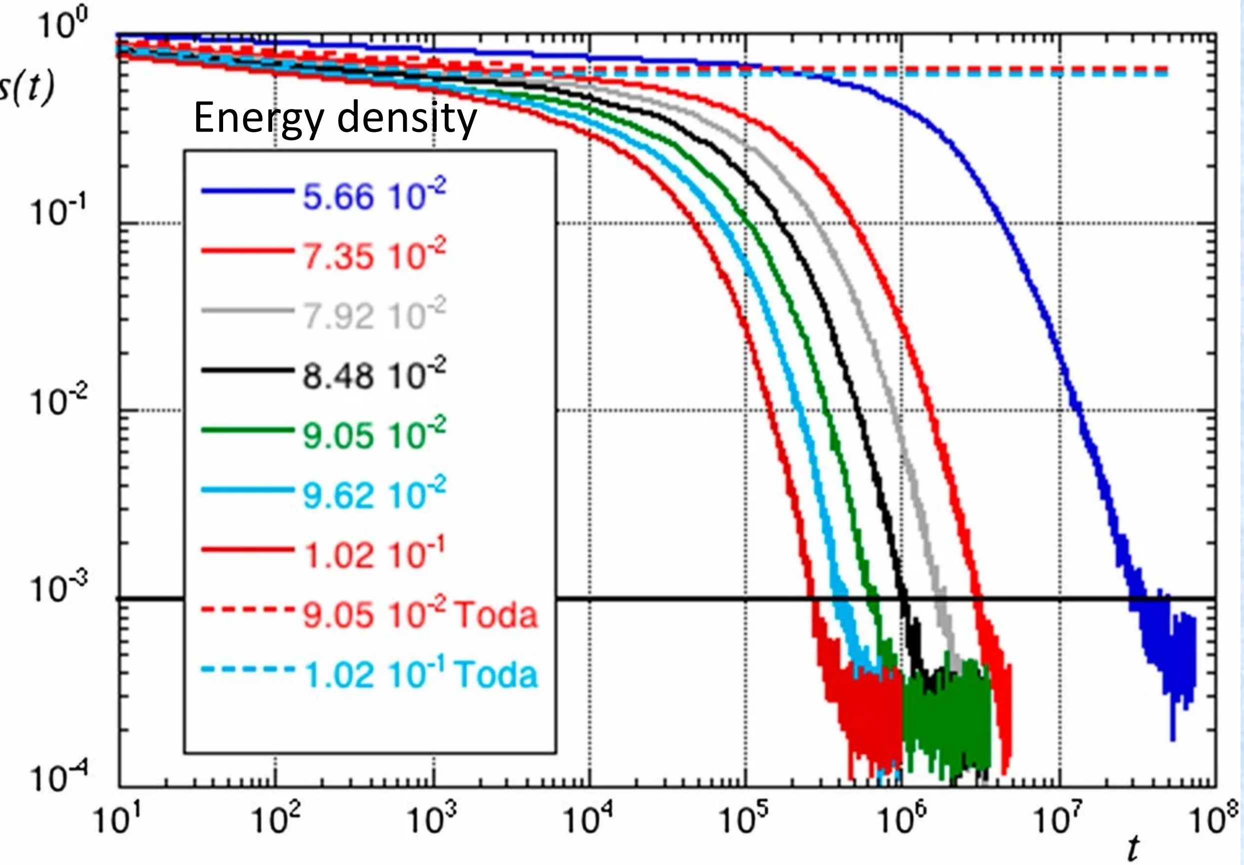
$W_c \approx 3.6$ | $W_c \geq 5.3$ | $W_c \geq 10$ | $W_c \geq 20$ | $W_c \geq 100$

2008 | 2018 | 2019 | 2020 | 2021 | 2022

$t_{\mathrm{Th}} \approx 3\ 10^5$ | $t_{\mathrm{Th}} \geq 10^8$ | $t_{\mathrm{Th}} \geq 10^{15}$ | $t_{\mathrm{Th}} \geq 10^{30}$ | $t_{\mathrm{Th}} \geq 10^{152}$

