From Energy-Conserving Quantum Circuits to Quantum Fisher Information

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Frontiers in Non-equilibrium Physics
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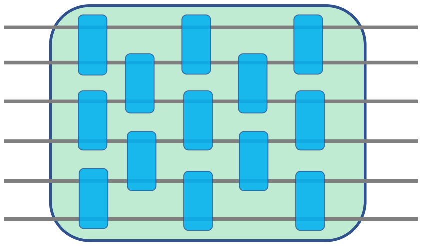


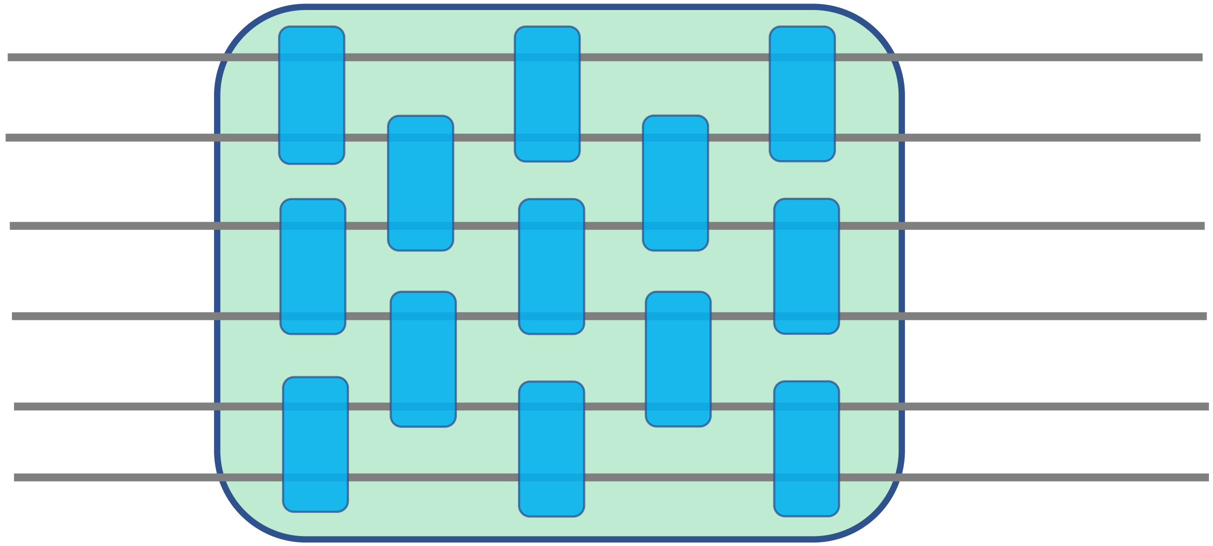
Part I. Symmetric Quantum Circuits

- Energy-Conserving Circuits
- New Conservation Laws in systems with SU(d) symmetry

IM, Nature Physics (2022).
IM, H. Liu, and A. Hulse, Phys Rev. Lett (2024).
IM, H. Liu, and A. Hulse, arXiv:2105.12877 (2021).
G Bai, IM, arXiv:2309.11051, to appear in Quantum Science and Technology (2024).
IM, arXiv:2302.12466, to appear in Phys Rev. Research (2024).

Part II. Irreversibility and Quantum Fisher Information Metrics

L. Gao, H. Li, IM, C Rouzé, arXiv:2302.02341, to appear in Comm. in Math. Physics (2024).



Universality: Any unitary transformation on a composite system can be realized *exactly* by a finite sequence of 2-local unitaries on the subsystems.

Universality with 3-qubit gates (Deutsch 1985)

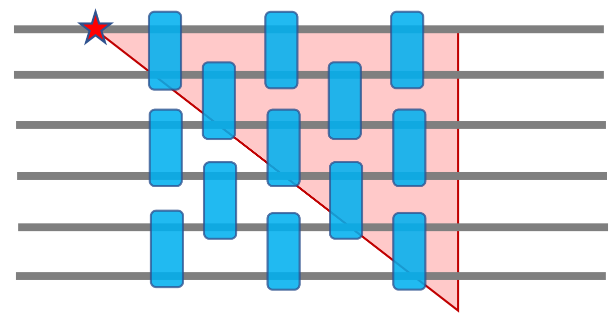
Universality with 2-qubits gates (DiVincenzo 1995), Almost all 2-qubits gates are universal (Lloyd 1995, Deutsch-Barenco-Ekert 1995)

What is the minimum energy that should be dissipated to the environment to calculate a Boolean function?

Classically, universality can be achieved with 3-bit reversible gates.

Universality as a statement about time evolution under general local Hamiltonians

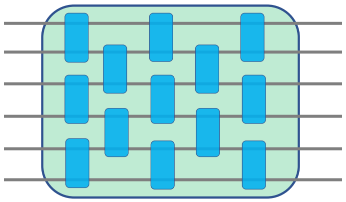
$$\frac{dV(t)}{dt} = -iH(t)V(t) \qquad V(0) = I$$



Locality of Hamiltonian puts various constraints on the **short-term dynamics**, e.g. finite speed of propagation of information, as highlighted by the Lieb-Robinson bound.

However, after a sufficiently long time, closed systems with local Hamiltonians can experience any arbitrary time evolution.

Questions

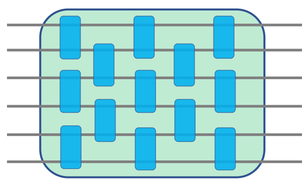


Does the universality of 2-local (k-local) unitaries remain valid in the presence of symmetries and conservation laws?