Age of the universe

FPU chain

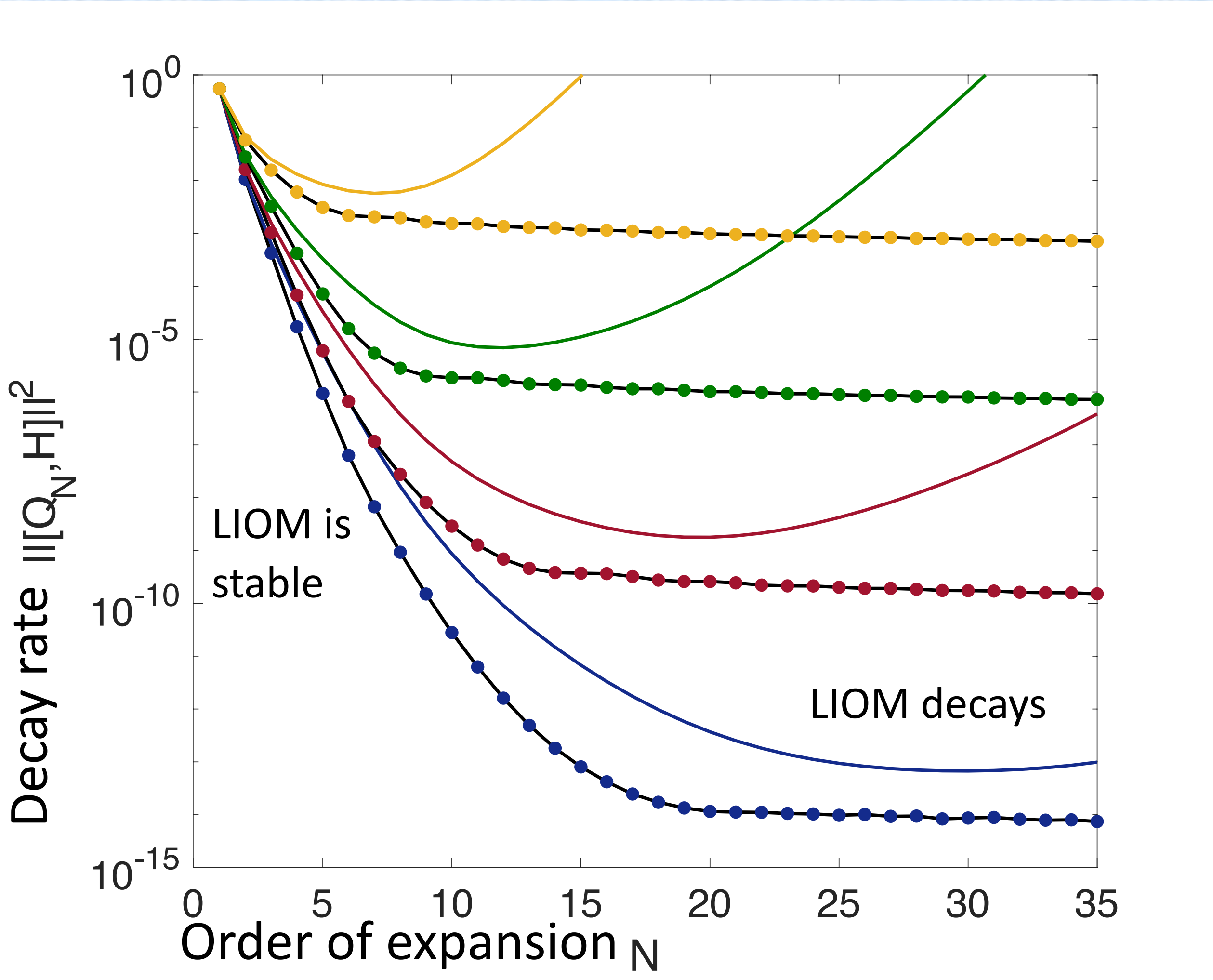


Can easily reach same astronomical times in classical systems (M. Onorato, L. Vozella, D. Proment, and Y. V. Lvov, PNAS 2015).

Glassy systems, classical disordered models, Floquet systems, …

Can do the variational minimization in the Krylov space instead of perturbative approach

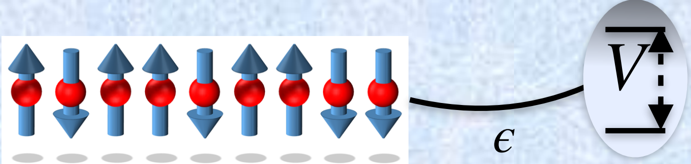
$$Q_{\text{var}} = S_0^z + \alpha_0 H_{\text{int}} + \alpha_1 \sigma_0^z [H_{\text{bath}}, H_{\text{int}}] + \alpha_2 [H_{\text{bath}}, [H_{\text{bath}}, H_{\text{int}}]] + \ldots, \quad \|[Q_{\text{var}}, H]\| = \min$$



The variational approach agrees with perturbative at small N and then crossovers to a very slow asymptotic regime.

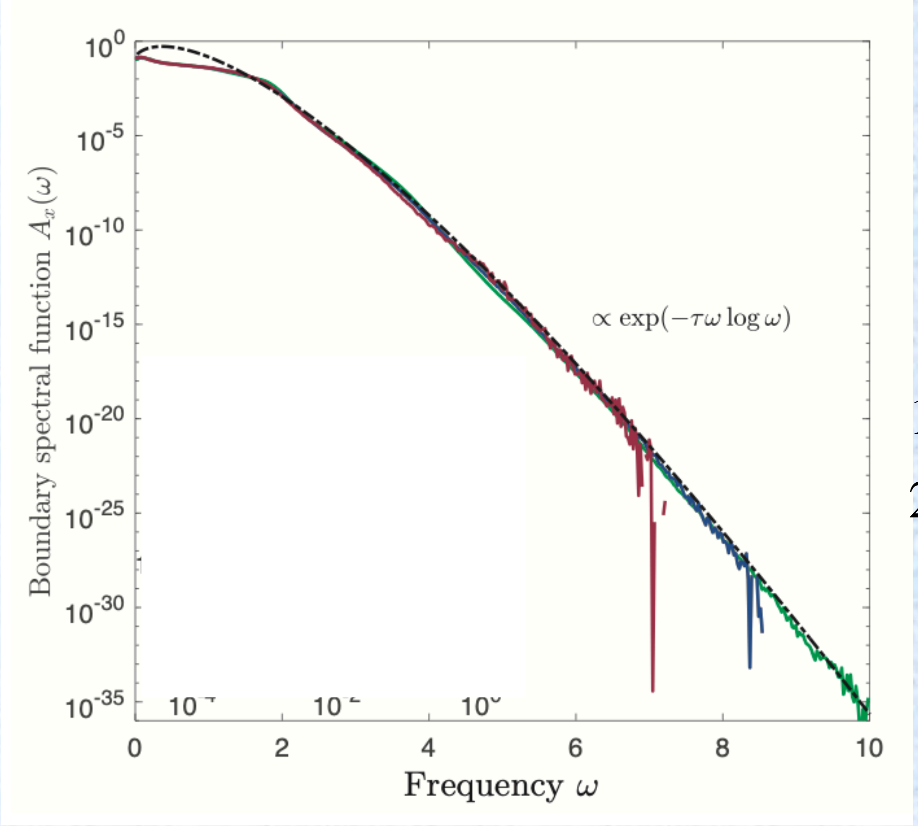
Many nearly degenerate solutions in the slow regime.

$$H = H_{\mathrm{bath}} + VS_0^z + \epsilon(S_0^x S_1^x + S_0^y S_1^y)$$



Rotating frame (interaction picture) with respect to VS_0^z (low-high frequency correspondence)

$$H_{\mathrm{rot}} = H_{\mathrm{bath}} + \epsilon(S_0^x S_1^x + S_0^y S_1^y)\cos(Vt) + \epsilon(S_0^x S_1^y - S_0^y S_1^x)\sin(Vt)$$



Long times: expect FGR decay of the magnetization:

$$\frac{dS_0^z}{dt} \approx -\Gamma(V)(S_0^z - 1/2)$$

From the universal operator spreading:

$$1D: \quad \Gamma(V) \sim \epsilon^2 \exp[-\tau V \log \tau V]$$

$$2D+: \quad \Gamma(V) \sim \epsilon^2 \exp[-\tau V]$$

D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, and E. Altman, PRX, 2019, C. Murthy, M. Srednicki, PRL 2019, A. Avdoshkin and A. Dymarsky, PRR 2020