Do local symmetric Hamiltonians generate all symmetric unitaries?

Motivations

The framework of Local Symmetric Quantum Circuits has become a standard tool in theoretical physics.
We need to understand them better!



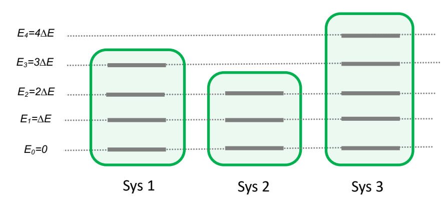
- Classification of symmetry-protected topological phases
  - Chen-Gu-Wen, *Classification of gapped symmetric phases in one-dimensional spin systems* (2011)
  - Chen-Gu-Liu-Wen, *Symmetry protected topological orders and the group cohomology of their symmetry group* (2013)
- Quantum chaos with conserved charges
  - Khemani-Vishwanath-Huse (2018)

Implications and Applications

1) Finding new conservation laws and constraints imposed on the dynamics of quantum systems by the presence of both symmetry and locality.

2) Symmetric unitaries are essential in Quantum Thermodynamics (athermality), Quantum Reference Frames, Resource Theory of Asymmetry, and Covariant Error-Correcting Codes. **How can we implement them?**

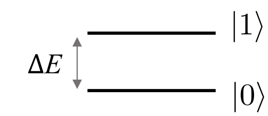
   - Is there any **hidden thermodynamic cost** for implementing a general energy-conserving unitary, using local energy-conserving unitaries?



$$H_{\text{tot}} = H_1 + H_2 + H_3$$

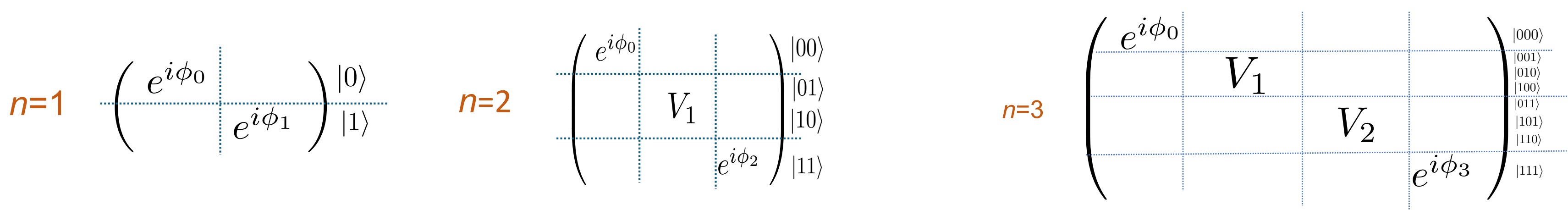
3) Synthesizing noise-resilient quantum circuits

Example:



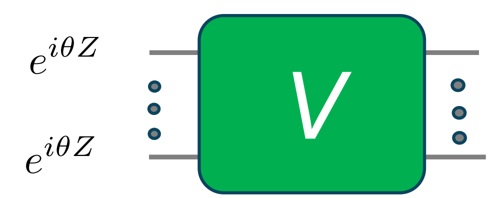
The intrinsic Hamiltonian of a qubit: $H = -\frac{\Delta E}{2} Z$

The unitary *V* on *n* qubits is energy-conserving if $V\big(\sum_{j=1}^{n} Z_j\big)V^\dagger = \sum_{j=1}^{n} Z_j$



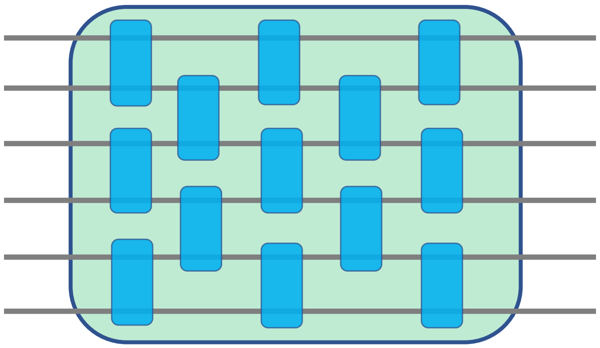
Equivalently: U(1)-invariant unitaries

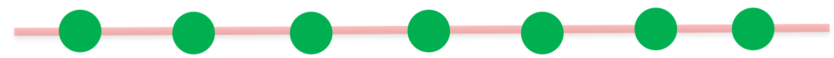
$$V(e^{i\theta Z})^{\otimes n} = (e^{i\theta Z})^{\otimes n} V \qquad \theta \in [0, 2\pi)$$



Examples: Unitaries generated by XX+YY interaction, CCZ gate, and Fredkin (controlled-SWAP) gate

Question: Can we realize 3-qubit U(1)-invariant unitaries using 2-qubit U(1)-invariant unitaries?





$$H(t) = \sum_r h_r(t) \qquad h_r(t) \text{ : } k\text{-local}$$

$$\forall t : \ [H(t), \sum_j Z_j] = 0$$

$$\frac{dV(t)}{dt} = -iH(t)V(t) \qquad V(0) = I$$

Set of unitaries that can be implemented by *k*-local symmetric unitaries

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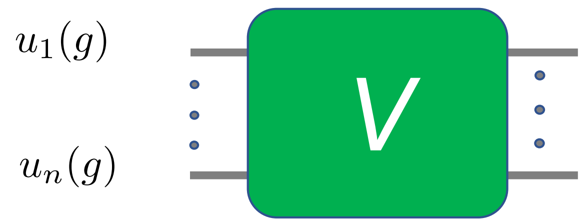
Set of unitaries generated by symmetric Hamiltonians that can be written as the sum of *k*-local terms

General Symmetries



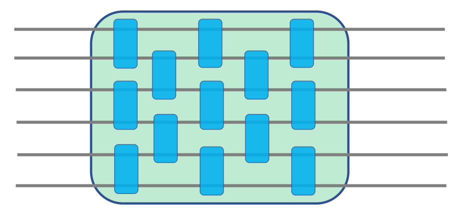
For a general group *G*, let $\{u_j(g) : g \in G\}$ be the representation of symmetry on site *j*.

We say a unitary *V* is **symmetric** with respect to the symmetry described by group *G*, if

$$\forall g \in G : \quad V \bigotimes_{j=1}^{n} u_j(g) = \bigotimes_{j=1}^{n} u_j(g) V$$

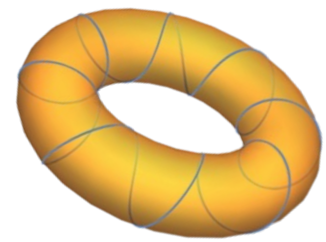
Or, equivalently

$$\forall g \in G : \quad [\bigotimes_{j=1}^{n} u_j(g)] V [\bigotimes_{j=1}^{n} u_j^{\dagger}(g)] = V$$



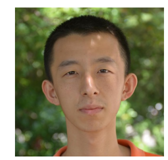
- **A general no-go theorem**

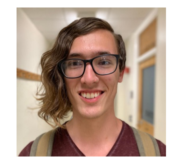
  In the case of continuous symmetries, it is not possible to implement generic symmetric unitaries, even approximately, using 2-local (k-local) symmetric unitaries on the subsystems. [IM, Nature Physics 2022].



Torus: Symmetric unitaries
Helix: The subgroup generated by *k*-local symmetric unitaries

- **Studied examples**
  - U(1) symmetry [IM, Nature Physics 2022]
  - SU(2) symmetry with qubits [IM, Hanqing Liu, Austin Hulse, Phys Rev. Lett 2024]
  - SU(d) symmetry with qudits [IM, Hanqing Liu, Austin Hulse, arXiv:2105.12877 (2021)]
    New conservation laws for d>2

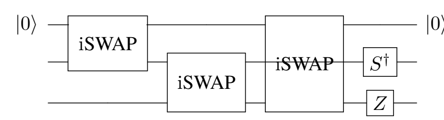


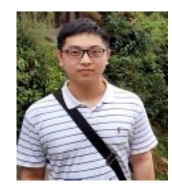


- **Conclusion:** The restrictions imposed by locality and symmetry vary significantly across different symmetry groups.

- **Theory of Abelian Quantum Circuits** [IM, arXiv:2302.12466 (2023), To appear in Phys Rev. Research]

- **Synthesis of Energy-Conserving Quantum Circuits with XY interaction**
  [Ge Bai, IM, arXiv:2309.11051 (2023), To appear in Q. Science and Technology]





Example: Energy-Conserving Unitaries (i.e., U(1) Symmetry)

The group generated by k-local U(1)-invariant unitaries

$$\dim(\mathcal{V}_{n,n}^{\mathrm{U}(1)}) - \dim(\mathcal{V}_{n,k}^{\mathrm{U}(1)}) = n - k$$

The maximum charge in the system minus the maximum charge that can participate in an interaction.

Example: *n*=3 qubits

Group of all U(1)-invariant unitaries**: 20 D**
Subgroup generated by *2*-local U(1)-invariant unitaries: **19 D**

Many useful energy-conserving unitaries are forbidden by these constraints (CCZ, Fredkin,..)

Example: SU(2) Symmetry

The group generated by k-local SU(2)-invariant unitaries

$$\dim(\mathcal{V}_{n,n}^{\mathrm{SU}(2)}) - \dim(\mathcal{V}_{n,k}^{\mathrm{SU}(2)}) = \lfloor \frac{n}{2} \rfloor - \lfloor \frac{k}{2} \rfloor$$

General groups

Grows unboundedly for continuous symmetries

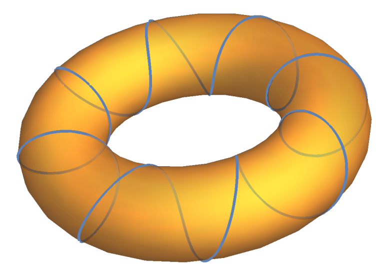
$$\dim(\mathcal{V}^G_{n,n}) - \dim(\mathcal{V}^G_{n,k}) \geq \text{Irreps}_G(n) - \text{Irreps}_G(k)$$

A rough Interpretation (For Abelian groups):
The maximum charge in the system minus the maximum charge that can participate in an interaction.

$\mathcal{V}^G_{n,n}$: The Lie group of all symmetric unitaries $\{V : VV^\dagger = I, [V, u(g)^{\otimes n}] = 0 \ : \forall g \in G\}$

$\mathcal{V}^G_{n,k}$: The subgroup generated by *k*-local symmetric unitaries

$\dim(\mathcal{V}^G_{n,k})$: Dimension of the corresponding manifold



$|\text{Irreps}_G(k)|$: Number of inequivalent irreducible representations of *G* appearing in the representation of symmetry on *k* subsystems. $\{u(g)^{\otimes k} : g \in G\}$

Application: Sensing the locality of interactions in nature

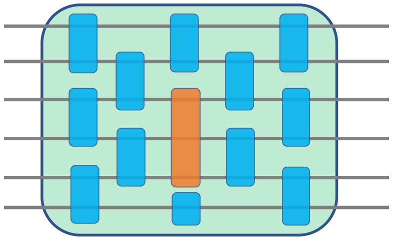
Consider a unitary time evolution under an unknown Hamiltonian.