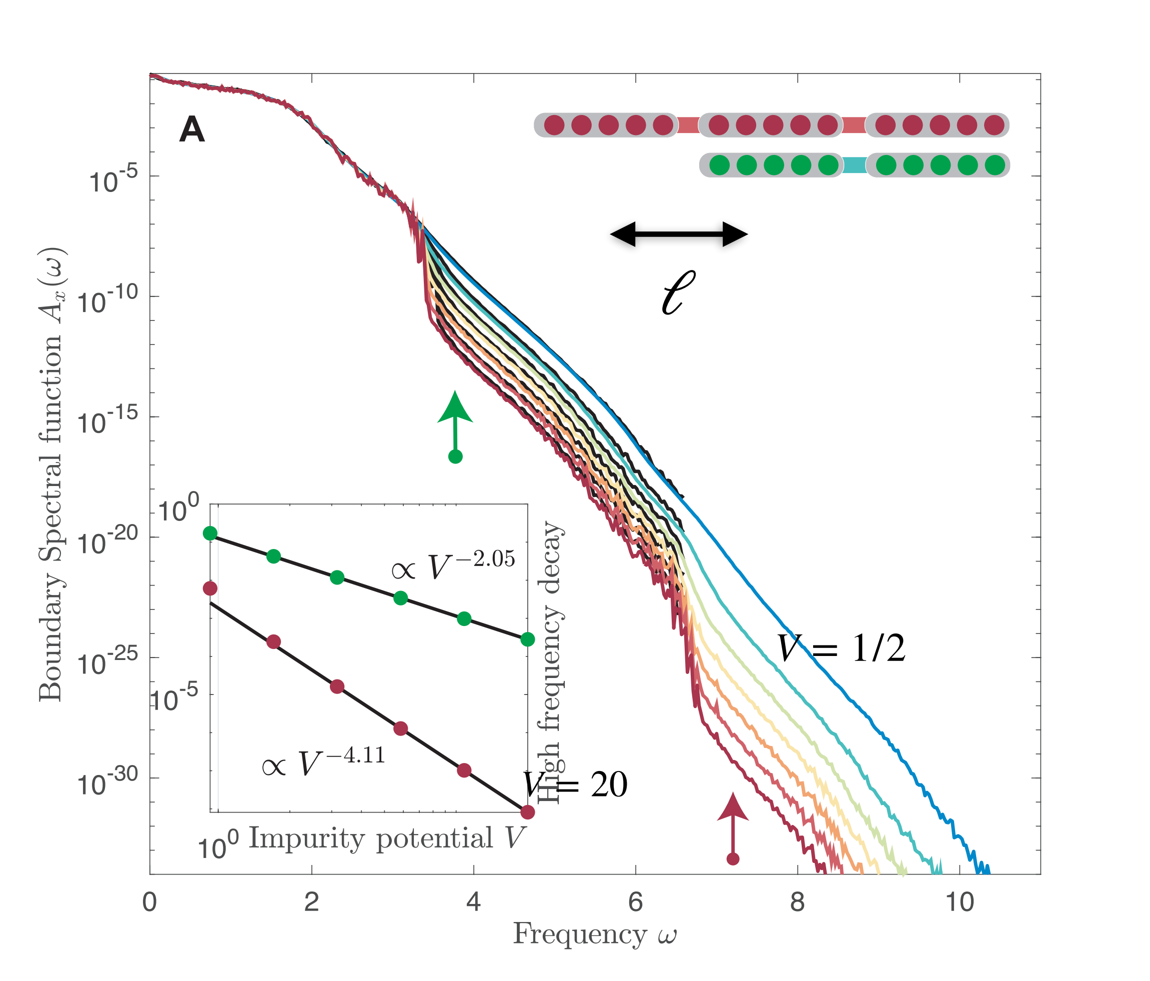
Non-perturbative in 1/V decay rate!

Finite systems: localization transition:

$$\Gamma(V_c) \approx \exp[-S(L)] = 2^{-L} \rightarrow V_c \approx \frac{L\log(2)}{\tau \log L}$$

Matches the LIOM decay rate.

Multiple impurities do SW transformation on all but one. Restrict Hilbert space.



FGR rate additionally suppressed by weak links $J_{\text{eff}} = J^2/V$

Same functional form as for a single impurity: $\tau \to \tau(1 + 1/\ell)$

$$\Gamma \approx \epsilon^2 \exp[-\tau(1 + 1/\ell)V \log(V)]$$

Disorder renormalizes τ by at most factor of 2.