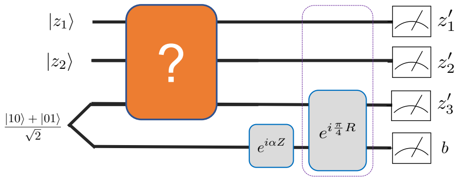
Question: By probing the outputs of this unitary evolution for different inputs, is it possible to detect the presence of 3-body (k-body) interactions in this unknown Hamiltonian?

No! 2-qubit gates are universal (DiVincenzo 1994)

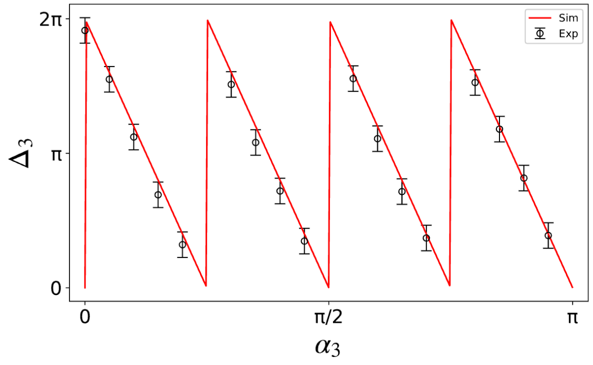
However, this becomes possible in the presence of symmetries!

L. Zhukas, Q. Wang, IM, C. Monroe, *Observation of the symmetry-protected signature of 3-body interactions,* under preparation.





IM, Nature Physics 2022



What are the constraints?

For qubit circuits with U(1) and SU(2) symmetry, the locality of gates only restricts the relative phases between sectors with inequivalent irreducible representations of symmetry, e.g., different Hamming weights (energies) in the case of U(1) symmetry.

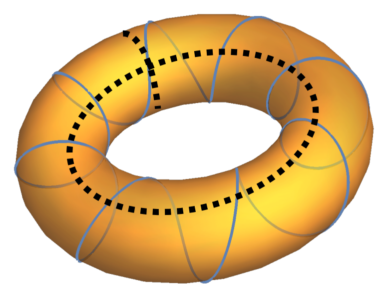
$$\begin{pmatrix} e^{i\phi_0} & & \\ & V_1 & \\ & & e^{i\phi_2} \end{pmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

Relative phases matter!

$$\mathrm{SWAP} = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

$$\mathrm{iSWAP} = \exp(\mathrm{i}\frac{\pi}{4}[X \otimes X + Y \otimes Y]) = \begin{pmatrix} 1 & & & \\ & & \mathrm{i} & \\ & \mathrm{i} & & \\ & & & 1 \end{pmatrix}$$

$$V = \bigoplus_{m=0}^{n} V_m \longrightarrow \arg(\det(V_m))$$



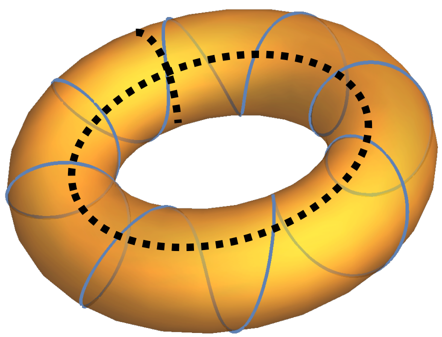
Characterizing the constraints on the relative phases

$$\begin{pmatrix} e^{i\phi_0} & & \\ & V_1 & \\ & & e^{i\phi_2} \end{pmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

U(1) Symmetry:

$$V = \bigoplus_{m=0}^{n} V_m \longrightarrow \arg(\det(V_m)) \longrightarrow \Phi_l \equiv \sum_{m=0}^{n} c_l(m) \times \arg(\det(V_m))$$

l-body phase of unitary V



$$= 0 \qquad k < l \leq n$$

- If *V* is realizable with *k*-local U(1)-invariant gates, they vanish for *k*>l.
- They are physically observable for *l*>0.

$$c_l(m) = \sum_{s=0}^{m} (-1)^s \binom{m}{s} \binom{n-m}{l-s}$$

Integer-valued polynomial of degree *l*

SU(2) Symmetry:

$$c_l(j) = \frac{1}{2^{l/2}(n-l)!} \sum_{r=0}^{l/2} \frac{(-4)^r r!(n-2r)!}{(l/2-r)!} \binom{\frac{n}{2}+j}{r} \binom{\frac{n}{2}-j}{r} \frac{\frac{n}{2}+j+1}{\frac{n}{2}+j+1-r}$$

Integer-valued polynomial of degree *l*/2 of *j*(*j*+1)

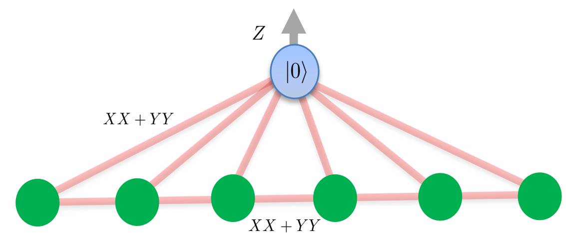
	Angular Momentum					
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$l=0$ body	1	1	1	1	1	1
$l=2$ body	-15	-11	-3	9	25	45
$l=4$ body	150	70	-42	-90	70	630
$l=6$ body	-1050	-210	462	-90	-1050	3150
$l=8$ body	4725	-315	-1323	2565	-3675	4725
$l=10$ body	-10395	3465	-2079	1485	-1155	945

n=10 qubits (spin half systems)

Circumventing the no-go theorem with ancilla qubits

U(1) Symmetry: In the case of the group U(1), this no-go theorem can be circumvented using an ancillary qubit.

$$\tilde{V}\left(|\psi\rangle \otimes |0\rangle_{\mathrm{a}}\right) = (V|\psi\rangle) \otimes |0\rangle_{\mathrm{a}}$$



Unitary $\tilde{V}$ can be implemented using 2-local U(1)-invariant unitaries.

Indeed, it can be realized with XX+YY Hamiltonian and single-qubit Pauli *Z*.

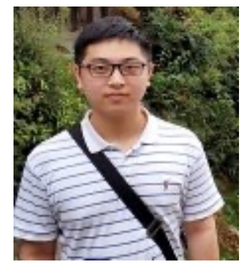
SU(2) Symmetry: Any rotationally-invariant unitary *V* on qubits can be implemented using the Heisenberg exchange interaction and 2 ancilla qubits, i.e.,

$$\widetilde{V}(|\psi\rangle \otimes |00\rangle_{ab}) = (V|\psi\rangle) \otimes |00\rangle_{ab}$$

where unitary $\tilde{V}$ can be implemented using the Heisenberg exchange interaction $X \otimes X + Y \otimes Y + Z \otimes Z$

Synthesis of Energy-Conserving Quantum Circuits with XY interaction

Ge Bai, IM, To appear in *Q. Science and Technology,*
arXiv:2309.11051 (2023)

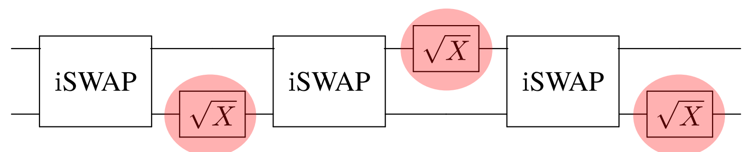


A simple example

$$\mathrm{SWAP} = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

$$\mathrm{iSWAP} = \exp(\mathrm{i}\frac{\pi}{4}[X \otimes X + Y \otimes Y]) = \begin{pmatrix} 1 & & & \\ & & \mathrm{i} & \\ & \mathrm{i} & & \\ & & & 1 \end{pmatrix}$$

Even though SWAP is energy-conserving, the standard method for implementing SWAP with XX+YY interaction requires non-energy-conserving unitaries.

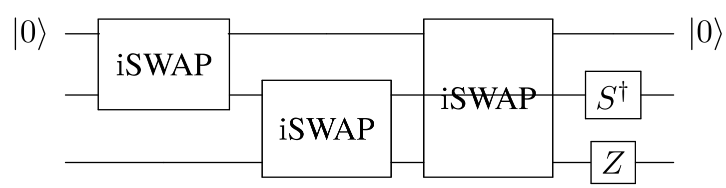


Schuch-Siewert, PRA 2003

Non-energy-conserving unitaries cannot be avoided.

IM, Nature Physics (2022)

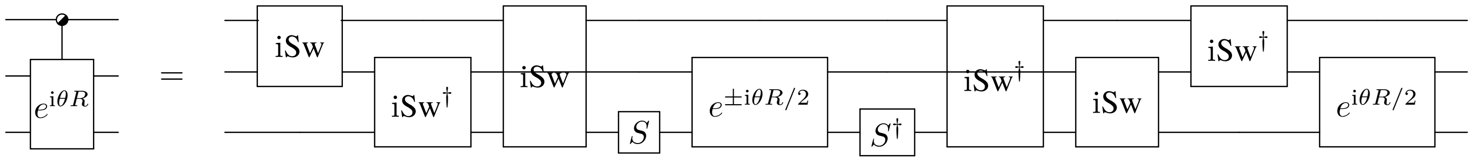
SWAP with energy-conserving gates and ancilla qubits

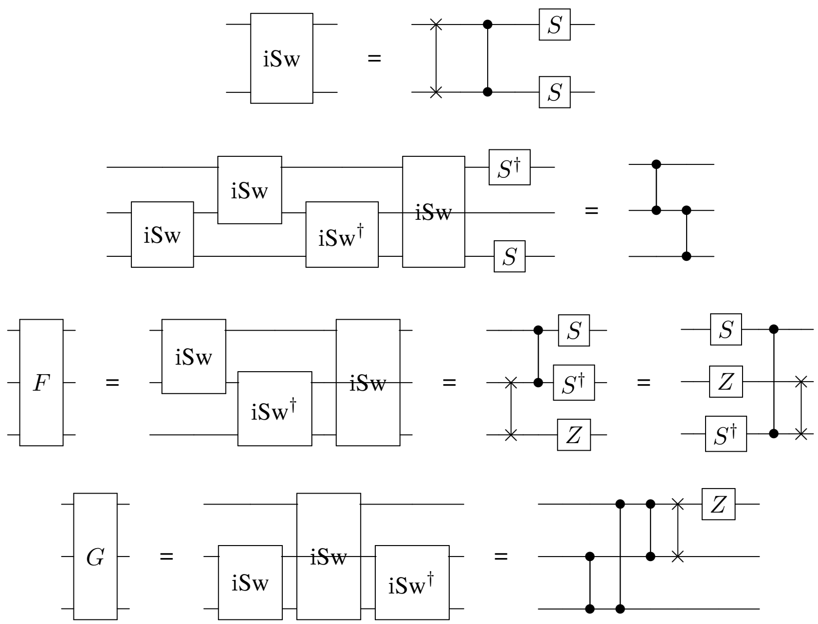


arXiv:2309.11051

Examples of useful circuit identities

$$R = (X \otimes X + Y \otimes Y)/2$$





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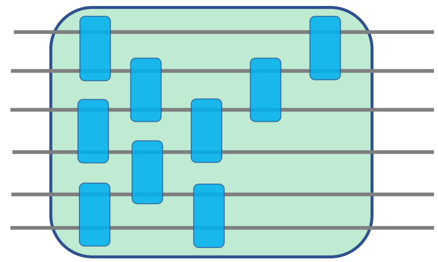
Approximate Implementation