$$V_c \geq \frac{L \log(2)}{2\tau(1 + 1/\ell)\log(L)}$$

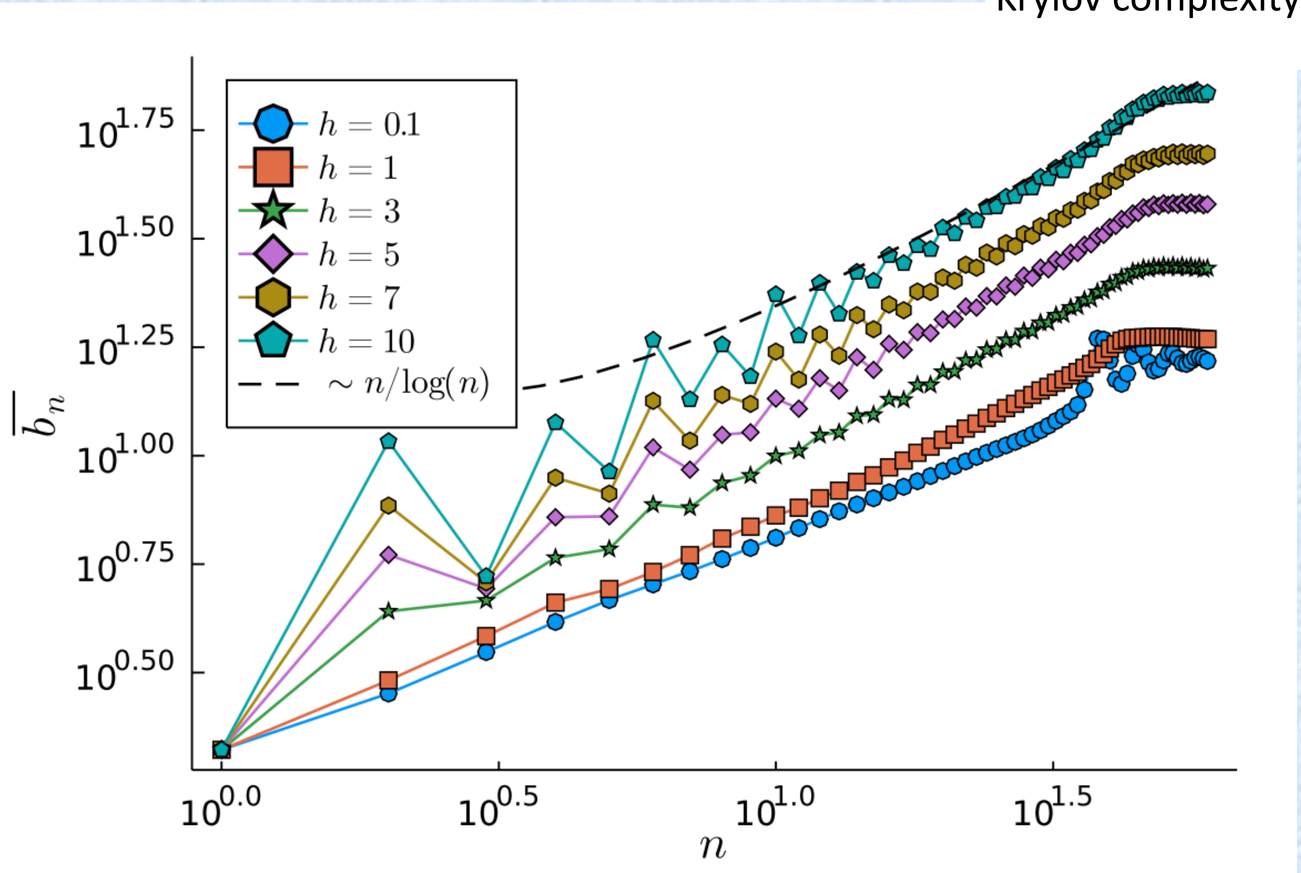
No special features in TD limit at any finite V.

Very small finite size effects in the spectral function and the FGR rate. Reach same conclusion from nested commutator norms: $\mathscr{L}^{2N+1}H_{\text{int}}$ contain some fraction of weak links.

$$R_k^2 = \|\mathscr{L}^k H_{\text{int}}\|^2 \sim \left(\frac{2k}{e\tau \log(2k)}\right)^{2k} \frac{1}{V^{2k/\ell}}$$

Disordered systems: expect same universal operator growth, which does not depend on assumptions about eigenstates (X. Cao, 2020).