Theorem: Any energy-conserving unitary on n>2 qubits can be realized with error $\epsilon > 0$ using $\mathcal{O}(4^n n^{3/2}(n + \log \epsilon^{-1})^{1.45})$ $\sqrt{\text{iSWAP}}$ gates alone with **two ancilla qubit.**

$$\exp(i\frac{\pi}{8}[X \otimes X + Y \otimes Y]) = \begin{pmatrix} 1 & & & \\ & \frac{1}{\sqrt{2}} & \frac{\text{i}}{\sqrt{2}} & \\ & \frac{\text{i}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ & & & 1 \end{pmatrix}$$



See arXiv:2309.11051 for explicit circuit synthesis techniques.

What about other symmetries?

Conjecture: Similar to the case of U(1) symmetry, for general symmetry groups the locality of interactions only impose constraints on the realizable relative phases between subspaces with inequivalent irreducible representation (charge) of symmetry.

No! For instance, for SU(d) symmetry with $d>2$ there are constraints even inside such subspaces.

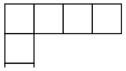
IM, Hanqing Liu, Austin Hulse, arXiv:2105.12877 (2021)

Additional constraints and new conservation laws

1) The unitary realized in a subspace with one irreducible representation (charge) of the symmetry dictates the realized unitaries in multiple other sectors with inequivalent representations of the symmetry.

2) In certain sectors, rather than all unitaries respecting the symmetry, the realizable unitaries are the symplectic or orthogonal subgroups of this group.

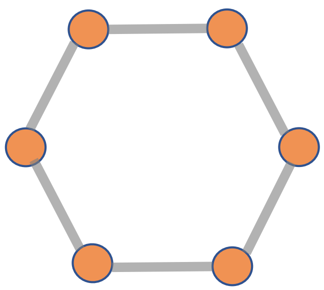
$|\psi\rangle = (|0\rangle \wedge |1\rangle \wedge |2\rangle) \otimes |0\rangle^{\otimes 3}$



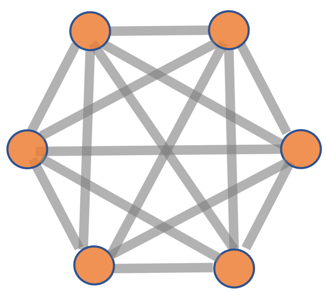
Example: 6 Qutrits with SU(3) symmetry

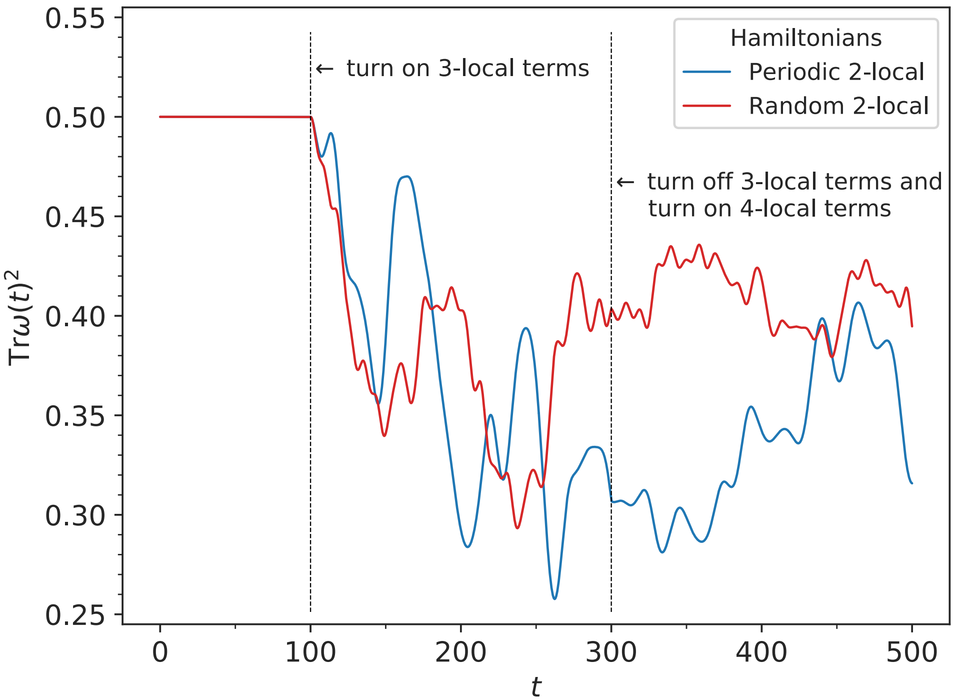
Consider a system with 6 qutrits that evolves under SU(3)-invariant Hamiltonians.

Periodic 2-local



Random 2-local





Single-particle purity as a function of time. $\mathrm{Tr}(\omega^2) = \frac{\mathrm{Tr}(\Omega^2[\psi])}{\mathrm{Tr}^2(\Omega[\psi])}$

$$\Omega_{ij}[\psi] \equiv \langle \Psi^f | c_j^\dagger c_i | \Psi^f \rangle : \quad i, j = 1, \cdots, n$$

Starting at time t = 100, we add the **3-body** term

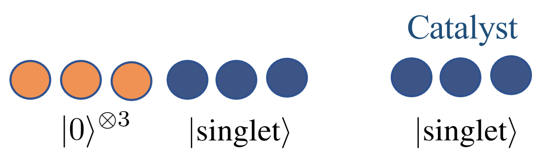
100<t<300 $\quad \mathbf{P}_{12}\mathbf{P}_{23} + \mathbf{P}_{23}\mathbf{P}_{12}$

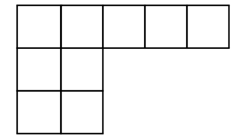
Then, at t=300 we turn off this and turn on **4-body**

300<t<500 $\quad \mathbf{P}(1234) + \mathbf{P}(4321)$

Example: 6 Qutrits with SU(3) symmetry

Total Hilbert space: $(\mathbb{C}^3)^{\otimes 6}$





$$|\text{singlet}\rangle = (|0\rangle \wedge |1\rangle \wedge |2\rangle) = \frac{1}{\sqrt{6}} \sum_{ijk=0}^{2} \epsilon_{ijk} |i\rangle|j\rangle|k\rangle \in (\mathbb{C}^3)^{\otimes 3}$$

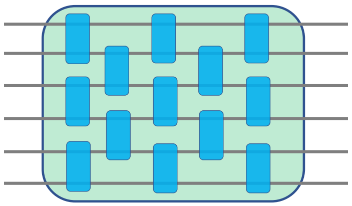
The joint state is restricted to a subspace with a single irrep of SU(3).

$$|\text{singlet}\rangle \otimes |0\rangle^{\otimes 3} \xrightarrow{\text{SU(3)-invariant unitaries}} \alpha|\text{singlet}\rangle \otimes |0\rangle^{\otimes 3} + \beta|0\rangle^{\otimes 3} \otimes |\text{singlet}\rangle$$

$$|\text{singlet}\rangle \otimes |0\rangle^{\otimes 3} \xrightarrow{\textbf{2-local}\ \text{SU(3)-invariant unitaries}} \begin{cases} |\text{singlet}\rangle \otimes |0\rangle^{\otimes 3} \\ |0\rangle^{\otimes 3} \otimes |\text{singlet}\rangle \end{cases}$$

$$\underbrace{|\text{singlet}\rangle}_{\text{Catalyst}} \otimes \Big[|\text{singlet}\rangle \otimes |0\rangle^{\otimes 3}\Big] \xrightarrow{\textbf{2-local}\ \text{SU(3)-invariant unitaries}} \Big[\alpha|\text{singlet}\rangle \otimes |0\rangle^{\otimes 3} + \beta|0\rangle^{\otimes 3} \otimes |\text{singlet}\rangle\Big] \otimes \underbrace{|\text{singlet}\rangle}_{\text{Catalyst}}$$

Random symmetric circuits: Not a 2-design



Consider the distribution of unitaries generated by circuits formed from random 2-local SU(d)-invariant unitaries.

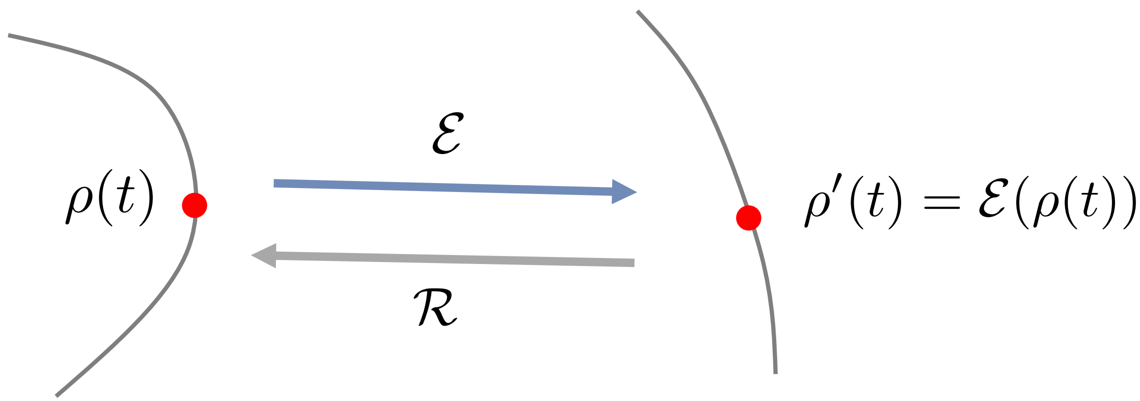
For *d*>2 circuits with random 2-local SU(*d*)-invariant unitaries are not 2-design for the Haar distribution over the group of SU(*d*)-invariant unitaries

$$\mathbb{E}_{V\sim\mu_{\text{2-loc}}}[V^{\otimes t}\otimes V^{*\otimes t}] \neq \mathbb{E}_{V\sim\mu_{\text{Haar}}}[V^{\otimes t}\otimes V^{*\otimes t}]$$

Part II. Irreversibility and Quantum Fisher Information Metrics

L Gao, H Li, IM, C Rouzé, *Sufficient statistic and recoverability via Quantum Fisher Information metrics,* arXiv:2302.02341 (2023).

Question: Can (Quantum) Fisher information always detect irreversibility?