Krylov complexity growth: $b_n \sim \dfrac{n}{\log n}$

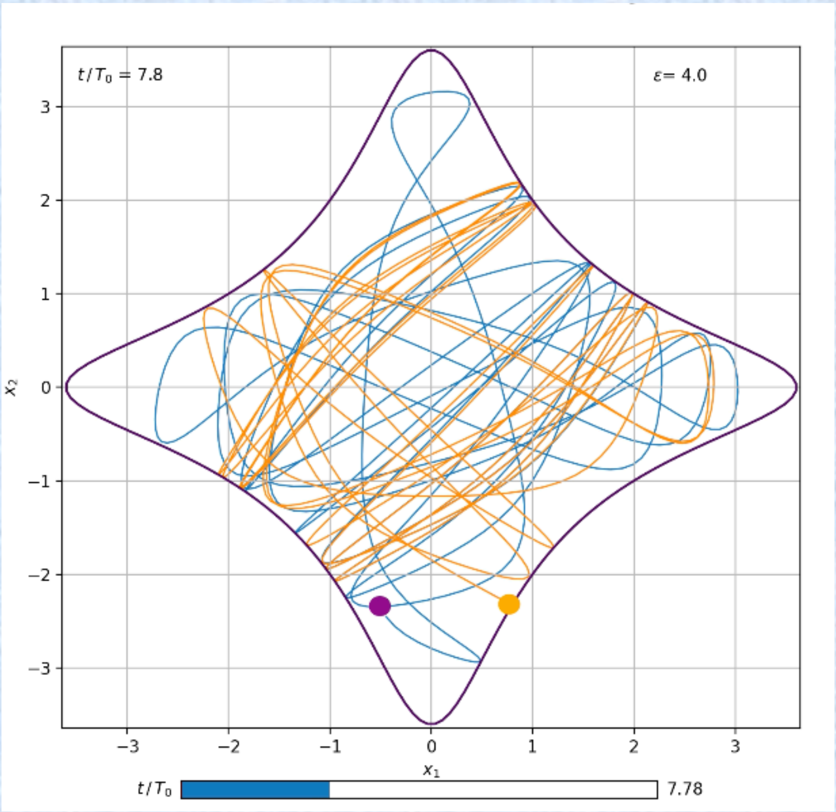


Disordered XXZ chain, h is the disorder strength. F. B. Trigueros and C.-J. Lin, arXiv 2021

At high orders disorder is irrelevant for the Lanczos coefficients. Have self averaging. Conclusion MBL is a glass: hard to distinguish from true localized phase, but there is no transition. Quantum Mechanics is not important: MBL is like a highly off-resonant transmission line.

Berry’s, BGS conjectures, ETH, RMT are measures of ergodicity (thermalization/mixing), not chaos! These conjectures are inconsistent with KAM regime (chaos without ergodicity).

M. Berry, private communication: “I hope you don't quote me as defining quantum chaos. I never did. Instead (responding to the 'no quantum chaos' people), I defined Quantum Chaology (see my papers “Quantum chaology’” 1987, “‘Quantum chaology, not quantum chaos”, 1989), in terms of semiclassical quantum phenomena associated with classical chaos. Such phenomena (RMT is an example) occur because the classical limit is singular (see ”‘Singular Limits”, 2002).”

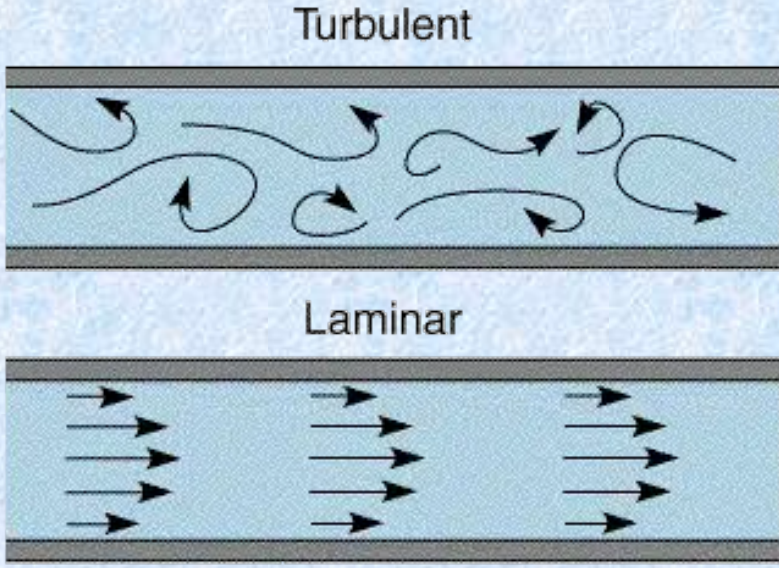


No traces of RMT even if we break all the symmetries (also Remy Dubertrand, private communication).

Mixing chaos and ergodicity lead to many confusions in theory works.

Everyday experience: chaos and thermalization are not the same.

Non-thermalizing (cascades), no RMT, no ETH, small Lyapunov exponents, ….



Thermalizing (ergodic, mixing), RMT, ETH, large Lyapunov exponents, ….

Image: Will Woodward, “Airway Resistance”, teachmephysiology.com