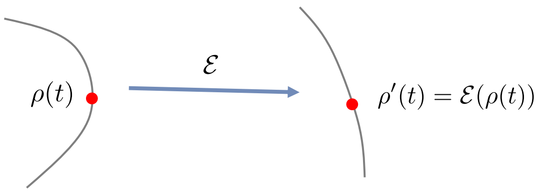
$$F(\rho(t)) \geq F(\rho'(t))$$

Decay of Quantum Fisher Information implies irreversibility.

Does the conservation of Quantum Fisher Information imply reversibility?

- Classically, yes!
- Quantum mechanically, it depends!

Classically, Fisher Information is the unique monotone Riemannian metric on the space of probability distributions.



Quantum mechanically, this is not the case!

The best known example of Quantum Fisher Information metrics is called Symmetric Logarithmic Derivative (SLD) Fisher information. It gives a tight Cramer-Rao bound, which is asymptotically achievable.

However, SLD Fisher information may remain conserved in irreversible processes.

“Regular” Fisher information metrics, on the other hand, always decreases in irreversible processes.

Example: Winger-Yanase-Dyson skew information

L. Gao, H. Li, IM, C Rouzé, arXiv:2302.02341, to appear in Comm. in Math. Physics (2024).

Example of regular QFI metrics: Winger-Yanase-Dyson skew information

For the family of states $e^{-iHt}\rho e^{iHt} \quad : t \in \mathbb{R}$

$$W_H^{(\alpha)}(\rho) = \mathrm{Tr}(\rho H^2) - \mathrm{Tr}(\rho^{\alpha} H \rho^{1-\alpha} H) \qquad : \quad 0 < \alpha < 1$$

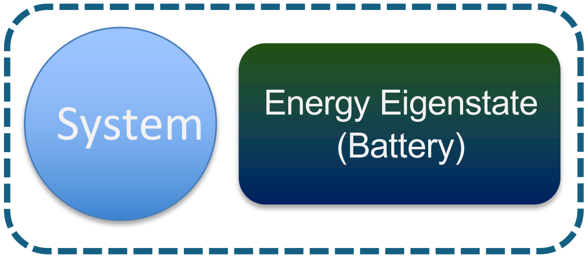
Example of measures of asymmetry.

IM, RW Spekkens, *Extending Noether's theorem by quantifying the asymmetry of quantum states,* Nature Communications 5, 3821 (2014).

R. Takagi , Skew informations from an operational view via resource theory of asymmetry (2019)

Applications: Reversibility under time-translationally-invariant operations

$$\mathcal{E}(e^{-iHt}\sigma e^{iHt}) = e^{-iHt}\mathcal{E}(\sigma)e^{iHt} \quad : \quad t \in \mathbb{R}$$



Energy-Conserving Unitary

If $\rho \xrightarrow{\mathrm{TI}} \rho'$ then $W^{(\alpha)}(\rho) \geq W^{(\alpha)}(\rho')$

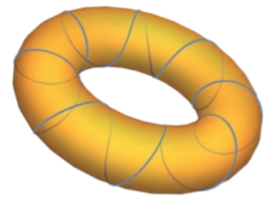
Furthermore, if the equality holds, then the process is reversible via TI operations, i.e., $\rho' \xrightarrow{\mathrm{TI}} \rho$

$$W_H^{(\alpha)}(\rho) = \mathrm{Tr}(\rho H^2) - \mathrm{Tr}(\rho^\alpha H \rho^{1-\alpha} H)$$

Applications of SLD and RLD QFI in the resource theories of Quantum Thermodynamics and Asymmetry

IM, Nature communications 11 (1), 25 (2020).
IM, PRL 129, 190502 (2022).
H. Tajima, N. Shiraishi, and K. Saito, PHYS REV. RES. (2020)

Thank you for your attention.



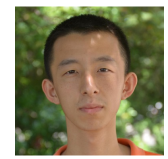
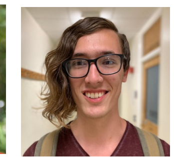
Summary of Part I

- **A general no-go theorem**

  In the case of continuous symmetries, it is not possible to implement generic symmetric unitaries, even approximately, using 2-local (k-local) symmetric unitaries on the subsystems. [IM, Nature Physics 2022].

- **Studied examples**
  - U(1) symmetry [IM, Nature Physics 2022]
  - SU(2) symmetry with qubits [IM, Hanqing Liu, Austin Hulse, PRL 2024]
  - SU(d) symmetry with qudits [IM, Hanqing Liu, Austin Hulse, arXiv:2105.12877 (2021)]

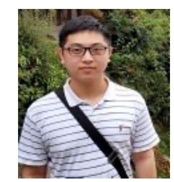
    New conservation laws for d>2

Conclusion: The restrictions imposed by locality and symmetry vary significantly across different symmetry groups.

- **Theory of Abelian Quantum Circuits** [IM, arXiv:2302.12466, To appear in Phys Rev. Research (2024)]
- **Synthesis of Energy-Conserving Quantum Circuits with XY interaction**

  [Ge Bai, IM, arXiv:2309.11051, To appear in Quantum Science and Technology (2024)]



Summary of Part II

The most popular Quantum Fisher Information metric, i.e., SLD QFI, cannot always detect irreversibility.
There exists a family of QFI metrics, including skew information, that can detect irreversibility.

L Gao, H Li, IM, C Rouzé, To appear in Comm in Mathematical Physics, arXiv:2302.02341 (2023). Regular QFI
IM, Nature communications 11 (1), 25 (2020). RLD QFI
IM, PRL 129, 190502 (2022). SLD QFI